FEEDBACK STABILIZATION OF THE 3-D NAVIER-STOKES EQUATIONS BASED ON AN EXTENDED SYSTEM

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Abstract We study the local exponential stabilization of the 3D Navier-Stokes equations in a bounded domain, around a given steady-state flow, by means of a boundary control. We look for a control so that the solution to the Navier-Stokes equation be a strong solution. In the 3D case, such solutions may exist if the Dirichlet control satisfies a compatibility condition with the initial condition. In order to determine a feedback law satisfying such a compatibility condition, we consider an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the boundary of the domain. We determine a linear feedback law by solving a linear quadratic control problem for the linearized extended system. We show that this feedback law also stabilizes the nonlinear extended system.

Keywords: Navier-Stokes equation, Feedback stabilization, Riccati equation.

1. Introduction

Let $\mathcal{O}$ and $\mathcal{B}$ two regular bounded domains of class C^∞ in $\mathbf{R}^3$ such that $\overline{\mathcal{B}} \subset \mathcal{O}$, $\Omega = \mathcal{O}\backslash\overline{\mathcal{B}}$, $\Gamma_e = \partial\mathcal{O}$ and $\Gamma_i = \partial\mathcal{B}$. We have $\Gamma_i \cap \Gamma_e = \emptyset$ and $\partial\Omega = \Gamma_i \cup \Gamma_e$. We consider the motion of an incompressible fluid around the bounded body $\mathcal{B}$ in Ω which is described by the couple (z_e, p_e), the velocity and the pressure, solution to the stationary Navier-Stokes equations

$$
\begin{aligned}
&-\Delta z_e + (z_e \cdot \nabla) z_e + \nabla p_e = 0 \text{ and } \nabla \cdot z_e = 0 \text{ in } \Omega, \\
&z_e = 0 \text{ on } \Gamma_i, \quad z_e = v_\infty \text{ on } \Gamma_e.
\end{aligned}
$$

According to [4], if $v_\infty \in H^{\frac{3}{2}}(\Gamma_i; \mathbf{R}^3)$ obeys $\int_{\Gamma_e} v_\infty \cdot n = 0$, such a stationary solution exists in $H^2(\Omega, \mathbf{R}^3) \times H^1(\Omega)/\mathbf{R}$. For an initial condition of the form $z_e + z_0$ and a Dirichlet boundary control u on Γ_i such that $\int_{\Gamma_i} u(t) \cdot n = 0$, the pair $(z + z_e, p + p_e)$ satisfies the instationary Navier-Stokes equations, and

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(z, p) obeys:

$$\partial_t z - \Delta z + (z \cdot \nabla) z_e + (z_e \cdot \nabla) z + (z \cdot \nabla) z + \nabla p = 0 \text{ in } Q, \quad (1)$$
$$\nabla \cdot z = 0 \text{ in } Q, \ z = u \text{ on } \Sigma_i, \ z = 0 \text{ on } \Sigma_e, \ z(0) = z_0. \quad (2)$$

In this setting $Q = \Omega \times (0, \infty)$, $\Sigma_i = \Gamma_i \times (0, \infty)$, $\Sigma_e = \Gamma_e \times (0, \infty)$ and n denotes the unit normal vector to Γ_i, exterior to Ω. We assume that z_e is an unstable solution of (1)-(2) corresponding to $z_0 = z_e$. Our goal is to find a Dirichlet boundary control u on Γ_i which stabilizes the instationary Navier-Stokes system (1)-(2) for initial data z_0 small enough in an appropriate functional space. To achieve this goal, the three dimensional case is hightly demanding in terms of velocity regularity: we need that $z \in L^2(0, \infty; \mathbf{H}^{\frac{3}{2}}(\Omega))$ to obtain a stabilization result. Therefore, we look for a control u regular enough to fit the expected smoothness of z and in particular, the initial compatibility condition $u(0) = z_0|_{\Gamma_i}$ should be satisfied. A way to obtain such compatibility condition is to characterize the trace u as the first component of $(u, \sigma) \in L^2_{loc}([0, \infty); L^2(\Gamma_i; \mathbf{R}^3)) \times L^2_{loc}([0, \infty))$, where (u, σ) is the solution to the time dependent equation:

$$\begin{aligned} \partial_t u &= \Delta_b u + \sigma n + g \quad \text{on } \Sigma_i, \\ u(0) &= z_{0|\Gamma_i}, \quad \text{and } \textstyle\int_{\Gamma_i} u(t) \cdot n = 0. \end{aligned}$$

Here Δ_b is a Laplace Beltrami operator and $g \in L^2(\Sigma_i; \mathbf{R}^3)$ is such that $g(t)$ obeys $\int_{\Gamma_i} g(t) \cdot n = 0$. Thus the state (z, u) now satisfy an extended system of two coupled equations with a distributed control g on Γ_i:

$$\partial_t z - \Delta z + (z \cdot \nabla) z_e + (z_e \cdot \nabla) z + (z \cdot \nabla) z + \nabla p = 0 \text{ in } Q, \quad (3)$$
$$\nabla \cdot z = 0 \text{ in } Q, \quad z = u \text{ on } \Sigma_i, \quad z = 0 \text{ on } \Sigma_e, \quad z(0) = z_0, \quad (4)$$
$$\partial_t u - \Delta_b u - \sigma n = g \text{ in } \Sigma_i, \quad \int_{\Gamma_i} u(t) \cdot n = 0, \quad u(0) = z_0|_{\Gamma_i}. \quad (5)$$

In a first step, we consider the linear problem derived from this last coupled system by dropping the nonlinear term $(z \cdot \nabla) z$. We introduce the velocity space $V^0_n(\Omega) = \{y \in L^2(\Omega; \mathbf{R}^3) \mid \nabla \cdot y = 0 \text{ in } \Omega, y \cdot n = 0 \text{ on } \partial\Omega\}$, and the orthogonal projector P from $L^2(\Omega; \mathbf{R}^3)$ into $V^0_n(\Omega)$. Next we rewrite the extended system as an evolution equation (see section **2.2**) involving a linear unbounded operator $\mathcal{A}$ which is studied in section **4**. Then we state a linear quadratic optimal control problem (see section **2.3**) which provides a distributed feedback controller for the extended system (see section **5**). Finally, we apply the feedback controller to the initial nonlinear system (see section **6**) and we show a local stabilization result.

2. Extended system and optimal control problem

2.1 Functional framework

Let us define the spaces of free divergence functions

$$\begin{aligned} V^s(\Omega) &= \{y \in H^s(\Omega; \mathbf{R}^3) \mid \nabla \cdot y = 0 \text{ in } \Omega,\ \textstyle\int_{\partial\Omega} y \cdot n = 0\}, \quad s \geq 0, \\ V_n^s(\Omega) &= \{y \in H^s(\Omega; \mathbf{R}^3) \mid \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega\}, \quad s \geq 0, \end{aligned}$$

and the corresponding trace spaces with a free mean normal component

$$V^s(\Gamma_i) = \{y \in H^s(\Gamma_i; \mathbf{R}^3) \mid \int_{\Gamma_i} y \cdot n = 0\}, \quad V^{-s}(\Gamma_i) = V^s(\Gamma_i)', \quad s \geq 0.$$

We denote by $V_0^s(\Omega)$ the interpolation space $[V^2(\Omega) \cap H_0^1(\Omega; \mathbf{R}^3)\,,\ V_n^0(\Omega)]_{1-s/2}$ for $0 \leq s \leq 2$ and $V^{-s}(\Omega) = V_0^s(\Omega)'$ its dual counterpart with respect to the pivot space $V_n^0(\Omega)$. It is well known that

$$\begin{aligned} V_0^s(\Omega) &= V_n^s(\Omega), \quad 0 \leq s < \tfrac{1}{2}, \\ V_0^{\frac{1}{2}}(\Omega) &= \{\, y \in V_n^{\frac{1}{2}}(\Omega) \mid \textstyle\int_\Omega \rho(x)^{-1} |y|^2 < +\infty \,\}, \\ V_0^s(\Omega) &= \{\, y \in V_n^s(\Omega) \mid y = 0 \text{ on } \partial\Omega \,\}, \quad \tfrac{1}{2} < s \leq 2, \end{aligned}$$

where $\rho(x)$ is the distance from x to $\partial\Omega$. Notice that, according to the above definition, we have $V_0^s(\Omega) = V^s(\Omega) \cap H_0^1(\Omega; \mathbf{R}^3)$ for $1 \leq s \leq 2$. Finally, for $0 < T \leq \infty$, and X_1 and X_2 two Banach spaces, we introduce the function space

$$W(0, T; X_1, X_2) = L^2(0, T; X_1) \cap H^1(0, T; X_2).$$

2.2 Abstract formulation of the extended system

In this section, we state an abstract weak formulation for the system

$$\partial_t z - \Delta z + (z \cdot \nabla) z_e + (z_e \cdot \nabla) z + \kappa (z \cdot \nabla) z + \nabla p = 0 \text{ in } Q, \quad (6)$$

$$\nabla \cdot z = 0 \text{ in } Q, \ z = u \text{ on } \Sigma_i, \ z = 0 \text{ on } \Sigma_e, \ z(0) = z_0 \in V^0(\Omega), \quad (7)$$

$$\partial_t u - \Delta_b u - \sigma n = g \text{ in } \Sigma_i, \quad u(0) = u_0 \in V^{-\frac{1}{2}}(\Gamma_i). \quad (8)$$

Equation (6) corresponds to the Oseen equation if $\kappa = 0$, and to the Navier-Stokes equation if $\kappa = 1$. Equation (6), the left hand side of (8) and (7) are satisfied in the sense of distributions. We observe that σ plays the role of the Lagrange multiplier associated with the constraint $\int_{\Gamma_i} u \cdot n$.

By using transformations developed in [6] we are going to give an equivalent formulation of (6)-(7)-(8). First we define the unbounded operator $(\mathcal{D}(A), A) = (V_0^2(\Omega), A(x, \partial))$ where $A(x, \partial) = P\Delta - P(\nabla z_e) - P(z_e \cdot \nabla)$. We choose $\lambda_0 > 0$ such that $\langle (\lambda_0 - A) y | y \rangle \geq \frac{1}{2} \|y\|_{V_0^1(\Omega)}$ for all $y \in V_0^1(\Omega)$, and we introduce the Dirichlet operator $D \in \mathcal{L}(V^0(\Gamma_i), V^0(\Omega))$ associated with $\lambda_0 - A$: $Du = w$ where $u \in V^{\frac{1}{2}}(\Omega)$ satisfies

$$\lambda_0 w - A(x,\partial)w = 0,\ \nabla\cdot w = 0 \ \text{ in } \Omega,\ w = u \text{ on } \Gamma_i \ \text{ and } \ w = 0 \text{ on } \Gamma_e.$$

Thus, for $z \in L^{12/5}(\Omega;\mathbf{R}^3)$ we define $b(z,z) \in L^2(0,T;\mathcal{D}(A^*)')$ by

$$\langle b(z,z)|v\rangle = \int_\Omega (\nabla v)\, z\cdot z, \quad \forall v \in \mathcal{D}(A^*).$$

Finally, we define the unbounded operator $(\mathcal{D}(A_b), A_b) = (V^2(\Gamma_i), P_b\Delta_b)$ in $V^0(\Gamma_i)$, where $P_b \in \mathcal{L}(L^2(\Gamma_i,\mathbf{R}^3), V^0(\Gamma_i))$ denotes the orthogonal projector from $L^2(\Gamma_i;\mathbf{R}^3)$ into $V^0(\Gamma_i)$.

DEFINITION 1 *We shall say that* $(z,u) \in L^2(0,T;V^0(\Omega))\times L^2(0,T;V^{-\frac{1}{2}}(\Gamma_i))$ *if* $\kappa = 0$, *or* $(z,u) \in L^2(0,T;V^0(\Omega)\cap L^{12/5}(\Omega,\mathbf{R}^3))\times L^2(0,T;V^{-\frac{1}{2}}(\Gamma_i))$ *if* $\kappa = 1$, *is a weak solution to (6)-(7)-(8) if and only if it obeys the system:*

$$\begin{aligned}(Pz)' &= APz + (\lambda_0 - A)PDu + \kappa b(z,z) \in L^2(0,T;\mathcal{D}(A^*)'), && (9)\\ u' &= A_b u + g \in L^2(0,T;\mathcal{D}(A_b)'), && (10)\\ Pz(0) &= Pz_0 \in V^0_n(\Omega), \quad u(0) = u_0 \in V^{-\frac{1}{2}}(\Gamma_i), && (11)\\ (I-P)z &= (I-P)Du \in L^2(0,T;V^0(\Omega)), \quad u_0\cdot n = z_0\cdot n. && (12)\end{aligned}$$

THEOREM 2 *Let* (z,p,u,σ) *be an element of* $W(0,T;V^1(\Omega),V^{-1}(\Omega))\times L^2(0,T;L^2(\Omega)/\mathbf{R})\times W(0,T;V^{\frac{1}{2}}(\Gamma_i),V^{-\frac{3}{2}}(\Gamma_i))\times L^2(0,T)$. *Then* (z,p,u,σ) *satisfies (6)-(7)-(8) if and only if* (z,u) *satisfy (9)-(10)-(11)-(12).*

According to [6], the right hand side of (12) is equivalent to $(I-P)z_0 = (I-P)Du_0$. Then (12) ensures that the couple (z,z_0) is entirely determined by its projected part (Pz,Pz_0) and the boundary values (u,u_0). In the following we only consider the new 'extended' state $Y = (Pz,u)$ and the initial condition $Y_0 = (Pz_0,u_0)$. We define $\mathcal{H}^0 = V^0_n(\Omega)\times V^{-\frac{1}{2}}(\Gamma_i)$ and an adequate unbounded operator $(\mathcal{D}(\mathcal{A}),\mathcal{A})$ in $\mathcal{H}^0$ - $\mathcal{A}$ is defined by (19), (20) and studied in section **4**. We introduce the bilinear operator B

$$B(Y_1,Y_2) = \begin{pmatrix} b(y_1 + (I-P)Du_1, y_2 + (I-P)Du_2 \\ 0 \end{pmatrix}. \qquad (13)$$

THEOREM 3 *Let* $(z,p,u,\sigma) \in W(0,T;V^1(\Omega),V^{-1}(\Omega))\times L^2(0,T;L^2(\Omega)/\mathbf{R})\times W(0,T;V^{\frac{1}{2}}(\Gamma_i),V^{-\frac{3}{2}}(\Gamma_i))\times L^2(0,T)$. *Then* (z,p,u,σ) *satisfies (6)-(7)-(8) if and only if (12) holds true and the state* $Y = (Pz,u)$ *satisfies*

$$Y' = \mathcal{A}Y + \kappa B(Y,Y) + \begin{pmatrix} 0 \\ g \end{pmatrix} \in L^2(0,T;\mathcal{D}(\mathcal{A}^*)'), \quad Y(0) = Y_0 \in \mathcal{H}^0. \qquad (14)$$

2.3 The extended system and the linear quadratic control problem

The feedback control law is obtained by studying the control problem

$$(\mathcal{Q})^{\infty}_{z_0,u_0} \quad \inf\{\, \mathcal{I}(g) \mid g \in L^2(0,\infty; V^0(\Gamma_i))\,\},$$

where

$$\mathcal{I}(g) = \frac{1}{2}\int_0^\infty \int_\Omega |\nabla z_g|^2 + \frac{1}{2}\int_0^\infty \int_{\Gamma_i} |g|^2,$$

and $(z_g, u_g) \in W(0,\infty; V^1(\Omega) \times V^{\frac{1}{2}}(\Gamma_i), V^{-1}(\Omega) \times V^{-\frac{3}{2}}(\Gamma_i))$ satisfies (6)-(7)-(8). If we introduce $\mathcal{C}Y = \nabla(y + (I-P)Du) \in \mathcal{Z}$ with $\mathcal{Z} = L^2(\Omega; \mathbf{R}^9)$ - $\mathcal{C}$ is studied in section **4** lemma 9 - we can rewrite $(\mathcal{Q})^{\infty}_{z_0,u_0}$ in the form:

$$(\mathcal{P}^{\infty}_{Y_0}) \quad \inf\{\mathcal{J}(g) \mid g \in L^2(0,\infty; V^0(\Gamma_i))\},$$

where

$$\mathcal{J}(g) = \frac{1}{2}\int_0^\infty \|\mathcal{C}Y_g\|^2_{\mathcal{Z}} + \frac{1}{2}\int_0^\infty \int_{\Gamma_i} |g|^2,$$

and Y_g satisfies (14) for $\kappa = 0$.

3. Main result

THEOREM 4 *Let* Π_2 *and* Π_3 *be the operator defined in (35). Consider the following coupled system,*

$$\partial_t z - \Delta z + (z\cdot\nabla)z_e + (z_e\cdot\nabla)z + (z\cdot\nabla)z + \nabla p = 0 \text{ in } Q, \tag{15}$$

$$\nabla\cdot z = 0 \text{ in } Q, \quad z = u \quad \text{in } \Sigma, \quad z = 0 \quad \text{in } \Sigma_e, \tag{16}$$

$$\partial_t u - \Delta_b u + \Pi_3 u - \sigma\, n = -\Pi_2 P z \text{ in } \Sigma, \tag{17}$$

$$z(0) = z_0 \in V^{\frac{1}{2}}(\Omega), \quad u(0) = u_0 \in V^0(\Gamma_i). \tag{18}$$

There exists $c_0 > 0$ *and* $\mu_0 > 0$ *such that, if* $\delta \in (0, \mu_0)$ *and*

$$(z_0, u_0) \in \mathcal{W}_\delta = \{(z_0,u_0) \in V^{\frac{1}{2}}(\Omega) \times V^0(\Gamma_i) \mid z_0 - Du_0 \in V_0^{\frac{1}{2}}(\Omega), \ \|u_0\|_{V^0(\Gamma_i)} + \|Pz_0\|_{V^{\frac{1}{2}}(\Omega)} \le c_0\delta\},$$

then, (15)-(16)-(17)-(18) admit a unique solution in the set

$$\mathcal{D}_\delta = \Big\{ (z,p,u,\sigma) \in W(0,+\infty; V^{\frac{3}{2}}(\Omega), V^{-\frac{1}{2}}(\Omega)) \times L^2(0,\infty; H^{\frac{1}{2}}(\Omega)/\mathbf{R}) \times W(0,+\infty; V^1(\Gamma_i), V^{-1}(\Gamma_i)) \times L^2(0,\infty) \mid \|z\|_{L^2(0,+\infty; V^{\frac{3}{2}}(\Omega))} + \|u\|_{L^2(0,+\infty;V^1(\Gamma_i))} + \|\sigma\|_{L^2(0,\infty)} \le \delta, \ \|p\|_{L^2(0,+\infty;H^{\frac{1}{2}}(\Omega))} \le \delta(1+\delta) \Big\}.$$

Moreover, (z, u) *obeys*

$$\|z(t)\|_{V^{\frac{1}{2}}(\Omega)} + \|u(t)\|_{V^0(\Gamma_i)} \le C(\|u_0\|_{V^0(\Gamma_i)} + \|Pz_0\|_{V^{\frac{1}{2}}(\Omega)})\, e^{-\eta t}, \quad t \ge 0.$$

4. The operator $\mathcal{A}$

The goals of this section are:

- to give a definition of the unbounded operator $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ in $\mathcal{H}^0$.
- to characterize the function spaces for which the mapping $Y \mapsto (Y' - \mathcal{A}Y, Y(0))$ is an isomorphism, in order to have optimal regularity results for the extended system (14) when $\kappa = 0$.
- to characterize the functional spaces for which the mapping $Q \mapsto (-Q' - \mathcal{A}^*Q, Q(T))$ is an isomorphism, in order to study the backward adjoint equation which appears in the characterization of the solution to $(\mathcal{P}_{Y_0}^{\infty})$ - see part **4**.

THEOREM 5
We define the unbounded operator $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ *in* $\mathcal{H}^0 = V_n^0(\Omega) \times V^{-\frac{1}{2}}(\Gamma_i)$ *by*

$$\mathcal{D}(\mathcal{A}) = \Big\{(y, u) \in V_n^2(\Omega) \times V^{\frac{3}{2}}(\Gamma_i) \mid (y - PDu) \in V_0^2(\Omega)\Big\}, \quad (19)$$

$$\mathcal{A} = \begin{pmatrix} A(x, \partial) & (\lambda_0 - A(x, \partial))PD \\ 0 & P_b\Delta_b \end{pmatrix}. \quad (20)$$

The domain $\mathcal{D}(\mathcal{A})$ *is dense in* $\mathcal{H}^0$*, and* $\mathcal{A}$ *generates an analytic semigroup in* $\mathcal{H}^0$*. Moreover, for* $0 \le \theta \le 1$*, the identifications below hold*

$$\begin{aligned} \mathcal{D}((\lambda_0 - \mathcal{A})^\theta) &= [\mathcal{D}(\mathcal{A}), \mathcal{H}^0]_{1-\theta}, \quad (21) \\ &= \{(y, u) \in V_n^{2\theta}(\Omega) \times V^{2\theta - \frac{1}{2}}(\Gamma_i) \mid (y - PDu) \in V_0^{2\theta}(\Omega)\}. \end{aligned}$$

The unbounded operator $(\mathcal{D}(\mathcal{A}^*), \mathcal{A}^*)$ *in* $\mathcal{H}_*^0 = V_n^0(\Omega) \times V^{\frac{1}{2}}(\Gamma_i)$ *is defined by*

$$\mathcal{D}(\mathcal{A}^*) = V_0^2(\Omega) \times V^{\frac{5}{2}}(\Gamma_i),$$

$$\mathcal{A}^* = \begin{pmatrix} A^*(x, \partial) & 0 \\ D^*(\lambda - A^*(x, \partial)) & P_b\Delta_b \end{pmatrix}.$$

It is the adjoint of $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ *with respect to the pivot space* $V_n^0(\Omega) \times V^0(\Gamma_i)$*. The domain* $\mathcal{D}(\mathcal{A}^*)$ *is dense in* $\mathcal{H}_*^0$ *and* $\mathcal{A}^*$ *generates an analytic semigroup in* $\mathcal{H}_*^0$*. Finally, for* $0 \le \theta \le 1$*, the identifications below hold*

$$\mathcal{D}((\lambda_0 - \mathcal{A}^*)^\theta) = [\mathcal{D}(\mathcal{A}^*), \mathcal{H}_*^0]_{1-\theta} = V_0^{2\theta}(\Omega) \times V^{\frac{1}{2} + 2\theta}(\Gamma_i). \quad (22)$$

Moreover, for $0 \leq \theta \leq 1$, we define the function spaces

$$\mathcal{H}^{2\theta} = [\mathcal{D}(\mathcal{A}), \mathcal{H}^0]_{1-\theta},\ \mathcal{H}_*^{2\theta} = [\mathcal{D}(\mathcal{A}^*), \mathcal{H}_*^0]_{1-\theta},$$
$$\mathcal{H}^{-2\theta} = (\mathcal{H}_*^{2\theta})',\ \mathcal{H}_*^{-2\theta} = (\mathcal{H}^{2\theta})'.$$

Then, as a consequence of the analyticity of the semigroups $(e^{\mathcal{A}t})_{t\geq 0}$ and $(e^{\mathcal{A}^*t})_{t\geq 0}$ respectively in $\mathcal{H}^0$ and in $\mathcal{H}_*^0$, we can state a general isomorphism Theorem (see [1], Chap.3, Thm 2.2, p.166):

THEOREM 6 *For every* $0 \leq \theta \leq 1$, *the mappings below are isomorphisms:*

$$\begin{array}{rcl} W(0,T;\mathcal{H}^{2\theta},\mathcal{H}^{2(\theta-1)}) & \rightarrow & L^2(0,T;\mathcal{H}^{2(\theta-1)}) \times [\mathcal{H}^{2\theta},\mathcal{H}^{2(\theta-1)}]_{\frac{1}{2}}, \\ Y & \mapsto & (Y' - \mathcal{A}Y, Y(0)), \end{array}$$

$$\begin{array}{rcl} W(0,T;\mathcal{H}_*^{2\theta},\mathcal{H}_*^{2(\theta-1)}) & \rightarrow & L^2(0,T;\mathcal{H}_*^{2(\theta-1)}) \times [\mathcal{H}_*^{2\theta},\mathcal{H}_*^{2(\theta-1)}]_{\frac{1}{2}}, \\ Q & \mapsto & (-Q' - \mathcal{A}^*Q, Q(T)). \end{array}$$

Next we determine the spaces $[\mathcal{H}^{2\theta},\mathcal{H}^{2(\theta-1)}]_{\frac{1}{2}}$ of initial conditions.

LEMMA 7 *For all* $0 \leq \theta \leq 1$ *the following characterization holds:*

$$[\mathcal{H}^{2\theta},\mathcal{H}^{2(\theta-1)}]_{\frac{1}{2}} = \mathcal{H}^{2\theta-1}. \tag{23}$$

Finally, a direct application of Theorem 6 with (23) ensures the existence of a unique solution Y to the extended linear system (14) when $\kappa = 0$.

THEOREM 8 *Let* $g \in L^2(0,T;V^0(\Gamma_i))$ *and* $Y_0 \in \mathcal{H}^0$. *There exists a unique solution* $Y \in L^2(0,T;\mathcal{H}^0)$ *to the extended system*

$$Y' = \mathcal{A}Y + \begin{pmatrix} 0 \\ g \end{pmatrix} \in L^2(0,T;\mathcal{D}(\mathcal{A}^*)'), \quad Y(0) = Y_0 \in \mathcal{H}^0. \tag{24}$$

Moreover, Y *belongs to* $W(0,T;\mathcal{H}^1,\mathcal{H}^{-1})$. *More generally, if we assume that* $Y_0 \in \mathcal{H}^{2\theta}$, $0 \leq \theta \leq \frac{1}{2}$, *then* Y *belongs to* $W(0,T;\mathcal{H}^{2\theta+1},\mathcal{H}^{2\theta-1})$.

We now treat the backward adjoint equation which appears in the characterization of the solution to $(\mathcal{P}_{Y_0}^{\infty})$.

LEMMA 9 *Let us define* $\mathcal{C} \in \mathcal{L}(\mathcal{H}^1, \mathcal{Z})$ *with* $\mathcal{Z} = L^2(\Omega, \mathbf{R}^9)$ *by* $\mathcal{C} : Y \rightarrow \nabla(y + (I-P)Du)$. *Then*

$$\|\mathcal{C}.\|_{\mathcal{Z}} \sim \|.\|_{\mathcal{H}^1} \quad \textit{and} \quad \mathcal{C}^*\mathcal{C} \in \mathcal{L}(\mathcal{H}^{\theta+1}, \mathcal{H}_*^{\theta-1}), \quad 0 \leq \theta \leq 1. \tag{25}$$

Then Theorem 6 with (25) leads to the following theorem.

THEOREM 10 *Let $Y \in L^2(0,T;\mathcal{H}^1)$. There exists a unique solution $Q \in L^2(0,T;\mathcal{H}^0_*)$ to the backward equation*

$$-Q' = \mathcal{A}^* Q + \mathcal{C}^* \mathcal{C} Y, \quad Q(T) = 0. \tag{26}$$

Moreover, Q belongs to $W(0,T;\mathcal{H}^1_*,\mathcal{H}^{-1}_*)$. More generally, if we assume that $Y \in L^2(0,T;\mathcal{H}^{2\theta})$, $\frac{1}{2} \le \theta \le 1$, then Q belongs to $W(0,T;\mathcal{H}^{2\theta}_*,\mathcal{H}^{2(\theta-1)}_*)$.

5. Resolution of the optimal control problem

5.1 The finite time horizon case

Let $0 < T < \infty$ be a finite time horizon. To deal with the optimal control problem $(\mathcal{P}^T_{Y_0})$ we first study the following problem:

$$(\mathcal{P}^T_\xi) \quad \inf\{\mathcal{J}_T(g) \mid g \in L^2(0,T;V^0(\Gamma_i))\}, \tag{27}$$

where

$$\mathcal{J}_T(g) = \frac{1}{2}\int_0^T \|\mathcal{C}Y_g\|^2_{\mathcal{Z}} + \frac{1}{2}\int_0^T \int_{\Gamma_i} |g|^2,$$

and $Y_g \in W(0,T;\mathcal{H},\mathcal{H}^{-1})$ is the solution to

$$Y' = \mathcal{A}Y + \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad Y(0) = \xi \in \mathcal{H}^0. \tag{28}$$

We introduce the projection operator $\Lambda : (f,g) \in \mathcal{H}^0_* \mapsto (0,g)$. The problem $(\mathcal{P}^T_\xi)$ admits a unique solution $(0, g_{\xi,T})$ where $(0, g_{\xi,T}) = -\Lambda Q_{\xi,T}$ and $(Y_{\xi,T}, Q_{\xi,T})$ is the unique solution to the system

$$(S^T) \quad \left\{ \begin{array}{lll} Y' & = \mathcal{A}Y - \Lambda Q, & Y(0) = \xi \in \mathcal{H}^0, \\ -Q' & = \mathcal{A}^*Q + \mathcal{C}^*\mathcal{C}Y, & Q(T) = 0. \end{array} \right.$$

Finally, we denote by $\Pi(T) \in \mathcal{L}(\mathcal{H}^0, \mathcal{H}^0_*)$, the mapping

$$\Pi(T) : \xi \mapsto Q_{\xi,T}(0).$$

5.2 The infinite time horizon case

Since, for every ξ and $0 < T < \infty$, the solution to $(\mathcal{P}^T_\xi)$ has been characterized, we are in position to study the optimal control problem $(\mathcal{P}^\infty_\xi)$ and the regularity of its solution in function of the regularity of ξ. The problem $(\mathcal{P}^\infty_\xi)$ is defined by

$$(\mathcal{P}^\infty_\xi) \quad \inf\{\mathcal{J}(g) \mid g \in L^2(0,\infty;V^0(\Gamma_i))\}, \tag{29}$$

where the following functional satisfies (28):

$$\mathcal{J}(g) = \frac{1}{2}\int_0^\infty \|CY_g\|_{\mathcal{Z}}^2 + \frac{1}{2}\int_0^\infty \int_{\Gamma_i} |g|^2, \quad Y_g \in W(0,\infty;\mathcal{H},\mathcal{H}^{-1}).$$

Using a null controllability result stated in [3] we can show that there exists a control $g \in L^2(0,\infty;V^0(\Gamma_i))$ such that $\mathcal{J}(g) < +\infty$. This gives us the existence of a unique solution g_ξ to $(\mathcal{P}_\xi^\infty)$.

THEOREM 11 *The problem $(\mathcal{P}_\xi^\infty)$ admits a unique solution g_ξ where $g_\xi = -\psi_\xi$ and $(Y_\xi, Q_\xi) = ((y_\xi, u_\xi), (\Phi_\xi, \psi_\xi)) \in W(0,\infty;\mathcal{H}^1,\mathcal{H}^{-1}) \times W(0,\infty;\mathcal{H}_*^1,\mathcal{H}_*^{-1})$ is the unique solution to the system:*

$$(S)\begin{cases} Y' &= \mathcal{A}Y - \Lambda Q, \qquad Y(0) = \xi \in \mathcal{H}^0, \\ -Q' &= \mathcal{A}^*Q + C^*CY, \qquad Q(\infty) = 0. \end{cases}$$

Moreover, there exists $\Pi \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}_*^0)$, $\Pi^* = \Pi$ such that

$$Q_\xi = \Pi Y_\xi, \quad \mathcal{J}(g_\xi) = \frac{1}{2}\langle \Pi\xi|\xi\rangle_{\mathcal{H}_*^0,\mathcal{H}^0}. \tag{30}$$

The control g_ξ has been calculated as the limit of $g_{\xi,T}$ when $T \to \infty$, where $g_{\xi,T}$ is the unique solution to $(\mathcal{P}_\xi^T)$.
We now focus on the properties of Π. First, from the limit $\Pi(T) \to \Pi$ as $T \to \infty$ we can show that Π satisfies an algebraic Riccati equation. Next, Theorems 8 and 10 for (S) lead to sharp regularity result for Π.

THEOREM 12 *Π satisfies a Riccati equation: $\forall(\xi,\zeta) \in \mathcal{H}^1 \times \mathcal{H}^1$,*

$$\langle \Pi\xi|\mathcal{A}\zeta\rangle_{\mathcal{H}_*^1,\mathcal{H}^{-1}} + \langle \mathcal{A}\xi|\Pi\zeta\rangle_{\mathcal{H}^{-1},\mathcal{H}_*^1} + (C\xi|C\zeta)_{\mathcal{Z}} - (\Lambda\Pi\xi|\Lambda\Pi\zeta)_{V^0(\Gamma_i)} = 0, \tag{31}$$

and the regularizing property $\quad \Pi \in \mathcal{L}(\mathcal{H}^{2\theta}, \mathcal{H}_*^{2\theta}), \quad 0 \le \theta \le \frac{1}{2}$.

Next we set $\mathcal{A}_\Pi = (\Lambda\Pi - \mathcal{A})$, so that the optimal trajectory Y_ξ satisfy

$$Y' + \mathcal{A}_\Pi Y = 0, \quad Y(0) = \xi \in \mathcal{H}^0. \tag{32}$$

From a classical result due to Datko in [5, Chap 4, Thm 4.1], we show the exponential stability of $e^{-\mathcal{A}_\Pi t}Y_0$. Let $\alpha > 0$, the positivity of $\mathcal{A}_\Pi$ allows us to define its fractional power $\mathcal{A}_\Pi^\alpha$ and we respectively identify $\mathcal{D}(\mathcal{A}_\Pi^\alpha)$ and $\mathcal{D}(\mathcal{A}_\Pi^{*\alpha})$ with $\mathcal{D}((\lambda_0 - \mathcal{A})^\alpha)$ and $\mathcal{D}((\lambda_0 - \mathcal{A}^*)^\alpha)$. Thus (21) and (22) ensures a characterization $\mathcal{D}(\mathcal{A}_\Pi^\alpha)$ and $\mathcal{D}(\mathcal{A}_\Pi^{*\alpha})$.

THEOREM 13 *The unbounded operator $(\mathcal{D}(-\mathcal{A}_\Pi), -\mathcal{A}_\Pi)$ generates an analytic exponentially stable semigroup in $\mathcal{H}^0$ and the characterizations below hold*

$$\mathcal{D}(\mathcal{A}_\Pi^\theta) = \mathcal{H}^{2\theta}, \quad \mathcal{D}(\mathcal{A}_\Pi^{*\theta}) = \mathcal{H}_*^{2\theta}, \quad 0 \le \theta \le 1.$$

Next we define new norms in the spaces $\mathcal{H}^\alpha$ and $\mathcal{H}^{1+\alpha}$ which are essential to study the stabilization of the Navier-Stokes equation (see section **6**).

DEFINITION 14 *We define* $\Pi^{(\alpha)} \in \mathcal{L}(\mathcal{H}^\alpha, \mathcal{H}_*^{-\alpha})$ *by* $\Pi^{(\alpha)} = \mathcal{A}_\Pi{}^{*\frac{\alpha}{2}}\,\Pi\,\mathcal{A}_\Pi{}^{\frac{\alpha}{2}}$.

THEOREM 15 *The operator* $\Pi^{(\alpha)}$ *has the regularizing property:*

$$\Pi^{(\alpha)} \in \mathcal{L}(\mathcal{H}^{2\theta+\alpha}, \mathcal{H}_*^{2\theta-\alpha}), \quad 0 \le \theta \le \frac{1}{2}. \tag{33}$$

DEFINITION 16 *We define the two mappings* $\mathcal{N}_\alpha$ *and* $\mathcal{R}_{1+\alpha}$ *by*

$$\begin{aligned}
\mathcal{N}_\alpha(\xi) &= (\langle \Pi^{(\alpha)}\xi|\xi\rangle_{\mathcal{H}_*^{-\alpha},\mathcal{H}^\alpha})^{\frac{1}{2}}, \quad \xi \in \mathcal{H}^\alpha,\\
\mathcal{R}_{1+\alpha}(\xi) &= (\langle \mathcal{A}_\Pi\xi|\Pi^{(\alpha)}\xi\rangle_{\mathcal{H}^{-1+\alpha},\mathcal{H}_*^{1-\alpha}})^{\frac{1}{2}}, \quad \xi \in \mathcal{H}^{1+\alpha}.
\end{aligned}$$

THEOREM 17 $\mathcal{N}_\alpha$ *and* $\mathcal{R}_{1+\alpha}$ *define norms respectively on* $\mathcal{H}^\alpha$ *and* $\mathcal{H}^{1+\alpha}$,

$$\mathcal{N}_\alpha(\cdot) \sim \|\cdot\|_{\mathcal{H}^\alpha}, \quad \mathcal{R}_{1+\alpha}(.) \sim \|\cdot\|_{\mathcal{H}^{1+\alpha}}. \tag{34}$$

We shall point out that the expression of $\mathcal{R}_{1+\alpha}(\xi)$ is explicitly given by

$$\langle \mathcal{A}_\Pi\xi|\Pi^{(\alpha)}\xi\rangle_{\mathcal{H}^{-1+\alpha},\mathcal{H}_*^{1-\alpha}} = \frac{1}{2}\|C\mathcal{A}_\Pi{}^{\frac{\alpha}{2}}\xi\|_{\mathcal{Z}}^2 + \frac{1}{2}\|\Lambda\Pi\mathcal{A}_\Pi{}^{\frac{\alpha}{2}}\xi\|_{V^0(\Gamma_i)}^2, \quad \forall \xi \in \mathcal{H}^{1+\alpha},$$

which follows from (31) in which we have replaced ξ and ζ by $\mathcal{A}_\Pi^{\frac{\alpha}{2}}\xi$.
We finish this section by giving the PDE formulation of the closed loop system (32).

THEOREM 18 *The operator* Π *can be rewritten as follows,*

$$\Pi = \begin{pmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{pmatrix}, \quad \begin{array}{l} \Pi_1 \in \mathcal{L}(V_n^0(\Omega)), \\ \Pi_2 \in \mathcal{L}(V_n^0(\Omega), V^{\frac{1}{2}}(\Gamma_i)), \\ \Pi_3 \in \mathcal{L}(V^{-\frac{1}{2}}(\Gamma_i), V^{\frac{1}{2}}(\Gamma_i)), \end{array} \tag{35}$$

where Π_1 *and* Π_3 *are positive, definite and self-adjoint operators. Then* $Y = (Pz, u) \in W(0,\infty; \mathcal{H}^1, \mathcal{H}^{-1})$ *satisfies (32) if and only if the element* (z, p, u, σ) *of* $W(0,\infty; V^1(\Omega), V^{-1}(\Omega)) \times L^2(0,\infty; L^2(\Omega)/\mathbf{R}) \times W(0,\infty; V^{\frac{1}{2}}(\Gamma_i), V^{-\frac{3}{2}}(\Gamma_i)) \times L^2(0,\infty)$ *satisfies*

$$\begin{aligned}
&\partial_t z - \Delta z + (z\cdot\nabla)z_e + (z_e\cdot\nabla)z + \nabla p = 0, \quad \nabla\cdot z = 0 \textit{ in } Q,\\
&\partial_t u - \Delta_b u + \Pi_3 u - \sigma n = -\Pi_2 Pz, \quad z = u \textit{ in } \Sigma_i, \quad z = 0 \textit{ in } \Sigma_e,\\
&z(0) = z_0 \in V^0(\Omega), \quad u(0) = u_0 \in V^{-\frac{1}{2}}(\Gamma_i), \quad z_0\cdot n = u_0\cdot n.
\end{aligned}$$

6. Stabilization of the Navier-Stokes equations

We now come back to the stabilization of the Navier-Stokes system. We now consider the nonlinear system (15)-(16)-(17)-(18) which we can rewrite in the abstract formulation

$$Y' + \mathcal{A}_\Pi Y = B(Y,Y), \quad Y(0) = Y_0 \in \mathcal{H}^{\frac{1}{2}}. \tag{36}$$

Finally, Theorem 4 is a direct consequence of the following result.

THEOREM 19 *There exists $c_0 > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0, \mu_0)$ and $Y_0 \in \mathcal{V}_\delta = \{Y \in \mathcal{H}^{\frac{1}{2}} | \quad \|Y\|_{\mathcal{H}^{\frac{1}{2}}} < c_0\delta\}$ then, (36) admit a unique solution in the set $\mathcal{S}_\delta = \{\, Y \in W(0,\infty; \mathcal{H}^{\frac{3}{2}}, \mathcal{H}^{-\frac{1}{2}}) | \, \|Y\|_{W(0,\infty;\mathcal{H}^{\frac{3}{2}},\mathcal{H}^{-\frac{1}{2}})} \leq \delta\,\}$. Moreover, there exists $\eta > 0$ such that*

$$\|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}} \leq C\|Y_0\|_{\mathcal{H}^{\frac{1}{2}}} e^{-\eta t}. \tag{37}$$

Proof. Here, we give a brief sketch of the proof of the stability result. We multiply the left hand side of (36) by $\Pi^{(\frac{1}{2})}Y(t)$. According to (34) with $\alpha = \frac{1}{2}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{N}^2_{\frac{1}{2}}(Y(t)) + \mathcal{R}^2_{\frac{3}{2}}(Y(t)) = \langle B(Y(t), Y(t)) | \Pi^{(\frac{1}{2})}Y(t)\rangle.$$

Then we invoke a classical estimation - see [2, Chap.6, 6.9 and 6.10] - to obtain $|\langle B(Y(t), Y(t))|\Pi^{(\frac{1}{2})}Y(t)\rangle| \leq C\|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}}\|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}}\|\Pi^{(\frac{1}{2})}Y(t)\|_{\mathcal{H}^{\frac{1}{2}}}$. According to (33) with $\alpha = \theta = \frac{1}{2}$ it yields

$$|\langle B(Y(t), Y(t))|\Pi^{(\frac{1}{2})}Y(t)\rangle| \leq C\|Y(t)\|_{\mathcal{H}^{\frac{1}{2}}}\|Y(t)\|^2_{\mathcal{H}^{\frac{3}{2}}}.$$

Thus (34) gives us $C_0 > 0$ such that

$$\frac{d}{dt}\mathcal{N}^2_{\frac{1}{2}}(Y(t)) + 2(1 - C_0\mathcal{N}_{\frac{1}{2}}(Y(t)))\mathcal{R}^2_{\frac{3}{2}}(Y(t) \leq 0. \tag{38}$$

It is obvious to see that if $\mathcal{N}_{\frac{1}{2}}(Y_0)$ is small enough, we can choose $\mathcal{N}_{\frac{1}{2}}(Y_0) < \frac{1}{4C_0}$, so that the mapping $t \longrightarrow \mathcal{N}_{\frac{1}{2}}(Y(t))$ be a nonincreasing function with values less than $\frac{1}{4C_0}$. This gives the existence of δ, and (37) follows from (38).

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