2 Numerical Bifurcation Techniques

2.1 Various Types of Bifurcation

Nonlinear phenomena relate to the processes that involve physical variables which are governed by nonlinear equations. The models which are described by these equations have been obtained by some approximate projection rationale from presumably more fundamental microscopic dynamics of the system. In some cases a reasonable projection may yield simple linear equations in some approximations.

To demonstrate the basic concepts of nonlinear dynamical systems, we consider a pair of first order coupled ordinary autonomous differential equations. The bases of the classification of these equations are well known and have received much attention in many text books on ordinary differential equations [1, 2].

$$\frac{dx_1}{dt} = f_1(x_1, x_2) \tag{2.1}$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2) \tag{2.2}$$

The equilibrium points are given by $f_1 = 0$ and $f_2 = 0$. Perturb the equilibrium point by Δx_1 and Δx_2, expand the resulting equations in the Taylor Series, and linearize the equations near this equilibrium point. The solutions of Δx_1 and Δx_2 are then given by

$$\Delta x_1 = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{2.3}$$

$$\Delta x_2 = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{2.4}$$

The constants C_1, C_2, C_3, C_4 are determined by the initial conditions. The exponents λ_1 and λ_2 are the eigenvalues of the Jacobian matrix

$$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and can be obtained by solving $|J - \lambda I| = 0$ (where a, b, c, d are the partial derivatives of f_1 and f_2 evaluated w.r.t. x_1 and x_2 at the equilibrium point).

$$\lambda_{1,2} = \frac{1}{2}[Tr(J) + \sqrt{\Delta}]$$

$$Tr(J) = a + d;\ \Delta = \text{discriminant} = Tr(J)^2 - 4\det(J)$$

There are a number of possibilities for the sign and character of λ_1 and λ_2, depending on the signs and relative magnitudes of $Tr(J)$ and $det(J)$. Different possible cases are briefly described below:

Case (i): $Tr(J) < 0, \det(J) > 0, \Delta > 0$: for these conditions λ_1 and λ_2 are both real and negative. The stationary state is stable and the perturbations decay. It belongs to stable node.

Case (ii): $Tr(J) > 0, \det(J) > 0, \Delta > 0$: λ_1 and λ_2 are both real and positive. The exponential terms in Eqs.2.3 and 2.4 increase monotonically with time. The perturbations grow exponentially. It belongs to unstable node.

Case (iii): $Tr(J) < 0, \det(J) > 0, \Delta < 0$: λ_1 and λ_2 are complex and the real part of λ_1 and λ_2 is negative. For this case the perturbations are given by

$$\Delta x = c_1 e^{\mathrm{Re}(\lambda t)} \cos(\mathrm{Im}(\lambda t) + \theta_1) \tag{2.3a}$$

$$\Delta y = c_2 e^{\mathrm{Re}(\lambda t)} \cos(\mathrm{Im}(\lambda t) + \theta_2) \tag{2.4a}$$

The decaying terms ensure a return to the original stationary state because of the cosine functions. This is a damped oscillatory motion. It belongs to stable focus.

Case (iv): $Tr(J) > 0, \det(J) > 0, \Delta < 0$: here λ_1 and λ_2 are complex and the real part of λ_1 and λ_2 is positive. The perturbations grow in a divergent oscillatory manner. It is an unstable focus.

Case (v): $Tr(J) > or < 0$, $\det(J) < 0$, $\Delta > 0$: λ_1 and λ_2 are real. $\lambda_1 = +ve$ and $\lambda_2 = -ve$. One of the exponential term in each of Δx_1 and Δx_2 decrease exponentially. The other with the positive root will increase with time. The growing term will eventually dominate and the system will move away from the stationary state. It leads to saddle point behavior.

