

1 Introduction

1.1 Problem Statements

In this book, we consider two kinds of dynamic optimization problems: optimal control problems and differential game problems.

In an optimal control problem for a dynamic system, the task is finding an admissible control trajectory $u : [t_a, t_b] \rightarrow \Omega \subseteq R^m$ generating the corresponding state trajectory $x : [t_a, t_b] \rightarrow R^n$ such that the cost functional $J(u)$ is minimized.

In a zero-sum differential game problem, one player chooses the admissible control trajectory $u : [t_a, t_b] \rightarrow \Omega_u \subseteq R^{m_u}$ and another player chooses the admissible control trajectory $v : [t_a, t_b] \rightarrow \Omega_v \subseteq R^{m_v}$. These choices generate the corresponding state trajectory $x : [t_a, t_b] \rightarrow R^n$. The player choosing u wants to minimize the cost functional $J(u, v)$, while the player choosing v wants to maximize the same cost functional.

1.1.1 The Optimal Control Problem

We only consider optimal control problems where the initial time t_a and the initial state $x(t_a) = x_a$ are specified. Hence, the most general optimal control problem can be formulated as follows:

Optimal Control Problem:

Find an admissible optimal control $u : [t_a, t_b] \rightarrow \Omega \subseteq R^m$ such that the dynamic system described by the differential equation

$$\dot{x}(t) = f(x(t), u(t), t)$$

is transferred from the initial state

$$x(t_a) = x_a$$

into an admissible final state

$$x(t_b) \in S \subseteq R^n ,$$

and such that the corresponding state trajectory $x(\cdot)$ satisfies the state constraint

$$x(t) \in \Omega_x(t) \subseteq R^n$$

at all times $t \in [t_a, t_b]$, and such that the cost functional

$$J(u) = K(x(t_b), t_b) + \int_{t_a}^{t_b} L(x(t), u(t), t) dt$$

is minimized.

Remarks:

- 1) Depending upon the type of the optimal control problem, the final time t_b is fixed or free (i.e., to be optimized).
- 2) If there is a nontrivial control constraint (i.e., $\Omega \neq R^m$), the admissible set $\Omega \subset R^m$ is time-invariant, closed, and convex.
- 3) If there is a nontrivial state constraint (i.e., $\Omega_x(t) \neq R^n$), the admissible set $\Omega_x(t) \subset R^n$ is closed and convex at all times $t \in [t_a, t_b]$.
- 4) Differentiability: The functions f , K , and L are assumed to be at least once continuously differentiable with respect to all of their arguments.

1.1.2 The Differential Game Problem

We only consider zero-sum differential game problems, where the initial time t_a and the initial state $x(t_a) = x_a$ are specified and where there is no state constraint. Hence, the most general zero-sum differential game problem can be formulated as follows:

Differential Game Problem:

Find admissible optimal controls $u : [t_a, t_b] \rightarrow \Omega_u \subseteq R^{m_u}$ and $v : [t_a, t_b] \rightarrow \Omega_v \subseteq R^{m_v}$ such that the dynamic system described by the differential equation

$$\dot{x}(t) = f(x(t), u(t), v(t), t)$$

is transferred from the initial state

$$x(t_a) = x_a$$

to an admissible final state

$$x(t_b) \in S \subseteq R^n$$

and such that the cost functional

$$J(u) = K(x(t_b), t_b) + \int_{t_a}^{t_b} L(x(t), u(t), v(t), t) dt$$

is minimized with respect to u and maximized with respect to v .

Remarks:

- 1) Depending upon the type of the differential game problem, the final time t_b is fixed or free (i.e., to be optimized).
- 2) Depending upon the type of the differential game problem, it is specified whether the players are restricted to open-loop controls $u(t)$ and $v(t)$ or are allowed to use state-feedback controls $u(x(t), t)$ and $v(x(t), t)$.
- 3) If there are nontrivial control constraints, the admissible sets $\Omega_u \subset R^{m_u}$ and $\Omega_v \subset R^{m_v}$ are time-invariant, closed, and convex.
- 4) Differentiability: The functions f , K , and L are assumed to be at least once continuously differentiable with respect to all of their arguments.

1.2 Examples

In this section, several optimal control problems and differential game problems are sketched. The reader is encouraged to wonder about the following questions for each of the problems:

- Existence: Does the problem have an optimal solution?
- Uniqueness: Is the optimal solution unique?
- What are the main features of the optimal solution?
- Is it possible to obtain the optimal solution in the form of a state feedback control rather than as an open-loop control?

Problem 1: Time-optimal, friction-less, horizontal motion of a mass point

State variables:

$$\begin{aligned}x_1 &= \text{position} \\x_2 &= \text{velocity}\end{aligned}$$

control variable:

$$u = \text{acceleration}$$

subject to the constraint

$$u \in \Omega = [-a_{\max}, +a_{\max}] .$$

Find a piecewise continuous acceleration $u : [0, t_b] \rightarrow \Omega$, such that the dynamic system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

is transferred from the initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} s_a \\ v_a \end{bmatrix}$$

to the final state

$$\begin{bmatrix} x_1(t_b) \\ x_2(t_b) \end{bmatrix} = \begin{bmatrix} s_b \\ v_b \end{bmatrix}$$

in minimal time, i.e., such that the cost criterion

$$J(u) = t_b = \int_0^{t_b} dt$$

is minimized.

Remark: $s_a, v_a, s_b, v_b,$ and a_{\max} are fixed.

For obvious reasons, this problem is often named “time-optimal control of the double integrator”. It is analyzed in detail in Chapter 2.1.4.

Problem 2: Time-optimal, horizontal motion of a mass with viscous friction

This problem is almost identical to Problem 1, except that the motion is no longer frictionless. Rather, there is a friction force which is proportional to the velocity of the mass.

Thus, the equation of motion (with $c > 0$) now is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -c \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) .$$

Again, find a piecewise continuous acceleration $u : [0, t_b] \rightarrow [-a_{\max}, a_{\max}]$ such that the dynamic system is transferred from the given initial state to the required final state in minimal time.

In contrast to Problem 1, this problem may fail to have an optimal solution. Example: Starting from stand-still with $v_a = 0$, a final velocity $|v_b| > a_{\max}/c$ cannot be reached.

Problem 3: Fuel-optimal, friction-less, horizontal motion of a mass point

State variables:

$$\begin{aligned} x_1 &= \text{position} \\ x_2 &= \text{velocity} \end{aligned}$$

control variable:

$$u = \text{acceleration}$$

subject to the constraint

$$u \in \Omega = [-a_{\max}, +a_{\max}] .$$

Find a piecewise continuous acceleration $u : [0, t_b] \rightarrow \Omega$, such that the dynamic system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

is transferred from the initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} s_a \\ v_a \end{bmatrix}$$

to the final state

$$\begin{bmatrix} x_1(t_b) \\ x_2(t_b) \end{bmatrix} = \begin{bmatrix} s_b \\ v_b \end{bmatrix}$$

and such that the cost criterion

$$J(u) = \int_0^{t_b} |u(t)| dt$$

is minimized.

Remark: $s_a, v_a, s_b, v_b, a_{\max}$, and t_b are fixed.

This problem is often named “fuel-optimal control of the double integrator”. The notion of fuel-optimality associated with this type of cost functional relates to the physical fact that in a rocket engine, the thrust produced by the engine is proportional to the rate of mass flow out of the exhaust nozzle. However, in this simple problem statement, the change of the total mass over time is neglected. — This problem is analyzed in detail in Chapter 2.1.5.

Problem 4: Fuel-optimal horizontal motion of a rocket

In this problem, the horizontal motion of a rocket is modeled in a more realistic way: Both the aerodynamic drag and the loss of mass due to thrusting are taken into consideration. State variables:

x_1 = position

x_2 = velocity

x_3 = mass

control variable:

u = thrust force delivered by the engine

subject to the constraint

$$u \in \Omega = [0, F_{\max}] .$$

The goal is minimizing the fuel consumption for a required mission, or equivalently, maximizing the mass of the rocket at the final time.

Thus, the optimal control problem can be formulated as follows:

Find a piecewise continuous thrust $u : [0, t_b] \rightarrow [0, F_{\max}]$ of the engine such that the dynamic system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{1}{x_3(t)} \{u(t) - \frac{1}{2}A\rho c_w x_2^2(t)\} \\ -\alpha u(t) \end{bmatrix}$$

is transferred from the initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} s_a \\ v_a \\ m_a \end{bmatrix}$$

to the (incompletely specified) final state

$$\begin{bmatrix} x_1(t_b) \\ x_2(t_b) \\ x_3(t_b) \end{bmatrix} = \begin{bmatrix} s_b \\ v_b \\ \text{free} \end{bmatrix}$$

and such that the equivalent cost functionals $J_1(u)$ and $J_2(u)$ are minimized:

$$J_1(u) = \int_0^{t_b} u(t) dt$$

$$J_2(u) = -x_3(t_b) .$$

Remark: $s_a, v_a, m_a, s_b, v_b, F_{\max}$, and t_b are fixed.

This problem is analyzed in detail in Chapter 2.6.3.

Problem 5: The LQ regulator problem

Find an unconstrained control $u : [t_a, t_b] \rightarrow R^m$ such that the linear time-varying dynamic system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

is transferred from the initial state

$$x(t_a) = x_a$$

to an arbitrary final state

$$x(t_b) \in R^n$$

and such that the quadratic cost functional

$$J(u) = \frac{1}{2}x^\top(t_b)Fx(t_b) + \frac{1}{2} \int_{t_a}^{t_b} \left(x^\top(t)Q(t)x(t) + u^\top(t)R(t)u(t) \right) dt$$

is minimized.

Remarks:

- 1) The final time t_b is fixed. The matrices F and $Q(t)$ are symmetric and positive-semidefinite and the matrix $R(t)$ is symmetric and positive-definite.
- 2) Since the cost functional is quadratic and the constraints are linear, automatically a linear solution results, i.e., the result will be a linear state feedback controller of the form $u(t) = -G(t)x(t)$ with the optimal time-varying controller gain matrix $G(t)$.
- 3) Usually, the LQ regulator is used in order to robustly stabilize a nonlinear dynamic system around a nominal trajectory:

Consider a nonlinear dynamic system for which a nominal trajectory has been designed for the time interval $[t_a, t_b]$:

$$\begin{aligned} \dot{X}_{\text{nom}}(t) &= f(X_{\text{nom}}(t), U_{\text{nom}}(t), t) \\ X_{\text{nom}}(t_a) &= X_a . \end{aligned}$$

In reality, the true state vector $X(t)$ will deviate from the nominal state vector $X_{\text{nom}}(t)$ due to unknown disturbances influencing the dynamic system. This can be described by

$$X(t) = X_{\text{nom}}(t) + x(t) ,$$

where $x(t)$ denotes the state error which should be kept small by hopefully small control corrections $u(t)$, resulting in the control vector

$$U(t) = U_{\text{nom}}(t) + u(t) .$$

If indeed the errors $x(t)$ and the control corrections can be kept small, the stabilizing controller can be designed by linearizing the nonlinear system around the nominal trajectory.

This leads to the LQ regulator problem which has been stated above. — The penalty matrices $Q(t)$ and $R(t)$ are used for shaping the compromise between keeping the state errors $x(t)$ and the control corrections $u(t)$, respectively, small during the whole mission. The penalty matrix F is an additional tool for influencing the state error at the final time t_b .

The LQ regulator problem is analyzed in Chapters 2.3.4 and 3.2.3. — For further details, the reader is referred to [1], [2], [16], and [25].

Problem 6: Goh's fishing problem

In the following simple economic problem, consider the number of fish $x(t)$ in an ocean and the catching rate of the fishing fleet $u(t)$ of catching fish per unit of time, which is limited by a maximal capacity, i.e., $0 \leq u(t) \leq U$. The goal is maximizing the total catch over a fixed time interval $[0, t_b]$.

The following reasonably realistic optimal control problem can be formulated:

Find a piecewise continuous catching rate $u : [0, t_b] \rightarrow [0, U]$, such that the fish population in the ocean satisfying the population dynamics

$$\dot{x}(t) = a\left(x(t) - \frac{x^2(t)}{b}\right) - u(t)$$

with the initial state

$$x(0) = x_a$$

and with the obvious state constraint

$$x(t) \geq 0 \quad \text{for all } t \in [0, t_b]$$

is brought up or down to an arbitrary final state

$$x(t_b) \geq 0$$

and such that the total catch is maximized, i.e., such that the cost functional

$$J(u) = - \int_0^{t_b} u(t) dt$$

is minimized.

Remarks:

- 1) $a > 0$, $b > 0$; x_a , t_b , and U are fixed.
- 2) This problem nicely reveals that the solution of an optimal control problem always is "as bad" as the considered formulation of the optimal control problem. This optimal control problem lacks any sustainability aspect: Obviously, the fish will be extinct at the final time t_b , if this is feasible. (Think of whaling or raiding in business economics.)
- 3) This problem has been proposed (and solved) in [18]. An even more interesting extended problem has been treated in [19], where there is a predator-prey constellation involving fish and sea otters. The competing sea otters must not be hunted because they are protected by law.

Goh's fishing problem is analyzed in Chapter 2.6.2.

Problem 7: Slender beam with minimal weight

A slender horizontal beam of length L is rigidly clamped at the left end and free at the right end. There, it is loaded by a vertical force F . Its cross-section is rectangular with constant width b and variable height $h(\ell)$; $h(\ell) \geq 0$ for $0 \leq \ell \leq L$. Design the variable height of the beam, such that the vertical deflection $s(\ell)$ of the flexible beam at the right end is limited by $s(L) \leq \varepsilon$ and the weight of the beam is minimal.

Problem 8: Circular rope with minimal weight

An elastic rope with a variable but circular cross-section is suspended at the ceiling. Due to its own weight and a mass M which is appended at its lower end, the rope will suffer an elastic deformation. Its length in the undeformed state is L . For $0 \leq \ell \leq L$, design the variable radius $r(\ell)$ within the limits $0 \leq r(\ell) \leq R$ such that the appended mass M sinks by δ at most and such that the weight of the rope is minimal.

Problem 9: Optimal flying maneuver

An aircraft flies in a horizontal plane at a constant speed v . Its lateral acceleration can be controlled within certain limits. The goal is to fly over a reference point (target) in any direction and as soon as possible.

The problem is stated most easily in an earth-fixed coordinate system (see Fig. 1.1). For convenience, the reference point is chosen at $x = y = 0$. The limitation of the lateral acceleration is expressed in terms of a limited angular turning rate $u(t) = \dot{\varphi}(t)$ with $|u(t)| \leq 1$.

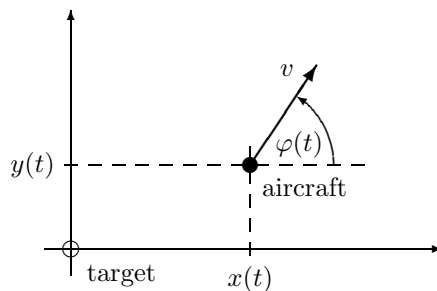


Fig. 1.1. Optimal flying maneuver described in earth-fixed coordinates.

Find a piecewise continuous turning rate $u : [0, t_b] \rightarrow [-1, 1]$ such that the dynamic system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v \cos \varphi(t) \\ v \sin \varphi(t) \\ u(t) \end{bmatrix}$$

is transferred from the initial state

$$\begin{bmatrix} x(0) \\ y(0) \\ \varphi(0) \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \\ \varphi_a \end{bmatrix}$$

to the partially specified final state

$$\begin{bmatrix} x(t_b) \\ y(t_b) \\ \varphi(t_b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \text{free} \end{bmatrix}$$

and such that the cost functional

$$J(u) = \int_0^{t_b} dt$$

is minimized.

Alternatively, the problem can be stated in a coordinate system which is fixed to the body of the aircraft (see Fig. 1.2).

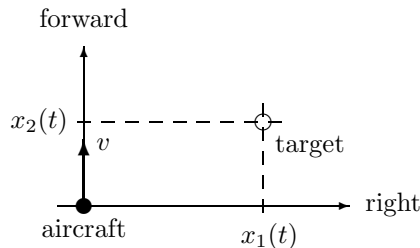


Fig. 1.2. Optimal flying maneuver described in body-fixed coordinates.

This leads to the following alternative formulation of the optimal control problem:

Find a piecewise continuous turning rate $u : [0, t_b] \rightarrow [-1, 1]$ such that the dynamic system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t)u(t) \\ -v - x_1(t)u(t) \end{bmatrix}$$

is transferred from the initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{1a} \\ x_{2a} \end{bmatrix} = \begin{bmatrix} -x_a \sin \varphi_a + y_a \cos \varphi_a \\ -x_a \cos \varphi_a - y_a \sin \varphi_a \end{bmatrix}$$

to the final state

$$\begin{bmatrix} x_1(t_b) \\ x_2(t_b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and such that the cost functional

$$J(u) = \int_0^{t_b} dt$$

is minimized.

Problem 10: Time-optimal motion of a cylindrical robot

In this problem, the coordinated angular and radial motion of a cylindrical robot in an assembly task is considered (Fig. 1.3). A component should be grasped by the robot at the supply position and transported to the assembly position in minimal time.

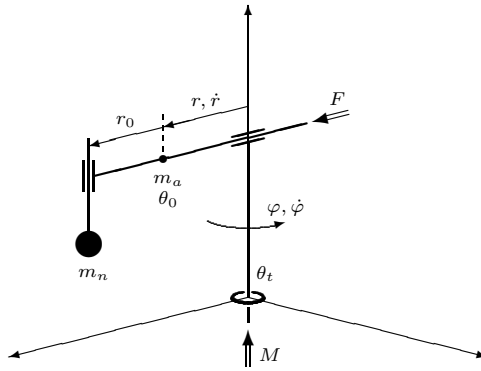


Fig. 1.3. Cylindrical robot with the angular motion φ and the radial motion r .

State variables:

$$x_1 = r = \text{radial position}$$

$$x_2 = \dot{r} = \text{radial velocity}$$

$$x_3 = \varphi = \text{angular position}$$

$$x_4 = \dot{\varphi} = \text{angular velocity}$$

control variables:

$$u_1 = F = \text{radial actuator force}$$

$$u_2 = M = \text{angular actuator torque}$$

subject to the constraints

$$|u_1| \leq F_{\max} \text{ and } |u_2| \leq M_{\max}, \text{ hence}$$

$$\Omega = [-F_{\max}, F_{\max}] \times [-M_{\max}, M_{\max}].$$

The optimal control problem can be stated as follows:

Find a piecewise continuous $u : [0, t_b] \rightarrow \Omega$ such that the dynamic system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ [u_1(t) + (m_a x_1(t) + m_n \{r_0 + x_1(t)\})x_4^2(t)] / (m_a + m_n) \\ x_4(t) \\ [u_2(t) - 2(m_a x_1(t) + m_n \{r_0 + x_1(t)\})x_2(t)x_4(t)] / \theta_{\text{tot}}(x_1(t)) \end{bmatrix}$$

where

$$\theta_{\text{tot}}(x_1(t)) = \theta_t + \theta_0 + m_a x_1^2(t) + m_n \{r_0 + x_1(t)\}^2$$

is transferred from the initial state

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} r_a \\ 0 \\ \varphi_a \\ 0 \end{bmatrix}$$

to the final state

$$\begin{bmatrix} x_1(t_b) \\ x_2(t_b) \\ x_3(t_b) \\ x_4(t_b) \end{bmatrix} = \begin{bmatrix} r_b \\ 0 \\ \varphi_b \\ 0 \end{bmatrix}$$

and such that the cost functional

$$J(u) = \int_0^{t_b} dt$$

is minimized.

This problem has been solved in [15].

Problem 11: The LQ differential game problem

Find unconstrained controls $u : [t_a, t_b] \rightarrow R^{m_u}$ and $v : [t_a, t_b] \rightarrow R^{m_v}$ such that the dynamic system

$$\dot{x}(t) = A(t)x(t) + B_1(t)u(t) + B_2v(t)$$

is transferred from the initial state

$$x(t_a) = x_a$$

to an arbitrary final state

$$x(t_b) \in R^n$$

at the fixed final time t_b and such that the quadratic cost functional

$$J(u, v) = \frac{1}{2}x^T(t_b)Fx(t_b) + \frac{1}{2} \int_{t_a}^{t_b} \left(x^T(t)Q(t)x(t) + u^T(t)u(t) - \gamma^2 v^T(t)v(t) \right) dt$$

is simultaneously minimized with respect to u and maximized with respect to v , when both of the players are allowed to use state-feedback control.

Remark: As in the LQ regulator problem, the penalty matrices F and $Q(t)$ are symmetric and positive-semidefinite.

This problem is analyzed in Chapter 4.2.

Problem 12: The homicidal chauffeur game

A car driver (denoted by “pursuer” P) and a pedestrian (denoted by “evader” E) move on an unconstrained horizontal plane. The pursuer tries to kill the evader by running him over. The game is over when the distance between the pursuer and the evader (both of them considered as points) diminishes to a critical value d . — The pursuer wants to minimize the final time t_b while the evader wants to maximize it.

The dynamics of the game are described most easily in an earth-fixed coordinate system (see Fig. 1.4).

State variables: x_p, y_p, φ_p , and x_e, y_e .

Control variables: $u \sim \dot{\varphi}_p$ (“constrained motion”) and v_e (“simple motion”).

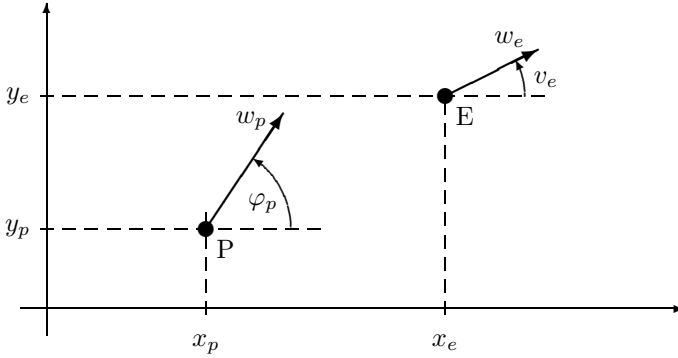


Fig. 1.4. The homicidal chauffeur game described in earth-fixed coordinates.

Equations of motion:

$$\begin{aligned} \dot{x}_p(t) &= w_p \cos \varphi_p(t) \\ \dot{y}_p(t) &= w_p \sin \varphi_p(t) \\ \dot{\varphi}_p(t) &= \frac{w_p}{R} u(t) & |u(t)| \leq 1 \\ \dot{x}_e(t) &= w_e \cos v_e(t) & w_e < w_p \\ \dot{y}_e(t) &= w_e \sin v_e(t) \end{aligned}$$

Alternatively, the problem can be stated in a coordinate system which is fixed to the body of the car (see Fig. 1.5).

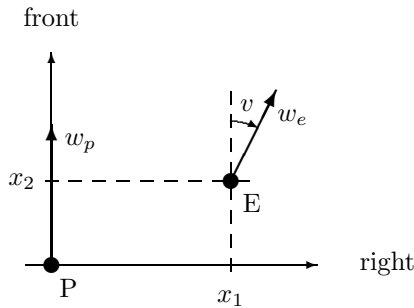


Fig. 1.5. The homicidal chauffeur game described in body-fixed coordinates.

This leads to the following alternative formulation of the differential game problem:

State variables: x_1 and x_2 .

Control variables: $u \in [-1, +1]$ and $v \in [-\pi, \pi]$.

Using the coordinate transformation

$$x_1 = (x_e - x_p) \sin \varphi_p - (y_e - y_p) \cos \varphi_p$$

$$x_2 = (x_e - x_p) \cos \varphi_p + (y_e - y_p) \sin \varphi_p$$

$$v = \varphi_p - v_e ,$$

the following model of the dynamics in the body-fixed coordinate system is obtained:

$$\dot{x}_1(t) = \frac{w_p}{R} x_2(t)u(t) + w_e \sin v(t)$$

$$\dot{x}_2(t) = -\frac{w_p}{R} x_1(t)u(t) - w_p + w_e \cos v(t) .$$

Thus, the differential game problem can finally be stated in the following efficient form:

Find two state-feedback controllers $u(x_1, x_2) \mapsto [-1, +1]$ and $v(x_1, x_2) \mapsto [-\pi, +\pi]$ such that the dynamic system

$$\dot{x}_1(t) = \frac{w_p}{R} x_2(t)u(t) + w_e \sin v(t)$$

$$\dot{x}_2(t) = -\frac{w_p}{R} x_1(t)u(t) - w_p + w_e \cos v(t)$$

is transferred from the initial state

$$x_1(0) = x_{10}$$

$$x_2(0) = x_{20}$$

to a final state with

$$x_1^2(t_b) + x_2^2(t_b) \leq d^2$$

and such that the cost functional

$$J(u, v) = t_b$$

is minimized with respect to $u(\cdot)$ and maximized with respect to $v(\cdot)$.

This problem has been stipulated and partially solved in [21]. The complete solution of the homicidal chauffeur problem has been derived in [28].

1.3 Static Optimization

In this section, some very basic facts of elementary calculus are recapitulated which are relevant for minimizing a continuously differentiable function of several variables, without or with side-constraints.

The goal of this text is to generalize these very simple necessary conditions for a constrained minimum of a function to the corresponding necessary conditions for the optimality of a solution of an optimal control problem. The generalization from constrained static optimization to optimal control is very straightforward, indeed. No “higher” mathematics is needed in order to derive the theorems stated in Chapter 2.

1.3.1 Unconstrained Static Optimization

Consider a scalar function of a single variable, $f : R \rightarrow R$. Assume that f is at least once continuously differentiable when discussing the first-order necessary condition for a minimum and at least k times continuously differentiable when discussing higher-order necessary or sufficient conditions.

The following conditions are *necessary* for a local minimum of the function $f(x)$ at x^o :

- $f'(x^o) = \frac{df(x^o)}{dx} = 0$
- $f^\ell(x^o) = \frac{d^\ell f(x^o)}{dx^\ell} = 0$ for $\ell = 1, \dots, 2k-1$
and $f^{2k}(x^o) \geq 0$ where $k = 1, \text{ or } 2, \text{ or } \dots$

The following conditions are *sufficient* for a local minimum of the function $f(x)$ at x^o :

- $f'(x^o) = \frac{df(x^o)}{dx} = 0$ and $f''(x^o) > 0$ or
- $f^\ell(x^o) = \frac{d^\ell f(x^o)}{dx^\ell} = 0$ for $\ell = 1, \dots, 2k-1$
and $f^{2k}(x^o) > 0$ for a finite integer number $k \geq 1$.

Nothing can be inferred from these conditions about the *existence* of a local or a global minimum of the function f !

If the range of admissible values x is restricted to a finite, closed, and bounded interval $\Omega = [a, b] \subset R$, the following conditions apply:

- If f is continuous, there exists at least one global minimum.

- Either the minimum lies at the left boundary a , and the lowest non-vanishing derivative is positive,
or
the minimum lies at the right boundary b , and the lowest non-vanishing derivative is negative,
or
the minimum lies in the interior of the interval, i.e., $a < x^o < b$, and the above-mentioned necessary and sufficient conditions of the unconstrained case apply.

Remark: For a function f of several variables, the first derivative f' generalizes to the Jacobian matrix $\frac{\partial f}{\partial x}$ as a row vector or to the gradient $\nabla_x f$ as a column vector,

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right], \quad \nabla_x f = \left(\frac{\partial f}{\partial x} \right)^T,$$

the second derivative to the Hessian matrix

$$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

and its positive-semidefiniteness, etc.

1.3.2 Static Optimization under Constraints

For finding the minimum of a function f of several variables x_1, \dots, x_n under the constraints of the form $g_i(x_1, \dots, x_n) = 0$ and/or $g_i(x_1, \dots, x_n) \leq 0$, for $i = 1, \dots, \ell$, the method of Lagrange multipliers is extremely helpful.

Instead of minimizing the function f with respect to the independent variables x_1, \dots, x_n over a constrained set (defined by the functions g_i), minimize the augmented function F with respect to its mutually completely independent variables $x_1, \dots, x_n, \lambda_1, \dots, \lambda_\ell$, where

$$F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_\ell) = \lambda_0 f(x_1, \dots, x_n) + \sum_{i=1}^{\ell} \lambda_i g_i(x_1, \dots, x_n).$$

Remarks:

- In shorthand, F can be written as $F(x, \lambda) = \lambda_0 f(x) + \lambda^T g(x)$ with the vector arguments $x \in R^n$ and $\lambda \in R^\ell$.

- Concerning the constant λ_0 , there are only two cases: it attains either the value 0 or 1.

In the singular case, $\lambda_0 = 0$. In this case, the ℓ constraints uniquely determine the admissible vector x^o . Thus, the function f to be minimized is not relevant at all. Minimizing f is not the issue in this case! Nevertheless, minimizing the augmented function F still yields the correct solution.

In the regular case, $\lambda_0 = 1$. The ℓ constraints define a nontrivial set of admissible vectors x , over which the function f is to be minimized.

- In the case of equality side constraints: since the variables $x_1, \dots, x_n, \lambda_1, \dots, \lambda_\ell$ are independent, the necessary conditions of a minimum of the augmented function F are

$$\frac{\partial F}{\partial x_i} = 0 \quad \text{for } i = 1, \dots, n \quad \text{and} \quad \frac{\partial F}{\partial \lambda_j} = 0 \quad \text{for } j = 1, \dots, \ell .$$

Obviously, since F is linear in λ_j , the necessary condition $\frac{\partial F}{\partial \lambda_j} = 0$ simply returns the side constraint $g_j = 0$.

- For an inequality constraint $g_i(x) \leq 0$, two cases have to be distinguished: Either the minimum x^o lies in the interior of the set defined by this constraint, i.e., $g_i(x^o) < 0$. In this case, this constraint is irrelevant for the minimization of f because for all x in an infinitesimal neighborhood of x^o , the strict inequality holds; hence the corresponding Lagrange multiplier vanishes: $\lambda_i^o = 0$. This constraint is said to be inactive. — Or the minimum x^o lies at the boundary of the set defined by this constraint, i.e., $g_i(x^o) = 0$. This is almost the same as in the case of an equality constraint. Almost, but not quite: For the corresponding Lagrange multiplier, we get the necessary condition $\lambda_i^o \geq 0$. This is the so-called “Fritz-John” or “Kuhn-Tucker” condition [7]. This inequality constraint is said to be active.

Example 1: Minimize the function $f = x_1^2 - 4x_1 + x_2^2 + 4$ under the constraint $x_1 + x_2 = 0$.

Analysis for $\lambda_0 = 1$:

$$F(x_1, x_2, \lambda) = x_1^2 - 4x_1 + x_2^2 + 4 + \lambda x_1 + \lambda x_2$$

$$\frac{\partial F}{\partial x_1} = 2x_1 - 4 + \lambda = 0$$

$$\frac{\partial F}{\partial x_2} = 2x_2 + \lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = x_1 + x_2 = 0 .$$

The optimal solution is:

$$\begin{aligned}x_1^o &= 1 \\x_2^o &= -1 \\ \lambda^o &= 2 .\end{aligned}$$

Example 2: Minimize the function $f = x_1^2 + x_2^2$ under the constraints $1 - x_1 \leq 0$, $2 - 0.5x_1 - x_2 \leq 0$, and $x_1 + x_2 - 4 \leq 0$.

Analysis for $\lambda_0 = 1$:

$$\begin{aligned}F(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) &= x_1^2 + x_2^2 \\ &\quad + \lambda_1(1 - x_1) + \lambda_2(2 - 0.5x_1 - x_2) + \lambda_3(x_1 + x_2 - 4) \\ \frac{\partial F}{\partial x_1} &= 2x_1 - \lambda_1 - 0.5\lambda_2 + \lambda_3 = 0 \\ \frac{\partial F}{\partial x_2} &= 2x_2 - \lambda_2 + \lambda_3 = 0 \\ \frac{\partial F}{\partial \lambda_1} &= 1 - x_1 \quad \begin{cases} = 0 & \text{and } \lambda_1 \geq 0 \\ < 0 & \text{and } \lambda_1 = 0 \end{cases} \\ \frac{\partial F}{\partial \lambda_2} &= 2 - 0.5x_1 - x_2 \quad \begin{cases} = 0 & \text{and } \lambda_2 \geq 0 \\ < 0 & \text{and } \lambda_2 = 0 \end{cases} \\ \frac{\partial F}{\partial \lambda_3} &= x_1 + x_2 - 4 \quad \begin{cases} = 0 & \text{and } \lambda_3 \geq 0 \\ < 0 & \text{and } \lambda_3 = 0 \end{cases}\end{aligned}$$

The optimal solution is:

$$\begin{aligned}x_1^o &= 1 \\x_2^o &= 1.5 \\ \lambda_1^o &= 0.5 \\ \lambda_2^o &= 3 \\ \lambda_3^o &= 0 .\end{aligned}$$

The third constraint is inactive.

1.4 Exercises

1. In all of the optimal control problems stated in this chapter, the control constraint Ω is required to be a time-invariant set in the control space R^m .

For the control of the forward motion of a car, the torque $T(t)$ delivered by the automotive engine is often considered as a control variable. It can be chosen freely between a minimal torque and a maximal torque, both of which are dependent upon the instantaneous engine speed $n(t)$. Thus, the torque limitation is described by

$$T_{\min}(n(t)) \leq T(t) \leq T_{\max}(n(t)) .$$

Since typically the engine speed is not constant, this constraint set for the torque $T(t)$ is not time-invariant.

Define a new transformed control variable $u(t)$ for the engine torque such that the constraint set Ω for u becomes time-invariant.

2. In Chapter 1.2, ten optimal control problems are presented (Problems 1–10). In Chapter 2, for didactic reasons, the general formulation of an optimal control problem given in Chapter 1.1 is divided into the categories A.1 and A.2, B.1 and B.2, C.1 and C.2, and D.1 and D.2. Furthermore, in Chapter 2.1.6, a special form of the cost functional is characterized which requests a special treatment.

Classify all of the ten optimal control problems with respect to these characteristics.

3. Discuss the geometric aspects of the optimal solution of the constrained static optimization problem which is investigated in Example 1 in Chapter 1.3.2.
4. Discuss the geometric aspects of the optimal solution of the constrained static optimization problem which is investigated in Example 2 in Chapter 1.3.2.
5. Minimize the function $f(x, y) = 2x^2 + 17xy + 3y^2$ under the equality constraints $x - y = 2$ and $x^2 + y^2 = 4$.