1

A Basic Problem

Before we start with the subject proper, it is perhaps useful to look at a concrete physical example, which can be easily built in the laboratory. It is a pendulum with a magnet at the end, which oscillates above three symmetrically arranged fixed magnets, which attract the oscillating magnet, as shown in Fig. 1.1. When one holds the magnet slightly eccentrically and let it go, it will dance around the three magnets, and finally settle at one of the three, when friction has slowed it down enough.

The interesting question is whether one can predict where it will land. That this is a difficult issue is visible to anyone who does the experiment, because the pendulum will hover above one of the magnets, "hesitate" and cross over to another one, and this will happen many times until the movement changes to a small oscillation around one of the magnets and ends the uncertainty of where it will go. Let us call the three magnets "red," "yellow," "blue"; one can ask for every initial position from which the magnet is started (with 0 speed) *where* it will eventually land. The result of the numerical simulation, to some resolution, is shown in Fig. 1.2. The incredible richness of this figure gives an inkling of the complexity of this problem, although we only deal with a simple classical pendulum.

Exercise 1.1. *Program the pendulum equation and check the figure. The equations for the potential U are, for $q \in \mathbb{R}^2$,*

$$U(q) = \frac{3}{8}|q|^2 - \sum_{j=0}^{2} V(q - q_j) , \tag{1.1}$$

where $q_j = \big(\cos(2\pi j/3), \sin(2\pi j/3)\big)$ and $V(q) = 1/|q|$. The equations of motion are

$$\dot{q} = p , \quad \dot{p} = -\gamma p - \nabla_q U(q) ,$$

where $\dot{q}$ is a shorthand for $\mathrm{d}q(t)/\mathrm{d}t$. The friction coefficient is $\gamma = 0.13$.

The domains of same color are very complicated, and the surprising thing about them is that their boundaries actually coincide: If x is in the boundary ∂R of the red region, it is also in the boundary of yellow and blue: $\partial R = \partial Y = \partial B$. (This fact has

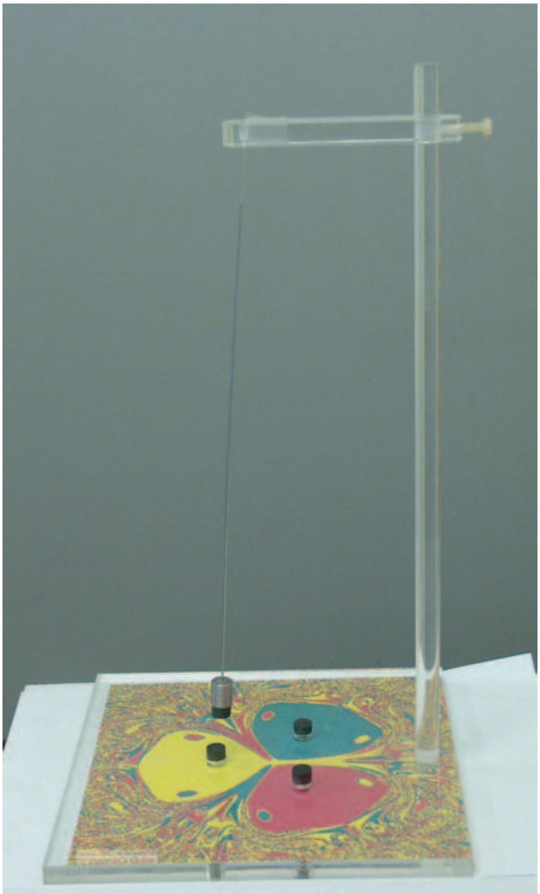

Fig. 1.1. Photograph of the pendulum with three magnets (The design is due to U. Smilansky.)

been proven for the simpler example of Fig. 3.6, and we conjecture the same result for the pendulum.)

The subject of this course is a generic understanding of such phenomena. While this example is not as clear as the one of the "crab" of Fig. 3.6, it displays a feature which will follow us throughout: **instability**. For the case at hand, there is exactly one unstable point in the problem, namely the center, and it is indicated in yellow in Fig. 1.3. Whenever the pendulum comes close to this point, it will have to "decide" on which side it will go: It may creep over the point, and the most minute change in the initial condition might change the final target color the pendulum will reach.

The aim of this book is to give an account of central concepts used to understand, describe, and analyze this kind of phenomena. We will describe the tools with which mathematicians and physicists study chaotic systems.

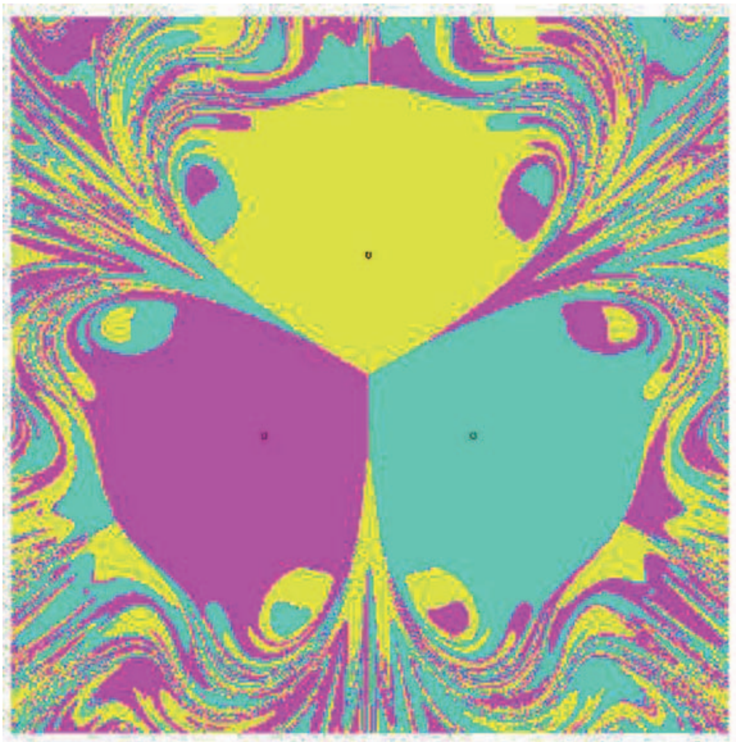

Fig. 1.2. The basins of attraction of the three magnets, color coded. The coordinates are the two components of the initial position: $q = (q_1, q_2)$. The three circles show the positions of the fixed magnets

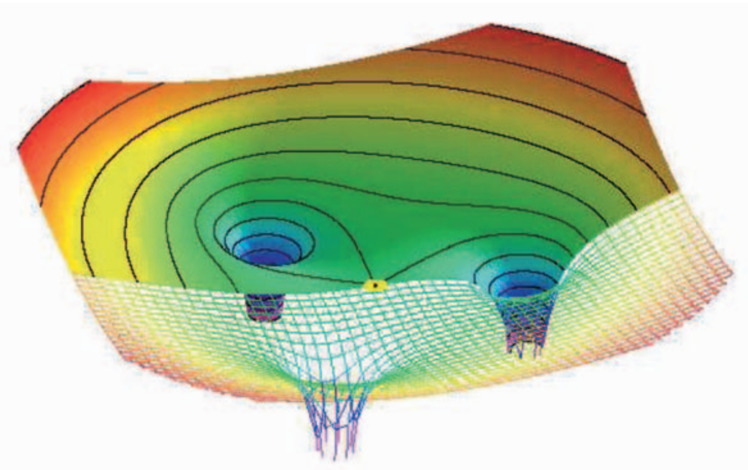

Fig. 1.3. A typical potential for the pendulum of Fig. 1.1. The equation used for the drawing is given in (1.1). The coordinates are the two components of the initial position: $q = (q_1, q_2)$, the height is the value of the potential