Chapter 2

# SURPLUS-SHARING LOCAL GAMES IN DYNAMIC EXCHANGE PROCESSES

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A well-known property of so-called "MDP processes"[1] is monotonicity in terms of the utilities of the agents, due to the sharing among the latter of a "surplus" of numeraire generated at each point of their trajectories. In this paper, we focus our attention on the somewhat neglected question of how this sharing takes place, and we propose to use game–theoretic concepts and methods for answering it. A byproduct of this enquiry is the formulation of a "nontâtonnement" process that seems to be of independent interest.

## 1. Introduction*

Call a "distribution profile" (for the surplus) the *n*-dimensional vector $\delta^N$ where $\delta_i^N \geqq 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n \delta_i^N = 1$, and denote by the number $\Theta^N(x) \geqq 0$ the amount of numeraire surplus generated at point $x$ of a trajectory of some MDP process.

Champsaur (1976) has demonstrated that for an *n*-consumer economy (with or without public goods), every Paretian utility vector individually rational with respect to some initial allocation can be reached by some MDP process with a *constant* distribution profile $\delta^N$ applied to the surplus at all points along the trajectory. A consequence of this theorem is that

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[1] For introductory surveys and early references, see Malinvaud (1972, Chapter 8) and Milleron (1972, Section V). The most recent contribution is Champsaur, Drèze and Henry (1977).

from the consumer's point of view, the choice of this fixed vector has relevance only with respect to the limit point of the process: as soon as $\delta^N$ is agreed upon—and this must occur before the process can start—the corresponding final outcome is determined. A further consequence is, however, that no *social* dynamics are actually taking place during the realization of the process; the only dynamics involved are that of the computer programme solving the appropriate differential equations.

In order that some form of social interaction be taken into account within the dynamics of MDP processes, one may consider that each allocation along a trajectory is a state of the economy that is actually taking place. In this perspective, the choice of a distribution profile for sharing the surplus at any allocation should naturally be considered as being determined by the bargaining power of the individuals and coalitions at that moment, and not any more by their power at the initial position of the economy. In other words, it is suggested here that the distribution profile $\delta^N$ be chosen at each point along the process, according to characteristics of the social conflict at that point. The "surplus-sharing local games" are developed below for systematically exploring this idea.

As early as 1971, Drèze and de la Vallée Poussin had put forward (in Section III of their paper) the idea of associating game-theoretic considerations with trajectories of their process. The object of their enquiry is different from ours, however: dealing with public goods, they define local games whose purpose is to characterize the behavior of the agents as far as preference revelation is concerned. These are essentially noncooperative games, and subsequent writings of Roberts (1979) and Schoumaker (1979) suggest calling them "incentives local games". By contrast, our "surplus-sharing" games are cooperative ones (they may be seen equivalently as games with or without side payments; see §(i) in Section 3.4 on this point).

On the other hand, the nature of our approach has proved most fruitful so far for the case of a pure exchange economy with private goods only: we shall thus consider here a particular type of MDP process adapted to pure exchange, leaving for a future occasion the extension to processes with public goods.

In Section 2, the economy and the process are defined. Surplus-sharing local games are introduced in Section 3: their imputations are shown to induce game-theoretical selections of profiles $\delta^N(x)$, at each point $x$ of some MDP trajectory, and solution concepts such as the core, the Shapley value and the nucleolus are proposed. In Section 4, we verify that processes determined by these solution concepts, i.e. with variable $\delta^N(x)$, have a uniquely determined solution and converge to a Pareto-efficient allocation. Section 5, finally, is devoted to interpretative considerations.

## 2. The economy and the process

Consider a pure exchange economy $\mathscr{E} = \{(\mathscr{X}_i, u_i(x_i), \omega_i) \mid i \in N\}$, where $N$ is the set of agents (indexed $i = 1, \ldots, n$; $n = |N|$), $\mathscr{X}_i \subseteq \mathbb{R}_+^H$ denotes $i$'s consumption set (of which $x_i$ is a typical element), and $\mathbb{R}_+^H$ is the $H$-dimensional commodity space, commodities being indexed by $h = 1, \ldots, H$; $u_i(x_i)$ is the utility function of agent $i$, defined on $\mathscr{X}_i$, and $\omega_i \in \mathbb{R}_+^H$ his initial endowment of commodities.

An allocation is a vector $x = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}_+^{nH}$ such that $x_i \in \mathscr{X}_i$, $\forall i \in N$. An allocation is feasible if $\sum_{i \in N} x_{ih} = \sum_{i \in N} \omega_{ih}$ for every $h$. Let $X$ denote the set of feasible allocations. An allocation $x$ is efficient if it is feasible and there exists no alternative allocation $x' \in X$ for which $u_i(x_i') \geqq u_i(x_i)$, $\forall i \in N$ with strict inequality for at least one $i$. Finally a feasible allocation $x$ is individually rational with respect to some other allocation $y \in X$ if it is such that $u_i(x_i) \geqq u_i(y_i)$, $\forall i \in N$.

*Assumption* 1. $\forall i \in N$, $\mathscr{X}_i = \mathbb{R}_+^H$.

*Assumption* 2. $\forall i \in N$, $\omega_i$ is in the interior of $\mathscr{X}_i$.

*Assumption* 3. $\forall i \in N$, $u_i(x_i)$ is strictly quasi-concave, twice continuously differentiable, and such that

$$u_{i1}( =_{\text{def}} \partial u_i/\partial x_{i_1}) > 0 \ \forall x_i \in \mathscr{X}_i,$$

where for $x_i$ on the boundary of $\mathscr{R}_+^H$ we take for $u_{i1}$ the one-sided derivative.

*Assumption* 4. $\forall i \in N$, $\{x_i | x_i \in \mathbb{R}_+^H \text{ and } u_i(x_i) \geqq u_i(\omega_i)\} \cap \partial\mathbb{R}_+^H = \varnothing$.

These assumptions are stronger than those usually made in this field (especially Assumption 4, which says that for each $i$, the indifference surface though $\omega_i$ does not touch the coordinate hyperplanes $x_{ih} = 0$). However, for testing the local games concept of Section 3, they offer the advantage of yielding a system of differential equations with classical properties. It is quite possible that similar results hold under weaker assumptions.

To define the process, let $t \in [0, +\infty)$ be a time variable and $x(t) \in \mathbb{R}_+^{nH}$ an allocation at some time $t \geqq 0$.

Define

$$\pi_{ih}[x_i(t)] = \left.\frac{\partial u_i/\partial x_{ih}}{\partial u_i/\partial x_{i1}}\right|_{x_i(t)} \qquad \begin{array}{l} \forall i \in N \\ h = 2, \ldots, H \\ t \in [0, +\infty), \end{array}$$

the marginal rate of substitution of agent $i$ between commodities $h$ and 1, evaluated at point $x_i(t)$. (We write only $\pi_{ih}$ when no confusion can arise as to this point and when the time argument is immaterial.) Define also

$$\bar{\pi}_h^N = (1/n)\sum_{i\in N} \pi_{ih} \quad h = 2, \ldots, H. \qquad \ldots(2.1)$$

As usual, a dot over a time-dependent variable will denote the operator *d/dt*.

For the above economy, a particular[2] version of the exchange planning process of Malinvaud (1972, Chapter 8) consists of the following system of differential equations:

*Process* M:

$$\left.\begin{aligned} \dot{x}_{ih} &= a(\pi_{ih} - \bar{\pi}_h^N)\ \forall i \in N;\ h = 2, \ldots, H, \\ \dot{x}_{i1} &= -\textstyle\sum_{h\neq 1} \pi_{ih}\dot{x}_{ih} + \delta_i^N a \sum_{h\neq 1} \sum_{j\in N} (\pi_{jh} - \bar{\pi}_h^N)^2\ \forall i \in N, \end{aligned}\right\} \quad \begin{aligned} &\ldots(2.2) \\ &\ldots(2.3) \end{aligned}$$

where $\delta_i^N \geqq 0\ \forall i \in N,\ \sum_{i\in N} \delta_i^N = 1,\ 0 < a < +\infty$, and $t \in [0, +\infty)$.

Using vector notation, the system is of the form $\dot{x} = f(x; \delta^N)$, with

$\delta^N = (\delta_1^N, \ldots, \delta_i^N, \ldots, \delta_n^N)$.

Since in (2.2–2.3) for every $i$ each variable $\pi_{ih}$, $h = 2, \ldots, H$ is a function of $x_i \in \mathscr{X}_i$, the function $f(\,\cdot\,; \delta^N)$ is defined on the product set $\mathscr{X} = \Pi_{i\in N}\mathscr{X}_i = \mathbb{R}_+^{nH}$, due to Assumption 1.

Given Assumptions 1–4, existence and uniqueness of a solution $x[t; \delta^N, x(0)]$, as well as convergence to an efficient allocation which is individually rational with respect to $x(0)$, are properties that can be inferred from the literature cited earlier; they also derive directly from the arguments made in Section 4 below.

*Remark* 2.1. Without loss of generality, we shall assume throughout that the speed of adjustment parameter $a$ is equal to 1 in process M. Henceforth, we shall thus ignore it, except for the observations made in Section 3.4.

## 3. Surplus-sharing local games

### 3.1 The need for cooperation in carrying out the process

Consider any allocation $x \in X$, and a process M defined at that point. For each agent $i$, the speed of utility changes entailed at $x$ by the process is of the form

[2] In Malinvaud's text, $\bar{\pi}_h^N$ is taken to be a weighted average. Some of the results below on local games will make clear the special interest of the simple average used here.

$$\dot{u}_i = u_{i1}(x_i)\delta_i^N \Theta^N(x) \geqq 0 \qquad \ldots(3.1)$$

where $\Theta^N(x)$ is defined by

$$\Theta^N(x) = \sum_{h \neq 1} \sum_{j \in N} (\pi_{jh} - \bar{\pi}_h^N)^2. \qquad \ldots(3.2)$$

It was suggested by Malinvaud that the last expression be interpreted as a "social surplus" accruing to the agents from the reallocation specified at point $x$. This surplus is measured in units of commodity 1 per unit of time; according to equation (2.3) its distribution among the members of $N$ is determined by the distribution profile $\delta^N$.

With this interpretation in mind, the question naturally arises whether all agents $i \in N$ will agree to carry out this exchange. One may think of at least two categories of reasons for explaining why some agents would refuse to trade: (i) they may object to the use of process M itself as a means for commodity reallocation (e.g. they would not reveal their coefficients $\pi_{ih}$); or (ii) while accepting that the process be used, they may object to the distribution profile $\delta^N$, which is seen by (3.1) to be the key factor in determining each agent's utility increase. In this paper, we propose to concentrate on this second kind of objection; we therefore assume, from this point on, that all agents agree on the use of process M.

For any subset (or "coalition") $S$ of agents, $S \subset N$, objecting to a given distribution profile can be rationalized in various ways: for instance, $\delta^N$ may be seen by some agents as "inequitable"; or, some agents may feel that they could do better on their own. In both cases, there is an underlying reference to the fact that alternatives are open to them, as well as to the possible outcomes of such alternatives. Given our assumption of general agreement on the use of process M, such alternatives seem to be naturally described, for each $S$, by the notion of a *process restricted to a coalition S*. In such a process, the coalition $S$ follows exactly the procedure described above, assuming only that $S$ replaces $N$ as the set of traders. Formally, the coalition $S$ proceeds according to the system:

*Process* $\mathrm{M}^S$:

$$\left.\begin{aligned} \dot{x}_{ih}^S &= (\pi_{ih} - \bar{\pi}_h^S)\ \forall i \in S;\ h = 2, \ldots, H, \\ \dot{x}_{i1} &= -\textstyle\sum_{h \neq 1} \pi_{ih}\dot{x}_{ih}^S + \delta_i^S \sum_{h \neq 1} \sum_{j \in S} (\pi_{jh} - \bar{\pi}_h^S)^2\ \forall i \in S. \end{aligned}\right\} \qquad \begin{aligned} \ldots(3.3) \\ \ldots(3.4) \end{aligned}$$

where

$$\bar{\pi}_h^S = (1/s)\sum_{j \in S} \pi_{jh};\ h = 2, \ldots, H;\ s = |S|;$$

$$\delta_i^S \geqq 0 \quad \forall i \in S \text{ and } \sum_{i \in S} \delta_i^S = 1.$$

Such a process can be defined for any non-empty coalition $S \subset N$ and any starting allocation $x \in X$. Just as in process M, there is in any process $M^S$ a social surplus $\Theta^S$ being generated, defined as

$$\Theta^S(x) = \sum_{h \neq 1} \sum_{j \in S} (\pi_{jh} - \bar{\pi}_h^S)^2, \qquad \ldots (3.5)$$

and shared among the members of $S$ according to some distribution profile $\delta^S = (\delta_1^S, \ldots, \delta_s^S)$.

Both $\Theta^S(x)$ and $\delta^S$ summarize the outcome of the alternatives considered by $S$ at $x$. Hence, the problem of finding at $x$ a profile $\delta^N(x)$ that raises no objection in this sense may be seen as one of sharing the surplus $\Theta^N(x)$ in such a way that for all agents $i \in N$, and for all subsets $S \subset N$, the alternatives represented by the surpluses $\Theta^S(x)$ and profiles $\delta^S$ be adequately taken into account. This points directly to a formulation in terms of $n$-person cooperative games.

### 3.2 Local games: definition, imputations and distribution profiles

*Definition* 3.1. For every allocation $x \in X$, the characteristic function $v(\cdot\,; x)$ defined by $v(S; x) = \Theta^S(x) S \subseteqq N$, defines a "*surplus-sharing local game*". Notice that $v(\cdot\,; x)$ is 0-normalized, i.e. $v(\{i\}; x) = 0 \ \forall i \in N$.

Let $y(x) \in \mathbb{R}_+^n$ denote an *imputation* for the game $v(\cdot\,; x)$, i.e. an $n$-vector such that $\sum_{i \in N} y_i(x) = \Theta^N(x)$ and $y_i(x) \geqq 0 \ \forall i \in N$. Let also $Y(x) \subset \mathbb{R}_+^n$ be the set of imputations for $v(\cdot\,; x)$.

It is easy to see that the selection of an imputation in $Y(x)$ amounts to selecting a distribution profile $\delta^N(x)$ for the surplus $\Theta^N(x)$. Indeed, given $y(x)$, one may define $\delta_i^N(x) \forall i \in N$ as

$$\delta_i^N(x) = \frac{y_i(x)}{\sum_{i \in N} y_i(x)} = \frac{y_i(x)}{\Theta^N(x)} \geqq 0, \qquad \ldots (3.6)$$

and clearly $\sum_{i \in N} \delta_i^N(x) = 1$. In other words, there is at any $x \in X$, a natural one-to-one mapping between the set of imputations $Y(x)$ and the set of distribution profiles for the surplus $\Theta^N(x)$. Thus, the game-theoretic approach to the problem of choosing at any $x$ a satisfactory distribution profile $\delta^N(x)$—or a set of such profiles—essentially consists of selecting a point—or a set of points—in the set of imputations of the side payment local game defined at $x$, that is, selecting a solution concept for this game.

### 3.3 Solution concepts for the local games $v(\cdot\,; x)$

Among the many conceivable reasons why a coalition $S$ might object to a given distribution profile $\delta^N(x)$—that is, to an imputation $y(x)$—a most intuitive one is the consideration of *what this coalition* *could achieve on its own*. In the general theory of $n$-person games, this idea has for a long time been referred to by the notion of "blocking"; in the side payment case, it may also be expressed (and be given a scalar measure) by the notion of "excess of a coalition with respect to an imputation". Formally, for any $y \in Y$ and $S \subseteqq N$, such excess is defined by

$$e(S,y) = v(S) - \sum_{i \in S} y_i,$$

with $e(S,y) > 0$ meaning that $S$ can improve upon (or "block") the imputation $y$, while $e(S,y) \leqq 0$ implies that $S$ cannot. From the excess notion, one is led to the definition of two well known solution concepts, viz. the *core*[3] due to Gillies (1959) and the *nucleolus*,[4] due to Schmeidler (1969).

Alternatively, an explicit set of axioms can be stated, which a solution should satisfy to be considered as acceptable: the *Shapley value*,[5] due to Shapley (1953), is such a solution concept, which is essentially cooperative in nature; the axioms from which it is derived are sometimes interpreted as reflecting some notion of *equity* in the sharing of $v(N)$ among the $n$ players.

As is established by the just quoted authors, the nucleolus always exists, and the Shapley value is always well defined for all side payment games; moreover, both are unique-point solution concepts. The core, on the other hand, may be empty; when it is not, it may consist of more than one imputation.

Consequently, both the nucleolus and the Shapley value can be used as solution concepts for the local games $v(\cdot\,;x)$. As far as the core is concerned, its non-emptiness for these games is established by the following result.

**Theorem 3.2.** *For every local game $v(\cdot\,;x)$, the imputation $\mu(x) \in Y(x)$ defined by*

$$\mu_i(x) = \sum_{h \neq 1} (\pi_{ih} - \bar{\pi}_h^N)^2 \quad \forall i \in N$$

*belongs to the core.*

[3] For a definition, see the proof of Theorem 3.2 below.

[4] The nucleolus may be defined as follows: given an imputation $y$, let $\theta(y)$ be a vector in $\mathbb{R}_+^{2n-1}$, whose components are the $2^n - 1$ excesses $\{e(S; y)\}$, $S \subseteq N$, $S \neq \emptyset$, arranged in non-decreasing order; consider then any pair of imputations, say $x$ and $y$: $\theta(x)$ is said to be larger than $\theta(y)$ in the lexicographic order [notation: $\theta(x) >_L \theta(y)$] if $\exists i \in \{1, 2, \ldots, 2^n - 1\}$ such that $\theta_i(x) > \theta_i(y)$ and $\theta_j(x) = \theta_j(y)$ $\forall j < i$; the nucleolus of a side payment game is the set of imputations $v \in Y$ such that the vector $\theta(v)$ is minimal in the lexicographic order on $\mathbb{R}_+^{2n-1}$.

[5] The Shapley value is the imputation defined by the function

$$\phi_i(v) = \Sigma_{s \subset N}((s-1)!(n-s)!/n!)[v(S) - v(S\backslash\{i\})],\ \forall i \in N.$$

*Proof*. In terms of excesses, the core is defined as the set of imputations $y \in Y$ such that for every coalition $S \subseteqq N, e(S,y) \leqq 0$. In the case of local games $v(\cdot\,;x)$, excesses with respect to the imputation $\mu(x)$ are of the form:

$$e(S,\ \mu(x)) = \Theta^S(x) - \sum_{i \in S} \mu_i(x)$$

$$= \sum_{h \neq 1} \left[ \sum_{i \in S} (\pi_{ih} - \bar{\pi}_h^S)^2 - \sum_{i \in S} (\pi_{ih} - \bar{\pi}_h^N)^2 \right] \quad \forall S \subseteqq N.$$

That this expression is non-positive follows from the fact that each sum $\sum_{i \in S} (\pi_{ih} - \bar{\pi}_h^S)^2$ within the square brackets is the second moment of the $\{\pi_{ih}\}_{i \in S}$ with respect to their arithmetic mean $\bar{\pi}_h^S$. Now, it is well known that in general, among all second moments, the minimal is the one taken with respect to the simple arithmetic mean. Therefore, $\sum_{i \in S} (\pi_{ih} - \bar{\pi}_h^N)^2$ cannot be smaller than $\sum_{i \in S} (\pi_{ih} - \bar{\pi}_h^S)^2$. ||

Besides the imputation $\mu(x)$, the nucleolus—denoted by $\nu(x)$—is another selection in the core of any local game $v(\cdot\,;x)$: indeed, according to Schmeidler's 1969 Theorem 4, the nucleolus always belongs to the core of a game, when the core is non-empty. The Shapley value, on the other hand, does not have this property.

### 3.4 Other aspects of the local games $v(\cdot\,;x)$

(i) The local games considered so far belong to the class of side payment games; however, they could also be formulated as non-side payment games, defined by

$$V^S(x) = \{\dot{u}^S \in \mathbb{R}_+^S | \exists \delta^S \text{ for which } u_{i1}\delta_i^S \Theta^S(x) \geqq \dot{u}_i^S \quad \forall i \in S\}, \forall S \subseteqq N.$$

$V^S(x)$ is thus the set of all vectors of speeds of utility increases achievable by coalition $S$. However, this formulation is basically equivalent to the side payment one, due to the obvious one-to-one correspondence between the set $Y(x)$ of imputations of the side payment game, and the Pareto surface of the non-side payment game: to $y(x) \in Y(x)$, there corresponds the point

$$\dot{u}^N = [u_{11}y_1(x), \ldots, u_{i1}y_i(x), \ldots, u_{n1}y_n(x)],$$

which is clearly on the Pareto surface of $V^N(x)$. In fact, the non-side payment local games mentioned here are a special case of Billera's 1970 $\pi$-hyperplane games.

(ii) From the argument used in the proof of Theorem 3.2, it appears that the games $v(\cdot\,;x)$ are in fact the sum of $H-1$ games whose characteristic function, defined by

$$v_h(S) = \sum_{i \in S} (\pi_{ih} - \bar{\pi}_h^S)^2,$$

is simply the variance, multiplied by $s$, of the collection of numbers $\{\pi_{ih}\}_{i \in S}$. From the properties of the variance, it can easily be derived that these games are superadditive, monotonic, and balanced in the sense of Shapley (1967).

(iii) If the assumption that the speed of adjustment $a = 1$ for every process $M^S$ were relaxed, and replaced by that of an arbitrary collection of positive coefficients $A = \{a^S\}_{S \subseteq N}$, the characteristic function of the local games would read $v(S; x, A) = a^S \Theta^S(x) \quad \forall S \subseteq N$. In this more general setting, it can be shown that the above results on the core imputation $\mu$ (and on balancedness) still hold, provided that the collection $A$ be such that

$$a^N \geqq \max_{S \subseteq N} \{a^S\}.$$

On the other hand, if the collection $A$ is such that $a^S = \rho s \; \forall S \subseteqq N$, where $0 < \rho < +\infty$ and $s = |S|$, the local games $v(\cdot\, ; x, A)$ appear to be convex in the sense of Shapley (1971); in this case the Shapley value also belongs to the core of the game. However, it is not easy to provide a convincing economic interpretation of adjustment speeds which vary with the size of each coalition, at least in a continuous time context. An attempt in that direction, and proofs of the preceding propositions can be found in Tulkens and Zamir (1976, pp. 27–33).

## 4. Strategically stable processes

Given the exchange process of Section 2, and the solution concepts exhibited in Section 3 for the local games, we are now in a position to combine the two. We want to describe processes whose trajectories are determined by these solution concepts.

Specifically, we suggest the following modification of process M: at each point $x(t)$ on the trajectory, the profile $\delta^N(t)$ is selected according to some solution concept of the local game $v(\cdot\, ; x(t))$ at that point. Given the three concepts discussed above, namely the nucleolus $\nu(x)$, the Shapley value $\phi(x)$, and the imputation $\mu(x)$ which is in the core, we can define three processes: process $M - \nu$, process $M - \phi$, and process $M - \mu$. The first of these is defined by the following differential equations:

$$\left.\begin{aligned} \dot{x}_{ih} &= \pi_{ih} - \bar{\pi}_h^N, \quad \forall i \in N,\, h = 2, \ldots, H, \qquad \ldots(4.1)\\ \dot{x}_{i1} &= -\sum_{h \neq 1} \pi_{ih}\dot{x}_{ih} + \delta_i^\nu(x)\Theta^N(x), \quad \forall i \in N, \qquad \ldots(4.2) \end{aligned}\right\}$$

where $\delta^{\nu}(x) = (\delta_1^{\nu}(x), \ldots, \delta_i^{\nu}(x), \ldots, \delta_n^{\nu}(x))$ is derived from the imputation $\nu(x)$ of the local game $v(\cdot\,; x)$ through the mapping (3.6), and $\Theta^N(x)$ is defined as in (3.2).

The processes $\mathrm{M} - \phi$ and $\mathrm{M} - \mu$ are defined by a system of equations similar to (4.1–4.2), the only difference being that $\delta^{\nu}(x)$ is replaced by $\delta^{\phi}(x)$ [respectively $\delta^{\mu}(x)$], derived through the mapping (3.6) from the Shapley value [respectively from the imputation $\mu(x)$] of the game $v(\cdot\,; x)$.

These three processes are well defined, since $\delta^{\nu}(x)$, $\delta^{\phi}(x)$, and $\delta^{\mu}(x)$ are uniquely determined for each $x$. Their solutions, if they exist, may be called "strategically stable", in the game-theoretic sense of the corresponding distribution profile.

Existence and uniqueness of solutions for these new processes follow from known theorems on systems of differential equations (see e.g. Nemytskii and Stepanov (1960), or Champsaur, Drèze and Henry (1977)): indeed, given Assumptions 1–4, the right-hand sides of (4.1–4.2) can be shown[6] to be Lipschitzian for all three processes (that the Lipschitz property holds in particular for $\delta^{\nu}(x)$ derives from the piecewise linearity of the nucleolus, viewed as a function $\nu(u(\cdot\,; x))$ of the characteristic function of the game, as shown by Charnes and Kortanek (1969) and Kohlberg (1971).

We can claim in addition that for each of these processes, the solution converges, as it does for the original process M, to a unique efficient allocation. To show this, one may either introduce a Lyapunov function such as $\sum_{i \in N} u_i(x_i(t))$, and apply Theorem 6.1 of Champsaur, Drèze and Henry (1977), or demonstrate directly the following rather easy steps[7]: (i) that the solution remains in a compact set, whence it has at least one accumulation point; (ii) that every accumulation point is an efficient allocation; (iii) that if the limit point of this process is an efficient allocation, it is unique.

## 5. Interpretative concluding remarks

### 5.1 Local vs. global approaches

In contrast with the "global" games usually associated with an economy, the local games presented in this paper are characterized by the myopic nature of their imputations. Indeed, they shift the attention from the utility levels of the agents at the point where the MDP process stops, to the rate at which these utilities increase at any one moment of

[6] This is done in Tulkens and Zamir (1976, Lemma 4.1 and Appendix 1).
[7] These steps are developed in Tulkens and Zamir (1976, Lemmas 4.2 and 4.5, and Theorem 4.6).

time, before the limit point is reached. This implies that as regards the limit point itself, no particular game theoretical property (other than Pareto efficiency and individual rationality with respect to the initial point) is to be expected from the solution of our strategically stable processes: for instance, the solution of processes such as M – $\nu$ or M – $\mu$, whose trajectory is determined by imputations in the core of the local games, does not, in general, converge to an allocation in the core *of the economy*.[8]

The interest of the local approach is to be found, instead, in the same reason why iterative processes of the MDP type have been conceived of, namely the ignorance by the agents of where an efficient allocation is located. Given this circumstance, myopic behavior finds a justification if, following some maximin rule of decision making under uncertainty, the agents always take the view that the next step may be the last.[9]

### 5.2 Choosing between solution concepts

As three "strategically stable" exchange processes have been exhibited, one may wonder whether one of them is, in some sense, "better" than the others. The answer to this question can only be a general one; indeed, each of the solution concepts $\nu$, $\phi$, or $\mu$ for the local games corresponds to a different characterization of the role of coalititions in the social process under consideration, *viz*.: minimizing the strongest objection in the case of the nucleolus, averaging out the contribution of each player to the various coalitions he can join in the case of the Shapley value, and impossibility of blocking in the case of core imputations. The fact that $\nu$ and $\phi$ are different imputations in general, and that $\phi$ may not belong to the core, shows that the corresponding criteria are to some extent incompatible. The necessary choice between them can only be based on assumptions pertaining to the collective behavior of members of coalitions, assumptions that neither economic analysis, nor game theory seem to be currently able to provide.

### 5.3 A price interpretation of process M $-$ $\mu$

A closer look at the structure of commodity exchanges under the process M $-$ $\mu$ leads, finally, to a further interpretation, that goes in a different direction. With every feasible allocation $x$, suppose that there is associated a strictly positive price vector $p(x) \in \mathbb{R}^H_+$, normalized by

[8] This point has been checked by means of several numerical examples that can be found in Appendix 2 of Tulkens and Zamir (1976).

[9] We owe this point to the editor.

assuming $p_1(\cdot) = 1$. Given any such vector (with the argument $x$ deleted unless it is necessary), the budget constraint of every consumer $i$, solved for $x_{i1}$, reads

$$x_{i1} = \omega_{i1} + \sum_{h \neq 1} p_h(\omega_{ih} - x_{ih}). \qquad \ldots(5.1)$$

Introducing this expression in the utility function $u_i(x_{i1}, \ldots, x_{iH})$ and differentiating yields

$$du_i = u_{i1}[\sum_{h \neq 1} (\pi_{ih} - p_h) dx_{ih}]. \qquad \ldots(5.2)$$

Assume now that each consumer $i$, when he holds the bundle $x_i$, and faces the price system $p$, seeks the local reallocation of his budget that maximizes his utility change $du_i$. It is well known that the vector with this property is the gradient vector, which in the case of (5.2) is proportional to the one defined by

$$dx_{ih} = (\pi_{ih} - p_h),\ h = 2, \ldots, H. \qquad \ldots(5.3)$$

For the numeraire, one immediately derives from (5.1) that

$$\begin{aligned} dx_{i1} &= -\sum_{h \neq 1} p_h dx_{ih} \\ &= -\sum_{h \neq 1} p_h(\pi_{ih} - p_h). \end{aligned} \qquad \ldots(5.4)$$

Now, if at every feasible allocation $x(t)$ the price system $p(x(t))$ is chosen such that

$$p_h(x(t)) = \bar{\pi}_h^N(x(t)),\ h = 2, \ldots, H,$$

the behavioral assumption just made implies that the consumers make local transactions specified by

$$\left.\begin{aligned} \dot{x}_{ih} &= (\pi_{ih} - \bar{\pi}_h^N),\ h = 2, \ldots, H;\ i \in N, \\ \dot{x}_{i1} &= -\textstyle\sum_{h \neq 1} \bar{\pi}_h^N(\pi_{ih} - \bar{\pi}_h^N),\ i \in N. \end{aligned}\right\} \qquad \begin{aligned} &\ldots(5.5) \\ &\ldots(5.6) \end{aligned}$$

But (5.5) is identical to (4.1), and (5.6) is easily seen to be equivalent to (4.2) with

$$\delta_i^{\mu}(x) = \mu_i(x)/\Theta^N(x)$$

for each $i$. These spontaneous transactions are exactly those determined by the process $M - \mu$.

Implicit in this process, there is thus a continuous price adjustment mechanism, which is reminiscent of the "Edgeworth barter process" formulated in discrete time by Uzawa (1962, Sections 2 and 3): indeed, in this author's "terminology, a "transaction rule" is specified by (5.5) and (5.6), and a "price adjustment function" is given at each $x(t)$ by

$$\begin{aligned}\dot{p}_h = \dot{\pi}_h^N &= \frac{1}{n}\sum_{i\in N}\dot{\pi}_{ih},\\ &= \frac{1}{n}\sum_{i\in N}\sum_{k=1}^{H}\frac{\partial\pi_{ih}}{\partial x_{ik}}\dot{x}_{ik},\\ &= \frac{1}{n}\sum_{i\in N}\sum_{k=1}^{H}(\pi_{ihk} - \pi_{ih}\pi_{i1k})\dot{x}_{ik},\ h = 2,\ldots, H,\end{aligned}$$

where

$$\pi_{ihk} =_{\text{def}} \frac{\partial^2 u_i}{\partial x_{ih}\partial x_{ik}} \cdot \left(\frac{\partial u_i}{\partial x_{i1}}\right)^{-1} \quad h, k = 1, 2, \ldots, H. \qquad \ldots(5.7)$$

Contrary to the Walrasian rule of excess demands (which are identically zero in this model), and more in a Marshallian spirit, this price adjustment rule depends exclusively upon preference characteritics of the agents.

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