SPECIAL CASES

Case (vi): $\det(J) = 0$ **:** here λ_1 and λ_2 are both real.

$$\text{For Tr}(J) > 0 \quad \begin{matrix} \lambda_1 > 0 \\ \lambda_2 = 0 \end{matrix}$$

$$\text{For Tr}(J) < 0 \quad \begin{matrix} \lambda_1 = 0 \\ \lambda_2 < 0 \end{matrix}$$

This leads to saddle node bifurcation or fold. To capture the true system behavior, we have to consider nonlinear terms.

Case (vii): $Tr(J) = 0, \det(J) > 0, \Delta < 0$: here $\lambda 1$ and $\lambda 2$ are both complex and the real part of these eigenvalues is zero. For this case also, to capture true system behavior, we have to consider the nonlinear terms. This may lead to Hopf bifurcation.

Except for three critical cases: (vi) $det(j) = 0$; (vii) $Tr(J) = 0$; $det(J) > 0$; and a special case where both $det(J) = 0$; $Tr(J) = 0$; the integral curves of the nonlinear system have the same behavior as those of linearized systems in the neighborhood of the equilibrium. These results are summarized with the values of the trace and determinant of the corresponding Jacobian matrix as shown in the phase diagram (Fig.2.1). For linear systems in R^3 [3] make sound classification and arrangement of phase portraits.

However, in the three critical cases mentioned before, the structure of orbits in the state space will change qualitatively. Such a qualitative change in called a bifurcation. This bifurcation may be due to variation of certain parameters in the system. The critical value of the parameter where the bifurcation occurs is the bifurcation value of the parameter.

The chapter is organized as follows: Section 2.2 describes the general principles involved in the study of bifurcation behavior of an n dimensional dynamical system. Sections 2.3, 2.4 and 2.5 discuss the continuation based numerical techniques that can be effectively used to identify various bifurcation points.

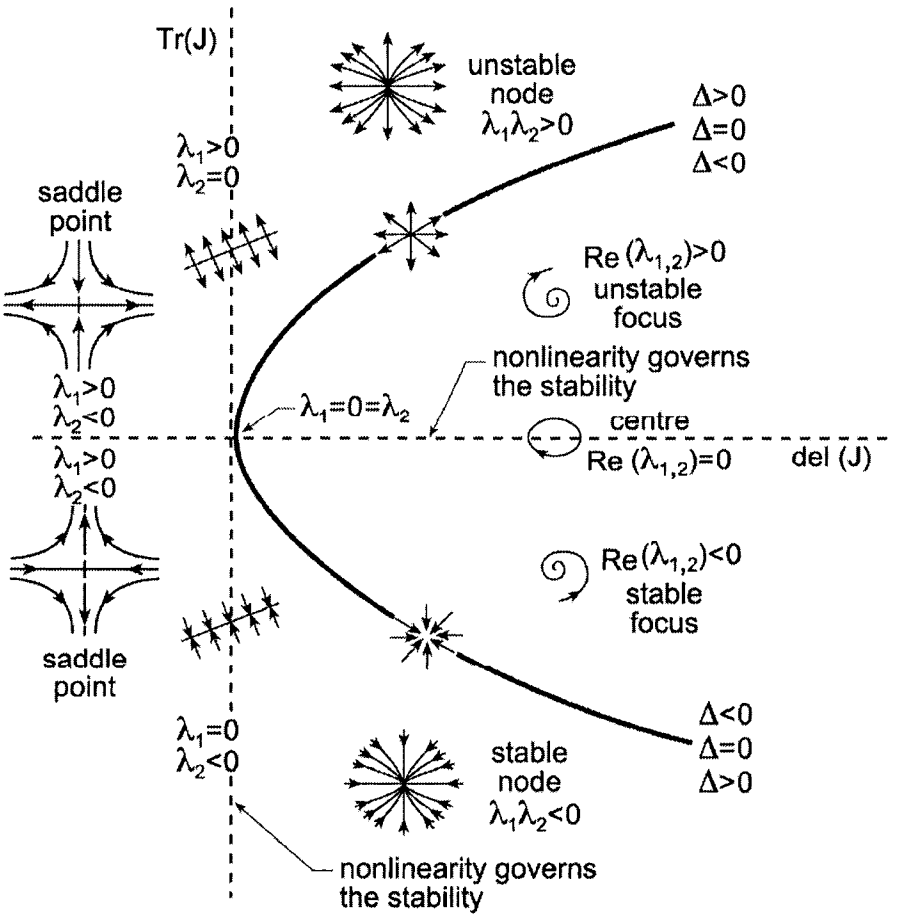


Fig.2.1 Phase diagram [4]

2.2 Bifurcation of Dynamical Systems

Consider a dynamical model of a system [5] described by autonomous differential equations of the vector form in n-dimensional space

$$\dot{x} = F(x, \lambda), x \in R^n, \lambda \in R^k \tag{2.5}$$

Here x denotes the state variables. For power system models these are: generator angles, generator angular velocities, load voltage magnitudes, or angles etc. λ is a vector of time invariant scalar parameters. At an equilibrium point (x_0, λ_0), the left hand term $\dot{x}$ of equation becomes zero, i.e., the steady state solution of Eq.2.5 satisfies the set of nonlinear algebraic equations $F(x_0, \lambda_0) = 0$. If the eigenvalues of the Jacobian $\partial F/\partial x$ become non-zero, then according to implicit function theorem the equilibria of Eq.2.5 can be expressed as the smooth function of $x = x(\lambda)$. The function $x(\lambda)$ is called the branch of equilibria. However if the Jacobian has an eigenvalue with zero real part occurring at some λ, say λ_c, the system $\dot{x} = F(x_c, \lambda_c)$ is structurally unstable and several branches of $x = x(\lambda)$ can come together at (x_c, λ_c) in R^{n+k}. The parameter set λ_c where the system loses its stability is called a bifurcation set. The point (x_c, λ_c) is called bifurcation point. (In general, in engineering systems a one-parameter family with k-1 relations between the parameters μ_1, μ_2, μ_3,. . . . can be represented as a curve, λ, in the k-dimensional parameter space.) Thus the principle of linear stability differentiates between two categories of equilibrium solutions. For the hyperbolic fixed points (where the eigenvalues have non-zero real parts), linear stability analysis suffices completely. For non-hyperbolic fixed points (the points where at least one eigenvalue has zero real part), a linear stability analysis is not applicable and a full nonlinear analysis has to be carried out. There are techniques available to simplify, without any significant loss of information, the representation of the flow in the nonlinear dynamical systems in the neighborhood of non-hyperbolic points. One of these techniques is the center manifold theory. This theory closes the gap left by Hartman-Grobman theorem (HGT). According to HGT, if the Jacobian $\partial F/\partial x$ has no eigenvalues with zero real part, then the family of trajectories near an equilibrium point (x_0, λ_0) of a nonlinear system $\dot{x} = F(x, \lambda)$, and those of the locally linearized system have the same topological structure, which means that in the neighborhood of (x_0, λ_0) there exist homeomorphic mappings which map trajectories of the nonlinear system into trajectories of the linear system. Should, however, an eigenvalue with a zero real part exist, the open question arises how this effects the flow in the neighborhood of the equi-

librium point. It is this gap left open by HGT that is closed by the center manifold theory.

2.2.1 Center manifold [6]

Let (x_0, λ_0) be the equilibrium point of $F(x, \lambda)$, and E^s, E^u and E^c the corresponding generalized eigenspaces of the Jacobian matrix $\partial F/\partial x \mid x_0$, where the real part of the eigenvalues (μ) defines the eigenspaces,

$$\mathrm{Re}(\mu) = \begin{cases} < 0 - E^s \\ = 0 - E^c \\ > 0 - E^u \end{cases}$$

Then there exist stable W^s, unstable W^u and center manifold W^c, which are tangential to E^s, E^u, E^c respectively at (x_0, λ_0). If one is interested in the long term behavior (i.e., $t \Rightarrow \infty$) the overall dynamics in the neighborhood of an equilibrium point are reproduced by the flow on the center manifold W^c. This reduction of the dynamics to those in the W^c subspace is the subject of center manifold theory. In order to calculate the flow of the reduced dynamics on W^c, the nonlinear vector field can be transformed to the following form. We can assume that unstable manifold W^u is empty. This makes the presentation simple, without loss of generality.

$$\dot{x}_c = A_c x_c + f(x_c, x_s); X_c \in R^{nc} \tag{2.6}$$

$$\dot{x}_s = A_s x_c + g(x_c, x_s); X_s \in R^{ns} \tag{2.7}$$

The matrix $A_c(n_c, n_c)$ contains n_c, eigenvalues with zero real parts. A_s, matrix (n_s, n_s) contains n_s eigenvalues with negative real parts. The nonlinear functions f and g should be continuously differentiable at least twice and vanish together with their first derivatives at the equilibrium point. X_c correspond to center manifold and are sometimes called active variables. X_s correspond to stable manifold and are called passive variables. Due to nonlinear couplings the influence of x_s in the equation for x_c cannot be ignored. Hence the correct way of analysis is to compute the center manifold.

$$x_s = h(x_c) \tag{2.8}$$

by expressing the dependence of x_s on x_c from Eq.2.8 and then to eliminate from Eq.2.6 to obtain the bifurcation equation

$$\dot{x}_c = A_c x_c + f(x_c, h(x_c)) \tag{2.9}$$

Then the equivalence theorem [5] states that for $t \Rightarrow \infty$, the dynamics of Eq.2.9 in the neighborhood of the equilibrium point is equivalent to the dynamics of the initial system $\dot{x} = F(x, \lambda)$ with λ fixed at the value λ. In order to solve Eq.2.9, one has to know the function $h(x_c)$. This can be obtained as follows

$$\frac{dx_s}{dt} = \frac{dh(x_c)}{dt} = \frac{\partial h}{\partial x_c}\frac{dx}{dt} \tag{2.10}$$

from Eqs.2.6 and 2.7, Eq.2.10 can be written as

$$A_s h(x_c) + g(x_c, h(x_c)) = \left(\frac{\partial h}{\partial x_c}\right)\left[A_c x_c + f(x_c, h(x_c))\right] \tag{2.11}$$

$$\text{or} \quad \left(\frac{\partial h}{\partial x_c}\right)\left[A_c x_c + f(x_c, h(x_c)\right] - A_s x_c - g(x_c, h_c)) = 0$$

The functions h and $(\partial h/\partial x_c)$ are zero at the equilibrium point. Eq.2.11 is in general a partial differential equation which cannot be solved exactly in most cases. But its solution can sometimes be approximated by a series expansion near the equilibrium point. The aforementioned reduction technique of the center manifold theory is similar to its physical counterpart in the slaving principle associated with the synergetic approach proposed by the physicist Herman Haken in the early seventies [7].

In summary, if x is non-hyperbolic then there exist invariant center manifolds tangential to the center subspace and its dimension is equal to the number of eigenvalues of the Jacobian matrix having zero real parts. Then the practically interesting local stability behavior is completely governed by the flow on the center manifold.

Effect of small perturbations of the critical parameters around the bifurcation point can also be studied by unfolding the center manifold. This can

be achieved via the method of normal forms [8, 9]. Normal forms play an essential role in bifurcation theory because they provide the simplest system of equations that describe the dynamics of the original system close to the bifurcation points. Even away from the bifurcation point Poincare's theory of normal forms reduces the initial nonlinear equations into the simplest possible forms without distorting the dynamic behavior in the neighborhood of fixed points or periodic solutions. The transformations, which yield to a reduction to normal forms, can be generated by developing the deviations from a state of equilibrium or from periodic motion into power series. Symbolic manipulation packages like MACSYMA, and MAPLE, are helpful in the development of normal forms. Application of normal form away form the bifurcation points to power system examples is given by [10, 11] and examples of the application of center-manifold theory to power systems are given by [12, 13, 14].

The number of possible types of bifurcation increases rapidly with increasing dimension of the parameter space. The bifurcations are organized hierarchically with increasing co-dimension, where co-dimension is the lowest dimension of a parameter space which is necessary to observe a given bifurcation phenomenon. In this book we discuss only the dynamical system with a single parameter variation. Changing this parameter may drive the system into a critical state at which (i) a real eigenvalue becomes zero or (ii) a pair of complex conjugate eigenvalues becomes imaginary. In case (i) new branches of stationary solutions usually arise and are called static bifurcations. (Typical static bifurcations are (i) saddle node or fold, (ii) trans-critical, and (iii) pitchfork.) Case (ii) may lead to the birth of a branch of periodic solutions called dynamic bifurcations. Typical dynamical bifurcation is Hopf.

In many practical engineering problems, identification of these bifurcations is important. For example, buckling load of elastic structures [15] and voltage collapse in power systems [12, 13, 16, and 17] is related to saddle-node bifurcations. Hopf bifurcation and bifurcation of periodic solutions are observed in chemical engineering [18], mechanical engineering [19, 20] and electrical engineering [21, 22, and 50] to name a few. The next section concentrates on the numerical identification of these bifurcations.

2.3 Detection of Bifurcation Points

2.3.1 Static bifurcations

The problem of determining the roots of nonlinear equations is of frequent occurrence in scientific work. Such equations arise typically in connection with equilibrium problems. When describing a real life problem, the nonlinear equations usually involve one or more parameters. Denoting one such parameter by λ, the nonlinear equations read:

$$F(x,\lambda)=0 \tag{2.12}$$

where F: $R^n \times R \rightarrow R^n$ is a mapping which is assumed smooth. In Eq.2.12, λ=0 usually corresponds to the base case solution. If a priori knowledge concerning zero points of F is available, it is advisable to calculate x via a Newton type algorithm defined by an iteration formula such as:

$$x_{i+1} = x_i - A_i^{-1} F(x_i,0) \qquad i = 0, 1, \ldots n \tag{2.13}$$

where A_i is some reasonable approximation of the Jacobian $F_x(x,0)$. However, if an adequate starting value for a Newton type iteration method is not available, we must seek other remedies. In Section 2.3.2, we will introduce how the lack of knowledge for an initial guess can be tackled by the homotopy method.

Because the systems $F(x, \lambda)$ =0 depends on λ, we speak of a family of nonlinear equations. Solutions now depend on the parameter λ, i.e., $x(\lambda)$. Upon varying the parameter λ, we will get a series of solutions. This is often called a solution curve. At each point corresponding to a certain λ_k, if we keep solving $F(x, \lambda)$ =0 via the conventional Newton type iteration, i.e. by formula (2.13), we may run into difficulty due to the singularity of the Jacobian $F_x(x, \lambda_k)$. The singularity occurs at a so-called turning point (or it is also identified with fold and saddle node) and when the equation is parameterized with respect to λ. In the subsequent sections, we will discuss the interesting topic of curve tracing via the continuations method. We will show how the problem of singularity of the Jacobian can be solved, namely, by switching the continuation parameter.

2.3.2 Homotopy Method

We center our discussion on obtaining a solution to a system of n nonlinear equations in n variables described by Eq.2.12 when λ is at a fixed value. Homotopy method (also some times called embedding method) first defines an easy problem for which a solution is known. Then it defines a path between the easy problem and the problem we actually want to solve. The easy problem, with which the homotopy method starts, is gradually transformed to the solution of the hard problem. Mathematically, this means that one has to define a homotopy or deformation: $R^n \times R \rightarrow R^n$ such that

$$H(x,0) = g(x), H(x,1) = F(x) \tag{2.14}$$

where g is a trivial smooth map having known zero points and H is also smooth. Typically one may choose a convex homotopy such as

$$H(x,t) = (1-t)g(x) + tF(x) \tag{2.15}$$

The problem $H(x,t)=0$ is then solved for values of t between 0 and 1. This is equivalent to tracing an implicitly defined curve $c(s) \in H^{-1}(0)$ (i.e. $H(c(s))=0$) for a starting point $(x_0,0)$ to a solution point $(x_n,1)$. Under certain conditions, $c(s)$ can be defined as (see Fig2.2):

$$x'(t) = -(H_x(t,x(t)))^{-1} H_t(t,x(t)) \tag{2.16}$$

If this succeeds, then a zero point of F is obtained, i.e. $H(x,1) = F(x)$. However, the reader may suspect that this is an unnatural approach, since Eq.2.16 seems to be a more complicated problem than to solve $H(c(s))=0$ as a stabilizer. This is the general idea in the continuation methods with a predictor and corrector tracing scheme.

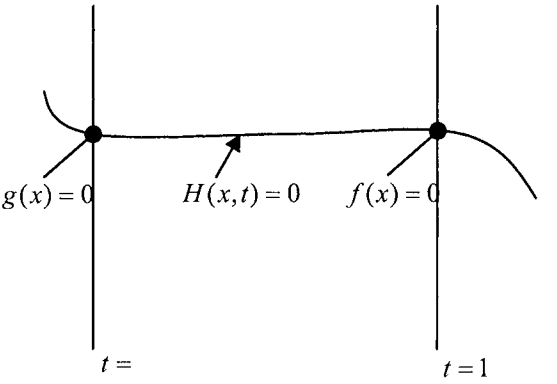


Fig.2.2 Homotopy solution

The relationship Eq.2.15, which embeds the original problem in a family of problems, gives an example of a homotopy that connects the two functions F and g. In general a homotopy can be any continuous connection between F and g. If such a map H exists, we say that F is homotropic to g. A simple two-dimensional nonlinear problem is given here to illustrate how the homotopy method works. The details of this method are given in [23].

Numerical example 1 :[24]

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1^2 - 3x_2^2 + 3 \\ x_1 x_2 + 6 \end{bmatrix}$$

$$[X]^T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Define the homotopy function as:

$$\begin{aligned} H(x,t) &= tF(x) + (1-t)g(x) \\ &= tF(x) + (1-t)F(x) - F(x_0) \\ &= F(x) + (t-1)F(x) \end{aligned}$$

Then we get a curve (from Eq.2.5) defined by:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = -\frac{1}{\Delta}\begin{bmatrix} x_1 & 6x_2 \\ -x_2 & 2x_1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = -\frac{1}{\Delta}\begin{bmatrix} x_1 & 42x_2 \\ -x_2 & 14x_1 \end{bmatrix}$$

where $\Delta=2x_1^2+6x_2^2$, with x_0= (1,1). After tracing the implicitly defined curve via some continuation method, we arrive at a solution when t = 1: x^*= (-2.961, 1.978). A real root of F is (-3, 2). Reasonably we can expect that Newton's method would work well with x^* as the initial guess. After one step of Newton-Raphson iteration, we get x_1= (-3.0003, 2.0003).

However, if we start the Newton's methods directly with the initial guess x_0= (1, 1), it takes more than 5 iterations to get the answer x_1. For a more complicated practical nonlinear problem, the conventional Newton's method might not work at all due to the poor selection of the initial values.

Whether or not the tracing of a curve can succeed depends on the continuation strategy employed. If the curve can be parameterized with respect to the parameter t, then the classical embedding algorithm [23] can be applied. In the following sections, we will discuss how a parameterization is done and how vital this procedure is in the continuation, or say the curve tracing process. Particularly, we will show how the continuation is carried on even when the curve is not parameterizeable with respect to a certain parameter.

2.3.3 Continuation methods

General description of different aspects of continuation methods with minimum mathematical details in curve tracing is given below. For detailed explanation and mathematical proofs of these methods, please refer to the mathematical references provided in this section. Brief but more pertinent exploration of applying the methodology to power system studies is given in Chapter 3. The system of nonlinear equations in the form of equation Eq.2.12 serves as a basis for discussion. One note to make here is that, for the tracing of a curve defined by Eq.2.15, the discussion is the same as for the curve defined by Eq.2.12. Here, x denotes an n-dimensional vector.

Continuation methods usually consist of the following [25]: predictor, parameterization strategy, corrector and step length control. Assume that at least one solution of equation Eq.2.12 has been calculated, for instance, by

the homotopy method. For the tracing of a curve defined by Eq.2.15, this corresponds to the assumption that g has a known zero point. The j^{th} continuation step starts from a solution (x_{j+1}, λ_j) of Eq.2.12 and attempts to calculate the next solution (x_{j+1}, λ_{j+1}), for the next λ, namely λ_{j+1}. With a predictor-corrector method, the step j to step j+1 is split into two parts, with (x_{j+1}, λ_j) produced in between by the prediction. In general, the predictor merely provides an initial guess for the corrector iterations that home in a solution of equations Eq.2.12. The distance between two consecutive solutions is called the step size. In addition to equation Eq.2.12, a relation that identifies the location of a solution on the branch is needed. This identification is closely related to the kind of parameterization strategy chosen to trace the curve.

In the curve tracing process, at some critical points (e.g. turning or fold points), the singularity of the Jacobian matrix F_x often causes trouble either in the prediction or in the correction process. This means that the current continuation parameter has become ill-suited for parameterizing the curve. One way of overcoming this difficulty at turning points is to parameterize the curve by arc length. The augmented Jacobian can be nonsingular throughout the tracing process. However, in practical power system analysis, we always want to get as much useful information as possible during the continuation process. The arc length usually has a geometrical rather than physical meaning, therefore we are often more interested in another important ODE-based predictor, i.e., the tangent parameterized at each step. This is deferred to Section 2.3.4.

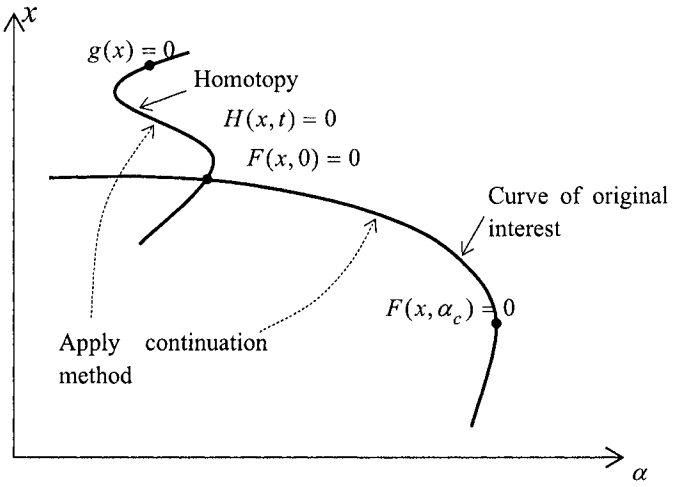


Fig.2.3 Homotopy vs. continuation

Fig.2.3 provides a conceptual view point of homotopy in combination with continuation. Homotopy can be used to get an initial point on the curve of original interest. Continuation method can use this solution to further trace the curve of original interest. As mentioned before homotopy method uses artificial parameter (t in Fig.2.3) to get a solution on the curve of original interest. Continuation method in general uses a natural or physical parameter for the continuation. The efficiency in curve tracing is closely related to the step length control strategy. It is not difficult to choose a workable step size in practice, though some trial and error is often required before the appropriate step size can be found. