2. On a New Method in the Plane Problem on Elastic Vibrations*

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1. The problem on vibrations of an elastic half-space bounded by the vacuum was posed by H. Lamb in his well-known article [1]. He considered a series of problems on vibrations under the action of different forces. In some cases he solved these problems completely, and in other cases he only presented formulas containing divergent Fourier integrals. First, H. Lamb considered the force periodic in time and spatial coordinates, and then he applied the Fourier integral to arrive at the general case.

In the present work we propose a new method, which allows us to solve some of H. Lamb's problems by means of simple calculations. Our method gives tools to determine displacements not only on the surface (as H. Lamb), but also inside the half-space.

The essential feature of our method is the reduction of a problem with three independent variables to one with one or two independent variables.

Two real variables can be reduced to one complex variable, and we can use the theory of functions of a complex variable to find the solution.

First, we consider the problem discussed by H. Lamb on vibrations of the half-space under the action of a vertical impact on the surface. Then, we discuss problems when the source of the force is located inside the elastic medium. Under some fundamental assumptions, we find a solution by reducing a number of independent variables. Obtained solutions satisfy initial and boundary conditions.

Our general reasoning allows us to study the reflection of elastic waves of special types on the plane.

For instance, we can solve the problem on vibration of an elastic layer.

2. Let us state the first problem on vibrations of the half-space under the action of a vertical impact on the surface.

Assume that the surface of the medium is the (x, z)-plane and suppose that the motion does not depend on the coordinate z. Then, our problem is reduced to the two-dimensional problem, which is very important later.

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For the components of the displacement u and v we have

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}, \tag{1}$$

and the functions φ and ψ must satisfy the equations

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{a^2}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right), \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{1}{b^2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right), \tag{2}$$

where

$$a = \sqrt{\frac{\rho}{\lambda + 2\mu}}, \quad b = \sqrt{\frac{\rho}{\mu}}. \tag{3}$$

Denote by ρ the density of the medium, λ and μ are the Lame elastic constants.

Suppose that $R(x,t)$ is the vertical force acting along the x-axis and normal to the surface $y = 0$. Then we have the boundary conditions

$$\left[2\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right]\bigg|_{y=0} = 0, \tag{4}$$

$$\left[\left(\frac{b^2}{a^2} - 2\right)\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) + 2\frac{\partial^2 \varphi}{\partial y^2} - 2\frac{\partial^2 \psi}{\partial x \partial y}\right]\bigg|_{y=0} = \frac{R(x,t)}{\mu}. \tag{5}$$

To consider the case of the impact concentrated at the point $x = 0$ at the moment $t = 0$, we pass to the limit.

Let

$$P_\varepsilon(x,t) = \frac{1}{\varepsilon^2} P\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right),$$

where $P(x,t)$ is a function continuous in the rectangle

$$-1 \le x \le 1, \quad 0 \le t \le 1,$$

$$P(x,t) \equiv 0 \quad \text{for} \quad |x| \ge 1 \quad \text{or} \quad \left|t - \frac{1}{2}\right| \ge \frac{1}{2}.$$

Let $\varphi_\varepsilon(x,y,t)$ and $\psi_\varepsilon(x,y,t)$ be solutions of equations (2) with conditions (4) and (5), where we replace $R(x,t)$ by $P_\varepsilon(x,t)$.

We consider the problem on vibrations under the action of the impact as the limiting case of the stated problem as $\varepsilon \to 0$.

Thus, we have

$$\varphi(x,y,t) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(x,y,t), \quad \psi(x,y,t) = \lim_{\varepsilon \to 0} \psi_\varepsilon(x,y,t).$$

The value of the impact is defined as

$$Q = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dx \int_0^{\varepsilon} P_\varepsilon(x,t)\,dt = \int_{-1}^{1} dx \int_0^1 P(x,t)\,dt.$$

After defining the functions φ_ε and ψ_ε, we have

$$\varphi_\varepsilon(kx, ky, kt) = \varphi_{\varepsilon/k}(x, y, t) \quad \text{and} \quad \psi_\varepsilon(kx, ky, kt) = \psi_{\varepsilon/k}(x, y, t).$$

This property of the functions φ_ε and ψ_ε is stipulated by the form of equations (2), conditions (4) and (5), and by the definition of $P_\varepsilon(x, t)$. Passing to the limit, we have

$$\varphi(kx, ky, kt) = \varphi(x, y, t) \quad \text{and} \quad \psi(kx, ky, kt) = \psi(x, y, t),$$

i.e., the functions φ and ψ are homogeneous of degree 0. Hence they depend on two variables

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t}. \tag{6}$$

Also, note the case when the potentials φ and ψ are homogeneous functions. Let $P(x)$ be an odd function for $-1 \leq x \leq 1$. In (5) we put

$$R(x, t) = 0 \quad \text{for} \quad t < 0 \quad \text{and} \quad R(x, t) = \frac{1}{\varepsilon^2} P\left(\frac{x}{\varepsilon}\right) \quad \text{for} \quad t > 0.$$

In this case, we have

$$\int_{-\varepsilon}^{\varepsilon} R(x, t)\, dx = \frac{1}{\varepsilon} \int_{-1}^{1} P(x)\, dx = 0,$$

and the moment with respect to $x = 0$ is equal to

$$\frac{2}{\varepsilon^2} \int_{0}^{\varepsilon} x P\left(\frac{x}{\varepsilon}\right) dx = 2 \int_{0}^{1} x P(x)\, dx = q.$$

As $\varepsilon \to 0$, we have the focused moment q applied at $t = 0$.

Therefore, we see that the case of homogeneous potentials can arise under different mechanical circumstances. In this connection, later we will see that a solution of the problem contains several arbitrary constants, defined by mechanical conditions of the problem. It should be noted that we again deal with nonuniqueness of the solution. Later we will have an equation on the boundary of the existence domain of an analytic function. This equation will express the fact that the real part of a linear operator must vanish on this function. Assuming that the mentioned operator vanishes everywhere, we will select the simplest solution of this equation. We will also be able to obtain other solutions of the problem. For this, we equate this operator to a regular function, whose real part has zero boundary value on the entire contour with the exception of a unique singular point of this function. We will not study the family of all solutions, but we hope to do it in a future paper.

Moving on to consideration of the functions φ and ψ, let us note a fact, which we will encounter later.

Using homogeneity of the functions φ and ψ, we reduce equations (2) to two equations with two independent variables. Furthermore, by suitable choice of these variables, we reduce these equations to the Laplace equation or the vibrating string equation. Indeed, if the functions φ and ψ depend only on quantities (6), then equations (2) take the form

$$\begin{aligned}
&(a^2\xi^2-1)\frac{\partial^2\varphi}{\partial\xi^2}+2a^2\xi\eta\frac{\partial^2\varphi}{\partial\xi\partial\eta}+(a^2\eta^2-1)\frac{\partial^2\varphi}{\partial\eta^2}+2a^2\xi\frac{\partial\varphi}{\partial\xi}+2a^2\eta\frac{\partial\varphi}{\partial\eta}=0,\\
&(b^2\xi^2-1)\frac{\partial^2\psi}{\partial\xi^2}+2b^2\xi\eta\frac{\partial^2\psi}{\partial\xi\partial\eta}+(b^2\eta^2-1)\frac{\partial^2\psi}{\partial\eta^2}+2b^2\xi\frac{\partial\psi}{\partial\xi}+2b^2\eta\frac{\partial\psi}{\partial\eta}=0.
\end{aligned} \tag{7}$$

Characteristics for the first equation in (7) are determined by the ordinary differential equation

$$(a^2\xi^2-1)d\eta^2-2a^2\xi\eta d\xi d\eta+(a^2\eta^2-1)d\xi^2=0$$

and by a similar equation for the second equation.

The last equation can be written in the form

$$a^2(\xi d\eta-\eta d\xi)^2-(d\xi^2+d\eta^2)=0.$$

Let ds be an element of the characteristic arc. Then we can write our equation in the form

$$\xi\frac{d\eta}{ds}-\eta\frac{d\xi}{ds}=\pm\frac{1}{a},$$

hence we see that the characteristics touch the circle

$$\xi^2+\eta^2=\frac{1}{a^2}.$$

The first equation in (7) is elliptic, if

$$\xi^2+\eta^2<\frac{1}{a^2}, \tag{8.1}$$

and hyperbolic, if

$$\xi^2+\eta^2>\frac{1}{a^2}. \tag{8.2}$$

In the last case, two families of characteristics are expressed by the equation

$$-C\xi\pm\sqrt{a^2-C^2}\eta+1=0,$$

where C is an arbitrary constant. This equation gives for C two complex conjugate values under condition (8.1). Let us begin our analysis with this case. Then, we have the imaginary characteristics

$$\frac{\xi}{\xi^2+\eta^2}\pm i\frac{\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}=C.$$

Putting

$$\sigma = \frac{\xi}{\xi^2+\eta^2}, \quad \tau = \frac{\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}, \tag{9.1}$$

we reduce the first equation in (7) to the Laplace equation

$$\frac{\partial^2\varphi}{\partial\sigma^2} + \frac{\partial^2\varphi}{\partial\tau^2} = 0. \tag{10.1}$$

Similarly, under condition (8.2), by the real transform

$$\sigma = \frac{\xi}{\xi^2+\eta^2}, \quad \tau = \frac{\eta\sqrt{a^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}, \tag{9.2}$$

we bring the first equation in (7) to the vibrating string equation

$$\frac{\partial^2\varphi}{\partial\sigma^2} - \frac{\partial^2\varphi}{\partial\tau^2} = 0. \tag{10.2}$$

In the second part of our work we discuss a more general and simple way of the reduction of equations (2) to canonical form (10.1) or (10.2).

3. Taking into account that the initial moment $t = 0$ of the action of our force corresponds to the rest of the half-space and that vibrations cannot propagate with a velocity more than the velocity of longitudinal vibrations, we can assert that a required solution will vanish outside the circle

$$\xi^2+\eta^2 = \frac{1}{a^2}. \tag{11.1}$$

Thus, to find the potential φ, we have to integrate equation (10.1).

As regards the search for the potential ψ, a should be replaced by b in all previous formulas. The characteristics of the second equation in (7) will be tangent to the circle

$$\xi^2+\eta^2 = \frac{1}{b^2}, \tag{11.2}$$

and this equation will be reduced to (10.2) outside this circle. If the point (ξ, η) is located not only outside circle (11.2), but also outside circle (11.1), then the value of ψ must also vanish at this point.

Note that at each point outside circle (11.2) ψ is a sum of two terms[1], each of which is constant along one of two characteristics passing through this point. Then we can assert that ψ can differ from zero outside circle (11.2) only on the intervals of tangents between the point of tangency and the axis

[1] The function ψ has the form

$$f_1\left(\frac{\xi+\eta\sqrt{b^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}\right) + f_2\left(\frac{\xi-\eta\sqrt{b^2(\xi^2+\eta^2)-1}}{\xi^2+\eta^2}\right). \text{ – } \textit{Ed.}$$

$\eta = 0$, and on such tangents, which have a projection on this axis less than $\frac{1}{a}$ by counting from the origin of coordinates.

Therefore, for the transverse wave, the front in the (ξ, η)-plane consists of the arc AB of circle (11.2) and two segments of tangents AA_1 and BB_1 such that $\overline{OA_1} = \overline{OB_1} = \frac{1}{a}$ (see Fig. 1). For the longitudinal wave, i.e., for the potential φ, the front consists only of semicircle (11.1). The shape of the front of the transverse wave (see Fig. 1) can be immediately obtained from the Fermat principle. It should be noted that vibrations propagate over the surface with the velocity $\frac{1}{a}$, and each point of this surface is a source of not only longitudinal, but also transverse vibrations. At the same time these transverse vibrations propagate inside with the velocity $\frac{1}{b}$.

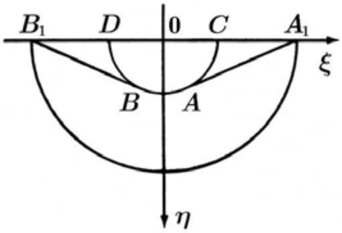

Fig. 1.

The equation of the straight line AA_1 in the (ξ, η)-plane is

$$a\xi + \sqrt{b^2 - a^2}\,\eta - 1 = 0. \tag{12.1}$$

Returning to the variables x, y, t, we obtain the rectilinear front

$$ax + \sqrt{b^2 - a^2}\,y - t = 0. \tag{12.2}$$

To study equation (10.1), we introduce the complex variable

$$\theta_1 = \sigma + i\tau = \frac{\xi}{\xi^2 + \eta^2} + i\frac{\eta\sqrt{1 - a^2(\xi^2 + \eta^2)}}{\xi^2 + \eta^2}.$$

This transform maps the semidisk

$$\xi^2 + \eta^2 < \frac{1}{a^2}, \quad \eta > 0,$$

onto the half-plane $\tau > 0$ of the complex variable θ_1, the diameter B_1A_1 onto the intervals $(-\infty, -a)$ and $(+a, +\infty)$ of the axis $\tau = 0$, and the semicircle B_1A_1 onto the interval $(-a, +a)$ of this axis (see Fig. 2). In the half-plane

$\tau > 0$ the potential φ is a harmonic function and can be expressed as the real part of an analytic function $\Phi(\theta_1) = \varphi + i\varphi^*$:

$$\varphi = \mathrm{Re}\,[\Phi(\theta_1)].$$

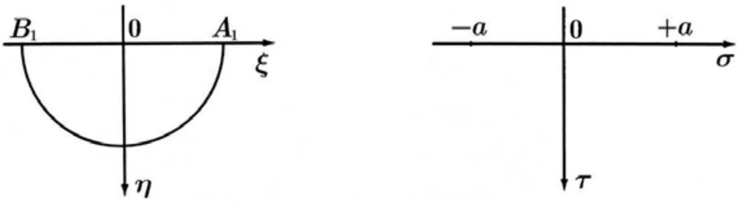

Fig. 2.

Similarly, introducing the complex variable

$$\theta_2 = \sigma + i\tau = \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{1-b^2(\xi^2+\eta^2)}}{\xi^2+\eta^2},$$

in the semidisk

$$\xi^2 + \eta^2 < \frac{1}{b^2}, \quad \eta > 0,$$

we can express the potential ψ as the real part of a function $\Psi(\theta_2) = \psi + i\psi^*$ analytic in the half-plane $\tau > 0$:

$$\psi = \mathrm{Re}\,[\Psi(\theta_2)].$$

The formulas

$$\begin{aligned}
\theta_1 &= \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{1-a^2(\xi^2+\eta^2)}}{\xi^2+\eta^2},\\
\theta_2 &= \frac{\xi}{\xi^2+\eta^2} + i\frac{\eta\sqrt{1-b^2(\xi^2+\eta^2)}}{\xi^2+\eta^2}
\end{aligned} \tag{13}$$

prove that the values of θ_1 and θ_2 coincide at the points of the diameter CD (see Fig. 1), which will be essential later.

It is easy to prove that on the plane θ_2 the points D and C correspond to the points $+b$ and $-b$ of the axis $\tau = 0$, and the points B and A correspond to the points $+a$ and $-a$ of this axis.

Let us now introduce the boundary conditions with respect to the new variables. For any $t > 0$, there are no stresses on the surface of the half-space.

Therefore, for φ and ψ we should take conditions (4) and (5) with $P(x,t)=0$. We obtain

$$\begin{aligned} D_1(\varphi,\psi) &= \left[2\frac{\partial^2\varphi}{\partial\xi\partial\eta} - \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}\right]\Bigg|_{y=0} = 0, \\ D_2(\varphi,\psi) &= \left[\left(\frac{b^2}{a^2}-2\right)\left(\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\eta^2}\right) + 2\frac{\partial^2\varphi}{\partial\eta^2} - 2\frac{\partial^2\psi}{\partial\xi\partial\eta}\right]\Bigg|_{y=0} = 0, \end{aligned} \tag{14}$$

where we denote by D_1 and D_2 the linear operators on the left side of our conditions. Differentiation with respect to ξ and η can be replaced by differentiation with respect to θ_1 and θ_2. It is easy to see that for $\eta=0$ we have

$$\frac{\partial\theta_1}{\partial\xi} = -\theta_1^2, \quad \frac{\partial^2\theta_1}{\partial\xi^2} = 2\theta_1^3, \quad \frac{\partial\theta_1}{\partial\eta} = -\theta_1\sqrt{a^2-\theta_1^2}, \quad \frac{\partial^2\theta_1}{\partial\eta^2} = -2\theta_1^3,$$

$$\frac{\partial^2\theta_1}{\partial\xi\partial\eta} = -\frac{2\theta_1^4 - a^2\theta_1^2}{\sqrt{a^2-\theta_1^2}},$$

where the square root has the negative imaginary value for $\theta_1 > a$.

We have similar expressions for θ_2. Conditions (14) take the form

$$\begin{aligned} &\operatorname{Re}\Bigg[2\theta\sqrt{a^2-\theta^2}\Phi''(\theta) + 2\frac{a^2-2\theta^2}{\sqrt{a^2-\theta^2}}\Phi'(\theta) \\ &\qquad -(2\theta^2-b^2)\Psi''(\theta) - 4\theta\Psi'(\theta)\Bigg]\Bigg|_{\tau=0} = 0, \\ &\operatorname{Re}\Bigg[(b^2-2\theta^2)\Phi''(\theta) - 4\theta\Phi'(\theta) \\ &\qquad -2\theta\sqrt{b^2-\theta^2}\Psi''(\theta) - 2\frac{b^2-2\theta^2}{\sqrt{b^2-\theta^2}}\Psi'(\theta)\Bigg]\Bigg|_{\tau=0} = 0. \end{aligned} \tag{15}$$

Since θ_1 and θ_2 coincide on the axis $\eta=0$, we denote the variables by θ without index.

Conditions (15) must be satisfied on the part that corresponds to the diameters of the semicircles.

Taking into account what we said about the correspondence between θ_1, θ_2, ξ and η, we see that conditions (15) must be satisfied on the intervals $\sigma \le -b$ and $\sigma \ge +b$. Note once again that the interval $-a \le \sigma \le +a$ of the variables θ_1 and θ_2 corresponds to the arcs of the semicircles, forming the front of propagation of longitudinal and transverse vibrations. Consequently, the functions φ and ψ, i.e., the real parts of Φ and Ψ, must vanish on this interval. Taking into account that all coefficients on the left sides of (15) are real for $-a \le \theta \le +a$, we can assert that conditions (15) must be also satisfied on the interval $-a \le \theta \le +a$.

Later, we show that these conditions must hold also on two intervals $-b \le \theta \le -a$ and $a \le \theta \le b$. For this purpose we consider an equation of hyperbolic type for ψ in the curvilinear triangles AA_1C and BB_1D (see Fig. 1). It is enough to consider the triangle BB_1D. Introducing the variables

$$\sigma = \frac{\xi}{\xi^2 + \eta^2}, \qquad \tau = \frac{\eta\sqrt{b^2(\xi^2+\eta^2) - 1}}{\xi^2 + \eta^2}, \tag{16}$$

for ψ we have the vibrating string equation

$$\frac{\partial^2 \psi}{\partial \sigma^2} - \frac{\partial^2 \psi}{\partial \tau^2} = 0,$$

whose solution is

$$\psi = f_1(\sigma + \tau) + f_2(\sigma - \tau).$$

Since ψ is equal to zero outside circle (11.1), as above, we can assert that the last expression for ψ contains at most one term different from zero on the pieces of the characteristics, made of segments of tangents between the arc BD and the axis $\eta = 0$.

The mentioned segments can be defined by the values of the real parameter θ_3,

$$\theta_3 = \frac{\xi}{\xi^2 + \eta^2} - \frac{\eta\sqrt{b^2(\xi^2+\eta^2) - 1}}{\xi^2 + \eta^2}, \quad a \le \theta_3 \le b, \tag{17}$$

and the function ψ depends only on θ_3 inside the triangle BB_1D. It is easy to see that the value of θ_3 coincides on each tangent with the corresponding value of θ_2 on the arc BD. Hence, in view of continuity of ψ, in the triangle BB_1D we should take

$$\psi = \mathrm{Re}\,[\Psi(\theta_3)].$$

On the interval B_1D of the axis $\eta = 0$ the values of θ_3 coincide with the values of θ_1.

Returning to conditions (14), we can express the derivatives with respect to ξ and η by the derivatives with respect to θ_1 and θ_3. These variables can be denoted by the same letter θ, and $a \le \theta \le b$.

Conditions (14) take the form

$$\begin{gathered}
\mathrm{Re}\left\{2\theta\sqrt{a^2-\theta^2}\,\Phi''(\theta) + 2\frac{a^2 - 2\theta^2}{\sqrt{a^2-\theta^2}}\Phi'(\theta)\right\} \\
-(2\theta^2 - b^2)\mathrm{Re}\,[\Psi''(\theta)] - 4\theta\mathrm{Re}\,[\Psi'(\theta)] = 0, \\
\mathrm{Re}\,\{(b^2 - 2\theta^2)\Phi''(\theta) - 4\theta\Phi'(\theta)\} - 2\theta\sqrt{b^2-\theta^2}\,\mathrm{Re}\,[\Psi''(\theta)] \\
-2\frac{b^2 - 2\theta^2}{\sqrt{b^2-\theta^2}}\mathrm{Re}\,[\Psi'(\theta)] = 0, \\
a \le \theta \le b.
\end{gathered}$$

Hence conditions (15) must hold on the interval $a \le \theta \le b$.

Considering the triangle AA_1C, we can similarly show that conditions (15) must hold also on the interval $-b \le \theta \le -a$. Thus, conditions (15) are established on the entire real axis of the plane θ.

The simplest conclusion from this fact is that the analytic functions on the left sides of conditions (15) are equal to imaginary constants. This conclusion is necessary, if we assume that the passage to the limit on the axis $\tau = 0$ is continuous everywhere. Thus, we obtain

$$-2\theta\sqrt{a^2-\theta^2}\varPhi''(\theta) - 2\frac{a^2-2\theta^2}{\sqrt{a^2-\theta^2}}\varPhi'(\theta) + (2\theta^2-b^2)\varPsi''(\theta) + 4\theta\varPsi'(\theta) = \alpha i,$$

$$(b^2-2\theta^2)\varPhi''(\theta) - 4\theta\varPhi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\varPsi''(\theta) - 2\frac{b^2-2\theta^2}{\sqrt{b^2-\theta^2}}\varPsi'(\theta) = \beta i,$$

where α and β are real constants.

Integrating the equations with respect to θ, we have

$$\begin{aligned} -2\theta\sqrt{a^2-\theta^2}\varPhi'(\theta) + (2\theta^2-b^2)\varPsi'(\theta) &= \alpha i\theta + C_1, \\ (b^2-2\theta^2)\varPhi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\varPsi'(\theta) &= \beta i\theta + C_2, \end{aligned} \tag{18}$$

hence,

$$\begin{aligned} \varPhi'(\theta) &= \frac{-(\alpha i\theta + C_1)2\theta\sqrt{b^2-\theta^2} - (\beta i\theta + C_2)(2\theta^2-b^2)}{(2\theta^2-b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}, \\ \varPsi'(\theta) &= \frac{(\alpha i\theta + C_1)(2\theta^2-b^2) - (\beta i\theta + C_2)2\theta\sqrt{a^2-\theta^2}}{(2\theta^2-b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}, \end{aligned} \tag{19}$$

where C_1 and C_2 are complex constants. Consider real values of θ on the interval $-a \le \theta \le +a$. As above, this interval corresponds to the front of the longitudinal wave and to a part of the front of the transverse wave. Consequently, the real parts of $\varPhi'(\theta)$ and $\varPsi'(\theta)$ must be equal to zero on the interval $-a \le \theta \le +a$. Hence C_1 and C_2 are pure imaginary.

To find the constants, we express the projections of the displacements u, v by the functions $\varPhi$ and $\varPsi$ by using (1). We have

$$u = \operatorname{Re}\left[\varPhi'(\theta_1)\frac{\partial\theta_1}{\partial x} + \varPsi'(\theta_2)\frac{\partial\theta_2}{\partial y}\right], \qquad v = \operatorname{Re}\left[\varPhi'(\theta_1)\frac{\partial\theta_1}{\partial y} - \varPsi'(\theta_2)\frac{\partial\theta_2}{\partial x}\right]. \tag{20}$$

The expressions for θ_1 and θ_2 give

$$\begin{aligned} \frac{\partial\theta_1}{\partial x} &= -\theta_1\frac{\partial\theta_1}{\partial t}, & \frac{\partial\theta_1}{\partial y} &= -\sqrt{a^2-\theta_1^2}\frac{\partial\theta_1}{\partial t}, \\ \frac{\partial\theta_2}{\partial x} &= -\theta_2\frac{\partial\theta_2}{\partial t}, & \frac{\partial\theta_2}{\partial y} &= -\sqrt{b^2-\theta_2^2}\frac{\partial\theta_2}{\partial t}, \end{aligned} \tag{21}$$

where the square roots are negative imaginary for θ_1 and $\theta_2 > b$. Indeed, for the variables θ_1 and θ_2 we have

$$\begin{aligned}\theta_1 &= \frac{xt}{x^2+y^2} + i\frac{y\sqrt{t^2-a^2(x^2+y^2)}}{x^2+y^2},\\ \theta_2 &= \frac{xt}{x^2+y^2} + i\frac{y\sqrt{t^2-b^2(x^2+y^2)}}{x^2+y^2}.\end{aligned} \tag{22}$$

Consider now values of u and v on the axis $x = 0$. We assume that the impact is concentrated at the point $x = 0$ and acts along the axis $x = 0$. Hence $u = 0$ on this axis. Obviously, θ_1, θ_2, $\dfrac{\partial\theta_1}{\partial t}$ and $\dfrac{\partial\theta_2}{\partial t}$ are pure imaginary on this axis. Consequently, $\dfrac{\partial\theta_1}{\partial x}$ is real, and $\dfrac{\partial\theta_2}{\partial y}$ is pure imaginary. From the first of equations (20) we can conclude that $C_1 = \beta = 0$. Denote C_2 by $-Ci$, where C is a real constant. Then, we can write

$$\begin{aligned}\varPhi'(\theta) &= i\frac{-2\alpha\theta^2\sqrt{b^2-\theta^2} + C(2\theta^2-b^2)}{F(\theta)},\\ \varPsi'(\theta) &= i\frac{\alpha\theta(2\theta^2-b^2) + C2\theta\sqrt{a^2-\theta^2}}{F(\theta)},\end{aligned} \tag{23}$$

where

$$F(\theta) = (2\theta^2-b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}. \tag{24}$$

Formulas (23) contain two real constants α and C. Consider the displacements u and v at a point of the axis $y = 0$ and assume that the time t tends to infinity. Under these assumptions, the variables θ_1 and θ_2 equal $\dfrac{t}{x}$ and tend to infinity. The expression

$$F(\theta) = (2\theta^2-b^2)^2 - 4\theta^4\left(1-\frac{a^2}{\theta^2}\right)^{1/2}\left(1-\frac{b^2}{\theta^2}\right)^{1/2} = (2a^2-2b^2)\theta^2 + \cdots$$

has order θ^2.

Using the expression for θ,

$$\theta = \frac{xt}{x^2+y^2} + i\frac{y\sqrt{t^2-c^2(x^2+y^2)}}{x^2+y^2}, \quad c^2 = a^2 \text{ or } b^2, \tag{25}$$

it easy to expand u and v in power series with respect to $\dfrac{1}{t}$. If $\alpha \neq 0$, then these series begin with a constant term, and we have the displacements different from zero as $t \to \infty$. This term is equal to zero for $\alpha = 0$. This fact forces us to put $\alpha = 0$. Then formulas (23) give us

$$\varPhi'(\theta) = iC\frac{2\theta^2-b^2}{F(\theta)}, \quad \varPsi'(\theta) = iC\frac{2\theta\sqrt{a^2-\theta^2}}{F(\theta)}. \tag{26}$$

The elementary potential ψ will be defined by the real part of the analytic function $\Psi(\theta)$ not only inside the semidisk

$$\xi^2 + \eta^2 < \frac{1}{b^2},$$

but also in two triangles, if we replace θ_2 by the variable θ_3 defined above.

4. The constant C in (26) depends on the concentrated impact Q. Assume that this constant is determined by the condition that Q is equal to 1. Also, assume that the force $Q(t)$ acts at the point $x = 0$ of the axis $y = 0$, where $Q(t)$ is a continuous function of t. Let $\varphi_0(x, y, t)$ and $\psi_0(x, y, t)$ be elementary potentials at the given point (x, y) at the moment t. We can construct these potentials by means of superposition of the effects of the action of the elementary impulses $Q(t - H)dH$ concentrated at the moment $t - H$, where the variable H belongs to the interval (H_0, ∞). We denote by H_0 the time interval necessary for the impulse to propagate to the point (x, y). For the longitudinal wave, H_0 is equal to $a\sqrt{x^2 + y^2}$. In the case of the transverse wave, the expression for H_0 depends on the position of the point (x, y). If this point is located inside the angle AOB (see Fig. 1), where the front of the transverse wave has the shape of a circular arc, then $H_0 = b\sqrt{x^2 + y^2}$. If, on the contrary, this point is located outside this angle, then we have $H_0 = ax + \sqrt{b^2 - a^2}\, y$. These expressions for H_0 follow immediately from equation (12.1) (in this case we assume that $x > 0$). Finally, using equations (20) and (26), we obtain two expressions for the components of the displacement:

$$u = C\mathrm{Im} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial\theta_1}{\partial x}}{F(\theta_1)} Q(t - H)\, dH$$

$$+ C\mathrm{Im} \int\limits_{b\sqrt{x^2+y^2}}^{\infty} \frac{2\theta_2\sqrt{a^2 - \theta_2^2}\frac{\partial\theta_2}{\partial y}}{F(\theta_2)} Q(t - H)\, dH, \tag{27.1}$$

$$v = C\mathrm{Im} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial\theta_1}{\partial y}}{F(\theta_1)} Q(t - H) dH$$

$$- C\mathrm{Im} \int\limits_{b\sqrt{x^2+y^2}}^{\infty} \frac{2\theta_2\sqrt{a^2 - \theta_2^2}\frac{\partial\theta_2}{\partial x}}{F(\theta_2)} Q(t - H)\, dH. \tag{27.2}$$

Expressions (27) are related to the case when (x, y) are located inside AOB, i.e., if $b^2x^2 \le a^2(x^2 + y^2)$. In the case $b^2x^2 \ge a^2(x^2 + y^2)$, we have

$$u = C\mathrm{Im} \int\limits_{a\sqrt{x^2+y^2}}^{\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial\theta_1}{\partial x}}{F(\theta_1)} Q(t - H)\, dH$$

$$+C\operatorname{Im}\int\limits_{ax+\sqrt{b^2-a^2}\,y}^{\infty}\frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial\theta_2}{\partial y}}{F(\theta_2)}Q(t-H)\,dH,\qquad(28.1)$$

$$v=C\operatorname{Im}\int\limits_{a\sqrt{x^2+y^2}}^{\infty}\frac{(2\theta_1^2-b^2)\frac{\partial\theta_1}{\partial y}}{F(\theta_1)}Q(t-H)\,dH$$

$$-C\operatorname{Im}\int\limits_{ax+\sqrt{b^2-a^2}\,y}^{\infty}\frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial\theta_2}{\partial x}}{F(\theta_2)}Q(t-H)\,dH.\qquad(28.2)$$

In these formulas we should take

$$\theta_2=\frac{Hx}{x^2+y^2}+i\frac{y\sqrt{H^2-b^2(x^2+y^2)}}{x^2+y^2}\quad\text{for}\quad H^2\geq b^2(x^2+y^2),$$

$$\theta_2=\frac{Hx}{x^2+y^2}-\frac{y\sqrt{b^2(x^2+y^2)-H^2}}{x^2+y^2}\quad\text{for}\quad H^2\leq b^2(x^2+y^2),$$

with the arithmetical square root. To determine the derivatives of θ with respect to x and y, one can use formulas (21). Obviously, we should assume that the behavior of the function $Q(t)$ as $t\to-\infty$ is such that the integrals mentioned above converge.

Formulas (27) and (28) coincide with the formulas derived in the work of S. L. Sobolev [2], but the method described here is simpler and allows to solve many other questions without any application of the Fourier integral. It is known that such application frequently leads to essential complexities in solving the problem.

The analysis of formulas (26), (27) and (28) was carried out in the mentioned work of S. L. Sobolev, nevertheless, we repeat some moments of this analysis here.

First of all, note that in the case of the concentrated impact, the components of u and v are infinite on circles (11.1) and (11.2). This fact follows from the expressions for the derivatives

$$\frac{\partial\theta}{\partial x}\quad\text{and}\quad\frac{\partial\theta}{\partial y}.$$

A unique exception are points on the axis $\eta=0$, where the displacement is equal to zero. The mentioned circumstance also take place on the parts AC and BD of circle (11.2), which does not compose the front of the disturbance propagation. At the moments corresponding to such parts, we have the beginning of a new phase of vibrations. On the lines

$$\pm a\xi+\sqrt{b^2-a^2}\eta=1,$$

which compose the front of the transverse wave, the derivatives $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ are infinite. This follows from the fact that $\Psi'(\theta)$ contains the factor $\sqrt{a^2-\theta^2}$, and the mentioned lines correspond to the case $\theta^2 = a^2$.

The regular functions $\Phi(\theta)$ and $\Psi(\theta)$ defined by (26) have two poles $\theta = \pm c$ on the real axis. These poles are roots of the equation

$$F(\theta) = 0. \tag{29}$$

It is easy to see that $\theta = c$ is a number reciprocal to the velocity of the surface waves, which were first studied by Lord Rayleigh. Taking into account that $\theta = \dfrac{ht}{x}$ on the real axis, we can assert that such poles give an infinite displacement propagating on the surface in two directions with the velocity $\dfrac{1}{c}$. With the exception of these poles, the functions $\Phi(\theta)$ and $\Psi(\theta)$ do not have any singular point.

The proof of this fact is contained, for example, in the work of V. D. Kupradze and S. L. Sobolev [3][2].

5. It is now easy to obtain formulas for the displacement also in the case when the force is distributed continuously along the axis $y = 0$. Let $f(x)$ be a density of this distribution. If the impact happens at the moment $t = 0$, then the formulas have the form

$$u(x,y,t) = C\operatorname{Im}\int_{-\infty}^{+\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial \theta_1}{\partial x}}{F(\theta_1)} f(\xi)\,d\xi$$

$$+ C\operatorname{Im}\int_{-\infty}^{+\infty} \frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial \theta_2}{\partial y}}{F(\theta_2)} f(\xi)\,d\xi, \tag{30.1}$$

$$v(x,y,t) = C\operatorname{Im}\int_{-\infty}^{+\infty} \frac{(2\theta_1^2 - b^2)\frac{\partial \theta_1}{\partial y}}{F(\theta_1)} f(\xi)\,d\xi$$

$$- C\operatorname{Im}\int_{-\infty}^{+\infty} \frac{2\theta_2\sqrt{a^2-\theta_2^2}\,\frac{\partial \theta_2}{\partial x}}{F(\theta_2)} f(\xi)\,d\xi, \tag{30.2}$$

where

$$\theta_1 = \frac{(x-\xi)t}{(x-\xi)^2 + y^2} + i\frac{y\sqrt{t^2 - a^2(x-\xi)^2 - a^2y^2}}{(x-\xi)^2+y^2},$$

[2] See corresponding reasoning in the paper [4] of Part I of this book (p. 148). – *Ed.*

$$\theta_2 = \begin{cases} \dfrac{(x-\xi)t}{(x-\xi)^2+y^2} + i\dfrac{y\sqrt{t^2-b^2(x-\xi)^2-b^2y^2}}{(x-\xi)^2+y^2} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad b^2(x-\xi)^2+b^2y^2<t^2, \\ \dfrac{(x-\xi)t}{(x-\xi)^2+y^2} - \dfrac{y\sqrt{b^2(x-\xi)^2+b^2y^2-t^2}}{(x-\xi)^2+y^2} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad b^2(x-\xi)^2+b^2y^2>t^2. \end{cases}$$

Note that the imaginary parts of all integrands in formulas (30) are equal to zero outside the fronts of the corresponding waves. Assume that the force is distributed not only along the axis $y = 0$, but the image of its action in time is of unconcentrated nature. Then, multiplying the elementary potentials by $Q(\xi, t-H)$, we have to integrate with respect to H as in (27), (28), and with respect to ξ as in (30). The lower limit of integration with respect to H in the first integral is

$$a\sqrt{(x-\xi)^2+y^2}.$$

In the second integral the lower limit is

$$b\sqrt{(x-\xi)^2+y^2}$$

for

$$b^2(x-\xi)^2 \le a^2[(x-\xi)^2+y^2],$$

and

$$a|x-\xi| + \sqrt{b^2-a^2}\,y$$

for

$$b^2(x-\xi)^2 \ge a^2[(x-\xi)^2+y^2].$$

6. All previous conclusions up to formulas (19) remain valid also in the case of a focused force acting along the axis $y = 0$. In this case, we need only to determine the constants in (19) somewhat differently. It is easy to see that in this case the component v must vanish at the points on the axis $x = 0$. Indeed, if we change the direction of the force acting along $y = 0$, then, by the symmetry principle, the component v must remain unchanged on the axis $x = 0$, at the same time u must change sign. On the other hand, the displacement vector can only change its direction. Whence $v = 0$. Arguing in the same way as above, by (26) we obtain the formulas

$$\Phi'(\theta) = -iC\frac{2\theta\sqrt{b^2-\theta^2}}{F(\theta)}, \qquad \Psi'(\theta) = iC\frac{2\theta^2-b^2}{F(\theta)}. \tag{31}$$

7. Before moving on to solving other problems, we present some general considerations, which were essential in the preceding discussion and will be even more important in the future. The essential moment in solving the problem is reducing the wave equation (2) for the potential

$$c^2 \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}, \qquad c^2 = a^2 \text{ or } b^2,$$

to the Laplace equation in new independent variables σ and τ. In the case $c^2 = b^2$, we obtained the solution of (2) with an arbitrary function of one variable, which we denoted above by $(\sigma - \tau)$. In the first case, the dependence of the complex variable $\theta = \sigma + i\tau$ on the original variables (x, y, t) is expressed by the formula

$$-\theta x - \sqrt{c^2 - \theta^2}\, y + t = 0. \tag{32}$$

If we consider the three-dimensional space S with the coordinates (x, y, t), then from the preceding computations it follows that equation (32) has complex roots inside the cone

$$c^2(x^2 + y^2) - t^2 = 0. \tag{33}$$

If we take a root θ of this equation with the positive imaginary part, then for the root $\sqrt{c^2 - \theta^2}$ in (32) we have to choose the negative imaginary value for $\theta > c$. Outside cone (33), i.e., for

$$c^2(x^2 + y^2) - t^2 > 0,$$

equation (32) has two real roots, and an arbitrary function of each of these roots satisfies equation (2).

We point out a more general class of solutions of equation (2), which is obtained by the reduction of this equation to the Laplace equation.

For the dependence of the new variable $\theta = \sigma + i\tau$ on the variables (x, y, t) we use a linear function of x, y, and t with coefficients, which are analytic functions of θ. Obviously, the coefficient at t may be taken equal to 1. This leads us to the relation

$$t + \chi_1(\theta)x + \chi_2(\theta)y = \chi(\theta). \tag{34}$$

Assume that in a domain of the space S this equation has a complex root $\theta = \sigma + i\tau$, which is a function of (x, y, t). Consider a solution of (2), depending only on σ and τ.

In this case, one can verify that equation (2) can be reduced to the form

$$\frac{\partial^2 \varphi}{\partial \sigma^2} + \frac{\partial^2 \varphi}{\partial \tau^2} = 0$$

under the condition

$$\chi_1^2(\theta) + \chi_2^2(\theta) = c^2.$$

This circumstance is a consequence of the geometric nature of the lines $\sigma = \text{const}$, $\tau = \text{const}$, which are the straight lines in our three-dimensional space S. However, since we do not use this fact, we will not discuss it in detail. Taking into account that a harmonic function is mapped to a harmonic function under the action of the conformal mapping, we can take $\chi_1(\theta)$ as a new complex variable. Then, in view of the condition mentioned above, we have

$$\chi_2(\theta) = \pm\sqrt{c^2 - \theta^2},$$

and we can reduce relation (34) to the form

$$t - \theta x \pm \sqrt{c^2 - \theta^2}\, y - \chi(\theta) = 0. \tag{35}$$

If this equation has a real root in a domain of the space S, then an arbitrary function of this root satisfies equation (2).

All these assertions can be verified by simple calculation.

We present the corresponding formulas, since they will be useful later.

Denote by δ the left side of equation (35) and by δ' the partial derivative $\dfrac{\partial \delta}{\partial \theta}$. We have

$$\frac{\partial \theta}{\partial x} = \frac{\theta}{\delta'}, \quad \frac{\partial \theta}{\partial y} = \mp\frac{\sqrt{c^2 - \theta^2}}{\delta'}, \quad \frac{\partial \theta}{\partial t} = -\frac{1}{\delta'}. \tag{36}$$

The second-order derivatives are

$$\begin{aligned} &\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left(\frac{\theta^2}{\delta'}\right), \quad \frac{\partial^2 \theta}{\partial y^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left(\frac{c^2 - \theta^2}{\delta'}\right), \\ &\frac{\partial^2 \theta}{\partial t^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left(\frac{1}{\delta'}\right), \quad \frac{\partial^2 \theta}{\partial x \partial y} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left(\frac{\mp\theta\sqrt{c^2 - \theta^2}}{\delta'}\right). \end{aligned} \tag{37}$$

By (36), if equation (35) has a real root θ in a domain of the space S, then this root satisfies the inequality $-c \le \theta \le +c$, and the function $\chi(\theta)$ must have real values.

Let us note also some formulas used later. Let θ be a complex root of (35), let $f(\theta)$ be an analytic function. Using (36) and (37), we obtain the following expressions for the derivatives of $f(\theta)$ with respect to (x, y, t):

$$\begin{aligned} &\frac{\partial^2 f}{\partial x^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{\theta^2}{\delta'}\right], \quad \frac{\partial^2 f}{\partial y^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{c^2 - \theta^2}{\delta'}\right], \\ &\frac{\partial^2 f}{\partial t^2} = \frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{1}{\delta'}\right], \quad \frac{\partial^2 f}{\partial x \partial y} = \mp\frac{1}{\delta'}\frac{\partial}{\partial \theta}\left[f'(\theta)\frac{\theta\sqrt{c^2 - \theta^2}}{\delta'}\right]. \end{aligned} \tag{38}$$

The same formulas remain valid for the function $f(\theta)$ of the real argument θ, if θ is a real root of equation (35).

8. Let us now discuss the two-dimensional problem on vibrations of the half-space under the action of a source of force F, located inside the half-space. As before, assume that the elastic half-plane is $y \ge 0$. Let $x = 0$, $y = f$ be the coordinates of the force source. We assume that the force action is concentrated at some moment. As above, we denote by t the time passed from this moment.

Introduce two functions $X(\alpha, t)$ and $Y(\alpha, t)$ defined on

$$0 \le \alpha \le 2\pi, \qquad 0 \le t \le 1.$$

Consider vibrations of the half-plane, being at rest at the moment $t = 0$, under the action of stresses

$$\frac{1}{\varepsilon^2} X\left(\alpha, \frac{t}{\varepsilon}\right) \quad \text{and} \quad \frac{1}{\varepsilon^2} Y\left(\alpha, \frac{t}{\varepsilon}\right)$$

applied at the points of a circle of radius ε with center $F(0, f)$, where the interval of the action of stresses is $0 \le t \le \varepsilon$. As $\varepsilon \to 0$, we have vibrations of the half-plane with a singularity at the point $F(0, f)$ and with potentials φ and ψ homogeneous in x, $(y - f)$, and t. A similar result is obtained if the moment is at $t = 0$. Note that the singularity of this type, generally speaking, is homogeneous. We assume that our source has such singularity.

In another work we hope to conduct a mechanical analysis of this concept of homogeneous singularity.

On the interval $0 \le t \le af$ there is no wave reflected from the plane $y = 0$ of the space S, and, as discussed above, the elementary potentials φ and ψ depend only on the ratios $\dfrac{x}{t}$ and $\dfrac{y-f}{t}$, i.e., they must remain constant on the straight lines of the space S, passing through the point $x = 0$, $y = f$, $t = 0$. Subsequently, these lines will be called the rays of the space S. First of all, we consider the case when the source F is the source of longitudinal waves, i.e., we assume that the potential ψ is equal to zero on the interval $0 \le t \le af$. The potential φ is not equal to zero only for

$$t^2 > a^2[x^2 + (y - f)^2],$$

i.e., inside the cone T_0 of the space S with apex F. The equation of the cone is

$$t^2 - a^2[x^2 + (y - f)^2] = 0. \tag{39}$$

We consider only the inner part of this cone, where $y \ge 0$ and $t > 0$.

Introduce the complex variable θ_1 determined, as in (35), by the equality

$$t - \theta_1 x + \sqrt{a^2 - \theta_1^2}(y - f) = 0,$$

i.e.,

$$\delta_1 = t - \theta_1 x + \sqrt{a^2 - \theta_1^2}\, y - \sqrt{a^2 - \theta_1^2}\, f = 0. \tag{40}$$

Then φ must be the real part of an analytic function of the complex variable θ_1

$$\varphi_1 = \operatorname{Re}[\varPhi_1(\theta_1)]. \tag{41}$$

Expression (40) sets in the correspondence to each ray inside the cone T_0 a value of θ_1, and φ_1 remains constant along each ray. Consider this correspondence in detail. Solving equation (40) with respect to θ_1, we obtain

$$\theta_1 = \frac{xt - i(y-f)\sqrt{t^2 - a^2[x^2 + (y-f)^2]}}{x^2 + (y-f)^2}, \tag{42}$$

where the radical is taken with "+" sign. The rays, located in the half-space $t > 0$ and crossing the plane $y = 0$, correspond to the complex values of θ_1 from the upper plane, i.e., with the positive imaginary part. Formula (42) establishes the law of the correspondence between the rays and the values of θ_1. The family of rays, forming the part of the cone where $t > 0$, corresponds to the entire complex plane with the cut $(-a, +a)$ along the real axis. However, the points of this cut correspond to the generators of the cone. The intervals $(-\infty, -a)$ and $(+a, +\infty)$ of the real axis of θ_1 correspond to the rays located on the plane $y = f$, the imaginary axis corresponds to the rays of the plane $x = 0$, and the upper half $(0, +i\infty)$ of this axis corresponds to the rays for which $y < f$, and which further intersect the plane $y = 0$. From the last fact and equation (40) it follows that in this equation the radical $\sqrt{a^2 - \theta_1^2}$ is positive for the values of θ_1 on the imaginary semiaxis $(0, +i\infty)$. This is equivalent to the assumption that the value of the radical $\sqrt{a^2 - \theta_1^2}$ is negative imaginary for $\theta_1 > a$.

The generators of the cone T_0 correspond to the front of propagation of vibrations. Consequently, φ_1 must vanish in the corresponding points, i.e., the function $\Phi_1(\theta_1)$ in (41) must be purely imaginary on the cut $(-a, +a)$. The points of the axis of the cone T_0 correspond to the source of different moments, and this axis corresponds to the point of the plane θ_1 at infinity. Since we know the source, we do the singularity of $\Phi_1(\theta)$ at infinity.

Thus, the function $\Phi_1(\theta)$ is determined. A more detailed analysis of different sources will be conducted later. Our assumption, that the potential φ remains constant along each ray emanating from the point $x = 0$, $y = f$, $t = 0$, leads us to the fact that the singularity of φ_1 in the force source takes place at all moments $t > 0$.

9. The given elementary potential φ_1 determines the motion when $t < af$. For $t > af$ we have to add two more potentials: one φ_2 for the longitudinal wave, and another ψ_1 for the transverse wave. We select these potentials in the same way as above, i.e., we assume that these potentials must remain constant along some rays of the space S. These rays are called the *reflected rays*. Beginning with the construction of φ_2, first of all, we note that φ_2 must be the real part of an analytic function:

$$\varphi_2 = \operatorname{Re}[\Phi_2(\theta_2)], \tag{43}$$

where θ_2 is defined by equation (35) for $c = a$. We choose the function $\chi(\theta)$ in this equation such that the values of θ_1 and θ_2 coincide for $y = 0$, i.e., we select $\chi(\theta)$ as in equation (40).

Then, for θ_2 we have the equation

$$\delta_2 = t - \theta_2 x - \sqrt{a^2 - \theta_2^2}\, y - \sqrt{a^2 - \theta_2^2}\, f = 0. \tag{44}$$

It is easy to verify that these reflected rays generate the cone

$$t^2 - a^2[x^2 + (y+f)^2)] \geq 0$$

with apex $(0, -f, 0)$. We select in equation (44) the opposite sign of the radical than in equation (40), so the rays going to the domain $t > 0$, $y > 0$, correspond to the complex values of θ_2 with the positive imaginary parts.

Constructing the potential ψ_1, we should put $c = b$ in equation (35). The term $\chi(\theta)$ is chosen in the same way as in equation (40). The sign of the radical in the coefficient at y should be taken such that the rays, along which y and t increase, correspond to the values of θ with the positive imaginary parts. It is easy to show that we should take "$-$" sign.

Then, for θ_3 we have the equation

$$\delta_3 = t - \theta_3 x - \sqrt{b^2 - \theta_3^2}\, y - \sqrt{a^2 - \theta_3^2}\, f = 0. \tag{45}$$

For $y = 0$ the values of θ_3 coincide with the values of θ_1 and θ_2.

The potential ψ_1 is the real part of an analytic function

$$\psi_1 = \mathrm{Re}\,[\Psi(\theta_3)]. \tag{46}$$

Before we construct the functions $\Phi_2(\theta_2)$ and $\Psi(\theta_3)$, let us point out the connection between the variables θ. For this purpose, consider the section of the main cone T_0 by the plane $y = 0$, where we have the reflection. In the section we have the hyperbola

$$t^2 - a^2(x^2 + f^2) = 0.$$

Each point (x, t) of the plane $y = 0$, located inside this hyperbola, for which

$$t^2 - a^2(x^2 + f^2) \geq 0 \quad \text{and} \quad t > 0,$$

corresponds to a complex value of θ_1 from the upper half-plane or the real axis. By the construction of equations (44) and (45), the values of θ_2 and θ_3 coinciding with the values of θ_1 correspond to the point (x, t). Thus, choosing the point (x, t), we define the complex values of θ_2 and θ_3 from the upper half-plane. Substituting these values into (44) and (45), we obtain two reflected rays in the space S. The potential φ_2 remains constant along one of these rays, and ψ_1 remains constant along another one. The values of y and t increase along these reflected rays. Hence the addition of the potentials φ_2 and ψ_1 does not influence the motion for $t < af$ and does not change the initial data. If we fix a point (x, y) and a moment t, then the corresponding values of θ_2

and θ_3 from the upper half-plane are obtained from equations (44) and (45). Obviously, it is impossible to define complex θ_2 and θ_3 for some points (x, y, t).

This corresponds to the fact that the reflected rays do not fill the entire domain $t > 0$, $y > 0$.

If, for example, the reflected ray of the potential φ_2 does not pass through the point (x, y, t), then we should not add the potential φ_2 at this point in order to construct the solution.

It is easy to verify that, by (35), the complex value of θ characterizes some direction in the space S without any dependence on the term $\chi(\theta)$. Thus, the above reasoning gives us the law of the correspondence between the directions of the incident and reflected rays. We will not discuss this anymore, since our goal is only the effective construction of the solution.

For the displacement components we have

$$\begin{aligned} u &= \operatorname{Re}\left[\varPhi_1'(\theta_1)\frac{\partial\theta_1}{\partial x} + \varPhi_2'(\theta_2)\frac{\partial\theta_2}{\partial x} + \varPsi'(\theta_3)\frac{\partial\theta_3}{\partial y}\right], \\ v &= \operatorname{Re}\left[\varPhi_1'(\theta_1)\frac{\partial\theta_1}{\partial y} + \varPhi_2'(\theta_2)\frac{\partial\theta_2}{\partial y} - \varPsi'(\theta_3)\frac{\partial\theta_3}{\partial x}\right]. \end{aligned} \tag{47}$$

Inside the hyperbola $t^2 - a^2(x^2 + f^2) = 0$ on the plane $y = 0$ the boundary conditions expressing the absence of stresses must hold. However, note that the variables θ_1, θ_2 and θ_3 coincide for $y = 0$. This allows us to omit index. Furthermore, let δ' without index denote the general value of the variables δ_1', δ_2' and δ_3' for $y = 0$. Using (38), we can write the boundary conditions in the form

$$\begin{aligned} &\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{-2\theta\sqrt{a^2-\theta^2}[\varPhi_1'(\theta)-\varPhi_2'(\theta)] + (b^2-2\theta^2)\varPsi'(\theta)}{\delta'}\right\}\Bigg|_{y=0} = 0, \\ &\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{(b^2-2\theta^2)[\varPhi_1'(\theta)+\varPhi_2'(\theta)] - 2\theta\sqrt{b^2-\theta^2}\varPsi'(\theta)}{\delta'}\right\}\Bigg|_{y=0} = 0. \end{aligned} \tag{48}$$

The expressions under the sign of the real part Re contain the complex variable θ, which can take arbitrary values from the upper half-plane, and the real variable x, which appears in the formula for δ.

First, note that θ can be expressed in terms of x and t. This follows from formula (42) for $y = 0$.

Thus, in the expression

$$\delta' = -x + \frac{\theta}{\sqrt{a^2-\theta^2}}\, f \tag{49}$$

we can replace x by means of the formula

$$x = \frac{t}{\theta} - \frac{\sqrt{a^2-\theta^2}\, f}{\theta}.$$

Hence we have one complex variable θ and one real parameter t under the sign of the real part on the left sides of (48).

Consider the interval $(-a, +a)$ of the real axis of the plane θ, corresponding to the generators of the cone T_0, i.e., to the front of propagation of vibrations on the plane $y = 0$. All three potentials φ_1, φ_2 and ψ must vanish on this front, i.e., the real parts of the functions Φ_1, Φ_2 and Ψ must be equal to zero on this interval. Obviously, we can make the same conclusion about the derivatives Φ_1', Φ_2', and Ψ'. Taking into account that the radical $\sqrt{a^2 - \theta^2}$ is real on this interval, we can assert that conditions (48) are satisfied for each positive real value of t on the interval $-a < \theta < +a$. Fix now a value of t and prove that conditions (48) will be satisfied for this value of t and for all θ from the upper half-plane. If t is fixed, and x is changing from

$$-\frac{\sqrt{t^2 - a^2 f^2}}{a}$$

to

$$+\frac{\sqrt{t^2 - a^2 f^2}}{a},$$

then the complex variable

$$\theta = \frac{xt}{x^2 + f^2} \pm i \frac{f\sqrt{t^2 - a^2(x^2 + f^2)}}{x^2 + f^2}$$

describes a curve l, issuing from a point A on the interval $(-a, +a)$ and arriving at another point B on the same interval. The curve l together with the interval AB of the real axis form a closed contour. By (48) for fixed t the expressions for θ and t along this contour have zero real parts. Then, these real parts must vanish on the entire upper half-plane of θ. Making the change of variables

$$t = \theta x + \sqrt{a^2 - \theta^2}\, f,$$

we can conclude that conditions (48), where δ' is defined by (49), must hold for an arbitrary value of x on the entire upper half-plane of θ. Let us prove that we then have

$$\begin{aligned} &-2\theta\sqrt{a^2 - \theta^2}[\Phi_1'(\theta) - \Phi_2'(\theta)] + (b^2 - 2\theta^2)\Psi'(\theta) = 0, \\ &(b^2 - 2\theta^2)[\Phi_1'(\theta) + \Phi_2'(\theta)] - 2\theta\sqrt{b^2 - \theta^2}\Psi'(\theta) = 0. \end{aligned} \tag{50}$$

Denoting by $\sigma_1(\theta)$ the left side of the first of these equalities and putting

$$\sigma_2(\theta) = \frac{\theta f}{\sqrt{a^2 - \theta^2}},$$

we can express the first condition in (48) in the form

$$\frac{\sigma_1'(\theta)[-x + \sigma_2(\theta)] - \sigma_2'(\theta)\sigma_1(\theta)}{[-x + \sigma_2(\theta)]^2} = Ci,$$

where C is a real constant depending only on x, and θ can take arbitrary values in the upper half-plane. From the last equality it follows that the coefficient at x and the term independent of x in the numerator of this fraction must vanish. Hence $\sigma_1(\theta) = 0$, i.e., the first equality in (50) holds. Similarly, we can prove the second equality.

Solving equations (50) with respect to $\Phi_2'(\theta)$ and $\Psi'(\theta)$, we obtain

$$\begin{aligned} \Phi_2'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Phi_1'(\theta), \\ \Psi'(\theta) &= -\frac{4\theta(2\theta^2-b^2)\sqrt{a^2-\theta^2}}{F(\theta)}\Phi_1'(\theta), \end{aligned} \tag{51}$$

where

$$F(\theta) = (2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}. \tag{52}$$

The fractions in (51) are real on the interval $-a < \theta < +a$ of the real axis. On the other hand, the real part of the function $\Phi_1(\theta)$ vanishes on this interval. Whence, by condition, the real parts of $\Phi_2'(\theta)$ and $\Psi'(\theta)$ also vanish on this interval. Integrating, we can choose additive constants in the expressions for the potentials $\Phi_2(\theta)$ and $\Psi(\theta)$ such that the real parts of $\Phi_2(\theta)$ and $\Psi(\theta)$ will also be equal to zero. By formulas (51) and (47), we can determine the displacement components.

10. Let us point out some consequences of the obtained formulas. Consider equation (40) having complex roots inside the cone T_0 and real roots from the interval $-a < \theta < +a$ on the generators of this cone. As is known, these generators correspond to the front of propagation of the longitudinal wave in the domain $t > 0$, $y > 0$ of the space S. Let $\theta_1 = \theta_0$ be a value from the interval $-a < \theta_1 < +a$, let λ_0 be a corresponding generator. If we put $\theta_1 = \theta_0$ in equation (40), then we have the equation of the plane tangent to the cone T_0 along λ_0. Therefore points (x, y, t) in the exterior of the cone T_0 correspond to real values of θ_1 from the interval $-a < \theta_1 < +a$. Then, $\delta_1'' = 0$ along each generator λ_0, i.e., the derivative of the left side of equation (40) is equal to zero. Hence the derivatives $\dfrac{\partial\theta_1}{\partial x}$ and $\dfrac{\partial\theta_1}{\partial y}$ are infinite along these generators, and we have infinite displacements on the front of propagation of the longitudinal wave. The study of equations (44) and (45) leads to a similar conclusion, and we have infinite displacements on the fronts of reflected waves.

Expanding the left side of equation (40) in powers of $(\theta_1 - \theta_0)$, we can conclude that $\dfrac{\partial\theta_1}{\partial x}$ and $\dfrac{\partial\theta_1}{\partial y}$ are infinite of the orders

$$\frac{1}{\sqrt{x - x_0}} \quad \text{and} \quad \frac{1}{\sqrt{y - y_0}},$$

respectively.

Let us now move on to finding asymptotic estimates of the obtained solution as $t \to \infty$. This will give us the phenomenon of surface waves in the clear form.

Let $\xi = x - \frac{t}{c}$, $\eta = y$, where c is a positive root of the equation $F(\theta) = 0$, i.e., $\dfrac{1}{c}$ is the known velocity of the Rayleigh wave [2]. Assuming that ξ and η remain bounded, let us construct the asymptotic expansions for θ_1 and θ_2 up to the terms of order $\dfrac{1}{t^2}$. It is easy to see that we have

$$\begin{aligned} \theta_1 &= c - \frac{c^2\xi}{t} - i\frac{c\sqrt{c^2-a^2}(\eta - f)}{t} + O\left(\frac{1}{t^2}\right), \\ \theta_2 &= c - \frac{c^2\xi}{t} + i\frac{c\sqrt{c^2-a^2}(\eta + f)}{t} + O\left(\frac{1}{t^2}\right). \end{aligned} \tag{53}$$

Hence,

$$\begin{aligned} \frac{\partial\theta_1}{\partial x} &= -\frac{c^2}{t} + O\left(\frac{1}{t^2}\right), \quad \frac{\partial\theta_1}{\partial y} = -i\frac{c\sqrt{c^2-a^2}}{t} + O\left(\frac{1}{t^2}\right), \\ \frac{\partial\theta_2}{\partial x} &= -\frac{c^2}{t} + O\left(\frac{1}{t^2}\right), \quad \frac{\partial\theta_2}{\partial y} = i\frac{c\sqrt{c^2-a^2}}{t} + O\left(\frac{1}{t^2}\right). \end{aligned} \tag{54}$$

By $F(c) = 0$, one can also verify that

$$F(\theta_1) = F'(c)\frac{-c^2\xi - ic\sqrt{c^2-a^2}(\eta - f)}{t} + O\left(\frac{1}{t^2}\right), \tag{55}$$

$$F(\theta_2) = F'(c)\frac{-c^2\xi + ic\sqrt{c^2-a^2}(\eta + f)}{t} + O\left(\frac{1}{t^2}\right). \tag{56}$$

Similarly, for θ_3 we obtain

$$\theta_3 = c - \frac{c^2\xi}{t} + i\frac{c\sqrt{c^2-b^2}\eta}{t} + i\frac{c\sqrt{c^2-a^2}\, f}{t} + O\left(\frac{1}{t^2}\right), \tag{57}$$

$$\frac{\partial\theta_3}{\partial x} = -\frac{c^2}{t} + O\left(\frac{1}{t^2}\right), \quad \frac{\partial\theta_3}{\partial y} = i\frac{c\sqrt{c^2-b^2}}{t} + O\left(\frac{1}{t^2}\right). \tag{58}$$

This allows us to write the asymptotic expansions for the displacement components up to the term of order $\dfrac{1}{t}$. Taking into account (47) and (51)[3], we have

[3] The authors use also the formula

$$F(\theta_3) = F'(c)\frac{-c^2\xi + ic\sqrt{c^2-b^2}\eta + ic\sqrt{c^2-a^2}f}{t} + O\left(\frac{1}{t^2}\right). \text{ – } \mathit{Ed.}$$

$$u = \mathrm{Re}\left\{ \frac{-(2c^2-b^2)^2 - 4c^2\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{F'(c)} \right.$$
$$\times \frac{-c}{-c\xi + i\sqrt{c^2-a^2}(\eta+f)} \Phi_1'(c) - \frac{i4c(2c^2-b^2)\sqrt{c^2-a^2}}{F'(c)}$$
$$\left. \times \frac{i\sqrt{c^2-b^2}}{-c\xi + i\sqrt{c^2-b^2}\eta + i\sqrt{c^2-a^2}\, f} \Phi_1'(c) \right\} + O\left(\frac{1}{t}\right), \tag{59.1}$$
$$v = \mathrm{Re}\left\{ \frac{-(2c^2-b^2)^2 - 4c^2\sqrt{c^2-a^2}\sqrt{c^2-b^2}}{F'(c)} \right.$$
$$\times \frac{i\sqrt{c^2-a^2}}{-c\xi + i\sqrt{c^2-a^2}(\eta+f)} \Phi_1'(c) + \frac{i4c(2c^2-b^2)\sqrt{c^2-a^2}}{F'(c)}$$
$$\left. \times \frac{-c}{-c\xi + i\sqrt{c^2-b^2}\eta + i\sqrt{c^2-a^2}\, f} \Phi_1'(c) \right\} + O\left(\frac{1}{t}\right). \tag{59.2}$$

Our analysis allows us to note that at infinity vibrations produce the wave propagating with the velocity $\dfrac{1}{c}$ with bounded amplitude. It is easy to see that this wave is a natural generalization of the Rayleigh wave[4].

In the case of the concentrated source of the force inside the medium, we see that the surface wave has nonperiodic nature. We should also mention that the exponential law of damping in the depth is not valid anymore. Obviously, the concept of wave length does not make sense.

11. Let us now move on to the source of transverse waves. As in the previous problem, we assume that this source is regular, i.e., the given elementary potential of the transverse waves ψ_1 is the real part of a regular analytic function

$$\psi_1 = \mathrm{Re}\,[\Psi_1(\theta_1)], \tag{60}$$

where the complex variable θ_1 is defined by an equation similar to (40),

$$\delta_1 = t - \theta_1 x + \sqrt{b^2-\theta_1^2}\, y - \sqrt{b^2-\theta_1^2}\, f = 0. \tag{61}$$

In this case, the cone T_0 is defined by the equation

$$t^2 - b^2[x^2 + (y-f)^2] = 0, \tag{62}$$

and the rays located inside this cone correspond to the plane of the complex variable θ_1 with the cut $(-b, +b)$ along the real axis. The values of θ_1 on this cut correspond to the generators of the cone. We look for the potential of longitudinal reflected waves in the form of the real part of a function analytic in the upper half-plane

$$\varphi = \mathrm{Re}\,[\Phi(\theta_2)], \tag{63}$$

[4] These waves were studied by S. L. Sobolev in his work cited above.

where θ_2 is defined by the equation

$$\delta_2 = t - \theta_2 x - \sqrt{a^2 - \theta_2^2}\, y - \sqrt{b^2 - \theta_2^2}\, f = 0. \tag{64}$$

As before, we chose an equation such that it coincides with equation (61) for $y = 0$. In the section of cone (62), by the plane $y = 0$, we have the hyperbola

$$t^2 - b^2(x^2 + f^2) = 0, \quad t > 0. \tag{65}$$

Each point P from the interior of this hyperbola corresponds to a complex value of θ_1 from the upper half-plane, and points of the hyperbola correspond to values of θ_1 on the interval $(-b, +b)$ of the real axis. To obtain a reflected ray $l_{x,t}$ of the potential φ of the longitudinal wave passing through a point P with coordinates (x, t) of the plane $y = 0$, we should take the corresponding value of θ_1 and substitute it for θ_2 into equation (64). This ray $l_{x,t}$ passes through the point P, and equation (64) defines its direction.

As we have already noted, the direction of the straight lines, obtained from equation (64), is completely defined by the first three terms on the left side of this equation. Hence the direction is the same as one obtained from equation (44) with the same value of θ. The straight lines of equation (44) form the already known cone with apex $x = 0$, $y = -f$, $t = 0$ and the apex angle equal to $\arctan \dfrac{1}{a}$. For this cone as well as for the cone T_0 from our problem, the values of θ from the upper half-plane correspond to the rays along which y and t increase simultaneously. When the value of θ tends to a point of the real interval $(-a, +a)$, the direction of the corresponding ray coincides with the direction of the corresponding generator of the cone. When θ tends to a point of the real axis outside the interval $(-a, +a)$, the ray direction is parallel to the plane $y = 0$ in the limit. In the present case, the points of hyperbola (62) correspond to the values of θ_1 on the interval $(-b, +b)$. Let A and B be the points of this hyperbola for $\theta_1 = \pm a$ (see Fig. 3).

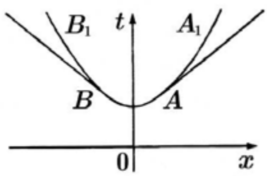

Fig. 3.

The arc AB of the hyperbola corresponds to the values of θ_1 from $-a < \theta_1 < +a$.

The infinite branches AA_1 and BB_1 correspond to the values of θ_1 from the intervals $a \le \theta_1 < b$ and $-b \le \theta_1 < -a$. The above reasoning leads us to the following conclusion: if a point $P(x,t)$ tends to a point on the arc AA_1 or BB_1, then the angle between the corresponding ray $l_{x,t}$ of the reflected longitudinal wave and the plane $y=0$ tends to zero. The limit for the points located on these arcs is on the plane $y=0$.

Substituting into (64) instead of θ_2 some value from the interval (a,b) or $(-b,-a)$, we obtain the equation of the ray $l_{x,t}$ passing through a point of the arc AA_1 or BB_1 and located on the plane $y=0$:

$$t-\theta_2 x-\sqrt{b^2-\theta_2^2}\,f=0.$$

It is easy to show that the last equation defines the tangents to hyperbola (65). Hence, for each point of the arcs AA_1 and BB_1 of the hyperbola, the corresponding ray of the reflected longitudinal potential is tangent to hyperbola (65) at this point.

Later we will see that the potential of the reflected longitudinal waves will be equal to zero only on the interval $(-a,+a)$ of the real axis, as in the case of the longitudinal source, but it will not be equal to zero on the intervals (a,b) and $(-b,-a)$. Also, it will not vanish in two domains of the plane bounded by the arcs AA_1 and BB_1 of the hyperbola and two tangents to the hyperbola at the points A and B. We denote these domains by (I) and (II). There is no incident transverse wave in these domains. To satisfy the boundary conditions, we have to define the potential ψ_2 of the reflected transverse wave not only inside hyperbola (65), but also outside this hyperbola in the domains (I) and (II).

We will see later how to do it. We now move on to the definition of ψ_2 inside the hyperbola, i.e., for complex values of θ from the upper half-plane. Here, ψ_2 is the real part of an analytic function

$$\psi_2=\operatorname{Re}\,[\Psi_2(\theta_3)], \tag{66}$$

where θ_3 is defined by the equation

$$\delta_3=t-\theta_3 x-\sqrt{b^2-\theta_3^2}\,y-\sqrt{b^2-\theta_3^2}\,f=0, \tag{67}$$

which defines the conical beam T_1 of rays with apex

$$F_1(x=0,\ y=-f,\ t=0)$$

and angle $\arctan\dfrac{1}{b}$ at the apex. We consider only those rays of this beam which pass through the domain $y>0$, $t>0$ of the space S.

Let us now write the boundary conditions for the mentioned points, i.e., for the values of θ from the upper half-plane. The values of θ_1, θ_2 and θ_3 coincide for $y=0$. Denoting by θ this common value, we have

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{2\theta\sqrt{a^2-\theta^2}\varPhi'(\theta)+(b^2-2\theta^2)[\varPsi_1'(\theta)+\varPsi_2'(\theta)]}{\delta'}\right\}\Bigg|_{y=0}=0,\\
&\operatorname{Re}\left\{\frac{1}{\delta'}\frac{\partial}{\partial\theta}\frac{(b^2-2\theta^2)\varPhi'(\theta)+2\theta\sqrt{b^2-\theta^2}[\varPsi_1'(\theta)-\varPsi_2'(\theta)]}{\delta'}\right\}\Bigg|_{y=0}=0,
\end{aligned}\tag{68}$$

where δ' is the derivative of the expression $(t-\theta x-\sqrt{b^2-\theta^2}\,f)$ with respect to θ.

As in the case of the source of longitudinal waves, from above we obtain

$$\begin{aligned}
&2\theta\sqrt{a^2-\theta^2}\varPhi'(\theta)+(b^2-2\theta^2)[\varPsi_1'(\theta)+\varPsi_2'(\theta)]=0,\\
&(b^2-2\theta^2)\varPhi'(\theta)+2\theta\sqrt{b^2-\theta^2}[\varPsi_1'(\theta)-\varPsi_2'(\theta)]=0.
\end{aligned}\tag{69}$$

Then, we can define the functions $\varPhi'(\theta)$ and $\varPsi_2'(\theta)$

$$\begin{aligned}
\varPhi'(\theta)&=\frac{4\theta(2\theta^2-b^2)\sqrt{b^2-\theta^2}}{F(\theta)}\varPsi_1'(\theta),\\
\varPsi_2'(\theta)&=\frac{-(2\theta^2-b^2)^2+4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\varPsi_1'(\theta).
\end{aligned}\tag{70}$$

The potential ψ_1 of the transverse waves propagating from the source must vanish on the wave front. It means that the real parts of the functions $\varPsi_1(\theta)$ and $\varPsi_1'(\theta)$ must be equal to zero for $-b\le\theta\le+b$. Taking into account that the fractions in (70) are real for $-a\le\theta\le+a$, we can assert that $\varPhi'(\theta)$ and $\varPsi_2'(\theta)$ have zero real part for $-a\le\theta\le+a$. This fact is not valid anymore on the intervals $a<\theta<b$ and $-b<\theta<-a$, since the indicated fractions contain the radical $\sqrt{a^2-\theta^2}$. Therefore, in the domains (I) and (II) of the plane $y=0$ the potential φ equal to the real part of $\varPhi(\theta)$ is not equal to zero. These domains are generated by the rays $l_{x,t}$ or the l_θ, corresponding to the real values of θ from the intervals (a,b) and $(-b,-a)$. If we substitute such value of θ for θ_3 into equation (67), we have the equation of some plane in the space S. The section of this plane by the plane $y=0$ is the ray l_θ. It is easy to see that this plane is tangent to the cone T_1 of the reflected transverse wave. Thus, we have the family of planes tangent to the cone T_1 along the generators passing through the points P of the arcs AA_1 and BB_1 of the hyperbola. Consider one of the planes tangent to the cone along the generator F_1P. Let θ be the real value of the parameter θ, corresponding to this generator F_1P. Denote by U_θ the domain of this tangent plane, bounded by the generator F_1P and the ray l_θ of the plane $y=0$, and located in the half-space $y>0$. The values of θ belong to the intervals (a,b) or $(-b,-a)$.

The domains U_θ fill a domain R in the space S. In this domain we define the potential ψ_2 as a function of the real variable θ. This function is constant in each U_θ. As already mentioned in Sect. 7, an arbitrary function of a real root θ of equation (67) in the domain R satisfies the wave equation (2) for $c = b$.

Our choice of U_θ allows us to assert that we did not break the initial conditions, since we have $t > af$ for U_θ. Similar circumstances will be valid for the future problems, and we will not discuss it anymore. As we will see later, our procedure always determines the potential continuously. Moving on to the effective computation of this potential, we have to choose a function of θ, which defines the potential ψ_2 in the domain R such that the boundary conditions are always satisfied in the domains (I) and (II) of the plane $y = 0$. The second formula in (70) gives us $\Psi_2'(\theta)$ on the intervals (a, b) and $(-b, -a)$. Integrating along the real axis, we obtain $\Psi_2(\theta)$. Obviously, one can put $\Psi_2(\pm a) = 0$. It is easy to prove that if the potential ψ_2 is equal to the real part of the indicated function $\Psi_2(\theta)$ on the planes U_θ, then the boundary conditions will be satisfied also in the domains (I) and (II) of the plane $y = 0$. Indeed, returning to equalities (69), we can assert that they hold also on the intervals $a \le \theta \le b$ and $-b \le \theta \le -a$. However, the real part of $\Psi_1'(\theta)$ is equal to zero on these intervals, and the coefficients of this function in equation (69) do not contain the radical $\sqrt{a^2 - \theta^2}$. Hence these coefficients are real. Taking into account once again the fact that δ' is also real, in the discussed case we have the condition in form (68) with $\Psi_1'(\theta) = 0$, i.e.,

$$\operatorname{Re}\left\{ \frac{1}{\delta'} \frac{\partial}{\partial\theta} \frac{2\theta\sqrt{a^2-\theta^2}\Phi'(\theta) + (b^2 - 2\theta^2)\Psi_2'(\theta)}{\delta'} \right\}\Bigg|_{y=0} = 0,$$

$$\operatorname{Re}\left\{ \frac{1}{\delta'} \frac{\partial}{\partial\theta} \frac{(b^2 - 2\theta^2)\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\Psi_2'(\theta)}{\delta'} \right\}\Bigg|_{y=0} = 0.$$

These equations show that the boundary conditions hold in the domains (I) and (II) of the plane $y = 0$. Thus, the problem is solved.

Let us note again that the value of ψ_2 on U_θ is equal to the value of this function along the generator F_1P, through which U_θ passes.

12. Let us now derive some consequences of the obtained results. As in Sect. 10, we can prove that the displacements are infinite on the fronts of the waves. We will not return to this point anymore.

If we cross the constructions made in the space S by the plane $t = \text{const}$, we obtain the fronts of the waves at the time moment t (see Fig. 4). Let us take sufficiently large t such that the plane $t = \text{const}$ to pass through the domain R of the space S. In this case, the front of the transverse waves consists of three parts. The first part is the arc AHB of the circle that is the section of the cone T_0 by the plane $t = \text{const}$. This is a wave propagating from the source. The second part is the arc $AEFB$ of the circle that is the section of the cone T_1 by the plane $t = \text{const}$. The third part consists of two lines CE and DF that are the sections U_{+a} and U_{-a} by the plane $t = \text{const}$. This last part is generated by the longitudinal waves propagating along the plane $y = 0$ with the velocity $\frac{1}{a}$. The points E and F are the points of the intersection of the plane $t = \text{const}$ with the generators of the cone T_0, corresponding to the values $\theta = \pm a$. The front of the longitudinal waves is the curve CGD enveloping the lines

$$\theta x + \sqrt{a^2 - \theta^2}\, y + \sqrt{b^2 - \theta^2}\, f = t, \qquad -a \le \theta \le +a, \quad t = \text{const}.$$

All these fronts propagate according to the Fermat principle. As in the previous case, one can give asymptotic representations of the displacements and to reveal the surface wave. The explanation is completely analogous to the above one.

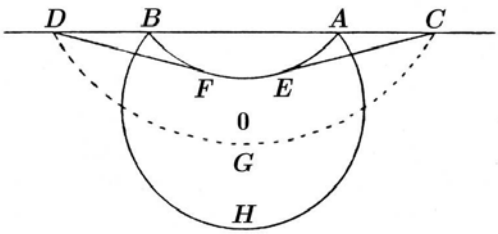

Fig. 4.

13. The presented approach can be applied not only to the two-dimensional problem on vibrations of the half-space, but it also gives the general law of reflection of a beam of rays of special type from a plane in the space S.

For this special type, the potential (longitudinal or transverse) is the real part of an analytic function of θ in the upper half-plane, where θ is a root of the equation

$$t - \theta x \pm \sqrt{c^2 - \theta^2}\, y - \chi(\theta) = 0, \quad c = a \text{ or } b.$$

As mentioned above, this form is equivalent to form (34). We will say that in this case vibrations have imaginary potentials.

The indicated analytic function satisfies also some boundary conditions. In the last cases it is necessary to consider real values of θ which correspond to planes in the space S. The potential must remain constant on each of these planes. We do not consider the entire plane, but rather only its part concluded between the reflective plane and the terminal position of the ray obtained when θ from the upper half-plane tends to the discussed real value corresponding to the plane. The presented method gives, for example, a solution of the problem on vibrations of a layer.

Let $2f$ be the thickness of the plane layer bounded by the lines $y = 0$ and $y = 2f$. Suppose that we have a source of longitudinal type at the point $x = 0$, $y = f$ with the singularity of the type described above. Let the potential of this source be given by the formula

$$\varphi = \mathrm{Re}\,[\Phi(\theta)], \tag{71}$$

where the analytic function $\Phi(\theta)$ is defined on the entire plane with the cut $(-a, +a)$ along the real axis; let the real part of $\Phi(\theta)$ be equal to zero on this cut. Consider the part Ω of the space S, bounded by the planes $y = 0$ and $y = 2f$. Denote by S_0 the first of these planes, and the second by S_1. If we are at rest for $t < 0$, then we have longitudinal vibrations with the given potential φ for $0 \le t < af$. The rays corresponding to this wave form the cone T_0 with apex $(x = 0,\ y = f,\ t = 0)$ and angle $\arctan \dfrac{1}{a}$ at the apex. At the moment $t = af$ we have reflected rays of longitudinal and transverse waves with respect to the planes S_0 and S_1. All these rays follow the direction of growth of t. Hence, in the domain Ω bounded by the planes $t = 0$ and $t = af$, the displacement is defined by the fundamental cone T_0. In expression (71), θ is defined by the equality

$$t - \theta x + \sqrt{a^2 - \theta^2}\, y - \sqrt{a^2 - \theta^2}\, f = 0.$$

Let φ_1 and ψ_1 be the potentials of the longitudinal and transverse waves reflected from the plane S_0, let φ_2 and ψ_2 be the analogous potentials for the reflection from S_1.

We have

$$\varphi_1 = \mathrm{Re}\,[\Phi_1(\theta_1)] \quad \text{and} \quad \psi_1 = \mathrm{Re}\,[\Psi_1(\theta_1')], \tag{72}$$

where θ_1 and θ_1' are complex values from the upper half-plane, defined by the equations

$$\begin{gathered} t - \theta_1 x - \sqrt{a^2 - \theta_1^2}\, y - \sqrt{a^2 - \theta_1^2}\, f = 0, \\ t - \theta_1' x - \sqrt{b^2 - {\theta_1'}^2}\, y - \sqrt{a^2 - {\theta_1'}^2}\, f = 0. \end{gathered} \tag{73}$$

Equations (51) allow us to obtain the functions $\Phi_1(\theta_1)$ and $\Psi_1(\theta_1)$ for values of the argument from the upper half-plane

$$\begin{aligned} \Phi_1'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2 - \theta^2}\sqrt{b^2 - \theta^2}}{F(\theta)}\Phi'(\theta), \\ \Psi_1'(\theta) &= \frac{-4\theta(2\theta^2 - b^2)\sqrt{a^2 - \theta^2}}{F(\theta)}\Phi'(\theta). \end{aligned} \tag{74}$$

In this case, the real parts of $\Phi_1'(\theta)$ and $\Psi_1'(\theta)$ are equal to zero on the interval $-a \le \theta \le +a$. These reflected rays pass through points of the plane S_0, located inside the hyperbola $t^2 - a^2(x^2 + f^2) = 0$. The rays fall on the plane S_1 above the line $t = 3af$.

Values of θ from the lower half-plane correspond to rays of the cone T_0, falling on the plane S_1. Let

$$\varphi_2 = \mathrm{Re}\,[\Phi_2(\theta_2)], \quad \psi_2 = \mathrm{Re}\,[\Psi_2(\theta_2')] \tag{75}$$

be the potentials of the longitudinal and transverse waves reflected from the plane S_1. Complex values of θ_2 and θ_2' from the lower half-plane must coincide with θ for $y = 2f$. It is easy to see that θ_2 and θ_2' are defined by the equations

$$\begin{aligned} &t - \theta_2 x - \sqrt{a^2 - \theta_2^2}\, y + 3\sqrt{a^2 - \theta_2^2}\, f = 0, \\ &t - \theta_2' x - \sqrt{b^2 - \theta_2'^2}\, y + 2\sqrt{b^2 - \theta_2'^2}\, f - \sqrt{a^2 - \theta_2'^2}\, f = 0. \end{aligned} \tag{76}$$

The derivatives of the functions Φ_2 and Ψ_2 are determined by the formulas obtained from (74) by the sign change in front of the radical $\sqrt{a^2 - \theta^2}$ in the second formula. The displacement of the layer, bounded by the planes $t = f$ and $t = 3f$ in the domain Ω, is determined by the potentials φ, φ_1, φ_2, ψ_1, and ψ_2. Further, we have to consider the reflection of the rays corresponding to the potentials φ_1, φ_2, ψ_1, and ψ_2. First, consider the potential φ_1. The corresponding rays reflected from the plane S_0 fall on the plane S_1 and create reflected rays of longitudinal and transverse vibrations. Let us introduce the corresponding potentials

$$\varphi_3 = \mathrm{Re}\,[\Phi_3(\theta_3)], \quad \psi_3 = \mathrm{Re}\,[\Psi_3(\theta_3')], \tag{77}$$

where the variables θ_3 and θ_3' from the upper half-plane must coincide for $y = 2f$ with θ_1 defined by the first equation in (73). It is easy to see that the equations on these variables have the form

$$\begin{aligned} &t - \theta_3 x + \sqrt{a^2 - \theta_3^2}\, y - 5\sqrt{a^2 - \theta_3^2}\, f = 0, \\ &t - \theta_3' x + \sqrt{b^2 - \theta_3'^2}\, y - 2\sqrt{b^2 - \theta_3'^2}\, f - 3\sqrt{a^2 - \theta_3'^2}\, f = 0. \end{aligned} \tag{78}$$

The functions Φ_3' and Ψ_3' are determined through Φ_1' by the formulas obtained from (74) by the sign change in front of the radical $\sqrt{a^2 - \theta^2}$ in the second formula.

Introduce the potentials φ_4 and ψ_4 for rays corresponding to the reflection of the beam of rays of transverse vibrations with the potential ψ_1 from the plane S_1,

$$\varphi_4 = \mathrm{Re}\,[\Phi_4(\theta_4)], \quad \psi_4 = \mathrm{Re}\,[\Psi_4(\theta_4')], \tag{79}$$

where θ_4 and θ_4' from the upper half-plane satisfy the equations

$$\begin{aligned} &t - \theta_4 x + \sqrt{a^2 - \theta_4^2}\, y - 2\sqrt{b^2 - \theta_4^2}\, f - 3\sqrt{a^2 - \theta_4^2}\, f = 0, \\ &t - \theta_4' x + \sqrt{b^2 - \theta_4'^2}\, y - 4\sqrt{b^2 - \theta_4'^2}\, f - \sqrt{a^2 - \theta_4'^2}\, f = 0. \end{aligned} \tag{80}$$

The functions Φ_4' and Ψ_4' are determined through Ψ_1' by the formulas obtained from (70) by the sign change in front of the radical in the first formula

$$\Phi_4'(\theta) = \frac{-4\theta(2\theta^2 - b^2)\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta),$$
$$\Psi_4'(\theta) = \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Psi_1'(\theta). \tag{81}$$

Note that the range of the argument of the function $\Psi_1'(\theta)$ consists of the upper half-plane and the interval $(-a, +a)$, and the real part of $\Psi_1'(\theta)$ vanish on this interval. Analogous results are valid for all remaining functions obtained by the reflection from the planes S_0 and S_1. It is completely clear how we should continue the calculations.

In the case of a source of transverse vibrations we have somewhat different circumstances.

14. Assume that the formula $\psi = \mathrm{Re}\,[\Psi(\theta)]$ gives us the potential of a source of transverse vibrations, where the variable θ is defined by the equation

$$t - \theta x + \sqrt{b^2-\theta^2}\, y - \sqrt{b^2-\theta^2}\, f = 0, \tag{82}$$

and the range of change of this variable is the entire complex plane with the cut $(-b, +b)$ along the real axis. We construct the potentials φ_1 and ψ_1 reflected from the plane S_0,

$$\varphi_1 = \mathrm{Re}\,[\Phi_1(\theta_1)], \quad \psi_1 = \mathrm{Re}\,[\Psi_1(\theta_1')], \tag{83}$$

where

$$\Phi_1'(\theta) = \frac{4\theta(2\theta^2 - b^2)\sqrt{b^2-\theta^2}}{F(\theta)}\Psi'(\theta),$$
$$\Psi_1'(\theta) = \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2-\theta^2}\sqrt{b^2-\theta^2}}{F(\theta)}\Psi'(\theta). \tag{84}$$

Real values of θ from $a \le |\theta| \le b$ correspond to rays of longitudinal vibrations in the plane S_0 and to the plane, where $\Psi_1(\theta)$ is equal to a constant defined by (84).

For θ_1 and θ_1' we have the equations

$$t - \theta_1 x - \sqrt{a^2-\theta_1^2}\, y - \sqrt{b^2-\theta_1^2}\, f = 0,$$
$$t - \theta_1' x - \sqrt{b^2-{\theta_1'}^2}\, y - \sqrt{b^2-{\theta_1'}^2}\, f = 0. \tag{85}$$

Further, consider the reflection of the obtained rays of longitudinal vibrations from the plane S_1. We have the potentials of reflected longitudinal and transverse vibrations φ_2 and ψ_2,

$$\varphi_2 = \mathrm{Re}\,[\Phi_2(\theta_2)], \quad \psi_2 = \mathrm{Re}\,[\Psi_2(\theta_2')]. \tag{86}$$

For the variables θ_2 and θ_2' we have the equations

$$
\begin{aligned}
&t - \theta_2 x + \sqrt{a^2 - \theta_2^2}\, y - \sqrt{b^2 - \theta_2^2}\, f - 3\sqrt{a^2 - \theta_2^2}\, f = 0, \\
&t - \theta_2' x + \sqrt{b^2 - {\theta_2'}^2}\, y - 3\sqrt{b^2 - {\theta_2'}^2}\, f - 2\sqrt{a^2 - {\theta_2'}^2}\, f = 0,
\end{aligned}
\tag{87}
$$

and the functions Φ_2' and Ψ_2' are defined by the formulas

$$
\begin{aligned}
\Phi_2'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2 - \theta^2}\sqrt{b^2 - \theta^2}}{F(\theta)}\Phi_1'(\theta), \\
\Psi_2'(\theta) &= \frac{4\theta\sqrt{a^2 - \theta^2}(2\theta^2 - b^2)}{F(\theta)}\Phi_1'(\theta).
\end{aligned}
\tag{88}
$$

The rays of longitudinal vibrations of the potential φ_1, corresponding to the real values of θ from $a \le |\theta| \le b$, remain in the plane S_0. Hence the range of θ_2 and θ_2' is the upper half-plane and the interval $(-a, +a)$. This fact is valid for the variable θ in (88), and the real parts of $\Phi_2'(\theta)$ and $\Psi_2'(\theta)$ are equal to zero on the interval $(-a, +a)$.

Consider now the reflection of rays of transverse vibrations with the potential ψ_1 from the plane S_1. Introduce the potentials for the reflected rays

$$
\varphi_3 = \mathrm{Re}\,[\Phi_3(\theta_3)], \quad \psi_3 = \mathrm{Re}\,[\Psi_3(\theta_3')],
\tag{89}
$$

where

$$
\begin{aligned}
&t - \theta_3 x + \sqrt{a^2 - \theta_3^2}\, y - 3\sqrt{b^2 - \theta_3^2}\, f - 2\sqrt{a^2 - \theta_3^2}\, f = 0, \\
&t - \theta_3' x + \sqrt{b^2 - {\theta_3'}^2}\, y - 5\sqrt{b^2 - {\theta_3'}^2}\, f = 0
\end{aligned}
\tag{90}
$$

and

$$
\begin{aligned}
\Phi_3'(\theta) &= \frac{-4\theta(2\theta^2 - b^2)\sqrt{b^2 - \theta^2}}{F(\theta)}\Psi_1'(\theta), \\
\Psi_3'(\theta) &= \frac{-(2\theta^2 - b^2)^2 + 4\theta^2\sqrt{a^2 - \theta^2}\sqrt{b^2 - \theta^2}}{F(\theta)}\Psi_1'(\theta).
\end{aligned}
\tag{91}
$$

Let us make some additional comments about real values of θ such that $a < |\theta| < b$.

For $\Phi_3'(\theta)$ and $\Psi_3'(\theta)$ we obtain values with nonzero real parts. From the first equation in (90) it follows that the rays of longitudinal vibrations corresponding to these values of θ are located in the plane S_1. The second equation defines a family of planes, on which $\Psi_3(\theta)$ remains constant. It is easy to verify the boundary conditions by considering the potentials in domains of the plane S_1, filled with the rays of longitudinal vibrations. Let us consider these domains in detail.

For every real value of θ from the inequality $a < |\theta| < b$, the equation of the corresponding ray located in the plane S_1 is

$$t - \theta x - 3\sqrt{b^2 - \theta^2}\, f = 0, \qquad y = 2f.$$

Substituting into the second equation in (90) θ for θ_3' and putting $y = 2f$, we obtain the same equation . Hence the indicated ray of longitudinal vibrations coincide with the section of the plane, where $\Psi_3(\theta)$ is constant. The same can be obtained putting $y = 2f$ in the second equation in (85), i.e., the plane, on which $\Psi_1(\theta)$ is constant, crosses the plane S_1 along the same ray l_θ. When the rays of the potential φ_3 reflect from the plane S_0, the range of θ is the upper half-plane with the interval $(-a, +a)$ of the real axis. The real part of $\Phi_3'(\theta)$ is equal to zero along this interval.

Using the same argument, we can obtain solutions of problems with different boundary conditions, for example, with the absence of the displacements, etc.

15. Using the above method in the case when the source is located inside the medium, it is easy to solve also the first problem in the very compact form: the two-dimensional problem on vibrations of the half-space under the action of an impact concentrated on the surface.

Let the source of vibrations be located at the point

$$O(x = 0,\ y = 0,\ t = 0)$$

of the space S, let the complex potentials $\Phi(\theta_1)$ and $\Psi(\theta_1')$ of longitudinal and transverse vibrations correspond to this source. Consider two cones T_1 and T_2 with apex O and angles $\arctan\dfrac{1}{a}$ and $\arctan\dfrac{1}{b}$ at the apex. Write down the equations for θ_1 and θ_1',

$$t - \theta_1 x - \sqrt{a^2 - \theta_1^2}\, y = 0, \tag{92}$$

$$t - \theta_1' x - \sqrt{b^2 - {\theta_1'}^2}\, y = 0. \tag{93}$$

Complex values of θ_1 from the upper half-plane correspond to rays passing through the point O and moving inside the cone T_1 in the domain $y > 0$, $t > 0$ of the space S. Real values of θ_1 such that $|\theta_1| > a$ correspond to rays located in the plane $y = 0$. Finally, real values of θ_1 from the interval $(-a, +a)$ correspond to generators of the cone T_1. Completely analogous correspondence will take place between rays inside the cone T_2 and complex values of θ_1'.

Let OA and OA_1 be generators of T_1 in the plane $y = 0$, let OB and OB_1 be generators for T_2. Using (38), one can write the condition that the stress is equal to zero inside the angle BOB_1 at all points of the plane $y = 0$

$$\begin{aligned} &\operatorname{Re}\left\{\frac{\partial}{\partial\theta}[2\theta\sqrt{a^2-\theta^2}\Phi'(\theta) + (b^2 - 2\theta^2)\Psi'(\theta)]\right\} = 0,\\ &\operatorname{Re}\left\{\frac{\partial}{\partial\theta}[(b^2 - 2\theta^2)\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\Psi'(\theta)]\right\} = 0. \end{aligned} \tag{94}$$

Note that points inside the angle BOB_1 correspond to real values of θ such that $|\theta| > b$. Consider now the angles AOB and A_1OB_1. Here we have the potential $\Phi_1(\theta)$ of longitudinal vibrations. To satisfy the boundary conditions, we have to apply transverse vibrations. This corresponds to the fact that longitudinal vibrations propagating over the surface generate transverse vibrations inside. In this case, the argument of the function $\Phi_1(\theta)$ takes real values from the intervals (a, b) and $(-b, -a)$. For these values of θ, equation (93) defines planes tangent to the cone T_2. Let us take the parts of these planes between the plane $y = 0$ and the generators of the cone T_2.

Denote by U_θ these parts. On each U_θ the potential of transverse vibrations must be constant, and we have to choose the functions $\omega(\theta)$ such that the boundary conditions are satisfied in the angles AOB and A_1OB_1. Since $|\theta| \le b$, we can write these conditions in the form

$$\begin{aligned} &\mathrm{Re}\left\{\frac{\partial}{\partial\theta}[2\theta\sqrt{a^2-\theta^2}\Phi'(\theta) + (b^2 - 2\theta^2)\omega'(\theta)]\right\} = 0,\\ &\mathrm{Re}\left\{\frac{\partial}{\partial\theta}[(b^2 - 2\theta^2)\Phi'(\theta) - 2\theta\sqrt{b^2-\theta^2}\omega'(\theta)]\right\} = 0. \end{aligned} \tag{95}$$

Since the potential is continuous, the value of $\omega(\theta)$ must coincide with the real part of $\Psi(\theta)$ on the generator of the cone T_2, along which U_θ touches the cone. Then conditions (95) coincide with conditions (94), i.e., conditions (94) must be satisfied also for $a \le |\theta| \le b$. Since velocity of vibrations cannot be greater than $\dfrac{1}{a}$, we must assume that the potentials of longitudinal and transverse vibrations must vanish for $-a \le \theta \le a$, i.e., conditions (94) will be also satisfied for these values of θ. Thus, these conditions must be satisfied on the entire real axis. Calculating the functions $\Phi(\theta)$, $\Psi(\theta)$, and the potentials

$$\varphi = \mathrm{Re}\,[\Phi(\theta)], \quad \psi = \mathrm{Re}\,[\Psi(\theta)],$$

we have to continue ψ into the exterior of the cone T_2 along the planes U_θ.

The establishment of conditions (94) for all real values of θ is the essential fact in solving the first problem.

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[5] Paper [1] of Part I of this book. – *Ed.*