Chapter 1

GEOMETRY OF DEDUCTION VIA GRAPHS OF PROOFS

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Abstract We are here concerned with the study of proofs from a geometric perspective. By first recalling the pioneering work of Statman in his doctoral thesis *Structural Complexity of Proofs* (1974), we review two recent research programmes which approach the study of structural properties of formal proofs from a geometric perspective: (i) the notion of *proof-net*, given by Girard in 1987 in the context of linear logic; and (ii) the notion of *logical flow graph* given by Buss in 1991 and used as a tool for studying the exponential blow up of proof sizes caused by the cut-elimination process, a recent programme (1996–2000) proposed by Carbone in collaboration with Semmes.

Statman's geometric perspective does not seem to have developed much further than his doctoral thesis, but the fact is that it looks as if the main idea, *i.e.* extracting structural properties of proofs in natural deduction (ND) using appropriate geometric intuitions, offers itself as a very promising one. With this in mind, and having at our disposal some interesting and rather novel techniques developed for *proof-nets* and *logical flow graphs*, we have tried to focus our investigation on a research for an alternative proposal for looking at the geometry of ND systems. The lack of symmetry in ND presents a challenge for such a kind of study. Of course, the obvious alternative is to look at multiple-conclusion

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calculi. We already have in the literature different approaches involving such calculi. For example, Kneale's (1958) *tables of development* (studied in depth by Shoesmith & Smiley (1978)) and Ungar's (1992) multiple-conclusion ND.

After surveying the main research programmes, we sketch a proposal which is similar to both Kneale's and Ungar's in various aspects, mainly in the presentation of a multiple conclusion calculus in ND style. Rather than just presenting yet another ND proof system, we emphasise the use of 'modern' graph-theoretic techniques in tackling the 'old' problem of adequacy of multiple-conclusion ND. Some of the techniques have been developed for *proof-nets* (e.g. splitting theorem, soundness criteria, sequentialisation), and have proved themselves rather elegant and useful indeed.

Keywords: proofs as graphs, natural deduction, multiple-conclusion, geometry of deduction

1. Motivation

In 1980's various studies in "Logic and Computation" were pursued with the intention of giving a logical treatment of computer programming issues. Some of these studies have brought in a number of interesting proof-theoretic developments, such as for example:

- the functional interpretation of logical connectives[1] via deductive systems which use some sort of labelling mechanism:

  (i) Martin-Löf's *Intuitionistic Type theory* [53], which contributed to a better understanding of the foundations of computer science from a type-theoretic perspective, drawing on the connections between constructive mathematics and computer programming;

  and

  (ii) the *Labelled Deductive Systems*, introduced by Gabbay [34], which, arising from the need of computer science applications to handle "meta-level" aspects of logical system in harmony with object-level, helped providing a more general alternative to the "formulae-as-types" paradigm;

- *Linear Logic*, introduced by Girard in [38]. Since then it has become very popular in the theoretical computer science research community. The novelty here is that the logic comes with new connectives forming a new logical system with various interesting features for computer science, such as the possibility of interpreting a sequent as the state of a system and the treatment of a formula as a resource.

In recent years, linear logic has been established as one of the most widely used formalisms for the study of the interface between logic and computation. One of its key aspects represents a rather interesting novelty for studying the geometry of deductions: the concept of *proof-nets*. The theory of proof-nets developed out of a comparison between the sequent calculus and natural deduction (ND) Gentzen systems [36] as well as from an analysis of the importance of studying the structural properties of proofs through a *geometrical* perspective.

