Preface to the German Edition

Complex analysis has produced a large number of deep recumbent and aesthetic results in its more than 200 year-old history. In the classical context complex analysis is the theory of complex differentiable functions of a complex variable, or also the theory of *holomorphic functions*. These are the solutions of a (2×2)-system of partial differential equations, which usually are called *Cauchy–Riemann differential equations (CRD)*.

Indeed, the algebra of the real quaternions of W.R. Hamilton has been available since 1843 and the real Clifford algebra of W.K. Clifford since 1878, but until the 1930s the prevailing view was that complex analysis is a purely two-dimensional theory. Only the group around the Swiss mathematician R. Fueter and the Romanian mathematicians G.C. Moisil and N. Teodorescu around 1930 started to develop a hypercomplex analysis in the algebra of real quaternions and in real Clifford algebras. In the late 1960s a group of Belgian mathematicians around R. Delanghe in Ghent created a rich higher dimensional analogy to complex analysis, called Clifford analysis. Since 1990 the number of relevant articles and books has increased significantly. Today intensive research is going on in Clifford analysis to which the more than 9000 entries in our database on the relevant literature testify. The database can be found on the CD attached to our book.

The purpose of the present textbook is to collect the essentials of classical complex analysis and to present its higher dimensional generalizations at a level suitable for university instruction. The typical users we have in mind are, first of all, students of mathematics and physics, but also students of any discipline requiring some sophisticated mathematics from the second year onward. The material to be covered is extensive, and we have attempted to make it as self-sufficient as possible within the limits of a modest size book. We have covered not only analytical and geometrical aspects but numerical procedures as well. Historical references outline the development of the field and present some of the personal characteristics of the most important personalities who have contributed to that history.

In the first chapter complex numbers, quaternions, and the Clifford numbers are introduced. We have emphasized the parallelism of our presentation. Quaternions and Clifford numbers take up more space than complex numbers. Besides the algebraic and geometrical properties we treat in particular also rotations and representations.

In Section 4 we illustrate the topological and analytical basic facts for the treatment of functions up to Riemann spheres. This section is deliberately kept short in view of its relationship to classical analysis. Section 5 treats some of the possible definitions of holomorphic functions. We keep this name also in higher dimensions, because the definitions are almost independent of dimension. The standard literature uses here mostly the concept of Weierstrass monogenic functions. However, it seems to us at least debatable whether this best describes the meaning of the definition (cf. end of Section 5). Also, the notion of holomorphic functions fits conceptually better that of meromorphic functions. Since the articles by H. Malonek the concept of holomorphic functions can be introduced also via local approximation by suitable linear functions, so even in that context the analogy holds in all dimensions. Section 6 is devoted to "simple" functions, namely powers and Möbius transforms. The polynomials named after R. Fueter are suitable as power functions in higher dimensions since they have many nice qualities. Unfortunately, the reduction of Fueter polynomials to the planar case leads to powers $(-iz)^n$ and not to z^n; however, the parallelism is still given. In particular L.V. Ahlfors has studied Möbius transforms in higher dimensions. Here too the clear comparability of all dimensions can be recognized.

We have put together the necessary aids for integration in Appendix 2, and a short introduction to the theory of differential forms in Appendix 1. We believe that this can be helpful, because in lectures to beginners these areas are often treated only very briefly, if at all. Indeed, we do not include the proof of Stokes' integral theorem as this would lead too far away from the subject. Then in Chapter III Cauchy' s integral theorem and the Borel–Pompeiu formula are easy consequences of Stokes' theorem. However, we also consider the boundary value formulae of Plemelj–Sokhotzki. Conclusions on Cauchy's integral formula follow. Moreover, the Teodorescu transform is examined and the Hodge decomposition of quaternionic Hilbert space is treated. The needed functional spaces are briefly introduced in Appendix 3.

Chapter IV is devoted to different areas of hypercomplex analysis. We firstly treat Taylor and Laurent series. The effort is clearly larger in higher dimensions than in the plane, but the similarity helps. Unfortunately, the Taylor series in dimensions greater than 2 are not orthogonal expansions. For quaternions orthogonal expansions are introduced, which are especially adequate for numerical purposes.

The elementary functions in the plane have no special difficulties. They are given more or less canonically. For all generalizations to higher dimensions, a royal way does not exist symptomatically. Different generalizations of the exponential function are pointed out, and one generalization given by the method of separation of variables is developed. This exponential function has some nice properties and is an appropriate kernel for the Fourier transform of quaternion-valued functions.

Section 12, which explores the local structure of holomorphic functions, shows that in higher dimensions this is still an active field of research. The pleasant qualities of zeros and isolated singularities in the plane at first sight get lost in space. There is still no suitable structure in which to understand all the relevant phenomena. At least the residue theorem can be transferred, and also first attempts for an argument principle were found.

Section 13 deals with special functions. The Gamma function and the Riemann Zeta function are treated, followed by considerations about automorphic functions and forms in $C\ell(n)$ which offer an insight into the latest research in this field.

Problems at the end of every section should help the reader toward a better understanding of the corresponding area. The use of the skewfield structure of the real quaternions allows one to formulate some statements more precisely and in a more readable form than in general Clifford algebras. Since applications in $\mathbb{R}^3$ and $\mathbb{R}^4$ are of special interest, we have sometimes waived the more general case of real Clifford algebras $C\ell(n)$.

Results of many colleagues working in the area of Clifford analysis are used in the presentation of higher dimensional results. We thank especially our colleagues Professor Krüger (Kaiserslautern/Germany), Professor Malonek (Aveiro/Portugal), and Professor Kraußhar (Leuven/Belgium), who helped us to write some of the sections. We discussed details with Professor Sommen (Ghent/Belgium) and Professor Shapiro (Mexico-City/Mexico). We thank Professor Jank (Aachen/Germany) and the editor Dr. Hempfling (Birkhäuser) for improving the typescript. Critical remarks by the referees of a first version of the book were of great value for us. M. Sprößig and T. Lahmer helped us by very carefully
long work for this book.

Weimar, Aachen and Freiberg, August 2005

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Preface to the English Edition

We thank the publisher for the opportunity to translate this book into English. Of course we have corrected all mistakes we found in the German edition, other changes have been made only rarely.
We thank in particular Professor E. Venturino (Torino/Italy) who translated one chapter of the book, but also improved the translation of the rest of the book. Some sections have been translated by M. Schneider and A. Schlichting who are students in Freiberg/Germany.

Weimar, Aachen, Freiberg, May 2007

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