2

Preliminary Linear Algebra

This chapter includes a rapid review of basic concepts of Linear Algebra. After defining fields and vector spaces, we are going to cover bases, dimension and linear transformations. The theory of simultaneous equations and triangular factorization are going to be discussed as well. The chapter ends with the fundamental theorem of linear algebra.

2.1 Vector Spaces

2.1.1 Fields and linear spaces

Definition 2.1.1 *A set $\mathbb{F}$ together with two operations*

$$\begin{cases} + : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F} \text{ Addition} \\ \cdot : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F} \text{ Multiplication} \end{cases}$$

is called a field if

1. a) $\alpha + \beta = \beta + \alpha$, $\forall \alpha, \beta \in \mathbb{F}$ *(Commutative)*
   b) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, $\forall \alpha, \beta, \gamma \in \mathbb{F}$ *(Associative)*
   c) $\exists$ *a distinguished element denoted by 0* $\ni$ $\forall \alpha \in \mathbb{F}$, $\alpha + 0 = \alpha$ *(Additive identity)*
   d) $\forall \alpha \in \mathbb{F} \; \exists - \alpha \in \mathbb{F} \; \ni \alpha + (-\alpha) = 0$ *(Existence of an inverse)*
2. a) $\alpha \cdot \beta = \beta \cdot \alpha$, $\forall \alpha, \beta \in \mathbb{F}$ *(Commutative)*
   b) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$, $\forall \alpha, \beta, \gamma \in \mathbb{F}$ *(Associative)*
   c) $\exists$ *an element denoted by 1* $\ni$ $\forall \alpha \in \mathbb{F}$, $\alpha \cdot 1 = \alpha$ *(Multiplicative identity)*
   d) $\forall \alpha \neq 0 \in \mathbb{F} \; \exists \alpha^{-1} \in \mathbb{F} \; \ni \alpha \cdot \alpha^{-1} = 1$ *(Existence of an inverse)*
3. $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$, $\forall \alpha, \beta, \gamma \in \mathbb{F}$ *(Distributive)*

Definition 2.1.2 *Let $\mathbb{F}$ be a field. A set V with two operations*

$$\begin{cases} + : V \times V \mapsto V & \textit{Addition} \\ \cdot : \mathbb{F} \times V \mapsto V & \textit{Scalar multiplication} \end{cases}$$

is called a vector space (linear space) over the field $\mathbb{F}$ if the following axioms are satisfied:

1. a) $u + v = u + v, \ \forall u, v \in V$
   b) $(u + v) + w = u + (v + w), \ \forall u, v, w \in V$
   c) $\exists$ *a distinguished element denoted by* $\theta \ni \forall v \in V, \ v + \theta = v$
   d) $\forall v \in V \ \exists$ *unique* $-v \in V \ \ni v + (-v) = \theta$
2. a) $\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u, \ \forall \alpha, \beta \in \mathbb{F}, \ \forall u \in V$
   b) $\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v), \ \forall \alpha \in \mathbb{F}, \ \forall u, v \in V$
   c) $(\alpha + \beta) \cdot u = (\alpha \cdot u) + (\beta \cdot u), \ \forall \alpha, \beta \in \mathbb{F}, \ \forall u \in V$
   d) $1 \cdot u = u, \ \forall u \in V$, *where 1 is the multiplicative identity of* $\mathbb{F}$

Example 2.1.3 $\mathbb{R}^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_n)^T : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\}$ *is a vector space over* $\mathbb{R}$ *with* $(\alpha_1, \alpha_2, \ldots, \alpha_n) + (\beta_1, \beta_2, \ldots, \beta_n) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$; $c \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n) = (c\alpha_1, c\alpha_2, \ldots, c\alpha_n)$; *and* $\theta = (0, 0, \ldots, 0)^T$.

Example 2.1.4 *The set of all m by n complex matrices is a vector space over* $\mathbb{C}$ *with usual addition and multiplication.*

Proposition 2.1.5 *In a vector space V,*

i. θ *is unique.*
ii. $0 \cdot v = \theta, \ \forall v \in V$.
iii. $(-1) \cdot v = -v, \ \forall v \in V$.
iv. $-\theta = \theta$.
v. $\alpha \cdot v = \theta \Leftrightarrow \alpha = 0$ *or* $v = \theta$.

Proof. Exercise. □

2.1.2 Subspaces

Definition 2.1.6 *Let V be a vector space over $\mathbb{F}$, and let $W \subset V$. W is called a subspace of V if W itself is a vector space over $\mathbb{F}$.*

Proposition 2.1.7 *W is a subspace of V if and only if it is closed under vector addition and scalar multiplication, that is*

$$w_1, w_2 \in W, \ \alpha_1, \alpha_2 \in \mathbb{F} \Leftrightarrow \alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 \in W.$$

Proof. (Only if: $\Rightarrow$) Obvious by definition.
(If: $\Leftarrow$) we have to show that $\theta \in W$ and $\forall w \in W, \ -w \in W$.

i. Let $\alpha_1 = 1$, $\alpha_2 = -1$, and $w_1 = w_2$. Then,

$$1 \cdot w_1 + (-1) \cdot w_1 = w_1 + (-w_1) = \theta \in W.$$

ii. Take any w. Let $\alpha_1 = -1$, $\alpha_2 = 0$, and $w_1 = w$. Then,

$$(-1) \cdot w + (0) \cdot w_2 = -w \in W. \quad \square$$

Example 2.1.8 *$S \subset \mathbb{R}^{2\times 3}$, consisting of the matrices of the form* $\begin{bmatrix} 0 & \beta & \gamma \\ \alpha & \alpha-\beta & \alpha+2\gamma \end{bmatrix}$ *is a subspace of* $\mathbb{R}^{2\times 3}$.

Proposition 2.1.9 *If W_1, W_2 are subspaces, then so is $W_1 \cap W_2$.*

Proof. Take $w_1, w_2 \in W_1 \cap W_2$, $\alpha_1, \alpha_2 \in \mathbb{F}$.

i. $w_1, w_2 \in W_1 \Rightarrow \alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 \in W_1$
ii. $w_1, w_2 \in W_2 \Rightarrow \alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 \in W_2$

Thus, $\alpha_1 w_1 + \alpha_2 w_2 \in W_1 \cap W_2$. $\square$

Remark 2.1.10 *If W_1, W_2 are subspaces, then $W_1 \cup W_2$ is not necessarily a subspace.*

Definition 2.1.11 *Let V be a vector space over $\mathbb{F}$, $X \subset V$. X is said to be linearly dependent if there exists a distinct set of $x_1, x_2, \ldots, x_k \in X$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{F}$ not all zero $\ni \sum_{i=1}^k \alpha_i x_i = \theta$. Otherwise, for any subset of size k,*

$$x_1, x_2, \ldots, x_k \in X, \ \sum_{i=1}^k \alpha_i x_i = \theta \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

In this case, X is said to be linearly independent.

We term an expression of the form $\sum_{i=1}^k \alpha_i x_i$ as linear combination. In particular, if $\sum_{i=1}^k \alpha_i = 1$, we call it affine combination. Moreover, if $\sum_{i=1}^k \alpha_i = 1$ and $\alpha_i \geq 0$, $\forall i = 1, 2, \ldots, k$, it becomes convex combination. On the other hand, if $\alpha_i \geq 0$, $\forall i = 1, 2, \ldots, k$; then $\sum_{i=1}^k \alpha_i x_i$ is said to be canonical combination.

Example 2.1.12 *In $\mathbb{R}^n$, let $E = \{e_i\}_{i=1}^n$ where $e_i^T = (0, \cdots 0, 1, 0, \cdots, 0)$ is the i^{th} canonical unit vector that contains 1 in its i^{th} position and 0s elsewhere. Then, E is an independent set since*

$$\theta = \alpha_1 e_1 + \cdots + \alpha_n e_n = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \Rightarrow \alpha_i = 0, \ \forall i.$$

Let $X = \{x_i\}_{i=1}^n$ where $x_i^T = (0, \cdots 0, 1, 1, \cdots, 1)$ is the vector that contains 0s sequentially up to position i, and it contains 1s starting from position i onwards. X is also linearly independent since

$$\theta = \alpha_1 x_1 + \cdots + \alpha_n x_n = \begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \\ \vdots \\ \alpha_1 + \cdots + \alpha_n \end{bmatrix} \Rightarrow \alpha_i = 0, \ \forall i.$$

Let $Y = \{y_i\}_{i=1}^n$ *where* $y_i^T = (0, \cdots 0, -1, 1, 0, \cdots, 0)$ *is the vector that contains -1 in* i^{th} *position, 1 in* $(i+1)^{st}$ *position, and 0s elsewhere.* Y *is not linearly independent since* $y_1 + \cdots + y_n = \theta$.

Definition 2.1.13 *Let* $X \subset V$. *The set*

$$Span(X) = \left\{ v = \sum_{i=1}^{k} \alpha_i x_i \in V : x_1, x_2, \ldots, x_k \in X; \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{F}; k \in \mathbb{N} \right\}$$

is called the span of X. *If the above linear combination is of the affine combination form, we will have the affine hull of* X; *if it is a convex combination, we will have the convex hull of* X; *and finally, if it is a canonical combination, what we will have is the cone of* X. *See Figure 2.1.*

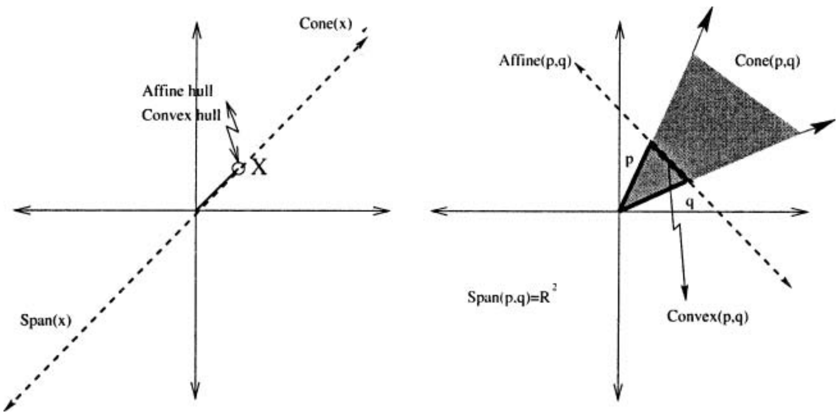

Fig. 2.1. The subspaces defined by $\{x\}$ and $\{p, q\}$.

Proposition 2.1.14 *Span(X) is a subspace of* V.

Proof. Exercise. □

2.1.3 Bases

Definition 2.1.15 *A set* X *is called a basis for* V *if it is linearly independent and spans* V.

Remark 2.1.16 *Since Span(X) $\subset V$, in order to show that it covers V, we only need to prove that $\forall v \in V$, $v \in$ Span(X).*

Example 2.1.17 *In $\mathbb{R}^n$, $E = \{e_i\}_{i=1}^n$ is a basis since E is linearly independent and $\forall \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{R}^n$, $\alpha = \alpha_1 e_1 + \cdots + \alpha_n e_n \in Span(E)$.*

$X = \{x_i\}_{i=1}^n$ is also a basis for $\mathbb{R}^n$ since $\forall \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{R}^n$, $\alpha = \alpha_1 x_1 + (\alpha_2 - \alpha_1)x_2 + \cdots + (\alpha_n - \alpha_{n-1})x_n \in Span(X)$.

Proposition 2.1.18 *Suppose $X = \{x_i\}_{i=1}^n$ is a basis for V over $\mathbb{F}$. Then,*

a) $\forall v \in V$ can be expressed as $v = \sum_{i=1}^n \alpha_i x_i$ where α_i's are unique.
b) Any linearly independent set with exactly n elements forms a basis.
c) All bases for V contain n vectors, where n is the dimension of V.

Remark 2.1.19 *Any vector space V of dimension n and an n–dimensional field $\mathbb{F}^n$ have an isomorphism.*

Proof. Suppose $X = \{x_i\}_{i=1}^n$ is a basis for V over $\mathbb{F}$. Then,

a) Suppose v has two different representations: $v = \sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^n \beta_i x_i$. Then, $\theta = v - v = \sum_{i=1}^N (\alpha_i - \beta_i)x_i \Rightarrow \alpha_i = \beta_i$, $\forall i = 1, 2, \ldots, n$. Contradiction, since X is independent.
b) Let $Y = \{y_i\}_{i=1}^n$ be linearly independent. Then, $y_1 = \sum \delta_i x_i$ ($\spadesuit$), where at least one $\delta_i \neq 0$. Without loss of generality, we may assume that $\delta_1 \neq 0$. Consider $X_1 = \{y_1, x_2, \ldots, x_n\}$. X_1 is linearly independent since $\theta = \beta_1 y_1 + \sum_{i=2}^n \beta_i x_i = \beta_1(\sum \delta_i x_i)^{(\spadesuit)} + \sum_{i=2}^n \beta_i x_i = \beta_1 \delta_1 x_1 + \sum_{i=2}^n (\beta_1 \delta_i + \beta_i)x_i \Rightarrow \beta_1 \delta_1 = 0$; $\beta_1 \delta_i + \beta_i = 0$, $\forall i = 2, \ldots, n \Rightarrow \beta_1 = 0$ ($\delta_1 \neq 0$); and $\beta_i = 0$, $\forall i = 2, \ldots, n$. Any $v \in V$ can be expressed as $v = \sum_{i=1}^n \gamma_i x_i = \gamma_1 x_1 + \sum_{i=2}^n \gamma_i x_i$
$v = \gamma_1(\delta_1^{-1} y_1 - \sum_{i=2}^n \delta_1^{-1} \delta_i x_i)^{(\spadesuit)} = (\gamma_1 \delta_1^{-1}) y_1 + \sum_{i=2}^n (\gamma_i - \gamma_1 \delta_1^{-1} \delta_i) x_i$.
Thus, $Span(X_1) = V$.
Similarly,
$X_2 = \{y_1, y_2, x_3, \ldots, x_n\}$ is a basis.
$\vdots$
$X_n = \{y_1, y_2, \ldots, y_n\} = Y$ is a basis.
c) Obvious from part b). □

Remark 2.1.20 *Since bases for V are not unique, the same vector may have different representations with respect to different bases. The aim here is to find the best (simplest) representation.*

2.2 Linear transformations, matrices and change of basis

2.2.1 Matrix multiplication

Let us examine another operation on matrices, matrix multiplication, with the help of a small example. Let $A \in \mathbb{R}^{3\times 4}$, $B \in \mathbb{R}^{4\times 2}$, $C \in \mathbb{R}^{3\times 2}$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = C = AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} \end{bmatrix}$$

Let us list the properties of this operation:

Proposition 2.2.1 *Let A, B, C, D be matrices and x be a vector.*

1. $(AB)x = A(Bx)$.
2. $(AB)C = A(BC)$.
3. $A(B + C) = AB + AC$ *and* $(B + C)D = BD + CD$.
4. $AB = BA$ *does not hold (usually* $AB \neq BA$*) in general.*
5. *Let I_n be a square n by n matrix that has 1s along the main diagonal and 0s everywhere else, called identity matrix. Then,* $AI = IA = A$.

2.2.2 Linear transformation

Definition 2.2.2 *Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$. The map $x \mapsto Ax$ describing a transformation $\mathbb{R}^n \mapsto \mathbb{R}^m$ with property (matrix multiplication)*

$$\forall x, y \in \mathbb{R}^n;\ \forall a, b \in \mathbb{R},\ \mathcal{A}(bx + cy) = b(\mathcal{A}x) + c(\mathcal{A}y)$$

is called linear.

Remark 2.2.3 *Every matrix A leads to a linear transformation $\mathcal{A}$. Conversely, every linear transformation $\mathcal{A}$ can be represented by a matrix A. Suppose the vector space V has a basis $\{v_1, v_2, \dots, v_n\}$ and the vector space W has a basis $\{w_1, w_2, \dots, w_m\}$. Then, every linear transformation $\mathcal{A}$ from V to W is represented by an m by n matrix A. Its entries a_{ij} are determined by applying $\mathcal{A}$ to each v_j, and expressing the result as a combination of the w's:*

$$\mathcal{A}v_j = \sum_{i=1}^{m} a_{ij} w_i,\ j = 1, 2, \dots, n.$$

Example 2.2.4 *Suppose $\mathcal{A}$ is the operation of integration of special polynomials if we take $1, t, t^2, t^3, \cdots$ as a basis where v_j and w_j are given by t^{j-1}. Then,*

$$\mathcal{A}v_j = \int t^{j-1}\, dt = \frac{t^j}{j} = \frac{1}{j} w_{j+1}.$$

For example, if dim $V = 4$ *and dim* $W = 5$ *then* $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$. *Let us try to integrate* $v(t) = 2t + 8t^3 = 0v_1 + 2v_2 + 0v_3 + 8v_4$*:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \Leftrightarrow \int (2t + 8t^3)\, dt = t^2 + 2t^4 = w_3 + 2w_5.$$

Proposition 2.2.5 *If the vector* x *yields coefficients of* v *when it is expressed in terms of basis* $\{v_1, v_2, \ldots, v_n\}$*, then the vector* $y = Ax$ *gives the coefficients of* $\mathcal{A}v$ *when it is expressed in terms of the basis* $\{w_1, w_2, \ldots, w_m\}$*. Therefore, the effect of* $\mathcal{A}$ *on any* v *is reconstructed by matrix multiplication.*

$$\mathcal{A}v = \sum_{i=1}^{m} y_i w_i = \sum_{i,j} a_{ij} x_j w_i.$$

Proof.

$$v = \sum_{j=1}^{n} x_j v_j \Rightarrow \mathcal{A}v = \mathcal{A}(\sum_{1}^{n} x_j v_j) = \sum_{1}^{n} x_j \mathcal{A}v_j = \sum_{j} x_j \sum_{i} a_{ij} w_i. \quad \square$$

Proposition 2.2.6 *If the matrices* A *and* B *represent the linear transformations* $\mathcal{A}$ *and* $\mathcal{B}$ *with respect to bases* $\{v_i\}$ *in* V*,* $\{w_i\}$ *in* W*, and* $\{z_i\}$ *in* Z*, then the product of these two matrices represents the composite transformation* $\mathcal{BA}$*.*

Proof. $\mathcal{A} : v \mapsto Av \quad \mathcal{B} : Av \mapsto BAv \Rightarrow \mathcal{BA} : v \mapsto BAv.$ $\square$

Example 2.2.7 *Let us construct* 3×5 *matrix that represents the second derivative* $\frac{d^2}{dt^2}$*, taking* P_4 *(polynomial of degree four) to* P_2*.*

$$t^4 \mapsto 4t^3,\ t^3 \mapsto 3t^2,\ t^2 \mapsto 2t,\ t \mapsto 1$$

$$\Rightarrow B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}.$$

Let $v(t) = 2t + 8t^3$*, then*

$$\frac{d^2 v(t)}{dt^2} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 48 \\ 0 \end{bmatrix} = 48t.$$

Proposition 2.2.8 *Suppose $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ are both bases for the vector space V, and let $v \in V$, $v = \sum_1^n x_j v_j = \sum_1^n y_j w_j$. If $v_j = \sum_1^n s_{ij} w_i$, then $y_i = \sum_1^n s_{ij} x_j$.*

Proof.

$$\sum_j x_j v_j = \sum_j \sum_i x_j s_{ij} w_i \text{ is equal to } \sum_i y_i w_i \sum_i \sum_j s_{ij} x_j w_i. \quad \square$$

Proposition 2.2.9 *Let $\mathcal{A} : V \mapsto V$. Let A_v be the matrix form of the transformation with respect to basis $\{v_1, v_2, \ldots, v_n\}$ and A_w be the matrix form of the transformation with respect to basis $\{w_1, w_2, \ldots, w_n\}$. Assume that $v_j = \sum_i s_{ij} w_j$. Then,*

$$A_v = S^{-1} A_w S.$$

Proof. Let $v \in V$, $v = \sum x_j v_j$. Sx gives the coefficients with respect to w's, then $A_w S x$ yields the coefficients of $\mathcal{A}v$ with respect to original w's, and finally $S^{-1} A_w S x$ gives the coefficients of $\mathcal{A}v$ with respect to original v's. $\square$

Remark 2.2.10 *Suppose that we are solving the system $Ax = b$. The most appropriate form of A is I_n so that $x = b$. The next simplest form is when A is diagonal, consequently $x_i = \frac{b_i}{a_{ii}}$. In addition, upper-triangular, lower-triangular and block-diagonal forms for A yield easy ways to solve for x. One of the main aims in applied linear algebra is to find a suitable basis so that the resultant coefficient matrix $A_v = S^{-1} A_w S$ has such a simple form.*

2.3 Systems of Linear Equations

2.3.1 Gaussian elimination

Let us take a system of linear m equations with n unknowns $Ax = b$. In particular,

$$\begin{array}{r} 2u + v + w = 1 \\ 4u + v = -2 \\ -2u + 2v + w = 7 \end{array} \Leftrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}.$$

Let us apply some elementary row operations:

S1. Subtract 2 times the first equation from the second,
S2. Subtract -1 times the first equation from the third,
S3. Subtract -3 times the second equation from the third.

The result is an equivalent but simpler system, $Ux = c$ where U is upper-triangular:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix}.$$

Definition 2.3.1 *A matrix U (L) is upper(lower)-triangular if all the entries below (above) the main diagonal are zero. A matrix D is called diagonal if all the entries except the main diagonal are zero.*

Remark 2.3.2 *If the coefficient matrix of a linear system of equations is either upper or lower triangular, then the solution can be characterized by backward or forward substitution. If it is diagonal, the solution is obtained immediately.*

Let us name the matrix that accomplishes S1 (E_{21}), subtracting twice the first row from the second to produce zero in entry (2,1) of the new coefficient matrix, which is a modified I_3 such that its (2,1)st entry is -2. Similarly, the elimination steps S2 and S3 can be described by means of E_{31} and E_{32}, respectively.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

These are called *elementary matrices.* Consequently,

$$E_{32}E_{31}E_{21}A = U \text{ and } E_{32}E_{31}E_{21}b = c,$$

where $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$ is lower triangular. If we undo the steps of *Gaussian elimination* through which we try to obtain an upper-triangular system $Ux = c$ to reach the solution for the system $Ax = b$, we have

$$A = E_{32}^{-1}E_{31}^{-1}E_{21}^{-1}U = LU,$$

where

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

is again lower-triangular. Observe that the entries below the diagonal are exactly the multipliers 2, -1, and -3 used in the elimination steps. We term L as the matrix form of the Gaussian elimination. Moreover, we have $Lc = b$. Hence, we have proven the following proposition that summarizes the Gaussian elimination or triangular factorization.

Proposition 2.3.3 *As long as pivots are nonzero, the square matrix A can be written as the product LU of a lower triangular matrix L and an upper triangular matrix U. The entries of L on the main diagonal are 1s; below the main diagonal, there are the multipliers l_{ij} indicating how many times of row j is subtracted from row i during elimination. U is the coefficient matrix, which appears after elimination and before back-substitution; its diagonal entries are the pivots.*

In order to solve $x = A^{-1}b = U^{-1}c = U^{-1}L^{-1}b$ we never compute inverses that would take n^3–many steps. Instead, we first determine c by forward-substitution from $Lc = b$, then find x by backward-substitution from $Ux = c$. This takes a total of n^2 operations. Here is our example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Remark 2.3.4 *Once factors U and L have been computed, the solution x' for any new right hand side b' can be found in the similar manner in only n^2 operations. For instance*

$$b' = \begin{bmatrix} 8 \\ 11 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -4 \end{bmatrix} \Rightarrow \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Remark 2.3.5 *We can factor out a diagonal matrix D from U that contains pivots, as illustrated below.*

$$U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} & \cdots & \frac{u_{1n}}{d_1} \\ & 1 & \frac{u_{23}}{d_2} & \cdots & \frac{u_{2n}}{d_2} \\ & & 1 & \cdots & \vdots \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}$$

Consequently, we have $A = LDU$, where L is lower triangular with 1s on the main diagonal, U is upper diagonal with 1s on the main diagonal and D is the diagonal matrix of pivots. LDU factorization is uniquely determined.

Remark 2.3.6 *What if we come across a zero pivot? We have two possibilities:*

Case (i) If there is a nonzero entry below the pivot element in the same column:
We interchange rows. For instance, if we are faced with

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

we will interchange row 1 and 2. The permutation matrix, $P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, represents the exchange. A permutation matrix P_{kl} is the modified identity *matrix of the same order whose rows k and l are interchanged. Note that $P_{kl} = P_{lk}^{-1}$ (exercise!). In summary, we have*

$$PA = LDU.$$

Case (ii) If the pivot column is entirely zero below the pivot entry:
The current matrix (so was A) is singular. Thus, the factorization is lost.

2.3.2 Gauss-Jordan method for inverses

Definition 2.3.7 *The left (right) inverse B of A exists if $BA = I$ $(AB = I)$.*

Proposition 2.3.8 $BA = I$ *and* $AC = I \Leftrightarrow B = C$.

Proof. $B(AC) = (BA)C \Leftrightarrow BI = IC \Leftrightarrow B = C$. □

Proposition 2.3.9 *If A and B are invertible, so is AB.*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I. \quad \square$$

Remark 2.3.10 *Let $A = LDU$. $A^{-1} = U^{-1}D^{-1}L^{-1}$ is never computed. If we consider $AA^{-1} = I$, one column at a time, we have $Ax_j = e_j, \forall j$. When we carry out elimination in such n equations simultaneously, we will follow the Gauss-Jordan method.*

Example 2.3.11 *In our example instance,*

$$[A|e_1e_2e_3] = \left[\begin{array}{rrr|rrr} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 3 & 1 \end{array}\right] = [U|L^{-1}] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{array}\right] = [I|A^{-1}].$$

2.3.3 The most general case

In this subsection, we are going to concentrate on the equation system, $Ax = b$, where we have n unknowns and m equations.

Axiom 2.3.12 *The system $Ax = b$ is solvable if and only if the vector b can be expressed as the linear combination of the columns of A (lies in Span[columns of A] or geometrically lies in the subspace defined by columns of A).*

Definition 2.3.13 *The set of non-trivial solutions $x \neq \theta$ to the homogeneous system $Ax = \theta$ is itself a vector space called the null space of A, denoted by $\mathcal{N}(A)$.*

Remark 2.3.14 *All the possible cases in the solution of the simple scalar equation $\alpha x = \beta$ are below:*

- $\alpha \neq 0$: $\forall \beta \in \mathbb{R}$, $\exists x = \frac{\beta}{\alpha} \in \mathbb{R}$ *(nonsingular case),*
- $\alpha = \beta = 0$: $\forall x \in \mathbb{R}$ *are the solutions (undetermined case),*
- $\alpha = 0, \beta \neq 0$: *there is no solution (inconsistent case).*

Let us consider a possible LU decomposition of a given $A \in \mathbb{R}^{m \times n}$ with the help of the following example:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

The final form of U is upper-trapezoidal.

Definition 2.3.15 *An upper-triangular (lower-triangular) rectangular matrix U is called upper-(lower-)trapezoidal if all the nonzero entries u_{ij} lie on and above (below) the main diagonal, $i \leq j$ ($i \geq j$). An upper-trapezoidal matrices has the following "echelon" form:*

$$\begin{bmatrix} \odot & * & * & * & * & * & * & * & * \\ 0 & \odot & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \odot & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \odot & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to obtain such an U, we may need row interchanges, which would introduce a permutation matrix P. Thus, we have the following theorem.

Theorem 2.3.16 *For any $A \in \mathbb{R}^{m \times n}$, there is a permutation matrix P, a lower–triangular matrix L, and an upper-trapezoidal matrix U such that $PA = LU$.*

Definition 2.3.17 *In any system $Ax = b \Leftrightarrow Ux = c$, we can partition the unknowns x_i as basic (dependent) variables those that correspond to a column with a nonzero pivot $\odot$, and free (nonbasic,independent) variables corresponding to columns without pivots.*

We can state all the possible cases for $Ax = b$ as we did in the previous remark without any proof.

Theorem 2.3.18 *Suppose the m by n matrix A is reduced by elementary row operations and row exchanges to a matrix U in echelon form. Let there be r nonzero pivots; the last $m - r$ rows of U are zero. Then, there will be r basic variables and $n - r$ free variables as independent parameters. The null space, $\mathcal{N}(A)$, composed of the solutions to $Ax = \theta$, has $n - r$ free variables.*

If $n = r$, then null space contains only $x = \theta$.

Solutions exist for every b if and only if $r = m$ (U has no zero rows), and $Ux = c$ can be solved by back-substitution.

If $r < m$, U will have $m - r$ zero rows. If one particular solution $\hat{x}$ to the first r equations of $Ux = c$ (hence to $Ax = b$) exists, then $\hat{x} + \alpha\dot{x}$, $\forall \dot{x} \in \mathcal{N}(A) \setminus \{\theta\}$, $\forall \alpha \in \mathbb{R}$ is also a solution.

Definition 2.3.19 *The number r is called the rank of A.*

2.4 The four fundamental subspaces

Remark 2.4.1 *If we rearrange the columns of A so that all basic columns containing pivots are listed first, we will have the following partition of U:*

$$A = [B|N] \to U = \left[\frac{U_B|U_N}{O}\right] \to V = \left[\frac{I_r|V_N}{O}\right]$$

where $B \in \mathbb{R}^{m\times r}$, $N \in \mathbb{R}^{m\times(n-r)}$, $U_B \in \mathbb{R}^{r\times r}$, $U_N \in \mathbb{R}^{r\times(n-r)}$, O is an $(m - r) \times n$ matrix of zeros, $V_N \in \mathbb{R}^{r\times(n-r)}$, and I_r is the identity matrix of order r. U_B is upper-triangular, thus non-singular.

If we continue from U and use elementary row operations to obtain I_r in the U_B part, like in the Gauss-Jordan method, we will arrive at the reduced row echelon form V.

2.4.1 The row space of A

Definition 2.4.2 *The row space of A is the space spanned by rows of A. It is denoted by $\mathcal{R}(A^T)$.*

$$\mathcal{R}(A^T) = Span(\{a_i\}_{i=1}^m) = \left\{y \in \mathbb{R}^m : y = \sum_{i=1}^m \alpha_i a_i\right\}$$
$$= \left\{d \in \mathbb{R}^m : \exists y \in \mathbb{R}^m \ni y^T A = d^T\right\}.$$

Proposition 2.4.3 *The row space of A has the same dimension r as the row space of U and the row space of V. They have the same basis, and thus, all the row spaces are the same.*

Proof. Each elementary row operation leaves the row space unchanged. □

2.4.2 The column space of A

Definition 2.4.4 *The column space of A is the space spanned by the columns of A. It is denoted by $\mathcal{R}(A)$.*

$$\mathcal{R}(A) = Span\left\{a^j\right\}_{j=1}^n = \left\{y \in \mathbb{R}^n : y = \sum_{j=1}^n \beta_j a^j\right\}$$
$$= \{b \in \mathbb{R}^n : \exists x \in \mathbb{R}^n \ni Ax = b\}.$$

Proposition 2.4.5 *The dimension of column space of A equals the rank r, which is also equal to the dimension of the row space of A. The number of independent columns equals the number of independent rows. A basis for $\mathcal{R}(A)$ is formed by the columns of B.*

Definition 2.4.6 *The rank is the dimension of the row space or the column space.*

2.4.3 The null space (kernel) of A

Proposition 2.4.7

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = \theta(Ux = \theta, Vx = \theta)\} = \mathcal{N}(U) = \mathcal{N}(V).$$

Proposition 2.4.8 *The dimension of $\mathcal{N}(A)$ is $n - r$, and a base for $\mathcal{N}(A)$ is the columns of* $T = \left[\frac{-V_N}{I_{n-r}}\right]$.

Proof.

$$Ax = \theta \Leftrightarrow Ux = \theta \Leftrightarrow Vx = \theta \Leftrightarrow x_B + V_N x_N = \theta.$$

The columns of $T = \left[\frac{-V_N}{I_{n-r}}\right]$ is linearly independent because of the last $(n-r)$ coefficients. Is their span $\mathcal{N}(A)$?
Let $y = \sum_j \alpha_j T^j$, $Ay = \sum_j \alpha_j(-V_N^j + V_N^j) = \theta$. Thus, $Span(\{T^j\}_{j=1}^{n-r}) \subseteq \mathcal{N}(A)$. Is $Span(\{T^j\}_{j=1}^{n-r}) \supseteq \mathcal{N}(A)$? Let $x = \left[\frac{x_B}{x_N}\right] \in \mathcal{N}(A)$. Then,

$$Ax = \theta \Leftrightarrow x_B + V_N x_N = \theta \Leftrightarrow x = \left[\frac{x_B}{x_N}\right] = \left[\frac{-V_N}{I_{n-r}}\right] x_N \in Span(\{T^j\}_{j=1}^{n-r})$$

Thus, $Span(\{T^j\}_{j=1}^{n-r}) \supseteq \mathcal{N}(A)$. □

2.4.4 The left null space of A

Definition 2.4.9 *The subspace of* $\mathbb{R}^m$ *that consists of those vectors* y *such that* $y^T A = \theta$ *is known as the left null space of* A.

$$\mathcal{N}(A^T) = \left\{ y \in \mathbb{R}^m : y^T A = \theta \right\}.$$

Proposition 2.4.10 *The left null space* $\mathcal{N}(A^T)$ *is of dimension* $m-r$, *where the basis vectors are the last* $m-r$ *rows of* $L^{-1}P$ *of* $PA = LU$ *or* $L^{-1}PA = U$.

Proof.

$$\bar{A} = [A|I_m] \to \bar{V} = \left[\begin{array}{c|c} \begin{array}{c|c} I_r & V_N \end{array} & \\ \hline O & L^{-1}P \end{array} \right]$$

Then, $(L^{-1}P) = \left[\frac{S_I}{S_{II}} \right]$, where S_{II} is the last $m-r$ rows of $L^{-1}P$. Then, $S_{II}A = \theta$. $\square$

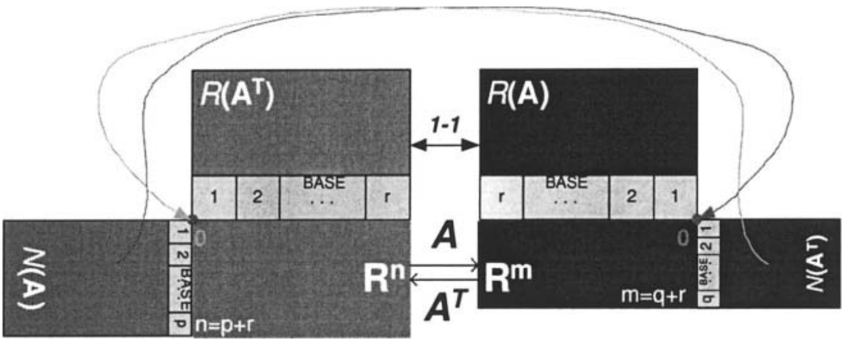

Fig. 2.2. The four fundamental subspaces defined by $A \in \mathbb{R}^{m \times n}$.

2.4.5 The Fundamental Theorem of Linear Algebra

Theorem 2.4.11 $\mathcal{R}(A^T)$= *row space of* A *with dimension* r;
$\mathcal{N}(A)$= *null space of* A *with dimension* $n-r$;
$\mathcal{R}(A)$= *column space of* A *with dimension* r;
$\mathcal{N}(A^T)$= *left null space of* A *with dimension* $m-r$;

Remark 2.4.12 *From this point onwards, we are going to assume that* $n \geq m$ *unless otherwise indicated.*

Problems

2.1. Graph spaces

Definition 2.4.13 *Let $GF(2)$ be the field with $+$ and $\times$ (addition and multiplication modulo 2 on $\mathbb{Z}^2$)*

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \quad \textit{and} \quad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

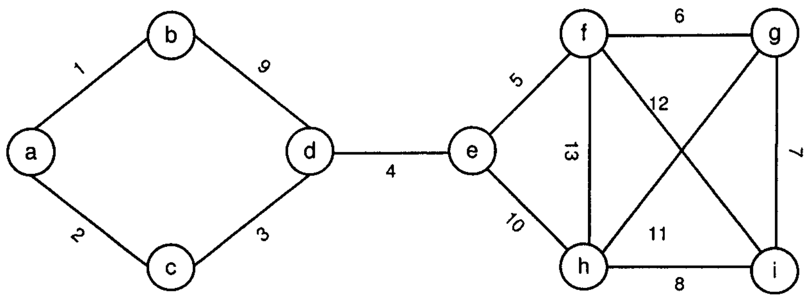

Fig. 2.3. The graph in Problem 2.1

Consider the node–edge incident matrix of the given graph $G = (V, E)$ over $GF(2)$, $A \in \mathbb{R}^{\|V\| \times \|E\|}$:

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \left[\begin{array}{ccccccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The addition $+$ operator helps to point out the end points of the path formed by the added edges. For instance, if we add the first and ninth columns of A, we will have $[1, 0, 0, 1, 0, 0, 0, 0, 0]^T$, which indicates the end points (nodes a and d) of the path formed by edges one and nine.

(a) Find the reduced row echelon form of A working over $GF(2)$. Interpret

the meaning of the bases.
(b) Let $T = \{1,2,3,4,5,6,7,8\}$ and $T^{\perp} = E \setminus T = \{9,10,11,12,13\}$.
Let $\bar{A} = \begin{bmatrix} I_8 & N \\ 0 & 0 \end{bmatrix}$. Let $Z = [I_8|N]$. For each row, $z_i, i \in T$, color the edges with non-zero entries. Interpret z_i
(c) Let $Y = \begin{bmatrix} N \\ I_5 \end{bmatrix}$. For each column y^j, $j \in T^{\perp}$, color the edges with non-zero entries. Interpret y_j.
(d) Find a basis for the four fundamental subspaces related with A.

2.2. Derivative of a polynomial

Let us concentrate on a $(n-k+1) \times (n+1)$ real valued matrix $A(n,k)$ that represents "taking k^{th} derivative of n^{th} order polynomial"

$$P(t) = a_0 + a_1 t + \cdots + a_n t^n.$$

(a) Let $n = 5$ and $k = 2$. Characterize bases for the four fundamental subspaces related with $A(5,2)$.
(b) Find bases for and the dimensions of the four fundamental subspaces related with $A(n,k)$.
(c) Find $B(n,k)$, the right inverse of $A(n,k)$. Characterize the meaning of the underlying transformation and the four fundamental subspaces.

2.3. As in Example 2.1.12, let $Y = \{y_i\}_{i=1}^n$ be defined as

$$y_i^T = (0, \cdots 0, -1, 1, 0, \cdots, 0),$$

the vector that contains -1 in i^{th} position, 1 in $(i+1)^{st}$ position, and 0s elsewhere. Let $A = [y_1|y_2|\cdots|y_n]$. Characterize the four fundamental subspaces of A.

Web material

http://algebra.math.ust.hk/matrix_linear_trans/02_linear_transform/lecture5.shtml
http://algebra.math.ust.hk/vector_space/11_changebase/lecture4.shtml
http://archives.math.utk.edu/topics/linearAlgebra.html
http://calculusplus.cuny.edu/linalg.htm
http://ceee.rice.edu/Books/CS/chapter2/linear43.html
http://ceee.rice.edu/Books/CS/chapter2/linear44.html
http://dictionary.reference.com/search?q=vector%20space
http://distance-ed.math.tamu.edu/Math640/chapter1/node6.html
http://distance-ed.math.tamu.edu/Math640/chapter4/node2.html
http://distance-ed.math.tamu.edu/Math640/chapter4/node4.html
http://distance-ed.math.tamu.edu/Math640/chapter4/node6.html

http://en.wikibooks.org/wiki/Algebra/Linear_transformations
http://en.wikibooks.org/wiki/Algebra/Vector_spaces
http://en.wikipedia.org/wiki/Examples_of_vector_spaces
http://en.wikipedia.org/wiki/Fundamental_theorem_of_linear_algebra
http://en.wikipedia.org/wiki/Gauss-Jordan_elimination
http://en.wikipedia.org/wiki/Gaussian_elimination
http://en.wikipedia.org/wiki/Linear_transformation
http://en.wikipedia.org/wiki/Vector_space
http://encyclopedia.laborlawtalk.com/Linear_transformation
http://eom.springer.de/L/l059520.htm
http://eom.springer.de/t/t093180.htm
http://eom.springer.de/v/v096520.htm
http://euler.mcs.utulsa.edu/~class_diaz/cs2503/Spring99/lab7/node8.html
http://everything2.com/index.pl?node=vector%20space
http://graphics.cs.ucdavis.edu/~okreylos/ResDev/Geometry/VectorSpaceAlgebra.html
http://kr.cs.ait.ac.th/~radok/math/mat5/algebra12.htm
http://math.postech.ac.kr/~kwony/Math300/chapter2P.pdf
http://math.rice.edu/~hassett/teaching/221fall05/linalg5.pdf
http://mathforum.org/workshops/sum98/participants/sinclair/outline.html
http://mathonweb.com/help/backgd3e.htm
http://mathworld.wolfram.com/Gauss-JordanElimination.html
http://mathworld.wolfram.com/GaussianElimination.html
http://mathworld.wolfram.com/LinearTransformation.html
http://mathworld.wolfram.com/VectorSpace.html
http://mizar.uwb.edu.pl/JFM/Vol1/vectsp_1.html
http://planetmath.org/encyclopedia/GaussianElimination.html
http://planetmath.org/encyclopedia/ProofOfMatrixInverseCalculationByGaussianElimination.html
http://planetmath.org/encyclopedia/VectorField.html
http://planetmath.org/encyclopedia/VectorSpace.html
http://rkb.home.cern.ch/rkb/AN16pp/node101.html
http://thesaurus.maths.org/mmkb/entry.html?action=entryById&id=2243
http://triplebuffer.devmaster.net/file.php?id=5&page=1
http://tutorial.math.lamar.edu/AllBrowsers/2318/LinearTransformations.asp
http://uspas.fnal.gov/materials/3_LinearAlgebra.doc
http://vision.unige.ch/~marchand/teaching/linalg/
http://web.mit.edu/18.06/www/Video/video-fall-99.html
http://www-math.cudenver.edu/~wbriggs/5718s01/notes2/notes2.html
http://www-math.mit.edu/~djk/18_022/chapter16/section01.html
http://www.absoluteastronomy.com/v/vector_space
http://www.amath.washington.edu/courses/352-spring-2001/Lectures/lecture7_print.pdf
http://www.answers.com/topic/linear-transformation
http://www.answers.com/topic/vector-space
http://www.biostat.umn.edu/~sudiptob/pubh8429/

MatureLinearAlgebra.pdf
http://www.bookrags.com/sciences/mathematics/vector-spaces-wom.html
http://www.cap-lore.com/MathPhys/Vectors.html
http://www.cartage.org.lb/en/themes/Sciences/Mathematics/Algebra/foci/topics/transformations/transformations.htm
http://www.cee.umd.edu/menufiles/ence203/fall01/Chapter%205c%20(Simultaneous%20Linear%2http://www.sosmath.com/matrix/system1/system1.html
http://www.cs.berkeley.edu/~demmel/cs267/lectureSparseLU/lectureSparseLU1.html
http://www.cs.cityu.edu.hk/~luoyan/mirror/mit/ocw.mit.edu/18/18.013a/f01/required-readings/chapter04/section02.html
http://www.cs.nthu.edu.tw/~cchen/CS2334/ch4.pdf
http://www.cs.ut.ee/~toomas_l/linalg/lin1/node6.html
http://www.cs.ut.ee/~toomas_l/linalg/lin1/node7.html
http://www.cse.buffalo.edu/~hungngo/classes/2005/Expanders/notes/LA-intro.pdf
http://www.dc.uba.ar/people/materias/ocom/apunte1.doc
http://www.eas.asu.edu/~aar/classes/eee598S98/4vectorSpaces.txt
http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/vector.html
http://www.ee.nchu.edu.tw/~minkuanc/courses/2006_01/LA/Lectures/Lecture%205%20-%202006.pdf
http://www.eng.fsu.edu/~cockburn/courses/eel5173_f01/four.pdf
http://www.everything2.com/index.pl?node_id=579183
http://www.fact-index.com/v/ve/vector_space_1.html
http://www.faqs.org/docs/sp/sp-129.html
http://www.fismat.umich.mx/~htejeda/aa/AS,L24.pdf
http://www.geometrictools.com/Books/GeometricTools/BookSample.pdf
http://www.krellinst.org/UCES/archive/classes/CNA/dir1.6/uces1.6.html
http://www.lehigh.edu/~brha/m43fall2004notes5_rev.pdf
http://www.library.cornell.edu/nr/bookcpdf/c2-2.pdf
http://www.ltcconline.net/greenl/courses/203/MatrixOnVectors/kernelRange.htm
http://www.ltcconline.net/greenl/courses/203/MatrixOnVectors/matrix_of_a_linear_transformatio.htm
http://www.ma.umist.ac.uk/tv/Teaching/Linear%20algebra%20B/Spring%202003/lecture2.pdf
http://www.math.byu.edu/~schow/work/GEnoP.htm
http://www.math.gatech.edu/~bourbaki/math2601/Web-notes/8.pdf
http://www.math.gatech.edu/~mccuan/courses/4305/notes.pdf
http://www.math.grin.edu/~stone/events/scheme-workshop/gaussian.html
http://www.math.harvard.edu/~elkies/M55a.99/field.html
http://www.math.hmc.edu/calculus/tutorials/lineartransformations/
http://www.math.hmc.edu/~su/pcmi/topics.pdf
http://www.math.jhu.edu/~yichen/teaching/2006spring/linear/review2.pdf
http://www.math.niu.edu/~beachy/aaol/fields.html
http://www.math.nps.navy.mil/~art/ma3046/handouts/Mat_Fund_Spa.pdf

http://www.math.poly.edu/courses/ma2012/Notes/GeneralLinearT.pdf
http://www.math.psu.edu/xu/451/HOMEWORK/computer6/node5.html
http://www.math.rutgers.edu/~useminar/basis.pdf
http://www.math.rutgers.edu/~useminar/lintran.pdf
http://www.math.sfu.ca/~lunney/macm316/hw05/node1.html
http://www.math.ubc.ca/~carrell/NB.pdf
http://www.math.uiuc.edu/documenta/vol-01/04.ps.gz
http://www.math.uiuc.edu/Software/magma/text387.html
http://www.math.uiuc.edu/~bergv/coordinates.pdf
http://www.mathcs.emory.edu/~rudolf/math108/summ1-2-3/node19.html
http://www.mathematik.uni-karlsruhe.de/mi2weil/lehre/stogeo2005s/media/cg.pdf
http://www.mathonweb.com/help/backgd3.htm
http://www.mathonweb.com/help/backgd3e.htm
http://www.mathreference.com/fld,intro.html
http://www.mathreference.com/la,lxmat.html
http://www.mathreference.com/la,xform.html
http://www.mathresource.iitb.ac.in/linear%20algebra/mainchapter6.2.html
http://www.maths.adelaide.edu.au/people/pscott/linear_algebra/lapf/24.html
http://www.maths.adelaide.edu.au/pure/pscott/linear_algebra/lapf/21.html
http://www.maths.nottingham.ac.uk/personal/sw/HG2NLA/gau.pdf
http://www.maths.qmul.ac.uk/~pjc/class_gps/ch1.pdf
http://www.mathwords.com/g/gaussian_elimination.htm
http://www.matrixanalysis.com/DownloadChapters.html
http://www.met.rdg.ac.uk/~ross/DARC/LinearVectorSpaces.html
http://www.numbertheory.org/courses/MP274/lintrans.pdf
http://www.phy.auckland.ac.nz/Staff/smt/453707/chap2.pdf
http://www.ping.be/~ping1339/lintf.htm
http://www.purplemath.com/modules/systlin5.htm
http://www.reference.com/browse/wiki/Vector_space
http://www.rsasecurity.com/rsalabs/node.asp?id=2370
http://www.sosmath.com/matrix/system1/system1.html
http://www.stanford.edu/class/ee387/handouts/lect07.pdf
http://www.swgc.mun.ca/~richards/M2051/M2051%20-March%2010%20-%20Vector%20Spaces%20and%20Subspaces.doc
http://www.ucd.ie/math-phy/Courses/MAPH3071/nummeth6.pdf
http://www.what-means.com/encyclopedia/Vector
http://www2.parc.com/spl/members/hhindi/reports/CvxOptTutPaper.pdf
http://xmlearning.maths.ed.ac.uk/eLearning/linear_algebra/binder.php?goTo=4-5-1-1
https://www.cs.tcd.ie/courses/baict/bass/4ict10/Michealmas2002/Handouts/12_Matrices.pdf