Another recent work which also presents a *geometrical* analysis in the study of structural properties of proofs has been developed by Carbone in collaboration with Semmes [14, 15, 16, 17, 18, 22]. Again in the context of "Logic and Computation", the analysis of Carbone and Semmes is motivated by questions which involve the middle ground between mathematical logic and computational complexity. In the beginning of the 1970's, Cook used the notion of satisfiability (a concept from logic) to study one of the most fundamental dichotomies in theoretical computer science: **P** versus **NP**. By the end of the decade Cook and Reckhow had established an important observation which puts emphasis on a relevant direction in complexity theory: **NP** is closed under complementation iff there is a propositional proof system in which all tautologies have a polynomial size proof [27]. This represents an important result linking mathematical logic and computational complexity since it relates classes of computational problems with proof systems. Motivated by questions such as the length of proofs in certain classical proof systems (in the style of Gentzen sequent calculus), Carbone set out to study the phenomenon of expansion of proofs, and for this purpose she found in concept of *logical flow graphs*, introduced by S. Buss [13], a rather convenient mathematical tool. Using the notion of *logical flow graph*, Carbone was able to obtain results such as, for example, providing an explanation for the exponential blow up of proof sizes caused by the cut-elimination process. With appropriate geometrical intuitions associated with the concept of *logical flow graph*, Carbone and Semmes developed a combinatorial model to study the evolution of graphs underlying proofs during the process of cut-elimination.

Now, if on the one hand we have

- Girard's proposal of studying the geometry of deductions through the concept of *proof-nets*, (in [40] he presents various arguments in defense of his programme, emphasizing the importance of "finding out the geometrical meaning of the *Hauptsatz*, *i.e.* what is hidden behind the somewhat boring syntactical manipulations it involves"),

on the other hand, there is

- Carbone's systematic use of *logical flow graph* in a geometrical study of the cut-elimination process, yielding a combinatorial model which uncovers the exponential expansion of proofs after cut-elimination.[2]

Although with different ends and means these two works concern the study of structural features of proofs by a geometric perspective. Back in the 1970's

Chapter 2

CHU'S CONSTRUCTION: A PROOF-THEORETIC APPROACH

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Abstract The essential interaction between classical and intuitionistic features in the system of linear logic is best described in the language of category theory. Given a symmetric monoidal closed category $\mathcal{C}$ with products, the category $\mathcal{C} \times \mathcal{C}^{op}$ can be given the structure of a $*$-autonomous category by a special case of the Chu construction. The main result of the paper is to show that the intuitionistic translations induced by Girard's trips determine the functor from the free $*$-autonomous category $\mathcal{A}$ on a set of atoms $\{P, P', \ldots\}$ to $\mathcal{C} \times \mathcal{C}^{op}$, where $\mathcal{C}$ is the free monoidal closed category with products and coproducts on the set of atoms $\{P_O, P_I, P'_O, P'_I, \ldots\}$ (a pair P_O, P_I in $\mathcal{C}$ for each atom P of $\mathcal{A}$).

Keywords: Chu spaces, proof-nets, linear logic

1. Preface

An essential aim of linear logic [16] is the study of the dynamics of proofs, essentially normalization (cut elimination), in a system enjoying the good proof-theoretic properties of *intuitionistic* logic, but where the dualities of *classical* logic hold. Indeed *classical linear logic* **CLL** has a denotational semantics and a game-theoretic semantics; proofs are formalized in a sequent calculus, but also in a system of *proof-nets* and in the latter representation cut elimination not only has the strong normalizability property, but is also confluent. Although Girard's main system of linear logic is *classical*, considerable attention in the literature has also been given to the system of *intuitionistic linear logic*

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ILL, where proofs are also formalized in a sequent calculus and in a natural deduction system. A better understanding of the relations between **CLL** and **ILL** is one of the goals to which the present work is intended as a contribution.

The fact that intuitionistic logic plays an important role in the architecture of linear logic is not surprising: as indicated in the introductory section of Girard's fundamental paper [16], a main source of inspiration for the system was its denotational semantics of coherent spaces, a refinement of Scott's semantics for the λ-calculus. Fundamental decisions about the system **CLL** were made so that **CLL** has a semantics of proofs in coherent spaces in the same way as intuitionistic logic has a semantics of proofs in Scott's domains. But linear logic is not just a refinement of intuitionistic logic, such as **ILL**: there are expectations that **CLL** may tell us something fundamental about classical logic as well, indeed, that through linear logic a deep level of analysis may have been reached from which the "unity of logic" can be appreciated [17]. Therefore the relations between classical and intuitionistic components of linear logic deserve careful investigation.

A natural points of view to look at this issue is *categorical logic*. It has been known for years that monoidal closed categories provide a model for *intuitionistic linear logic*, though a fully adequate formulation of the syntax and of the categorical semantics of **ILL** especially with respect to the exponentials, has required considerable subtlety and effort [4, 5, 6]. It is also well known that $*$-autonomous categories give a model for *classical linear logic* [3]. The appendix to [2] provides a method, due to Barr's student Chu, to construct $*$-autonomous categories starting from monoidal closed ones.

In our proof-theoretic investigation we encounter a special case of Chu's construction, namely $\mathbf{Chu}(\mathcal{C}, \top)$ where $\mathcal{C}$ is a symmetric monoidal closed category with terminal object $\top$. More specifically, given the free $*$-autonomous category $\mathcal{A}$ on a set of objects (propositional variables) $\{P, P', \ldots\}$ and given the symmetric monoidal closed category $\mathcal{C}$ with products, free on the set $\{P_O, P_I, P'_O, P'_I, \ldots\}$ (a pair P_O, P_I in $\mathcal{C}$ for each atom P of $\mathcal{A}$), the category $\mathcal{C} \times \mathcal{C}^{op}$ can be given the structure of a $*$-autonomous category by Chu's construction. Indeed, since the dualizing object is the terminal object, $\mathbf{Chu}(\mathcal{C}, \top)$ is just $\mathcal{C} \times \mathcal{C}^{op}$ and the pullback needed to internalize the homsets is in fact a product. Here the tensor product $(X, X^{op}) \otimes (Y, Y^{op})$ must be an object of the form $(X \otimes Y, (X \multimap Y^{op}) \times (Y \multimap X^{op}))$ and the identity of the tensor must be $(\mathbf{1}, \top)$. Dually, the *par* $(X, X^{op}) \wp (Y, Y^{op})$ is defined as $((X^{op} \multimap Y) \times (Y^{op} \multimap X), X^{op} \otimes Y^{op})$ and the identity of the *par* must be $(\top, \mathbf{1})$. Now since $\mathcal{A}$ is free, there is a functor F of $*$-autonomous categories from $\mathcal{A}$ to $(\mathcal{C} \times \mathcal{C}^{op})$ taking P to (P_O, P_I). This is well-known, but so far no familiar construction had been shown to correspond to the functor F given by the abstract theory. The main contribution of this paper is to show that a familiar proof-theoretic construction, namely *Girard's trips* [16] on a proof-net, represent the action of such a functor on the morphisms of $\mathcal{A}$. Of course one could state the same result using Danos–Regnier graphs, as it was done in [8], but as we shall see a simpler definition of orientations is possible in terms of Girard's trips.

The key idea is simple enough and may be illustrated as a logical translation of formulas and proofs in **CMALL** into formulas and proofs in **IMALL**. In the translation a **CMALL** sequent S: $\vdash \Gamma, A$ becomes *polarized*: a selected formula-occurrence A is mapped to a *positive* formula-occurrence A_O in the succedent of an intuitionistic sequent S' (the *output* part of a logical computation); all other formula-occurrences C in Γ are mapped to *negative* C_I in the antecedent of S' (the *input* part). The polarized occurrences of an atom A become A_O, A_I, just two copies of A. Negation changes the polarity. For other complex polarized formulas, the polarization of the immediate subformulas is uniquely determined – for instance, $(A \wp B)_I$ becomes $A_I \otimes B_I$ – except in the cases of $(A \wp B)_O$ and $(A \otimes B)_I$. In these cases we take the product (logically, the *with*) of two possible choices (the "switches" in a proof-net): for instance, $(A \wp B)_O$ is encoded as $(A_I \multimap B_O) \& (B_I \multimap A_O)$. The intuitive motivation is clear: $A \wp B$ has a reading simultaneously as the internalization of the function space $\mathrm{Hom}_{\mathcal{A}}(A^\perp, B)$ and of the function space $\mathrm{Hom}_{\mathcal{A}}(B^\perp, A)$. The fact that the translation is functorial here means, roughly, that it is defined independently on the formulas (objects) and on the proofs (morphisms) and that it admits the rule of Cut (composition of morphisms); it is also compatible with cut-elimination. In this form the result can be easily proved within the formalisms of the sequent calculi for **CMALL** and **IMALL**. However, when we ask questions about the *faithfulness* and *fullness* of such a functor, thus also asking questions about the identity of proofs in linear logic, we find it convenient to consider the more refined syntax of proof-nets.

On the other hand, proof-nets are also useful to highlight the geometric aspect of certain logical properties; indeed ideas related to the present result have already proved quite useful in the study of what is sometimes called the *géometrie du calcul* (*geometry of computations*). Our own investigation has been motivated by the desire to understand and clarify the notion of a proof-net and the present result appears to reward many collective efforts in this direction. Given a proof-structure, i.e., a directed graph where edges are labeled with formulas, a *correctness criterion* characterizes those proof-structures which correspond to proofs in the sequent calculus. Girard's original condition (*"there are no short trips"*) [16] is *exponential* in time on the size of the proof-structure, but other *quadratic* criteria were found soon after (among others, one was given in [7]). Thus it is natural to ask *what additional information is contained in the construction of Girard's trips other than the correctness*

Chapter 3

TWO PARADIGMS OF LOGICAL COMPUTATION IN AFFINE LOGIC?

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Abstract We propose a notion of *symmetric reduction* for a system of proof-nets for *Multiplicative Affine Logic with Mix* (**MAL** + Mix) (namely, multiplicative linear logic with the mix-rule the unrestricted weakening-rule). We prove that such a reduction has the strong normalization and Church–Rosser properties. A notion of irrelevance in a proof-net is defined and the *possibility* of cancelling the irrelevant parts of a proof-net without erasing the entire net is taken as one of the *correctness conditions*; therefore purely *local* cut-reductions are given, minimizing cancellation and suggesting a paradigm of *"computation without garbage collection"*. Reconsidering Ketonen and Weyhrauch's decision procedure for affine logic [15, 4], the use of the mix-rule is related to the non-determinism of classical proof-theory. The question arises, whether these features of classical cut-elimination are really irreducible to the familiar paradigm of cut-elimination for intuitionistic and linear logic.

Keywords: affine logic, proof-nets

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1. Introduction

1. *Classical Multiplicative Affine Logic* is classical multiplicative linear logic with the unrestricted rule of *weakening*, but without the rule of *contraction*. Classical affine logic is a much simpler system than classical logic, but it provides similar challenges for *logical computation*, both in the sense of *proof-search* and of *proof normalization* (or *cut-elimination*). For instance, the problem of *confluence* of cut-elimination (the *Church–Rosser property*) is already present in affine logic, but here we do not have the problem of *non-termination*. Affine logic is also simpler than linear logic from the point of view of proof-search: e.g., propositional linear logic is undecidable, yet becomes decidable when the unrestricted rule of weakening is added. Provability in *constant-only multiplicative linear logic* is NP-complete, yet it is decidable in linear time for *constant-only multiplicative affine logic*, as it is shown below.

The tool we will use here, proof-nets for affine logic, is older than the notion of a proof-net for linear logic. In a 1984 paper [15], J. Ketonen and R. Weyhrauch presented a decision procedure for first-order affine logic (called then *direct logic*) which essentially consists in building cut-free proof-nets, using the unification algorithm to determine the axioms. The 1984 paper is sketchy and it has been corrected (see [3, 4], where the relation between the decision procedure and proof-nets for **MLL**$^-$ are discussed), but it contains the main ideas exploited in the present paper, namely, the construction of proof-nets *free from irrelevance* through *basic chains*. Yet neither the 1984 paper nor its 1992 revisitation contained a treatment of cut-elimination.[1]

2. The problem of non-confluence for classical affine logic is non-trivial: the following well-known example (given in Lafont's Appendix to [14]) reminds us that the Church–Rosser property is non-deterministic under the familiar *asymmetric* cut-reductions.

Example 1

$$\dfrac{\dfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma \end{matrix}}{\vdash \Gamma, A}\,w_1 \qquad \dfrac{\begin{matrix} d_2 \\ \vdots \\ \vdash \Delta \end{matrix}}{\vdash \Delta, \neg A}\,w_2}{\vdash \Gamma, \Delta} \quad \text{reduces to} \quad \dfrac{\dfrac{\begin{matrix} d_1 \\ \vdots \\ \vdash \Gamma \end{matrix}}{\mathit{weakenings}}}{\vdash \Gamma, \Delta} \quad \text{or to} \quad \dfrac{\dfrac{\begin{matrix} d_2 \\ \vdots \\ \vdash \Delta \end{matrix}}{\mathit{weakenings}}}{\vdash \Gamma, \Delta}$$

Asymmetric reductions.

Indeed classical logic gives no justification for choosing between the two indicated reductions, the first commuting the cut-rule with the *left* application of the weakening-rule ("pushing d_2 up into d_1", thus erasing d_2), the second commuting the cut-rule with the *right* application of the weakening-rule ("pushing d_1 up into d_2", thus erasing d_1). Therefore the cut-elimination process in **MAL**, *a fortiori* in **LK**, is non-deterministic and non-confluent.

Compare this with normalization in intuitionistic logic. In the typed λ calculus a *cut / left weakening* pair corresponds to substitution of $t : A$ for a variable $x : A$ which does not occur in $u : B$; such a substitution is unambiguously defined as $u[t/x] = u$. Moreover in Prawitz's natural deduction **NJ** [19] the rule corresponding to *weakening-right* is the rule *"ex falso quodlibet"* and the normalization step for such a rule involves a form of η-expansion:

$$\dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{A \wedge B} \quad \textit{reduces to} \quad \dfrac{\dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{A} \quad \dfrac{\begin{array}{c} d \\ \vdots \\ \bot \end{array}}{B}}{A \wedge B}$$

Such a reduction does *not* yield cancellation. Thus the cut-elimination procedure for the intuitionistic sequent calculus **LJ** inherits one sensible reduction strategy from natural deduction: *"push the left derivation up into the right one"*. In the case of a *weakening / cut* pair it is always the *left* dedution to be erased.

3. Here we are interested in exploring an obvious remark: for classical logic in addition to the *asymmetric* reductions of Example 1, there is a *symmetric* possibility, the "*Mix*" of d_1 and d_2.

Example 1 cont.

$$\dfrac{\dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array}}{\vdash \Gamma, A}\, w_1 \quad \dfrac{\begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\vdash \Delta, \neg A}\, w_2}{\vdash \Gamma, \Delta} \quad \text{reduces to} \quad \dfrac{\begin{array}{c} d_1 \\ \vdots \\ \vdash \Gamma \end{array} \quad \begin{array}{c} d_2 \\ \vdots \\ \vdash \Delta \end{array}}{\vdash \Gamma, \Delta}\, Mix$$

Symmetric reduction.

Instead of choosing a direction where to "push up" the cut-rule, we do both asymmetric reductions, using the mix-rule.

The idea is loosely related to a procedure well-known in the literature for the case when both cut-formulas result from a contraction-rule, with the name *cross-cut reduction.* Let d_1 and d_2 be derivations of the left and right premises of the cut-rule: