1

Fourier Series

One of the most useful tools of mathematical analysis is Fourier series, named after the French mathematical physicist Jean Baptiste Joseph Fourier (1768–1830). Fourier analysis is ubiquitous in almost all fields of physical sciences.

In 1822, Fourier in his work on heat flow made a remarkable assertion that every function $f(x)$ with period 2π can be represented by a trigonometric infinite series of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1.1}$$

We now know that, with very little restrictions on the function, this is indeed the case. An infinite series of this form is called a Fourier series. The series was originally proposed for the solutions of partial differential equations with boundary (and/or initial) conditions. While it is still one of the most powerful methods for such problems, as we shall see in later chapters, its usefulness has been extended far beyond the problem of heat conduction. Fourier series is now an essential tool for the analysis of all kinds of wave forms, ranging from signal processing to quantum particle waves.

1.1 Fourier Series of Functions with Periodicity 2π

1.1.1 Orthogonality of Trigonotric Functions

To discuss Fourier series, we need the following integrals. If m and n are integers, then

$$\int_{-\pi}^{\pi} \cos mx \, \mathrm{d}x = 0, \tag{1.2}$$

$$\int_{-\pi}^{\pi} \sin mx \, \mathrm{d}x = 0, \tag{1.3}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, \mathrm{d}x = 0, \tag{1.4}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x = \begin{cases} 0 & m \neq n, \\ \pi & m = n \neq 0, \\ 2\pi & m = n = 0, \end{cases} \tag{1.5}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \begin{cases} 0 & m \neq n, \\ \pi & m = n. \end{cases} \tag{1.6}$$

The first two integrals are trivial, either by direct integration or by noting that any trigonometric function integrated over a whole period will give zero since the positive part will cancel the negative part. The rest of the integrals can be shown by using the trigonometry formulas for products and then integrating. An easier way is to use the complex forms

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{imx} + \mathrm{e}^{-imx}}{2} \, \frac{\mathrm{e}^{inx} - \mathrm{e}^{-inx}}{2i} \mathrm{d}x.$$

We can see the results without actually multiplying out. All terms in the product are of the form $\mathrm{e}^{\mathrm{i}kx}$, where k is an integer. Since

$$\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x = \frac{1}{\mathrm{i}k} \left[\mathrm{e}^{\mathrm{i}kx}\right]_{-\pi}^{\pi} = 0,$$

it follows that all integrals in the product are zero. Similarly

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{imx} + \mathrm{e}^{-imx}}{2} \, \frac{\mathrm{e}^{inx} + \mathrm{e}^{-inx}}{2} \mathrm{d}x$$

is identically zero except $n = m$, in that case

$$\int_{-\pi}^{\pi} \cos mx \cos mx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}2mx} + 2 + \mathrm{e}^{-\mathrm{i}2mx}}{4} \mathrm{d}x$$
$$= \int_{-\pi}^{\pi} \frac{1}{2} \left[1 + \cos 2mx\right] \mathrm{d}x = \begin{cases} \pi & m \neq 0, \\ 2\pi & m = 0. \end{cases}$$

In the same way we can show that if $n \neq m$,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = 0$$

and if $n = m$,

$$\int_{-\pi}^{\pi} \sin mx \sin mx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1}{2} \left[1 - \cos 2mx\right] \mathrm{d}x = \pi.$$

This concludes the proof of (1.2)–(1.6).

In general, if any two members ψ_n, ψ_m of a set of functions $\{\psi_i\}$ satisfy the condition

$$\int_a^b \psi_n(x)\psi_m(x)\mathrm{d}x = 0 \quad \text{if } n \neq m, \tag{1.7}$$

then ψ_n and ψ_m are said to be orthogonal, and (1.7) is known as the orthogonal condition in the interval between a and b. The set $\{\psi_i\}$ is an orthogonal set over the same interval.

Thus if the members of the set of trigonometric functions are

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \cos 3x,\ \sin 3x, \ldots,$$

then this is an orthogonal set in the interval from $-\pi$ to π.

1.1.2 The Fourier Coefficients

If $f(x)$ is a periodic function of period 2π, i.e.,

$$f(x + 2\pi) = f(x)$$

and it is represented by the Fourier series of the form (1.1), the coefficients a_n and b_n can be found in the following way.

We multiply both sides of (1.1) by $\cos mx$, where m is an positive integer

$$f(x)\cos mx = \frac{1}{2}a_0 \cos mx + \sum_{n=1}^{\infty}(a_n \cos nx \cos mx + b_n \sin nx \cos mx).$$

This series can be integrated term by term

$$\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos mx\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,\mathrm{d}x + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,\mathrm{d}x.$$

From the integrals we have discussed, we see that all terms associated with b_n will vanish and all terms associated with a_n will also vanish except the term with $n = m$, and that term is given by

$$\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x = \begin{cases} \dfrac{1}{2}a_0 \displaystyle\int_{-\pi}^{\pi} \mathrm{d}x = a_0\pi & \text{for } m = 0, \\ a_m \displaystyle\int_{-\pi}^{\pi} \cos^2 mx\,\mathrm{d}x = a_m\pi & \text{for } m \neq 0. \end{cases}$$

These relations permit us to calculate any desired coefficient a_m including a_0 when the function $f(x)$ is known.

The coefficients b_m can be similarly obtained. The expansion is multiplied by $\sin mx$ and then integrated term by term. Orthogonality relations yield

$$\int_{-\pi}^{\pi} f(x) \sin mx \, \mathrm{d}x = b_m \pi.$$

Since m can be any integer, it follows that a_n(including a_0) and b_n are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x, \tag{1.8}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x. \tag{1.9}$$

These coefficients are known as the Euler formulas for Fourier coefficients, or simply as the *Fourier coefficients*.

In essence, Fourier series decomposes the periodic function into cosine and sine waves. From the procedure, it can be observed that:

- The first term $\frac{1}{2}a_0$ represents the average value of $f(x)$ over a period 2π.
- The term $a_n \cos nx$ represents the cosine wave with amplitude a_n. Within one period 2π, there are n complete cosine waves.
- The term $b_n \sin nx$ represents the sine wave with amplitude b_n, and n is the number of complete sine wave in one period 2π.
- In general a_n and b_n can be expected to decrease as n increases.

1.1.3 Expansion of Functions in Fourier Series

Before we discuss the validity of the Fourier series, let us use the following example to show that it is possible to represent a periodic function with period 2π by a Fourier series, provided enough terms are taken.

Suppose we want to expand the square-wave function, shown in Fig. 1.1, into a Fourier series.

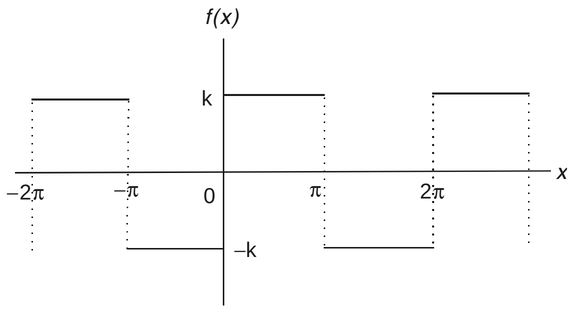


Fig. 1.1. A square-wave function

This function is periodic with period 2π. It can be defined as

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x).$$

To find the coefficients of the Fourier series of this function

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

it is always a good idea to calculate a_0 separately, since it is given by simple integral. In this case

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = 0$$

can be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π is zero. For the rest of the coefficients, they are given by (1.8) and (1.9). To carry out these integrations, we have to split each of them into two integrals because $f(x)$ is defined by two different formulas on the intervals $(-\pi, 0)$ and $(0, \pi)$. From (1.8)

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,\mathrm{d}x + \int_{0}^{\pi} k\cos nx\,\mathrm{d}x\right] \\ &= \frac{1}{\pi}\left\{\left[-k\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[k\frac{\sin nx}{n}\right]_{0}^{\pi}\right\} = 0. \end{aligned}$$

From (1.9)

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,\mathrm{d}x + \int_{0}^{\pi} k\sin nx\,\mathrm{d}x\right] \\ &= \frac{1}{\pi}\left\{\left[k\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[-k\frac{\cos nx}{n}\right]_{0}^{\pi}\right\} = \frac{2k}{n\pi}(1-\cos n\pi) \\ &= \frac{2k}{n\pi}(1-(-1)^n) = \begin{cases} \dfrac{4k}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

With these coefficients, the Fourier series becomes

$$\begin{aligned} f(x) &= \frac{4k}{\pi}\sum_{n\ \mathrm{odd}} \frac{1}{n}\sin nx \\ &= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \end{aligned} \tag{1.10}$$

Alternatively this series can be written as

$$f(x) = \frac{4k}{\pi}\sum_{n=1} \frac{1}{2n-1}\sin(2n-1)x.$$

To examine the convergence of this series, let us define the partial sums as

$$S_N = \frac{4k}{\pi} \sum_{n=1}^{N} \frac{1}{2n-1} \sin(2n-1)x.$$

In other words, S_N is the sum the first N terms of the Fourier series. S_1 is simply the first term $\frac{4k}{\pi} \sin x$, S_2 is the sum of the first two terms $\frac{4k}{\pi}(\sin x + \frac{1}{3} \sin 3x)$, etc.

In Fig. 1.2a, the first three partial sums are shown in the right column, the individual terms in these sums are shown in the left column. It is seen that S_N gets closer to $f(x)$ as N increases, although the contributions of the

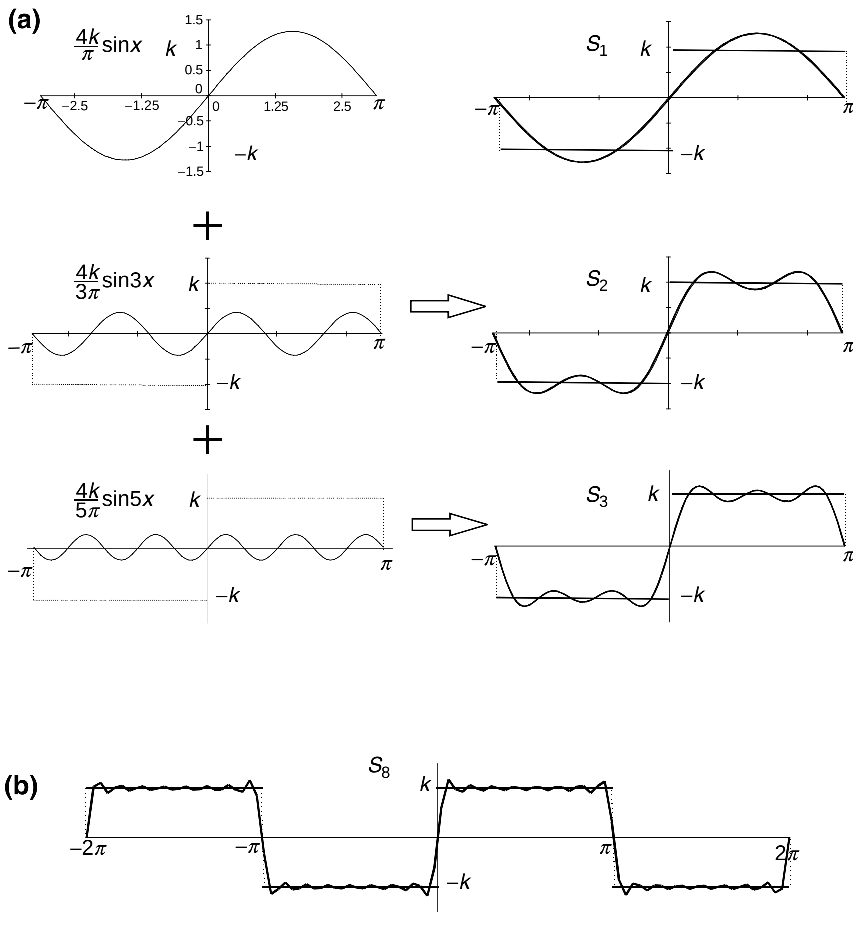


Fig. 1.2. The convergence of a Fourier series expansion of a square-wave function. (**a**) The first three partial sums are shown in the *right*; the individual terms in these sums are shown in the *left*. (**b**) The sum of the first eight terms of the Fourier series of the function

individual terms are steadily decreasing as n gets larger. In Fig. 1.2b, we show the result of S_8. With eight terms, the partial sum already looks very similar to the square-wave function. We notice that at the points of discontinuity $x = -\pi, x = 0$, and $x = \pi$, all the partial sums have the value zero, which is the average of the values of k and $-k$ of the function. Note also that as x approaches a discontinuity of $f(x)$ from either side, the value of $S_N(x)$ tends to overshoot the value of $f(x)$, in this case $-k$ or $+k$. As N increases, the overshoots (about 9% of the discontinuity) are pushed closer to the points of discontinuity, but they will not disappear even if N goes to infinity. This behavior of a Fourier series near a point of discontinuity of its function is known as *Gibbs' phenomenon.*

1.2 Convergence of Fourier Series

1.2.1 Dirichlet Conditions

The conditions imposed on $f(x)$ to make (1.1) valid are stated in the following theorem.

Theorem 1.2.1. *If a periodic function $f(x)$ of period 2π is bounded and piecewise continuous, and has a finite number of maxima and minima in each period, then the trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x, \quad n = 0, 1, 2, \ldots$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x, \quad n = 1, 2, \ldots$$

converges to $f(x)$ where $f(x)$ is continuous, and it converges to the average of the left- and right-hand limits of $f(x)$ at points of discontinuity.

A proof of this theorem may be found in G.P. Tolstov, *Fourier Series*, Dover, New York, 1976.

As long as $f(t)$ is periodic, the choice of the symmetric upper and lower integration limits $(-\pi, \pi)$ is not essential. Any interval of 2π, such as $(x_0, x_0 + 2\pi)$ will give the same result.

The conditions of convergence were first proved by the German mathematician P.G. Lejeune Dirichlet (1805–1859), and therefore known as Dirichlet conditions. These conditions impose very little restrictions on the function. Furthermore, these are only sufficient conditions. It is known that certain function that does not satisfy these conditions can also be represented by the

Fourier series. The minimum necessary conditions for its convergence are not known. In any case, it can be safely assumed that functions of interests in physical problems can all be represented by their Fourier series.

1.2.2 Fourier Series and Delta Function

(For those who have not yet studied complex contour integration, this section can be skipped.)

Instead of proving the convergence theorem, we will use a delta function to explicitly demonstrate that the Fourier series

$$S_\infty(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges to $f(x)$.

With a_n and b_n given by (1.8) and (1.9), $S_\infty(x)$ can be written as

$$\begin{aligned}
S_\infty(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\mathrm{d}x' + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(x')\cos nx'\mathrm{d}x'\right)\cos nx \\
&\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(x')\sin nx'\mathrm{d}x'\right)\sin nx \\
&= \int_{-\pi}^{\pi} f(x')\left[\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}(\cos nx'\cos nx + \sin nx'\sin nx)\right]\mathrm{d}x' \\
&= \int_{-\pi}^{\pi} f(x')\left[\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos n(x'-x)\right]\mathrm{d}x'.
\end{aligned}$$

If the cosine series

$$D(x'-x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos n(x'-x)$$

behaves like a delta function $\delta(x'-x)$, then $S_\infty(x) = f(x)$ because

$$\int_{-\pi}^{\pi} f(x')\delta(x'-x)\mathrm{d}x' = f(x) \quad \text{for} \quad -\pi < x < \pi.$$

Recall that the delta function $\delta(x'-x)$ can be defined as

$$\delta(x'-x) = \begin{cases} 0 & x' \neq x \\ \infty & x' = x \end{cases},$$

$$\int_{-\pi}^{\pi} \delta(x'-x)\mathrm{d}x' = 1 \quad \text{for} \quad -\pi < x < \pi.$$

Now we will show that indeed $D(x'-x)$ has these properties. First, to ensure the convergence, we write the cosine series as

$$D(x'-x) = \lim_{\gamma\to 1^-} D_\gamma(x'-x),$$
$$D_\gamma(x'-x) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \gamma^n \cos n(x'-x)\right],$$

where the limit $\gamma \to 1^-$ means that γ approaches one from below, i.e., γ is infinitely close to 1, but is always less than 1. To sum this series, it is advantageous to regard $D_\gamma(x'-x)$ as the real part of the complex series

$$D_\gamma(x'-x) = \mathrm{Re}\left[\frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \gamma^n \mathrm{e}^{\mathrm{i}n(x'-x)}\right)\right].$$

Since

$$\frac{1}{1-\gamma\mathrm{e}^{\mathrm{i}(x'-x)}} = 1 + \gamma\mathrm{e}^{\mathrm{i}(x'-x)} + \gamma^2\mathrm{e}^{\mathrm{i}2(x'-x)} + \cdots,$$
$$\frac{\gamma\mathrm{e}^{\mathrm{i}(x'-x)}}{1-\gamma\mathrm{e}^{\mathrm{i}(x'-x)}} = \gamma\mathrm{e}^{\mathrm{i}(x'-x)} + \gamma^2\mathrm{e}^{\mathrm{i}2(x'-x)} + \gamma^3\mathrm{e}^{\mathrm{i}3(x'-x)} + \cdots,$$

so

$$\frac{1}{2} + \sum_{n=1}^{\infty} \gamma^n \mathrm{e}^{\mathrm{i}n(x'-x)} = \frac{1}{2} + \frac{\gamma\mathrm{e}^{\mathrm{i}(x'-x)}}{1-\gamma\mathrm{e}^{\mathrm{i}(x'-x)}}$$
$$= \frac{1+\gamma\mathrm{e}^{\mathrm{i}(x'-x)}}{2(1-\gamma\mathrm{e}^{\mathrm{i}(x'-x)})} = \frac{1+\gamma\mathrm{e}^{\mathrm{i}(x'-x)}}{2(1-\gamma\mathrm{e}^{\mathrm{i}(x'-x)})}\,\frac{1-\gamma\mathrm{e}^{-\mathrm{i}(x'-x)}}{1-\gamma\mathrm{e}^{-\mathrm{i}(x'-x)}}$$
$$= \frac{1-\gamma^2+\gamma\mathrm{e}^{\mathrm{i}(x'-x)}-\gamma\mathrm{e}^{-\mathrm{i}(x'-x)}}{2[1-\gamma(\mathrm{e}^{\mathrm{i}(x'-x)}+\mathrm{e}^{-\mathrm{i}(x'-x)})+\gamma^2]} = \frac{1-\gamma^2+\mathrm{i}2\gamma\sin(x'-x)}{2[1-2\gamma\cos(x'-x)+\gamma^2]}.$$

Thus

$$D_\gamma(x'-x) = \mathrm{Re}\left[\frac{1-\gamma^2+\mathrm{i}2\gamma\sin(x'-x)}{2\pi[1-2\gamma\cos(x'-x)+\gamma^2]}\right]$$
$$= \frac{1-\gamma^2}{2\pi[1-2\gamma\cos(x'-x)+\gamma^2]}.$$

Clearly, if $x' \neq x$,

$$D(x'-x) = \lim_{\gamma\to 1} \frac{1-\gamma^2}{2\pi[1-2\gamma\cos(x'-x)+\gamma^2]} = 0.$$

If $x' = x$, then $\cos(x' - x) = 1$, and

$$\frac{1-\gamma^2}{2\pi[1-2\gamma\cos(x'-x)+\gamma^2]} = \frac{1-\gamma^2}{2\pi[1-2\gamma+\gamma^2]}$$
$$= \frac{(1-\gamma)(1+\gamma)}{2\pi[1-\gamma]^2} = \frac{1+\gamma}{2\pi(1-\gamma)}.$$

It follows that

$$D(x'-x) = \lim_{\gamma\to 1}\frac{1+\gamma}{2\pi(1-\gamma)} \to \infty, \quad x' = x.$$

Furthermore

$$\int_{-\pi}^{\pi} D_\gamma(x'-x)\mathrm{d}x' = \frac{1-\gamma^2}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}x'}{(1+\gamma^2)-2\gamma\cos(x'-x)}.$$

We have shown in the chapter on the theory of residue (see Example 3.5.2 of Volume 1) that

$$\oint \frac{\mathrm{d}\theta}{a-b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a > b.$$

With a substitution $x' - x = \theta$,

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}x'}{(1+\gamma^2)-2\gamma\cos(x'-x)} = \oint\frac{\mathrm{d}\theta}{(1+\gamma^2)-2\gamma\cos\theta}.$$

As long as γ is not exactly one, $1+\gamma^2 > 2\gamma$, so

$$\oint\frac{\mathrm{d}\theta}{(1+\gamma^2)-2\gamma\cos\theta} = \frac{2\pi}{\sqrt{(1+\gamma^2)^2-4\gamma^2}} = \frac{2\pi}{1-\gamma^2}.$$

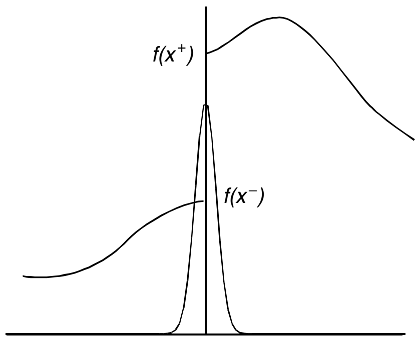
Therefore

$$\int_{-\pi}^{\pi} D_\gamma(x'-x)\mathrm{d}x' = \frac{1-\gamma^2}{2\pi}\frac{2\pi}{1-\gamma^2} = 1.$$

This concludes our proof that $D(x'-x)$ behaves like the delta function $\delta(x'-x)$. Therefore if $f(x)$ is continuous, then the Fourier series converges to $f(x)$,

$$S_\infty(x) = \int_{-\pi}^{\pi} f(x')D(x'-x)\mathrm{d}x' = f(x).$$

Suppose that $f(x)$ is discontinuous at some point x, and that $f(x^+)$ and $f(x^-)$ are the limiting values as we approach x from the right and from the left. Then in evaluating the last integral, half of $D(x'-x)$ is multiplied by $f(x^+)$ and half by $f(x^-)$, as shown in the following figure.



Therefore the last equation becomes

$$S_\infty(x) = \frac{1}{2}[f(x^+) + f(x^-)].$$

Thus at points where $f(x)$ is continuous, the Fourier series gives the value of $f(x)$, and at points where $f(x)$ is discontinuous, the Fourier series gives the mean value of the right and left limits of $f(x)$.

1.3 Fourier Series of Functions of any Period

1.3.1 Change of Interval

So far attention has been restricted to functions of period 2π. This restriction may easily be relaxed. If $f(t)$ is periodic with a period $2L$, we can make a change of variable

$$t = \frac{L}{\pi}x$$

and let

$$f(t) = f\left(\frac{L}{\pi}x\right) \equiv F(x).$$

By this definition,

$$f(t+2L) = f\left(\frac{L}{\pi}x + 2L\right) = f\left(\frac{L}{\pi}[x+2\pi]\right) = F(x+2\pi).$$

Since $f(t)$ is a periodic function with a period $2L$

$$f(t+2L) = f(t)$$

it follows that:

$$F(x+2\pi) = F(x).$$

So $F(x)$ is periodic with a period 2π.

We can expand $F(x)$ into a Fourier series, then transform back to a function of t

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.11}$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx \, \mathrm{d}x,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx \, \mathrm{d}x.$$

Since $x = \dfrac{\pi}{L}t$ and $F(x) = f(t)$, (1.11) can be written as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}t + b_n \sin\frac{n\pi}{L}t\right) \tag{1.12}$$

and the coefficients can also be expressed as integrals over t. Changing the integration variable from x to t with $\mathrm{d}x = \dfrac{\pi}{L}\mathrm{d}t$, we have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t, \tag{1.13}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t. \tag{1.14}$$

Kronecker's method. As a practical matter, very often $f(t)$ is in the form of t^k, $\sin kt$, $\cos kt$, or e^{kt} for various integer values of k. We will have to carry out the integrations of the type

$$\int t^k \cos\frac{n\pi t}{L}\mathrm{d}t, \quad \int \sin kt \cos\frac{n\pi t}{L}\mathrm{d}t.$$

These integrals can be evaluated by repeated integration by parts. The following systematic approach is helpful in reducing the tedious details inherent in such computation. Consider the integral

$$\int f(t)g(t)\mathrm{d}t$$

and let

$$g(t)\mathrm{d}t = \mathrm{d}G(t), \quad \text{then} \quad G(t) = \int g(t)\mathrm{d}t.$$

With integration by parts, one gets

$$\int f(t)g(t)\mathrm{d}t = f(t)G(t) - \int f'(t)G(t)\mathrm{d}t.$$

Continuing this process, with

$$G_1(t) = \int G(t)\mathrm{d}t, \quad G_2(t) = \int G_1(t)\mathrm{d}t, \ldots, G_n(t) = \int G_{n-1}(t)\mathrm{d}t,$$

we have

$$\int f(t)g(t)\mathrm{d}t = f(t)G(t) - f'(t)G_1(t) + \int f''(t)G_1(t)\mathrm{d}t \tag{1.15}$$

$$= f(t)G(t) - f'(t)G_1(t) + f''(t)G_2(t) - f'''(t)G_3(t) + \cdots . \tag{1.16}$$

This procedure is known as Kronecker's method.

Now if $f(t) = t^k$, then

$$f'(t) = kt^{k-1}, \ldots, f^k(t) = k!, \; f^{k+1}(t) = 0,$$

the above expression will terminate. Furthermore, if $g(t) = \cos \frac{n\pi t}{L}$, then

$$G(t) = \int \cos \frac{n\pi t}{L}\mathrm{d}t = \left(\frac{L}{n\pi}\right) \sin \frac{n\pi t}{L},$$

$$G_1(t) = \left(\frac{L}{n\pi}\right) \int \sin \frac{n\pi t}{L}\mathrm{d}t = -\left(\frac{L}{n\pi}\right)^2 \cos \frac{n\pi t}{L},$$

$$G_2(t) = -\left(\frac{L}{n\pi}\right)^3 \sin \frac{n\pi t}{L}, \quad G_3(t) = \left(\frac{L}{n\pi}\right)^4 \cos \frac{n\pi t}{L}, \ldots .$$

Similarly, if $g(t) = \sin \frac{n\pi t}{L}$, then

$$G(t) = \int \sin \frac{n\pi t}{L}\mathrm{d}t = -\left(\frac{L}{n\pi}\right) \cos \frac{n\pi t}{L}, \quad G_1(t) = -\left(\frac{L}{n\pi}\right)^2 \sin \frac{n\pi t}{L},$$

$$G_2(t) = \left(\frac{L}{n\pi}\right)^3 \cos \frac{n\pi t}{L}, \quad G_3(t) = \left(\frac{L}{n\pi}\right)^4 \sin \frac{n\pi t}{L}, \ldots .$$

Thus

$$\int_a^b t^k \cos \frac{n\pi t}{L}\mathrm{d}t = \left[\frac{L}{n\pi} t^k \sin \frac{n\pi t}{L} + \left(\frac{L}{n\pi}\right)^2 kt^{k-1} \cos \frac{n\pi t}{L} - \left(\frac{L}{n\pi}\right)^3 k(k-1)t^{k-2} \sin \frac{n\pi t}{L} + \cdots \right]_a^b \tag{1.17}$$

and

$$\int_a^b t^k \sin \frac{n\pi t}{L}\mathrm{d}t = \left[-\frac{L}{n\pi} t^k \cos \frac{n\pi t}{L} + \left(\frac{L}{n\pi}\right)^2 kt^{k-1} \sin \frac{n\pi t}{L} + \left(\frac{L}{n\pi}\right)^3 k(k-1)t^{k-2} \cos \frac{n\pi t}{L} + \cdots \right]_a^b . \tag{1.18}$$

If $f(t) = \sin kt$, then

$$f'(t) = k \cos kt, \quad f''(t) = -k^2 \sin kt.$$

we can use (1.15) to write

$$\int_a^b \sin kt \cos \frac{n\pi}{L} t \,\mathrm{d}t = \left[\frac{L}{n\pi} \sin kt \sin \frac{n\pi t}{L} + k \left(\frac{L}{n\pi} \right)^2 \cos kt \cos \frac{n\pi t}{L} \right]_a^b$$
$$+k^2 \left(\frac{L}{n\pi} \right)^2 \int_a^b \sin kt \cos \frac{n\pi}{L} t \,\mathrm{d}t.$$

Combining the last term with the left-hand side, we have

$$\left[1 - k^2 \left(\frac{L}{n\pi} \right)^2 \right] \int_a^b \sin kt \cos \frac{n\pi}{L} t \,\mathrm{d}t$$
$$= \left[\frac{L}{n\pi} \sin kt \sin \frac{n\pi t}{L} + k \left(\frac{L}{n\pi} \right)^2 \cos kt \cos \frac{n\pi t}{L} \right]_a^b$$

or

$$\int_a^b \sin kt \cos \frac{n\pi}{L} t \,\mathrm{d}t$$
$$= \frac{(n\pi)^2}{(n\pi)^2 - (kL)^2} \left[\frac{L}{n\pi} \sin kt \sin \frac{n\pi t}{L} + k \left(\frac{L}{n\pi} \right)^2 \cos kt \cos \frac{n\pi t}{L} \right]_a^b.$$

Clearly, integrals such as

$$\int_a^b \sin kt \sin \frac{n\pi}{L} t \,\mathrm{d}t, \quad \int_a^b \cos kt \cos \frac{n\pi}{L} t \,\mathrm{d}t, \quad \int_a^b \cos kt \sin \frac{n\pi}{L} t \,\mathrm{d}t,$$
$$\int_a^b \mathrm{e}^{kt} \cos \frac{n\pi}{L} t \,\mathrm{d}t, \quad \int_a^b \mathrm{e}^{kt} \sin \frac{n\pi}{L} t \,\mathrm{d}t$$

can similarly be integrated.

Example 1.3.1. Find the Fourier series for $f(t)$ which is defined as

$$f(t) = t \quad \text{for} \quad -L < t \le L, \quad \text{and} \quad f(t + 2L) = f(t).$$

Solution 1.3.1.

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right),$$
$$a_0 = \frac{1}{L} \int_{-L}^{L} t \,\mathrm{d}t = 0,$$
$$a_n = \frac{1}{L} \int_{-L}^{L} t \cos \frac{n\pi t}{L} \mathrm{d}t = \frac{1}{L} \left[\frac{L}{n\pi} t \sin \frac{n\pi t}{L} + \left(\frac{L}{n\pi} \right)^2 \cos \frac{n\pi t}{L} \right]_{-L}^{L} = 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} t \sin\frac{n\pi t}{L}\mathrm{d}t$$
$$= \frac{1}{L}\left[-\frac{L}{n\pi}t\cos\frac{n\pi t}{L} + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi t}{L}\right]_{-L}^{L} = -\frac{2L}{n\pi}\cos n\pi.$$

Thus

$$f(t) = \frac{2L}{\pi}\sum_{n=1}^{\infty} -\frac{1}{n}\cos n\pi \sin\frac{n\pi t}{L} = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{L}$$
$$= \frac{2L}{\pi}\left(\sin\frac{\pi t}{L} - \frac{1}{2}\sin\frac{2\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} - \cdots\right). \quad (1.19)$$

The convergence of this series is shown in Fig. 1.3, where S_N is the partial sum defined as

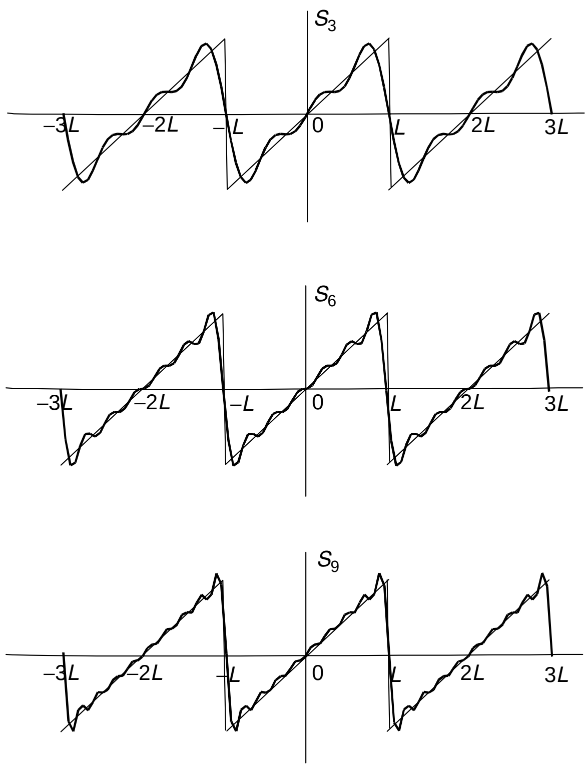


Fig. 1.3. The convergence of the Fourier series for the periodic function whose definition in one period is $f(t) = t,\ -L < t < L$. The first N terms approximations are shown as S_N

$$S_N = \frac{2L}{\pi} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi t}{L}.$$

Note the increasing accuracy with which the terms approximate the function. With three terms, S_3 already looks like the function. Except for the Gibbs' phenomenon, a very good approximation is obtained with S_9.

Example 1.3.2. Find the Fourier series of the periodic function whose definition in one period is

$$f(t) = t^2 \text{ for } -L < t \leq L, \quad \text{and} \quad f(t+2L) = f(t).$$

Solution 1.3.2. The Fourier coefficients are given by

$$a_0 = \frac{1}{L} \int_{-L}^{L} t^2 \mathrm{d}t = \frac{1}{L}\frac{1}{3}[L^3 - (-L)^3] = \frac{2}{3}L^2.$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} t^2 \cos \frac{n\pi t}{L} \mathrm{d}t, \quad n \neq 0 \\ &= \frac{1}{L} \left[\frac{L}{n\pi} t^2 \sin \frac{n\pi t}{L} + \left(\frac{L}{n\pi} \right)^2 2t \cos \frac{n\pi t}{L} - \left(\frac{L}{n\pi} \right)^3 2 \sin \frac{n\pi t}{L} \right]_{-L}^{L} \\ &= \frac{2L}{(n\pi)^2} \left[L \cos n\pi + L \cos(-n\pi) \right] = \frac{4L^2}{n^2\pi^2} (-1)^n . \end{aligned}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} t^2 \sin \frac{n\pi t}{L} \mathrm{d}t = 0.$$

Therefore the Fourier expansion is

$$\begin{aligned} f(t) &= \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi t}{L} \\ &= \frac{L^2}{3} - \frac{4L^2}{\pi^2} \left(\cos \frac{\pi}{L} t - \frac{1}{4} \cos \frac{2\pi}{L} t + \frac{1}{9} \cos \frac{3\pi}{L} t + \cdots \right). \end{aligned} \tag{1.20}$$

With the partial sum defined as

$$S_N = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \cos \frac{n\pi t}{L},$$

we compare S_3 and S_6 with $f(t)$ in Fig. 1.4.

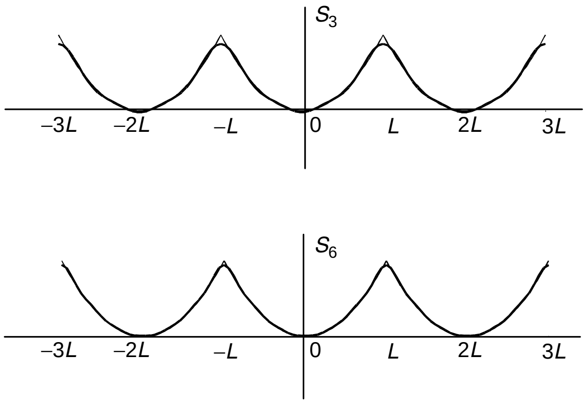


Fig. 1.4. The convergence of the Fourier expansion of the periodic function whose definition in one period is $f(t) = t^2, -L < t \leq L$. The partial sum of S_3 is already a very good approximation

It is seen that S_3 is already a very good approximation of $f(t)$. The difference between S_6 and $f(t)$ is hardly noticeable. This Fourier series converges much faster than that of the previous example. The difference is that $f(t)$ in this problem is continuous not only within the period but also in the extended range, whereas $f(t)$ in the previous example is discontinuous in the extended range.

Example 1.3.3. Find the Fourier series of the periodic function whose definition in one period is

$$f(t) = \begin{cases} 0 & -1 < t < 0 \\ t & 0 < t < 1 \end{cases}, \quad f(t+2) = f(t). \tag{1.21}$$

Solution 1.3.3. The periodicity $2L$ of this function is 2, so $L = 1$, and the Fourier series is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\pi t) + b_n \sin(n\pi t)]$$

with

$$a_0 = \int_{-1}^{1} f(t)\mathrm{d}t = \int_0^1 t\,\mathrm{d}t = \frac{1}{2},$$

$$a_n = \int_{-1}^{1} f(t)\cos(n\pi t)\mathrm{d}t = \int_0^1 t\cos(n\pi t)\mathrm{d}t,$$

$$b_n = \int_{-1}^{1} f(t)\sin(n\pi t)\mathrm{d}t = \int_0^1 t\sin(n\pi t)\mathrm{d}t.$$

Using (1.17) and (1.18), we have

$$
\begin{aligned}
a_n &= \left[\frac{1}{n\pi} t \sin n\pi t + \left(\frac{1}{n\pi}\right)^2 \cos n\pi t\right]_0^1 = \left(\frac{1}{n\pi}\right)^2 \cos n\pi - \left(\frac{1}{n\pi}\right)^2 \\
&= \frac{(-1)^n - 1}{(n\pi)^2}, \\
b_n &= \left[-\frac{1}{n\pi} t \cos n\pi t + \left(\frac{1}{n\pi}\right)^2 \sin n\pi t\right]_0^1 = -\frac{1}{n\pi} \cos n\pi = -\frac{(-1)^n}{n\pi}.
\end{aligned}
$$

Thus the Fourier series for this function is $f(t) = S_\infty$, where

$$
S_N = \frac{1}{4} + \sum_{n=1}^{N} \left[\frac{(-1)^n - 1}{(n\pi)^2} \cos n\pi t - \frac{(-1)^n}{n\pi} \sin n\pi t\right].
$$

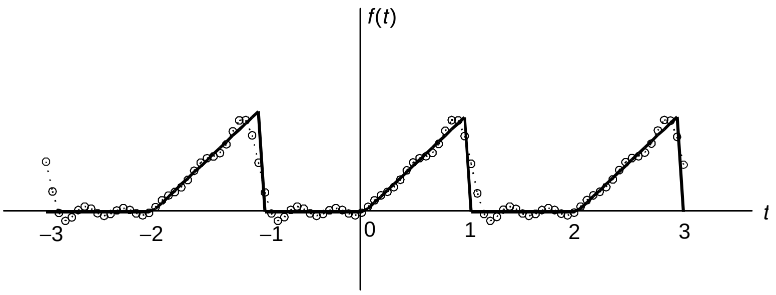


Fig. 1.5. The periodic function of (1.21) is shown together with the partial sum S_5 of its Fourier series. The function is shown as the *solid line* and S_5 as a *line of circles*

In Fig. 1.5 this function (shown as the solid line) is approximated with S_5 which is given by

$$
\begin{aligned}
S_5 = \frac{1}{4} &- \frac{2}{\pi^2} \cos \pi t - \frac{2}{9\pi^2} \cos 3\pi t - \frac{2}{25\pi^2} \cos 5\pi t \\
&+ \frac{1}{\pi} \sin \pi t - \frac{1}{2\pi} \sin 2\pi t + \frac{1}{3\pi} \sin 3\pi t - \frac{1}{4\pi} \sin 4\pi t + \frac{1}{5\pi} \sin 5\pi t.
\end{aligned}
$$

While the convergence in this case is not very fast, but it is clear that with sufficient number of terms, the Fourier series can give an accurate representation of this function.

1.3.2 Fourier Series of Even and Odd Functions

If $f(t)$ is a even function, such that

$$f(-t) = f(t),$$

then its Fourier series contains cosine terms only. This can be seen as follows. The b_n coefficients can be written as

$$b_n = \frac{1}{L}\int_{-L}^{0} f(s)\sin\left(\frac{n\pi}{L}s\right)\mathrm{d}s + \frac{1}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t. \tag{1.22}$$

If we make a change of variable and let $s = -t$, the first integral on the right-hand side becomes

$$\begin{aligned}\frac{1}{L}\int_{-L}^{0} f(s)\sin\left(\frac{n\pi}{L}s\right)\mathrm{d}s &= \frac{1}{L}\int_{L}^{0} f(-t)\sin\left(-\frac{n\pi}{L}t\right)\mathrm{d}(-t)\\ &= \frac{1}{L}\int_{L}^{0} f(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t,\end{aligned}$$

since $\sin(-x) = -\sin(x)$ and $f(-x) = f(x)$. But

$$\frac{1}{L}\int_{L}^{0} f(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t = -\frac{1}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)\mathrm{d}t,$$

which is the negative of the second integral on the right-hand side of (1.22). Therefore $b_n = 0$ for all n.

Following the same procedure and using the fact that $\cos(-x) = \cos(x)$, we find

$$\begin{aligned}a_n &= \frac{1}{L}\int_{-L}^{0} f(s)\cos\left(\frac{n\pi}{L}s\right)\mathrm{d}s + \frac{1}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t\\ &= \frac{1}{L}\int_{L}^{0} f(-t)\cos\left(-\frac{n\pi}{L}\right)\mathrm{d}(-t) + \frac{1}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t\\ &= -\frac{1}{L}\int_{L}^{0} f(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t + \frac{1}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t\\ &= \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\mathrm{d}t. \end{aligned}\tag{1.23}$$

Hence

$$f(t) = \frac{1}{L}\int_{0}^{L} f(t')\mathrm{d}t' + \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_{0}^{L} f(t')\cos\left(\frac{n\pi}{L}t'\right)\mathrm{d}t'\right]\cos\frac{n\pi}{L}t. \tag{1.24}$$

Similarly, if $f(t)$ is an odd function

$$f(-t) = -f(t),$$

then

$$f(t) = \sum_{n=1}^{\infty} \left[\frac{2}{L} \int_0^L f(t') \sin\left(\frac{n\pi}{L} t'\right) \mathrm{d}t' \right] \sin \frac{n\pi}{L} t. \tag{1.25}$$

In the previous examples, the periodic function in Fig. 1.3 is an odd function, therefore its Fourier expansion is a sine series. In Fig. 1.4, the function is an even function, so its Fourier series is a cosine series. In Fig. 1.5, the periodic function has no symmetry, therefore its Fourier series contains both cosine and sine terms.

Example 1.3.4. Find the Fourier series of the function shown in Fig. 1.6.

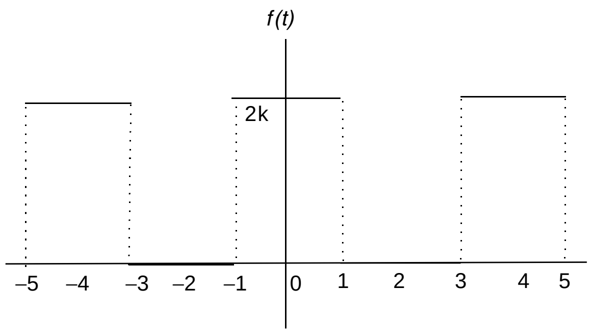


Fig. 1.6. An even square-wave function

Solution 1.3.4. The function shown in Fig. 1.6 can be defined as

$$f(t) = \begin{cases} 0 \text{ if } & -2 < t < -1 \\ 2k \text{ if } & -1 < t < 1 \\ 0 \text{ if } & 1 < t < 2 \end{cases}, \quad f(t) = f(t+4).$$

The period of the function $2L$ is equal to 4, therefore $L = 2$. Furthermore, the function is even, so the Fourier expansion is a cosine series, all coefficients for the sine terms are equal to zero

$$b_n = 0.$$

The coefficients for the cosine series are given by

$$a_0 = \frac{2}{2} \int_0^2 f(t)\mathrm{d}t = \int_0^1 2k\, \mathrm{d}t = 2k,$$

$$a_n = \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}\mathrm{d}t = \int_0^1 2k\cos\frac{n\pi t}{2}\mathrm{d}t = \frac{4k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus the Fourier series of $f(t)$ is

$$f(t) = k + \frac{4k}{\pi}\left(\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - \cdots\right). \tag{1.26}$$

It is instructive to compare Fig. 1.6 with Fig. 1.1. Figure 1.6 represents an even function whose Fourier expansion is a cosine series, whereas the function associated with Fig. 1.1 is an odd function and its Fourier series contains only sine terms. Yet they are clearly related. The two figures can be brought to coincide with each other if (a) we move y-axis in Fig. 1.6 one unit to the left (from $t = 0$ to $t = -1$), (b) make a change of variable so that the periodicity is changed from 4 to 2π, (c) shift Fig. 1.6 downward by an amount of k.

The changes in the Fourier series due to these operations are as follows. First let $t' = t + 1$, so that $t = t' - 1$ in (1.26),

$$f(t) = k + \frac{4k}{\pi}\left(\cos\frac{\pi}{2}(t'-1) - \frac{1}{3}\cos\frac{3\pi}{2}(t'-1) + \frac{1}{5}\cos\frac{5\pi}{2}(t'-1) - \cdots\right).$$

Since

$$\cos\frac{n\pi}{2}(t'-1) = \cos\left(\frac{n\pi}{2}t' - \frac{n\pi}{2}\right) = \begin{cases} \sin\frac{n\pi}{2}t' & n = 1, 5, 9, \ldots \\ -\sin\frac{n\pi}{2}t' & n = 3, 7, 11, \ldots \end{cases},$$

$f(t)$ expressed in terms of t' becomes

$$f(t) = k + \frac{4k}{\pi}\left(\sin\frac{\pi}{2}t' + \frac{1}{3}\sin\frac{3\pi}{2}t' + \frac{1}{5}\sin\frac{5\pi}{2}t' - \cdots\right) = g(t').$$

We call this expression $g(t')$, it still has a periodicity of 4. Next let us make a change of variable $t' = 2x/\pi$, so that the function expressed in terms of x will have a period of 2π,

$$\begin{aligned} g(t') &= k + \frac{4k}{\pi}\left(\sin\frac{\pi}{2}\left(\frac{2x}{\pi}\right) + \frac{1}{3}\sin\frac{3\pi}{2}\left(\frac{2x}{\pi}\right) + \frac{1}{5}\sin\frac{5\pi}{2}\left(\frac{2x}{\pi}\right) - \cdots\right) \\ &= k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x - \cdots\right) = h(x). \end{aligned}$$

Finally, shifting it down by k, we have

$$h(x) - k = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x - \cdots\right).$$

This is the Fourier series (1.10) for the odd function shown in Fig. 1.1.

1.4 Fourier Series of Nonperiodic Functions in Limited Range

So far we have considered only periodic functions extending from $-\infty$ to $+\infty$. In physical applications, often we are interested in the values of a function only in a limited interval. Within that interval the function may not be periodic. For example, in the study of a vibrating string fixed at both ends. There is no condition of periodicity as far as the physical problem is concerned, but there is also no interest in the function beyond the length of the string. Fourier analysis can still be applied to such problem, since we may continue the function outside the desired range so as to make it periodic.

Suppose that the interval of interest in the the function $f(t)$ shown in Fig. 1.7a is between 0 and L. We can extend the function between $-L$ and 0 any way we want. If we extend it first symmetrically as in part (b), then to the entire real line by the periodicity condition $f(t+2L) = f(t)$, a Fourier series consisting of only cosine terms can be found for the even function. An extension as in part (c) will enable us to find a Fourier sine series for the odd function. Both series would converge to the given $f(t)$ in the interval from 0 to L. Such series expansions are known as half-range expansions. The following examples will illustrate such expansions.

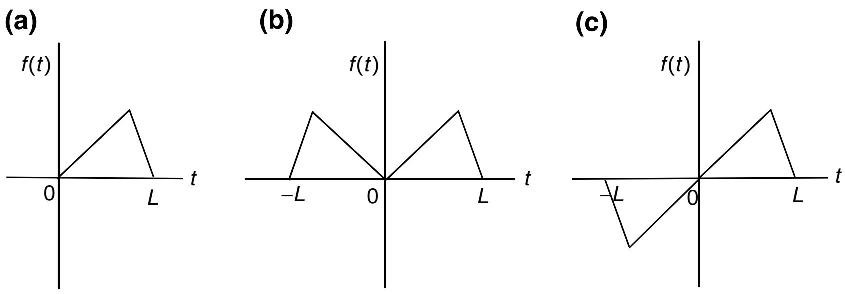


Fig. 1.7. Extension of a function. (**a**) The function is defined only between 0 and L. (**b**) A symmetrical extension yields an even function with a periodicity of $2L$. (**c**) An antisymmetrical extension yields an odd function with a periodicity of $2L$

Example 1.4.1. The function $f(t)$ is defined only over the range $0 < t < 1$ to be

$$f(t) = t - t^2.$$

Find the half-range cosine and sine Fourier expansions of $f(t)$.

Solution 1.4.1. (a) Let the interval (0,1) be half period of the symmetrically extended function, so that $2L = 2$ or $L = 1$. A half-range expansion of this even function is a cosine series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1} a_n \cos n\pi t$$

with

$$a_0 = 2\int_0^1 (t - t^2)\mathrm{d}t = \frac{1}{3},$$

$$a_n = 2\int_0^1 (t - t^2)\cos n\pi t\,\mathrm{d}t, \quad n \neq 0.$$

Using the Kronecker's method, we have

$$\int_0^1 t\cos n\pi t\,\mathrm{d}t = \left[\frac{1}{n\pi}t\sin n\pi t + \left(\frac{1}{n\pi}\right)^2 \cos n\pi t\right]_0^1$$

$$= \left(\frac{1}{n\pi}\right)^2 (\cos n\pi - 1),$$

$$\int_0^1 t^2\cos n\pi t\,\mathrm{d}t = \left[\frac{1}{n\pi}t^2\sin n\pi t + \left(\frac{1}{n\pi}\right)^2 2t\cos n\pi t - \left(\frac{1}{n\pi}\right)^3 2\sin n\pi t\right]_0^1$$

$$= 2\left(\frac{1}{n\pi}\right)^2 \cos n\pi,$$

so

$$a_n = 2\int_0^1 (t - t^2)\cos n\pi t\,\mathrm{d}t = -2\left(\frac{1}{n\pi}\right)^2 (\cos n\pi + 1).$$

With these coefficients, the half-range Fourier cosine expansion is given by S_∞^{even}, where

$$S_N^{\text{even}} = \frac{1}{6} - \frac{2}{\pi^2}\sum_{n=1}^{N}\frac{(\cos n\pi + 1)}{n^2}\cos n\pi t$$

$$= \frac{1}{6} - \frac{1}{\pi^2}\left(\cos 2\pi t + \frac{1}{4}\cos 4\pi t + \frac{1}{9}\cos 6\pi t + \cdots\right).$$

The convergence of this series is shown in Fig. 1.8a.

(b) A half-range sine expansion would be found by forming an antisymmetric extension. Since it is an odd function, the Fourier expansion is a sine series

$$f(t) = \sum_{n=1} b_n \sin \pi t$$

with

$$b_n = 2\int_0^1 (t - t^2)\sin n\pi t\,\mathrm{d}t.$$

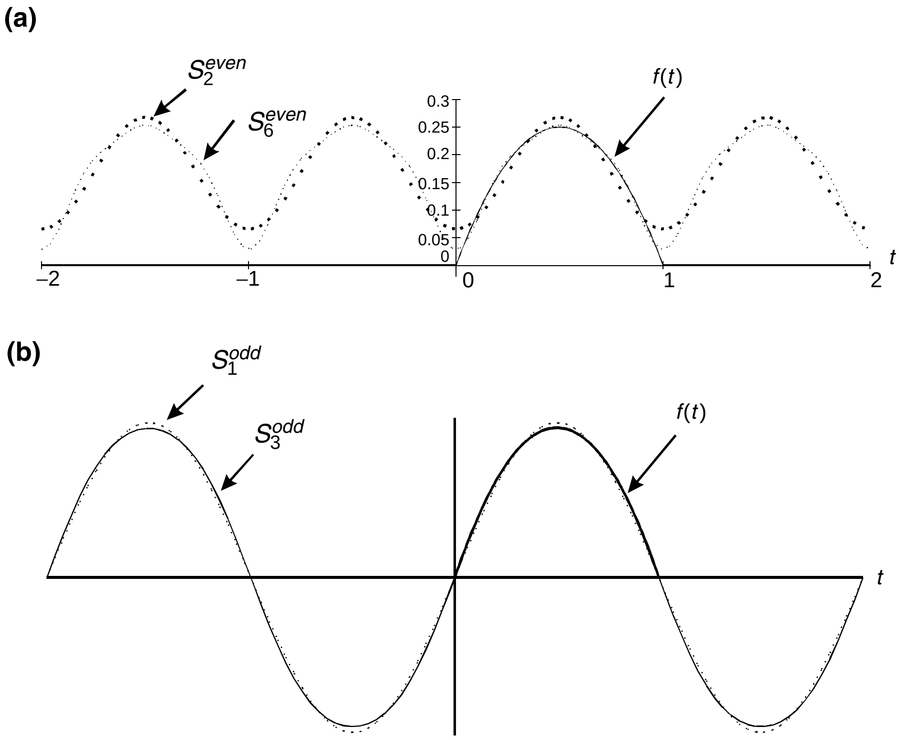


Fig. 1.8. Convergence of the half-range expansion series. The function $f(t) = t - t^2$ is given between 0 and 1. Both cosine and sine series converge to the function within this range. But outside this range, cosine series converges to an even function shown in (**a**) and sine series converges to an odd function shown in (**b**). S_2^{even} and S_6^{even} are two- and four-term approximations of the cosine series. S_1^{odd} and S_3^{odd} are one- and two-term approximations of the sine series

Now

$$\int_0^1 t \sin n\pi t \, \mathrm{d}t = \left[-\frac{1}{n\pi} t \cos n\pi t + \left(\frac{1}{n\pi} \right)^2 \sin n\pi t \right]_0^1 = -\frac{1}{n\pi} \cos n\pi,$$

$$\int_0^1 t^2 \sin n\pi t \, \mathrm{d}t = \left[-\frac{1}{n\pi} t^2 \cos n\pi t + \left(\frac{1}{n\pi} \right)^2 2t \sin n\pi t + \left(\frac{1}{n\pi} \right)^3 2 \cos n\pi t \right]_0^1$$

$$= -\frac{1}{n\pi} \cos n\pi + 2 \left(\frac{1}{n\pi} \right)^3 \cos n\pi - 2 \left(\frac{1}{n\pi} \right)^3,$$

so

$$b_n = 2 \int_0^1 (t - t^2) \sin n\pi t \, \mathrm{d}t = 4 \left(\frac{1}{n\pi} \right)^3 (1 - \cos n\pi).$$

Therefore the half-range sine expansion is given by S_∞^{odd}, with

$$S_N^{\text{odd}} = \frac{4}{\pi^3}\sum_{n=1}^{N}\frac{(1-\cos n\pi)}{n^3}\sin n\pi t$$
$$= \frac{8}{\pi^3}\left(\sin \pi t + \frac{1}{27}\sin 3\pi t + \frac{1}{125}\sin 5\pi t + \cdots\right).$$

The convergence of this series is shown in Fig. 1.8b.

It is seen that both the cosine and sine series converge to $t - t^2$ in the range between 0 and 1. Outside this range, the cosine series converges to an even function, and the sine series converges to an odd function. The rate of convergence is also different. For the sine series in (b), with only one term, S_1^{odd} is already very close to $f(t)$. With only two terms, S_3^{odd} (three terms if we include the $n = 2$ term that is equal to zero) is indistinguishable from $f(t)$ in the range of interest. The convergence of the cosine series in (a) is much slower. Although the four-term approximation S_6^{even} is much closer to $f(t)$ than the two-term approximation S_2^{even}, the difference between S_6^{even} and $f(t)$ in the range of interest is still noticeable.

This is generally the case that if we make extension smooth, greater accuracy results for a particular number of terms.

Example 1.4.2. A function $f(t)$ is defined only over the range $0 \le t \le 2$ to be $f(t) = t$. Find a Fourier series with only sine terms for this function.

Solution 1.4.2. One can obtain a half-range sine expansion by antisymmetrically extending the function. Such a function is described by

$$f(t) = t \quad \text{for} \quad -2 < t \le 2, \quad \text{and} \quad f(t+4) = f(t).$$

The Fourier series for this function is given by (1.19) with $L = 2$

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{2}.$$

However, this series does not converge to 2, the value of the function at $t = 2$. It converges to 0, the average value of the right- and left-hand limit of the function at $t = 2$, as shown in Fig. 1.3.

We can find a Fourier sine series that converges to the correct value at the end points, if we consider the function

$$f(t) = \begin{cases} t & \text{for } 0 < t \le 2, \\ 4 - t & \text{for } 2 < t \le 4. \end{cases}$$

An antisymmetrical extension will give us an odd function with a periodicity of 8 ($2L = 8, \;\; L = 4$). The Fourier expansion for this function is a sine series

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{4}$$

with

$$b_n = \frac{2}{4}\int_0^4 f(t)\sin\frac{n\pi t}{4}\mathrm{d}t$$
$$= \frac{2}{4}\int_0^2 t\sin\frac{n\pi t}{4}\mathrm{d}t + \frac{2}{4}\int_2^4 (4-t)\sin\frac{n\pi t}{4}\mathrm{d}t.$$

Using the Kronecker's method, we have

$$b_n = \frac{1}{2}\left[-\frac{4}{n\pi}t\cos\frac{n\pi t}{4} + \left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi t}{4}\right]_0^2 + 2\left[-\frac{4}{n\pi}\cos\frac{n\pi t}{4}\right]_2^4$$
$$-\frac{1}{2}\left[-\frac{4}{n\pi}t\cos\frac{n\pi t}{4} + \left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi t}{4}\right]_2^4$$
$$= \left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi}{2}.$$

Thus

$$f(t) = \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi}{2}\sin\frac{n\pi t}{4}$$
$$= \frac{16}{\pi^2}\left[\sin\frac{\pi t}{4} - \frac{1}{9}\sin\frac{3\pi t}{4} + \frac{1}{25}\sin\frac{5\pi t}{4} - \cdots\right]. \tag{1.27}$$

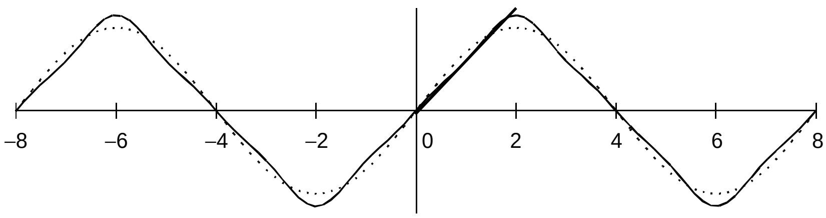


Fig. 1.9. Fourier series for a function defined in a limited range. Within the range $0 \le t \le 2$, the series (1.27) converges to $f(t) = t$. Outside this range the series converges to a odd periodic function with a periodicity of 8

Within the range of $0 \le t \le 2$, this sine series converges to $f(t) = t$. Outside this range, this series converges to an odd periodic function shown in Fig. 1.9. It converges much faster than the series in (1.19). The first term, shown as dashed line, already provides a reasonable approximation. The difference between the three-term approximation and the given function is hardly noticeable.

As we have seen, for a function that is defined only in a limited range, it is possible to have many different Fourier series. They all converge to the function in the given range, although their rate of convergence may be different. Fortunately, in physical applications, the question of which series we should use for the description the function is usually determined automatically by the boundary conditions.

From all the examples so far, we make the following observations:

- If the function is discontinuous at some point, the Fourier coefficients are decreasing as $1/n$.
- If the function is continuous but its first derivative is discontinuous at some point, the Fourier coefficients are decreasing as $1/n^2$.
- If the function and its first derivative are continuous, the Fourier coefficients are decreasing as $1/n^3$.

Although these comments are based on a few examples, they are generally valid (see the Method of Jumps for the Fourier Coefficients). It is useful to keep them in mind when calculating Fourier coefficients.

1.5 Complex Fourier Series

The Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}t + b_n \sin\frac{n\pi}{p}t\right)$$

can be put in the complex form. Since

$$\cos\frac{n\pi}{p}t = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}(n\pi/p)t} + \mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right),$$

$$\sin\frac{n\pi}{p}t = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}(n\pi/p)t} - \mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right),$$

it follows:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}a_n + \frac{1}{2\mathrm{i}}b_n\right)\mathrm{e}^{\mathrm{i}(n\pi/p)t} + \left(\frac{1}{2}a_n - \frac{1}{2\mathrm{i}}b_n\right)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right].$$

Now if we define c_n as

$$\begin{aligned} c_n &= \frac{1}{2}a_n + \frac{1}{2\mathrm{i}}b_n \\ &= \frac{1}{2}\frac{1}{p}\int_{-p}^{p} f(t)\cos\left(\frac{n\pi}{p}t\right)\mathrm{d}t + \frac{1}{2\mathrm{i}}\frac{1}{p}\int_{-p}^{p} f(t)\sin\left(\frac{n\pi}{p}t\right)\mathrm{d}t \end{aligned}$$

$$= \frac{1}{2p}\int_{-p}^{p} f(t)\left[\cos\left(\frac{n\pi}{p}t\right) - \mathrm{i}\sin\left(\frac{n\pi}{p}t\right)\right]\mathrm{d}t$$
$$= \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}t,$$

$$c_{-n} = \frac{1}{2}a_n - \frac{1}{2\mathrm{i}}b_n$$
$$= \frac{1}{2}\frac{1}{p}\int_{-p}^{p} f(t)\cos\left(\frac{n\pi}{p}t\right)\mathrm{d}t - \frac{1}{2\mathrm{i}}\frac{1}{p}\int_{-p}^{p} f(t)\sin\left(\frac{n\pi}{p}t\right)\mathrm{d}t$$
$$= \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{\mathrm{i}(n\pi/p)t}\mathrm{d}t$$

and

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2}\frac{1}{p}\int_{-p}^{p} f(t)\mathrm{d}t,$$

then the series can be written as

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n\mathrm{e}^{\mathrm{i}(n\pi/p)t} + c_{-n}\mathrm{e}^{\mathrm{i}(n\pi/p)t}\right]$$
$$= \sum_{n=-\infty}^{\infty} c_n\mathrm{e}^{\mathrm{i}(n\pi/p)t} \tag{1.28}$$

with

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}t \tag{1.29}$$

for positive n, negative n, or $n = 0$.

Now the Fourier series appears in complex form. If $f(t)$ is a complex function of real variable t, then the complex Fourier series is a natural one. If $f(t)$ is a real function, it can still be represented by the complex series (1.28). In that case, c_{-n} is the complex conjugate of c_n ($c_{-n} = c_n^*$).

Since

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n), \quad c_{-n} = \frac{1}{2}(a_n + \mathrm{i}b_n),$$

if follows that:

$$a_n = c_n + c_{-n}, \quad b_n = \mathrm{i}(c_n - c_{-n}).$$

Thus if $f(t)$ is an even function, then $c_{-n} = c_n$. If $f(t)$ is an odd function, then $c_{-n} = -c_n$.

Example 1.5.1. Find the complex Fourier series of the function

$$f(t) = \begin{cases} 0 & -\pi < t < 0, \\ 1 & 0 < t < \pi. \end{cases}$$

Solution 1.5.1. Since the period is 2π, so $p = \pi$, and the complex Fourier series is given by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{int}$$

with

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}t = \frac{1}{2},$$

$$c_n = \frac{1}{2\pi}\int_0^{\pi} \mathrm{e}^{-int}\mathrm{d}t = \frac{1-\mathrm{e}^{-in\pi}}{2\pi n\mathrm{i}} = \begin{cases} 0 & n = \text{even}, \\ \frac{1}{\pi ni} & n = \text{odd}. \end{cases}$$

Therefore the complex series is

$$f(t) = \frac{1}{2} + \frac{1}{\mathrm{i}\pi}\left(\cdots - \frac{1}{3}\mathrm{e}^{-\mathrm{i}3t} - \mathrm{e}^{-\mathrm{i}t} + \mathrm{e}^{\mathrm{i}t} + \frac{1}{3}\mathrm{e}^{\mathrm{i}3t} + \cdots\right).$$

It is clear that

$$c_{-n} = \frac{1}{\pi(-n)\mathrm{i}} = \frac{1}{\pi n(-\mathrm{i})} = c_n^*$$

as we expect, sine $f(t)$ is real. Furthermore, since

$$\mathrm{e}^{int} - \mathrm{e}^{-int} = 2\mathrm{i}\sin nt,$$

the Fourier series can be written as

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right).$$

This is also what we expected, since $f(t) - \frac{1}{2}$ is an odd function, and

$$a_n = c_n + c_{-n} = \frac{1}{\pi n\mathrm{i}} + \frac{1}{\pi(-n)\mathrm{i}} = 0,$$

$$b_n = \mathrm{i}(c_n - c_{-n}) = \mathrm{i}\left(\frac{1}{\pi n\mathrm{i}} - \frac{1}{\pi(-n)\mathrm{i}}\right) = \frac{2}{\pi n}.$$

Example 1.5.2. Find the Fourier series of the function defined as

$$f(t) = \mathrm{e}^t \quad \text{for} \quad -\pi < t < \pi, \quad f(t+2\pi) = f(t).$$

Solution 1.5.2. This periodic function has a period of 2π. We can express it as the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

However, the complex Fourier coefficients are easier to compute, so we first express it as a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nt}$$

with

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^t \mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t = \frac{1}{2\pi}\left[\frac{1}{1-\mathrm{i}n}\mathrm{e}^{(1-\mathrm{i}n)t}\right]_{-\pi}^{\pi}.$$

Since

$$\begin{aligned}
\mathrm{e}^{(1-\mathrm{i}n)\pi} &= \mathrm{e}^{\pi}\mathrm{e}^{-\mathrm{i}n\pi} = (-1)^n \mathrm{e}^{\pi}, \\
\mathrm{e}^{-(1-\mathrm{i}n)\pi} &= \mathrm{e}^{-\pi}\mathrm{e}^{\mathrm{i}n\pi} = (-1)^n \mathrm{e}^{-\pi}, \\
\mathrm{e}^{\pi} - \mathrm{e}^{-\pi} &= 2\sinh\pi,
\end{aligned}$$

so

$$c_n = \frac{(-1)^n}{2\pi(1-\mathrm{i}n)}(\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}) = \frac{(-1)^n}{\pi}\frac{1+\mathrm{i}n}{1+n^2}\sinh\pi.$$

Now

$$\begin{aligned}
a_n &= c_n + c_{-n} = \frac{(-1)^n}{\pi}\frac{2}{1+n^2}\sinh\pi, \\
b_n &= \mathrm{i}(c_n - c_{-n}) = -\frac{(-1)^n}{\pi}\frac{2n}{1+n^2}\sinh\pi.
\end{aligned}$$

Thus, the Fourier series is given by

$$\mathrm{e}^x = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nt - n\sin nt).$$

1.6 The Method of Jumps

There is an effective way of computing the Fourier coefficients, known as the method of jumps. As long as the given function is piecewise continuous, this method enables us to find Fourier coefficients by graphical techniques.

Suppose that $f(t)$, shown in Fig. 1.10, is a periodic function with a period $2p$. It is piecewise continuous. The locations of the discontinuity are at $t_1, t_2, \ldots, t_{N-1}$, counting from left to right. The two end points t_0 and t_N may or may not be points of discontinuity. Let $f(t_i^+)$ be the right-hand limit of the function as t approaches t_i from the right, and $f(t_i^-)$, the left-hand limit. At each discontinuity t_i, except at two end points t_0 and $t_N = t_0 + 2p$, we define a jump J_i as

$$J_i = f(t_i^+) - f(t_i^-).$$

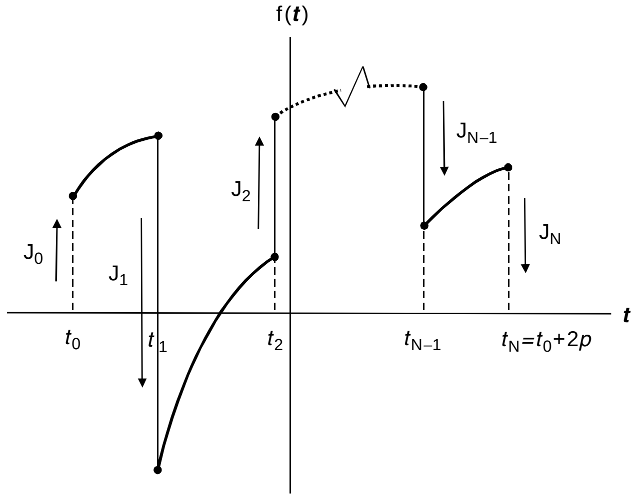


Fig. 1.10. One period of a periodic piecewise continuous function $f(t)$ with period $2p$

At t_0, the jump J_0 is defined as

$$J_0 = f(t_0^+) - 0 = f(t_0^+)$$

and at t_N, the jump J_N is

$$J_N = 0 - f(t_N^-) = -f(t_N^-).$$

These jumps are indicated by the arrows in Fig. 1.10. It is seen that J_i will be positive if the jump at t_i is up and negative if the jump is down. Note that at t_0, the jump is from zero to $f\left(t_0^+\right)$, and at t_N, the jump is from $f\left(t_N^-\right)$ to zero.

We will now show that the coefficients of the Fourier series can be expressed in terms of these jumps.

The coefficients of the complex Fourier series, as seen in (1.29), is given by

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}t.$$

Let us define the integral as

$$\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}t = I_n[f(t)].$$

So $c_n = \frac{1}{2p} I_n[f(t)]$.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{p}{\mathrm{i}n\pi}f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]=-\frac{p}{\mathrm{i}n\pi}\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}(n\pi/p)t}+f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t},$$

so

$$f\left(t\right)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}t=\mathrm{d}\left[-\frac{p}{\mathrm{i}n\pi}f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]+\frac{p}{\mathrm{i}n\pi}\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}f\left(t\right),$$

it follows that:

$$I_n\left[f\left(t\right)\right]=\int_{-p}^{p}\mathrm{d}\left[-\frac{p}{\mathrm{i}n\pi}f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]+\frac{p}{\mathrm{i}n\pi}\int_{-p}^{p}\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}f(t).$$

Note that

$$\int_{-p}^{p}\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\mathrm{d}f(t)=\int_{-p}^{p}\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}t=I_n\left[f'\left(t\right)\right],$$

and

$$\int_{-p}^{p}\mathrm{d}\left[-\frac{p}{\mathrm{i}n\pi}f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]=-\frac{p}{\mathrm{i}n\pi}\left[\int_{t_0}^{t_1}+\int_{t_1}^{t_2}+\cdots+\int_{t_{N-1}}^{t_N}\right]\times\mathrm{d}\left[f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right].$$

Since

$$\int_{t_0}^{t_1}\mathrm{d}\left[f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]=f(t_1^-)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_1}-f(t_0^+)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_0},$$

$$\int_{t_1}^{t_2}\mathrm{d}\left[f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]=f(t_2^-)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_2}-f(t_1^+)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_1},$$

$$\int_{t_{N-1}}^{t_N}\mathrm{d}\left[f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]=f(t_N^-)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_N}-f(t_{N-1}^+)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_{N-1}},$$

we have

$$\int_{-p}^{p}\mathrm{d}\left[-\frac{p}{\mathrm{i}n\pi}f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}\right]=\frac{p}{\mathrm{i}n\pi}f(t_0^+)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_0}$$
$$+\frac{p}{\mathrm{i}n\pi}[f(t_1^+)-f(t_1^-)]\mathrm{e}^{-\mathrm{i}(n\pi/p)t_1}$$
$$+\cdots-\frac{p}{\mathrm{i}n\pi}f(t_N^-)\mathrm{e}^{-\mathrm{i}(n\pi/p)t_N}=\frac{p}{\mathrm{i}n\pi}\sum_{k=0}^{k=N}J_k\mathrm{e}^{-\mathrm{i}(n\pi/p)t_k}.$$

Thus

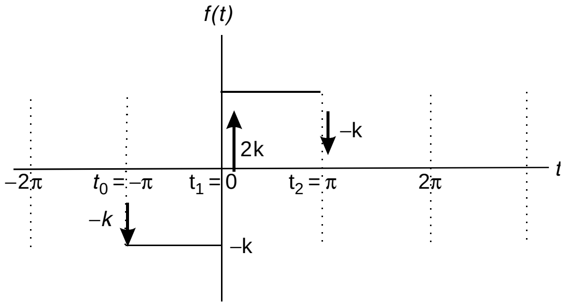
$$I_n[f(t)]=\frac{p}{\mathrm{i}n\pi}\sum_{k=0}^{k=N}J_k\mathrm{e}^{-\mathrm{i}(n\pi/p)t_k}+\frac{p}{\mathrm{i}n\pi}I_n[f'(t)].$$

Clearly, $I_n[f'(t)]$ can be evaluated similarly as $I_n[f(t)]$. This formula can be used iteratively to find the Fourier coefficient c_n for nonzero n, since $c_n = I_n[f(t)]/2p$. Together with c_0, which is given by a simple integral, these coefficients determine all terms of the Fourier series. For many practical functions, their Fourier series can be simply obtained from the jumps at the points of discontinuity. The following examples will illustrate how quickly this can be done with the sketches of the function and its derivatives.

Example 1.6.1. Use the method of jumps to find the Fourier series of the periodic function $f(t)$, one of its periods is defined on the interval of $-\pi < t < \pi$ as

$$f(t) = \begin{cases} k & \text{for } -\pi < t < 0 \\ -k & \text{for } \ 0 < t < \pi \end{cases}.$$

Solution 1.6.1. The sketch of this function is



The period of this function is 2π, therefore $p = \pi$. It is clear that all derivatives of this function are equal to zero, thus we have

$$c_n = \frac{1}{2\pi} I_n[f(t)] = \frac{1}{\mathrm{i}2\pi n} \sum_{k=0}^{2} J_k \mathrm{e}^{-\mathrm{i}(n\pi/p)t_k}, \quad n \neq 0,$$

where

$$t_0 = -\pi, \quad t_1 = 0, \quad t_2 = \pi$$

and

$$J_0 = -k, \quad J_1 = 2k, \quad J_2 = -k.$$

Hence

$$\begin{aligned} c_n &= \frac{1}{\mathrm{i}2\pi n}[-k\mathrm{e}^{\mathrm{i}n\pi} + 2k - k\mathrm{e}^{-\mathrm{i}n\pi}] \\ &= \frac{k}{\mathrm{i}2\pi n}[2 - 2\cos(n\pi)] = \begin{cases} 0 & n = \text{even} \\ \frac{2k}{\mathrm{i}n\pi} & n = \text{odd} \end{cases}. \end{aligned}$$

It follows that:

$$a_n = c_n + c_{-n} = 0,$$
$$b_n = \mathrm{i}(c_n - c_{-n}) = \begin{cases} 0 & n = \text{even} \\ \frac{4k}{n\pi} & n = \text{odd} \end{cases}.$$

Furthermore,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = 0.$$

Therefore the Fourier series is given by

$$f(t) = \frac{4k}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right).$$

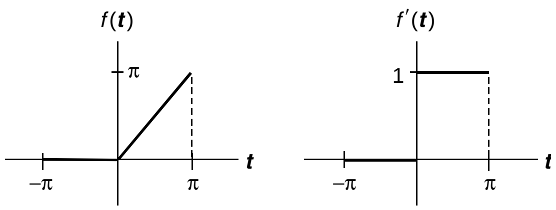
Example 1.6.2. Use the method of jumps to find the Fourier series of the following function:

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}, \quad f(t+2\pi) = f(t).$$

Solution 1.6.2. The first derivative of this function is

$$f'(t) = \begin{cases} 0 & -\pi < t < 0, \\ 1 & 0 < t < \pi \end{cases}$$

and higher derivatives are all equal to zero. The sketches of $f(t)$ and $f'(t)$ are shown as follows:



In this case

$$p = \pi, \quad t_0 = -\pi, \quad t_1 = 0, \quad t_2 = \pi.$$

Thus

$$I_n[f(t)] = \frac{1}{\mathrm{i}n}\sum_{k=0}^{2} J_k \mathrm{e}^{-\mathrm{i}nt_k} + \frac{1}{\mathrm{i}n} I_n[f'(t)],$$

where

$$J_0 = 0, \quad J_1 = 0, \quad J_2 = -\pi,$$

and

$$I_n[f'(t)] = \frac{1}{\mathrm{i}n}\sum_{k=0}^{2} J'_k \mathrm{e}^{-\mathrm{i}nt_k}$$

with

$$J'_0 = 0, \quad J'_1 = 1, \quad J'_2 = -1.$$

It follows that:

$$I_n[f(t)] = \frac{1}{\mathrm{i}n}(-\pi)\mathrm{e}^{-\mathrm{i}n\pi} + \frac{1}{\mathrm{i}n}\left[\frac{1}{\mathrm{i}n}(1-\mathrm{e}^{-\mathrm{i}n\pi})\right]$$

and

$$c_n = \frac{1}{2\pi}I_n[f(t)] = -\frac{1}{\mathrm{i}2n}\mathrm{e}^{-\mathrm{i}n\pi} - \frac{1}{2\pi n^2}(1-\mathrm{e}^{-\mathrm{i}n\pi}), \quad n \neq 0.$$

In addition

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} t\,\mathrm{d}t = \frac{\pi}{4}.$$

Therefore the Fourier coefficients a_n and b_n are given by

$$\begin{aligned} a_n = c_n + c_{-n} &= \frac{1}{\mathrm{i}2n}(-\mathrm{e}^{-\mathrm{i}n\pi} + \mathrm{e}^{\mathrm{i}n\pi}) + \frac{1}{2\pi n^2}(\mathrm{e}^{-\mathrm{i}n\pi} + \mathrm{e}^{\mathrm{i}n\pi}) - \frac{1}{\pi n^2} \\ &= \frac{1}{n}\sin n\pi + \frac{1}{\pi n^2}\cos n\pi - \frac{1}{\pi n^2} = \begin{cases} -\frac{2}{\pi n^2} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}, \end{aligned}$$

$$\begin{aligned} b_n = \mathrm{i}(c_n - c_{-n}) &= \mathrm{i}\left[-\frac{1}{\mathrm{i}2n}(\mathrm{e}^{-\mathrm{i}n\pi} + \mathrm{e}^{\mathrm{i}n\pi}) + \frac{1}{2\pi n^2}(\mathrm{e}^{-\mathrm{i}n\pi} - \mathrm{e}^{\mathrm{i}n\pi})\right] \\ &= -\frac{1}{n}\cos n\pi + \frac{1}{\pi n^2}\sin n\pi = \begin{cases} \frac{1}{n} & n = \text{odd} \\ -\frac{1}{n} & n = \text{even} \end{cases}. \end{aligned}$$

So the Fourier series can be written as

$$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}\frac{1}{(2n-1)^2}\cos(2n-1)t - \sum_{n=1}\frac{(-1)^n}{n}\sin nt.$$

1.7 Properties of Fourier Series

1.7.1 Parseval's Theorem

If the periodicity of a periodic function $f(t)$ is $2p$, the Parseval's theorem states that

$$\frac{1}{2p}\int_{-p}^{p}[f(t)]^2\mathrm{d}t = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

where a_n and b_n are the Fourier coefficients. This theorem can be proved by expressed $f(t)$ as the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}\right),$$

and carrying out the integration. However, the computation is simpler if we first work with the complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}(n\pi/p)t},$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}(n\pi/p)t}.$$

With these expressions, the integral can be written as

$$\frac{1}{2p}\int_{-p}^{p}[f(t)]^2\mathrm{d}t = \frac{1}{2p}\int_{-p}^{p} f(t)\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}(n\pi/p)t}\mathrm{d}t$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{\mathrm{i}(n\pi/p)t}\mathrm{d}t.$$

Since

$$c_{-n} = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}((-n)\pi/p)t} = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{\mathrm{i}(n\pi/p)t}\mathrm{d}t,$$

it follows that:

$$\frac{1}{2p}\int_{-p}^{p}[f(t)]^2\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} c_n c_{-n} = c_0^2 + 2\sum_{n=1}^{\infty} c_n c_{-n}.$$

If $f(t)$ is a real function, then $c_{-n} = c_n^*$. Since

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n), \quad c_n^* = \frac{1}{2}(a_n + \mathrm{i}b_n),$$

so

$$c_n c_{-n} = c_n c_n^* = \frac{1}{4}\left[a_n^2 - (\mathrm{i}b_n)^2\right] = \frac{1}{4}(a_n^2 + b_n^2).$$

Therefore

$$\frac{1}{2p}\int_{-p}^{p}[f(t)]^2\mathrm{d}t = c_0^2 + 2\sum_{n=1}^{\infty} c_n c_{-n} = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

This theorem has an interesting and important interpretation. In physics we learnt that the energy in a wave is proportional to the square of its amplitude. For the wave represented by $f(t)$, the energy in one period will be proportional to $\int_{-p}^{p}[f(t)]^2\mathrm{d}t$. Since $a_n \cos\frac{n\pi t}{p}$ also represents a wave, so the energy in this pure cosine wave is proportional to

$$\int_{-p}^{p}\left(a_n \cos\frac{n\pi}{p}t\right)^2 \mathrm{d}t = a_n^2 \int_{-p}^{p} \cos^2\frac{n\pi t}{p}\mathrm{d}t = pa_n^2$$

so the energy in the pure sine wave is

$$\int_{-p}^{p}\left(b_n \sin\frac{n\pi}{p}t\right)^2 \mathrm{d}t = b_n^2 \int_{-p}^{p} \sin^2\frac{n\pi t}{p}\mathrm{d}t = pb_n^2.$$

From the Parseval's theorem, we have

$$\int_{-p}^{p}[f(t)]^2\mathrm{d}t = p\frac{1}{2}a_0^2 + p\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

This says that the total energy in a wave is just the sum of the energies in all the Fourier components. For this reason, Parseval's theorem is also called "energy theorem."

1.7.2 Sums of Reciprocal Powers of Integers

An interesting application of Fourier series is that it can be used to sum up a series of reciprocal powers of integers. For example, we have shown that the Fourier series of the square-wave

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x)$$

is given by

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

At $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right),$$

thus

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}.$$

This is a famous result obtained by Leibniz in 1673 from geometrical considerations. It became well known because it was the first series involving π ever discovered.

The Parseval's theorem can also be used to give additional results. In this problem,

$$[f(t)]^2 = k^2, \quad a_n = 0, \quad b_n = \begin{cases} \frac{4k}{\pi n} & n = \text{odd} \\ 0 & n = \text{even} \end{cases},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)]^2 \mathrm{d}t = k^2 = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 = \frac{1}{2}\left(\frac{4k}{\pi}\right)^2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right).$$

So we have

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

In the following example, we will demonstrate that a number of such sums can be obtained with one Fourier series.

Example 1.7.1. Use the Fourier series for the function whose definition is

$$f(x) = x^2 \text{ for } -1 < x < 1, \quad \text{and} \quad f(x+2) = f(x),$$

to show that

$$\text{(a)} \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}, \quad \text{(b)} \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\text{(c)} \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}, \quad \text{(d)} \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution 1.7.1. The Fourier series for the function is given by (1.20) with $L = 1:$

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

(a) Set $x = 0$, so we have

$$x^2 = 0, \quad \cos n\pi x = 1.$$

Thus

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

or

$$-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{1}{3}.$$

It follows that:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

(b) With $x = 1$, the series becomes

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi.$$

Since $\cos n\pi = (-1)^n$, we have

$$1 - \frac{1}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

or

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

(c) Integrating both sides from 0 to 1/2,

$$\int_0^{1/2} x^2 \mathrm{d}x = \int_0^{1/2}\left[\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x\right]\mathrm{d}x$$

we get

$$\frac{1}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{3}\left(\frac{1}{2}\right) + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

or

$$-\frac{1}{8} = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi}{2}.$$

Since

$$\sin\frac{n\pi}{2} = \begin{cases} 0 & n = \text{even}, \\ 1 & n = 1, 5, 9, \ldots, \\ -1 & n = 3, 7, 11, \ldots \end{cases}$$

the sum can be written as

$$-\frac{1}{8} = -\frac{4}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right).$$

It follows that:

$$\frac{\pi^3}{32} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}.$$

(d) Using the Parseval's theorem, we have

$$\frac{1}{2}\int_{-1}^{1}\left(x^2\right)^2\mathrm{d}x = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{4}{\pi^2}\frac{(-1)^n}{n^2}\right]^2.$$

Thus

$$\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

It follows that:

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

This last series played an important role in the theory of black-body radiation, which was crucial in the development of quantum mechanics.

1.7.3 Integration of Fourier Series

If a Fourier series of $f(x)$ is integrated term-by-term, a factor of $1/n$ is introduced into the series. This has the effect of enhancing the convergence. Therefore we expect the series resulting from term-by-term integration will converge to the integral of $f(x)$. For example, we have shown that the Fourier series for the odd function $f(t) = t$ of period $2L$ is given by

$$t = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t.$$

We expect a term-by-term integration of the right-hand side of this equation to converge to the integral of t. That is

$$\int_0^t x \, \mathrm{d}x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^t \sin \frac{n\pi}{L} x \, \mathrm{d}x.$$

The result of this integration is

$$\frac{1}{2} t^2 = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[-\frac{L}{n\pi} \cos \frac{n\pi}{L} x \right]_0^t$$

or

$$t^2 = \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} - \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos \frac{n\pi}{L} t.$$

Since

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12},$$

we obtain

$$t^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi}{L} t.$$

This is indeed the correct Fourier series converging to t^2 of period $2L$, as seen in (1.20) .

Example 1.7.2. Find the Fourier series of the function whose definition in one period is

$$f(t) = t^3, \quad -L < t < L.$$

Solution 1.7.2. Integrating the Fourier series for t^2 in the required range term-by-term

$$\int t^2 \mathrm{d}t = \int \left[\frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi}{L} t \right] \mathrm{d}t,$$

we obtain

$$\frac{1}{3} t^3 = \frac{L^2}{3} t + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{L}{n\pi} \sin \frac{n\pi}{L} t + C.$$

We can find the integration constant C by looking at the values of both sides of this equation at $t = 0$. Clearly $C = 0$. Furthermore, since in the range of $-L < t < L$,

$$t = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t,$$

therefore the Fourier series of t^3 in the required range is

$$t^3 = \frac{2L^3}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t + \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin \frac{n\pi}{L} t.$$

1.7.4 Differentiation of Fourier Series

In differentiating a Fourier series term-by-term, we have to be more careful. A term-by-term differentiation will cause the coefficients a_n and b_n to be multiplied by a factor n. Since it grows linearly, the resulting series may not even converge. Take, for example

$$t = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t.$$

This equation is valid in the range of $-L < t < L$, as seen in (1.19). The derivative of t is of course equal to 1. However, a term-by-term differentiation of the Fourier series on the right-hand side

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t \right] = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi}{L} t$$

does not even converge, let alone equal to 1.

In order to see under what conditions, if any, that the Fourier series of the function $f(t)$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}t + b_n \sin \frac{n\pi}{L}t \right)$$

can be differentiated term-by-term, let us first assume that $f(t)$ is continuous within the range $-L < t < L$, and the derivative of the function $f'(t)$ can be expanded in another Fourier series

$$f'(t) = \frac{1}{2}a'_0 + \sum_{n=1}^{\infty} \left(a'_n \cos \frac{n\pi}{L}t + b'_n \sin \frac{n\pi}{L}t \right).$$

The coefficients a'_n are given by

$$\begin{aligned} a'_n &= \frac{1}{L} \int_{-L}^{L} f'(t) \cos \frac{n\pi}{L}t \, \mathrm{d}t \\ &= \frac{1}{L} \left[f(t) \cos \frac{n\pi}{L}t \right]_{-L}^{L} + \frac{n\pi}{L^2} \int_{-L}^{L} f(t) \sin \frac{n\pi}{L}t \, \mathrm{d}t \\ &= \frac{1}{L}[f(L) - f(-L)] \cos n\pi + \frac{n\pi}{L}b_n. \end{aligned} \tag{1.30}$$

Similarly

$$b'_n = \frac{1}{L}[f(L) - f(-L)] \sin n\pi - \frac{n\pi}{L}na_n. \tag{1.31}$$

On the other hand, differentiating the Fourier series of the function term-by-term, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}t + b_n \sin \frac{n\pi}{L}t \right) \right] \\ &= \sum_{n=1}^{\infty} \left(-a_n \frac{n\pi}{L} \sin \frac{n\pi}{L}t + b_n \frac{n\pi}{L} \cos \frac{n\pi}{L}t \right). \end{aligned}$$

This would simply give coefficients

$$a'_n = \frac{n\pi}{L}b_n, \quad b'_n = -\frac{n\pi}{L}a_n. \tag{1.32}$$

Thus we see that the derivative of a function is not, in general, given by differentiating the Fourier series of the function term-by-term. However, if the function satisfies the condition

$$f(L) = f(-L), \tag{1.33}$$

then a'_n and b'_n given by (1.30) and (1.31) are identical to those given by (1.32). We call (1.33) the "head equals tail" condition. Once this condition is satisfied, a term-by-term differentiation of the Fourier series of the function will converge to the derivative of the function. Note that if the periodic function $f(t)$ is continuous everywhere, this condition is automatically satisfied.

Now it is clear why (1.19) cannot be differentiated term-by-term. For this function

$$f(L) = L \neq -L = f(-L),$$

the "head equals tail" condition is not satisfied. In the following example, the function satisfies this condition. Its derivative is indeed given by the result of the term-by-term differentiation.

Example 1.7.3. The fourier series for t^2 in the range $-L < t < L$ is given by (1.20)

$$\frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi}{L} t = t^2.$$

It satisfies the "head equals tail" condition, as shown in Fig. 1.4. Show that a term-by-term differentiation of this series is equal to $2t$.

Solution 1.7.3.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi}{L} t\right] &= \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{\mathrm{d}}{\mathrm{d}t} \cos \frac{n\pi}{L} t \\ &= \frac{4L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t \end{aligned}$$

which is the Fourier series of $2t$ in the required range, as seen in (1.19) .

1.8 Fourier Series and Differential Equations

Fourier series play an important role in solving partial differential equations, as we shall see in many examples in later chapters. In this section, we shall confine ourselves with some applications of Fourier series in solving nonhomogeneous ordinary differential equations.

1.8.1 Differential Equation with Boundary Conditions

Let us consider the following nonhomogeneous differential equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 4x = 4t,$$

$$x(0) = 0, \quad x(1) = 0.$$

We want to find the solution between $t = 0$ and $t = 1$. Previously we have learned that the general solution of this equation is the sum of the complementary function x_c and the particular solution x_p. That is

$$x = x_\mathrm{c} + x_\mathrm{p},$$

where x_c is the solution of the homogeneous equation

$$\frac{\mathrm{d}^2 x_\mathrm{c}}{\mathrm{d}t^2} + 4x_\mathrm{c} = 0$$

with two arbitrary constants, and x_p is the particular solution of

$$\frac{\mathrm{d}^2 x_\mathrm{p}}{\mathrm{d}t^2} + 4x_\mathrm{p} = 4t$$

with no arbitrary constant. It can be easily verified that in this case

$$x_\mathrm{c} = A\cos 2t + B\sin 2t,$$
$$x_\mathrm{p} = t.$$

Therefore the general solution is

$$x(t) = A\cos 2t + B\sin 2t + t.$$

The two constants A and B are determined by the boundary conditions. Since

$$x(0) = A = 0,$$
$$x(1) = A\cos 2 + B\sin 2 + 1 = 0,$$

Thus

$$A = 0, \quad B = -\frac{1}{\sin 2}.$$

Therefore the exact solution that satisfies the boundary conditions is given by

$$x(t) = t - \frac{1}{\sin 2}\sin 2t.$$

This function in the range of $0 \leq t \leq 1$ can be expanded into a half-range Fourier sine series

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t,$$

where

$$b_n = 2\int_0^1 \left(t - \frac{1}{\sin 2}\sin 2t\right)\sin n\pi t \,\mathrm{d}t.$$

We have already shown that

$$\int_0^1 t\sin n\pi t\,\mathrm{d}t = \frac{(-1)^{n+1}}{n\pi}.$$

With integration by parts twice, we find

$$\int_0^1 \sin 2t \sin n\pi t \,\mathrm{d}t = \left[-\frac{1}{n\pi}\sin 2t\cos n\pi t + \frac{2}{(n\pi)^2}\cos 2t \sin n\pi t\right]_0^1 + \frac{4}{(n\pi)^2}\int_0^1 \sin 2t \sin n\pi t \,\mathrm{d}t.$$

Combining the last term with left-hand side and putting in the limits, we get

$$\int_0^1 \sin 2t \sin n\pi t \,\mathrm{d}t = \frac{(-1)^{n+1} n\pi}{[(n\pi)^2 - 4]}\sin 2.$$

It follows that:

$$b_n = 2\left[\frac{(-1)^{n+1}}{n\pi} - \frac{1}{\sin 2}\frac{(-1)^{n+1} n\pi}{[(n\pi)^2-4]}\sin 2\right] = (-1)^{n+1}\frac{8}{n\pi[4-(n\pi)^2]}. \quad (1.34)$$

Therefore the solution that satisfies the boundary conditions can be written as

$$x(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n[4-(n\pi)^2]}\sin n\pi t.$$

Now we shall show that this result can be obtained directly from the following Fourier series method. First we expand the solution, whatever it is, into a half-range Fourier sine series

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t.$$

This is a valid procedure because no matter what the solution is, we can always antisymmetrically extend it to the interval $-1 < t < 0$ and then to the entire real line by the periodicity condition $x(t+2) = x(t)$. The Fourier series representing this odd function with a periodicity of 2 is given by the above expression. This function is continuous everywhere, therefore it can be differentiated term-by-term. Furthermore, the boundary conditions, $x(0) = 0$ and $x(1) = 1$, are automatically satisfied by this series.

When we put this series into the differential equation, the result is

$$\sum_{n=1}^{\infty}\left[-(n\pi)^2 + 4\right] b_n \sin n\pi t = 4t.$$

This equation can be regarded as the function $4t$ expressed in a Fourier sine series. The coefficients $[-(n\pi)^2 + 4]b_n$ are given by

$$[-(n\pi)^2 + 4]\; b_n = 2\int_0^1 4t \sin n\pi t \,\mathrm{d}t = 8\frac{(-1)^{n+1}}{n\pi}.$$

It follows that:

$$b_n = \frac{8(-1)^{n+1}}{n\pi[4-(n\pi)^2]},$$

which is identical to (1.34). Therefore we will get the exactly same result as before.

This shows that the Fourier series method is convenient and direct. Not every boundary value problem can be handled in this way, but many of them can. When the problem is solved by the Fourier series method, often the solution is actually in a more useful form.

Example 1.8.1. A horizontal beam of length L, supported at each end is uniformly loaded. The deflection of the beam $y(x)$ is known to satisfy the equation

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = \frac{w}{EI},$$

where w, E, and I are constants (w is load per unit length, E is the Young's modulus, I is the moment of inertia). Furthermore, $y(t)$ satisfies the following four boundary conditions

$$y(0) = 0, \quad y(L) = 0,$$
$$y''(0) = 0, \quad y''(L) = 0.$$

(This is because there is no deflection and no moment at either end.) Find the deflection curve of the beam $y(x)$.

Solution 1.8.1. The function may be conveniently expanded in a Fourier sine series

$$y(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x.$$

The four boundary conditions are automatically satisfied. This series and its derivatives are continuous, therefore it can be repeatedly term-by-term differentiated. Putting it in the equation, we have

$$\sum_{n=1}^{\infty} b_n \left(\frac{n\pi}{L}\right)^4 \sin\frac{n\pi}{L}x = \frac{w}{EI}.$$

This means that $b_n(n\pi/L)^4$ is the coefficients of the Fourier sine series of w/EI. Therefore

$$b_n \left(\frac{n\pi}{L}\right)^4 = \frac{2}{L}\int_0^L \frac{w}{EI}\sin\frac{n\pi}{L}x\,\mathrm{d}x = -\frac{2}{L}\frac{w}{EI}\frac{L}{n\pi}\left(\cos n\pi - 1\right).$$

It follows that:

$$b_n = \begin{cases} \dfrac{4wL^4}{EI}\dfrac{1}{(n\pi)^5} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}.$$

Therefore

$$y(x) = \frac{4wL^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin\frac{(2n-1)n\pi x}{L}.$$

This series is rapidly convergent due to the fifth power of n in the denominator.

1.8.2 Periodically Driven Oscillator

Consider a damped spring–mass system driven by an external periodic forcing function. The differential equation describing this motion is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t). \tag{1.35}$$

We recall that if the external forcing function $F(t)$ is a sine or cosine function, then the steady state solution of the system is an oscillatory motion with the same frequency of the input function. For example, if

$$F(t) = F_0 \sin \omega t,$$

then

$$x_{\mathrm{p}}(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\sin(\omega t - \alpha), \tag{1.36}$$

where

$$\alpha = \tan^{-1}\frac{c\omega}{k - m\omega^2}.$$

However, if $F(t)$ is periodic with frequency ω, but is not a sine or cosine function, then the steady state solution will contain not only a term with the input frequency ω, but also other terms of multiples of this frequency. Suppose that the input forcing function is given by a square-wave

$$F(t) = \begin{cases} 1 & 0 < t < L \\ -1 & -L < t < 0 \end{cases}, \quad F(t + 2L) = F(t). \tag{1.37}$$

This square-wave repeats itself in the time interval of $2L$. The number of times that it repeats itself in 1 s is called frequency ν. Clearly $\nu = 1/(2L)$. Recall that the angular frequency ω is defined as $2\pi\nu$. Therefore

$$\omega = 2\pi\frac{1}{2L} = \frac{\pi}{L}.$$

Often ω is just referred to as frequency.

Now as we have shown, the Fourier series expansion of $F(t)$ is given by

$$F(t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}t,$$

$$b_n=\begin{cases}\dfrac{4}{n\pi} & n=\text{odd},\\ 0 & n=\text{even}.\end{cases}$$

It is seen that the first term is a pure sine wave with the same frequency as the input square-wave. We called it the fundamental frequency $\omega_1 (\omega_1=\omega)$. The other terms in the Fourier series have frequencies of multiples of the fundamental frequency. They are called harmonics (or overtones). For example, the second and third harmonics have, respectively, frequencies of $\omega_2=2\pi/L=2\omega$ and $\omega_3=3\pi/L=3\omega$. (In this terminology, there is no first harmonic.)

With the input square-wave $F(t)$ expressed in terms of its Fourier series in (1.35), the response of the system is also a superposition of the harmonics, since (1.35) is a linear differential equation. That is, if x_n is the particular solution of

$$m\frac{\mathrm{d}^2 x_n}{\mathrm{d}t}+c\frac{\mathrm{d}x_n}{\mathrm{d}t}+kx_n=b_n\sin\omega_n t,$$

then the solution to (1.35) is

$$x_{\mathrm{p}}=\sum_{n=1}^{\infty} x_n.$$

Thus it follows from (1.36) that with the input forcing function given by the square-wave, the steady state solution of the spring–mass system is given by

$$x_{\mathrm{p}}=\sum_{n=1}^{\infty}\frac{b_n\sin(\omega_n t-\alpha_n)}{\sqrt{(k-m\omega_n^2)^2+(c\omega_n)^2}},$$

where

$$\omega_n=\frac{n\pi}{L}=n\omega,\quad \alpha_n=\tan^{-1}\frac{c\omega_n}{k-m\omega_n^2}.$$

This solution contains not only a term with the same input frequency ω, but also other terms with multiples of this frequency. If one of these higher frequencies is close to the natural frequency of the system ω_0 ($\omega_0=\sqrt{k/m}$), then the particular term containing that frequency may play the dominant role in the system response. This is an important problem in vibration analysis. The input frequency may be considerably lower than the natural frequency of the system, yet if that input is not purely sinusoidal, it could still lead to resonance. This is best illustrated with a specific example.

Example 1.8.2. Suppose that in some consistent set of units, $m = 1$, $c = 0.2$, $k = 9$, and $\omega = 1$, and the input $F(t)$ is given by (1.37). Find the steady state solution $x_p(t)$ of the spring–mass system.

Solution 1.8.2. Since $\omega = \pi/L = 1$, so $L = \pi$ and $\omega_n = n$. As we have shown, the Fourier series of $F(t)$ is

$$F(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right).$$

The steady-state solution is therefore given by

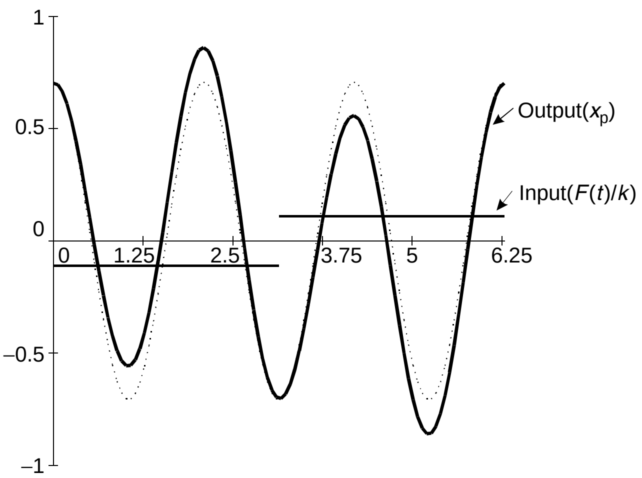
$$x_p(t) = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\frac{\sin(nt-\alpha_n)}{\sqrt{(9-n^2)^2+(0.2n)^2}},$$

$$\alpha_n = \tan^{-1}\frac{0.2n}{9-n^2}, \quad 0 \le \alpha_n \le \pi.$$

Carrying out the calculation, we find

$$x_p(t) = 0.1591\sin(t-0.0250) + 0.7073\sin(3t-1.5708) + 0.0159\sin(5t-3.0792) + \cdots.$$

The following figure shows $x_p(t)$ in comparison with the input force function. In order to have the same dimension of distance, the input force is expressed in terms of the "static distance" $F(t)/k$. The term $0.7073\sin(3t-1.5708)$ is shown as the dotted line. It is seen that this term dominates the response of the system.This is because the term with $n = 3$ in the Fourier series of $F(t)$



has the same frequency as the natural frequency of the system ($\sqrt{k/m} = 3$). Thus near resonance vibrations occur, with the mass completing essentially three oscillations for every single oscillation of the external input force.

An interesting demonstration of this phenomenon on a piano is given in the Feynman Lecture on Physics, Vol. I, Chap. 50.

Let us label the two successive Cs near the middle of the keyboard by C, C', and the Gs just above by G, G'. The fundamentals will have relative frequencies as follows:

$$\begin{array}{ll} C \;-\; 2 & G \;-\; 3 \\ C' \;-\; 4 & G' - \;6 \end{array}$$

These harmonic relationships can be demonstrated in the following way. Suppose we press C' slowly – so that it does not sound but we cause the damper to be lifted. If we sound C, it will produce its own fundamental and some harmonics. The second harmonic will set the strings of C' into vibration. If we now release C (keeping C' pressed) the damper will stop the vibration of the C strings, and we can hear ($softly$) the note of C' as it dies away. In a similar way, the third harmonic of C can cause a vibration of G'.

This phenomenon is as interesting as important. In a mechanical or electrical system that is forced with a periodic function having a frequency smaller than the natural frequency of the system, as long as the forcing function is not purely sinusoidal, one of its overtones may resonate with the system. To avoid the occurrence of abnormally large and destructive resonance vibrations, one must not allow any overtone of the input function to dominate the response of the system.

Exercises

1. Show that if m and n are integers then

(a) $$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\mathrm{d}x = \begin{cases} \dfrac{L}{2} & n = m, \\ 0 & n \neq m. \end{cases}$$

(b) $$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\mathrm{d}x = \begin{cases} \dfrac{L}{2} & n = m, \\ 0 & n \neq m. \end{cases}$$

(c) $$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\mathrm{d}x = 0, \quad \text{all} \quad n, m.$$

(d) $$\int_0^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\mathrm{d}x$$
$$= \begin{cases} 0 & n, m \text{ both even or both odd}, \\ \dfrac{L}{\pi}\dfrac{2n}{n^2 - m^2} & n \text{ even, } m \text{ odd; or } n \text{ odd, } m \text{ even}. \end{cases}$$

2. Find the Fourier series of the following functions:

(a) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 2 & 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x),$

(b) $f(x) = \begin{cases} 1 & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ -1 & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}, \quad f(x+2\pi) = f(x),$

(c) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x).$

Ans. (a) $f(x) = 1 + \dfrac{4}{\pi}\left[\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x - \cdots\right],$

(b) $f(x) = \dfrac{4}{\pi}\left[\cos x - \dfrac{1}{3}\cos 3x + \dfrac{1}{5}\cos 5x - \cdots\right],$

(c) $f(x) = \dfrac{1}{\pi} - \dfrac{2}{\pi}\left[\dfrac{1}{3}\cos 2x + \dfrac{1}{15}\cos 4x + \dfrac{1}{35}\cos 6x - \cdots\right]$
$+\dfrac{1}{2}\sin x.$

3. Find the Fourier series of the following functions:

(a) $f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \end{cases}, \quad f(t+4) = f(t),$

(b) $f(t) = t^2, \quad 0 < t < 2, \qquad f(t+2) = f(t).$

Ans. (a) $f(t) = \dfrac{4}{\pi}\left[\sin\dfrac{\pi t}{2} + \dfrac{1}{3}\sin\dfrac{3\pi t}{2} + \dfrac{1}{5}\sin\dfrac{5\pi t}{2}\cdots\right],$

(b) $f(t) = \dfrac{4}{3} + \dfrac{4}{\pi^2}\sum \dfrac{1}{n^2}\cos n\pi t + \dfrac{4}{\pi}\sum \dfrac{1}{n}\sin n\pi t.$

4. Find the half-range Fourier cosine and sine expansions of the following functions:

(a) $f(t) = 1, \quad 0 < t < 2.$
(b) $f(t) = t, \quad 0 < t < 1.$
(c) $f(t) = t^2, \quad 0 < t < 3.$

Ans. (a) 1; $\frac{4}{\pi}\sum\frac{1}{2n-1}\sin\frac{(2n-1)\pi t}{2}$,

(b) $\frac{1}{2}-\frac{4}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos(2n-1)\pi t$; $\frac{2}{\pi}\sum\frac{(-1)^{n+1}}{n}\sin n\pi t$,

(c) $$3+\frac{36}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos\frac{n\pi t}{3};\ \frac{18}{\pi^3}\left[\left(\frac{\pi^2}{1}-\frac{4}{1^3}\right)\sin\frac{\pi t}{3}\right.$$
$$-\frac{\pi^2}{2}\sin\frac{2\pi t}{3}+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin\frac{3\pi t}{3}$$
$$\left.-\frac{\pi^2}{4}\sin\frac{4\pi t}{3}+\cdots\right].$$

5. The output from an electronic oscillator takes the form of a sine wave $f(t)=\sin t$ for $0<t\le\pi/2$, it then drops to zero and starts again. Find the complex Fourier series of this wave form.
Ans.
$$\sum_{n=-\infty}^{\infty}\frac{2}{\pi}\frac{4n\mathrm{i}-1}{16n^2-1}\mathrm{e}^{\mathrm{i}4nt}.$$

6. Use the method of jumps to find the half-range cosine series of the function $g(t)=\sin t$ defined in the interval of $0<t<\pi$.

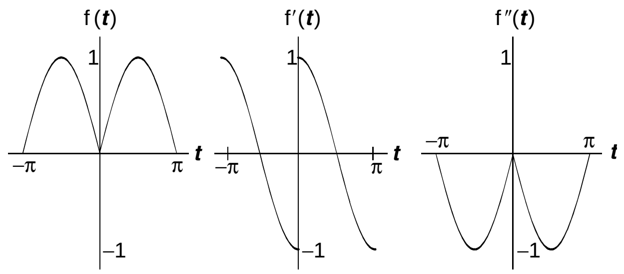
Hint: For a cosine series, we need an even extension of the function. Let

$$f(t)=\begin{cases} g(t)=\sin t & 0<t<\pi,\\ g(-t)=-\sin t & -\pi<t<0.\end{cases}$$

Its derivatives are

$$f'(t)=\begin{cases}\cos t & 0<t<\pi\\ -\cos t & -\pi<t<0\end{cases},\quad f''(t)=-f(t).$$

The sketches of the function and its derivatives are shown as follows:



Ans.

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}\cos 2t + \frac{1}{15}\cos 4t + \frac{1}{35}\cos 6t + \cdots\right).$$

7. Use the method of jumps to find the half range (a) cosine and (b) sine Fourier expansions of $g(t)$, which is defined only over the range $0 < t < 1$ as

$$g(t) = t - t^2, \quad 0 < t < 1.$$

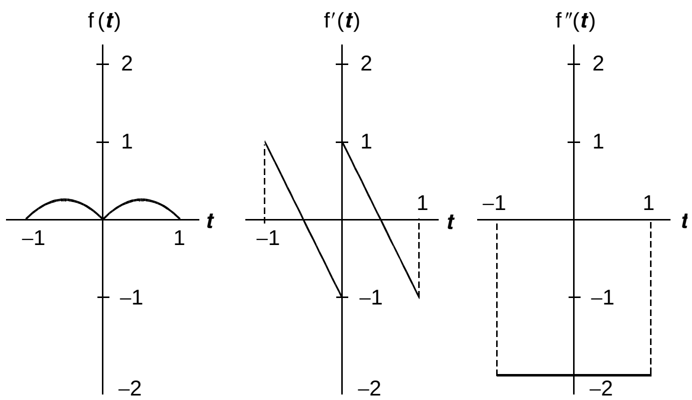
Hint: (a) For the half-range cosine expansion, the function must be symmetrically extended to negative t. That is, we have to expand into a Fourier series the even function $f(t)$ defined as

$$f(t) = \begin{cases} g(t) = t - t^2 & 0 < t < 1, \\ g(-t) = -t - t^2 & -1 < t < 0. \end{cases}$$

The first and second derivatives of this function are given by

$$f'(t) = \begin{cases} 1 - 2t & 0 < t < 1 \\ -1 - 2t & -1 < t < 0 \end{cases}, \quad f''(t) = -2$$

and all higher derivatives are zero. The sketches of this function and its derivatives are as follows:



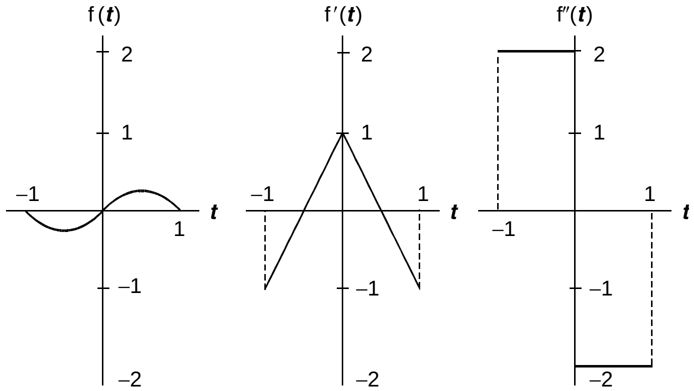
(b) For the half-range sine expansion, an antisymmetric extension of $g(t)$ to negative t is needed. Let

$$f(t) = \begin{cases} g(t) = t - t^2 & 0 < t < 1, \\ -g(-t) = t + t^2 & -1 < t < 0. \end{cases}$$

The first and second derivatives of this function are given by

$$f'(t) = \begin{cases} 1-2t & 0<t<1, \\ 1+2t & -1<t<0, \end{cases} \qquad f''(t) = \begin{cases} -2 & 0<t<1, \\ 2 & -1<t<0 \end{cases}$$

and all higher derivatives are zero. The sketches of these functions are shown below



Ans. (a) $f(t) = \frac{1}{6} - \frac{1}{\pi^2}\left(\cos 2t + \frac{1}{4}\cos 4t + \frac{1}{9}\cos 6t + \cdots\right).$

(b) $f(t) = \frac{8}{\pi^3}\left(\sin \pi t + \frac{1}{27}\sin 3\pi t + \frac{1}{125}\sin 5\pi t + \cdots\right).$

8. Do problem 3 with the method of jumps.

9. (a) Find the half-range cosine expansion of the following function:

   $f(t) = t, \quad 0 < t < 2.$

   (b) Sketch the function (from $t = -8$ to 8) that this Fourier series represents.

   (c) What is the periodicity of this function.

   Ans.

$$f(t) = 1 + \frac{4}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}(\cos n\pi - 1)\cos\frac{n\pi}{2}t; \quad \text{period} = 4.$$

10. (a) Find the half-range cosine expansion of the following function:

$$f(t) = \begin{cases} t & 0 < t \leq 2 \\ 4 - t & 2 \leq t < 4 \end{cases}.$$

(b) Sketch the function (from $t = -8$ to 8) this Fourier series represents.

(c) What is the periodicity of this function.

Ans.

$$f(t) = 1 - \frac{8}{\pi^2} \sum_1^{\infty} \frac{1}{n^2} (1 + \cos n\pi - 2 \cos \frac{n\pi}{2}) \cos \frac{n\pi}{4} t; \quad \text{period} = 8.$$

11. (a) Show that the Fourier series in the two preceding problems are identical to each other.

(b) Compare the two sketches to find out the reason why this is so.
Ans. Since they represent the same function, both Fourier series can be expressed as

$$f(t) = 1 - \frac{8}{\pi^2} (\cos \frac{\pi t}{2} + \frac{1}{9} \cos \frac{3\pi t}{2} + \frac{1}{25} \cos \frac{5\pi t}{2} + \cdots).$$

12. Use the Fourier series for

$$f(t) = t \quad \text{for} \quad -1 < t < 1, \quad \text{and} \quad f(t+2) = f(t)$$

to show that

(a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$,

(b) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$.

13. Use the Fourier series shown in Fig. 1.5 to show that

(a) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$,

(b) $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$.

Hint: (a) Set $t = 0$. (b) Use Parseval's theorem and $\sum 1/n^2 = \pi^2/6$.

14. Use

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

to show that

$$1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots = \frac{7\pi^4}{720}.$$

15. An odd function $f(t)$ of period of 2π is to be approximated by a Fourier series having only N terms. The so called "square deviation" is defined to be

$$\varepsilon = \int_{-\pi}^{\pi} \left[f(t) - \sum_{n=1}^{N} b_n \sin nt \right]^2 \mathrm{d}t.$$

It is a measure of the error of this approximation. Show that for ε to be minimum, b_n must be given by the Fourier coefficient

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \; \mathrm{d}t.$$

Hint: Set $\dfrac{\partial \varepsilon}{\partial b_n} = 0.$

16. Show that for $-\pi \le x \le \pi$

(a) $\cos kx = \dfrac{\sin k\pi}{k\pi} + \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2k \sin k\pi}{\pi(k^2 - n^2)} \cos nx,$

(b) $\cot k\pi = \dfrac{1}{\pi} \left(\dfrac{1}{k} - \displaystyle\sum_{n=1}^{\infty} \frac{2k}{n^2 - k^2} \right).$

17. Find the steady-state solution of

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = f(t),$$

where $f(t) = t, \quad -\pi \le t < \pi$, and $f(t + 2\pi) = f(t)$.

Ans.

$$x_{\mathrm{p}} = \sum \frac{(-1)^n 2(n^2 - 3)}{n(n^4 - 2n^2 + 9)} \sin nt + \sum \frac{(-1)^n 4}{n^4 - 2n^2 + 9} \cos nt.$$

18. Use the Fourier series method to solve the following boundary value problem

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = \frac{Px}{EIL}$$

$$y(0) = 0, \qquad y(L) = 0,$$

$$y''(0) = 0, \qquad y''(L) = 0.$$

($y(x)$ is the deflection of a beam bearing a linearly increasing load given by Px/L)
Ans.

$$y(x) = \frac{2PL^4}{\pi^4 EI} \sum \frac{(-1)^{n+1}}{n^5} \sin \frac{n\pi x}{L}.$$

19. Find the Fourier series for

(a) $f(t) = t$ for $-\pi < t < \pi$, and $f(t + 2\pi) = f(t)$,
(b) $f(t) = |t|$ for $-\pi < t < \pi$, and $f(t + 2\pi) = f(t)$.

Show that the series resulting from a term-by-term differentiation of the series in (a) does not converge to $f'(t)$, whereas the series resulting from a term-by-term differentiation of the series in (b) converges to $f'(t)$. Why?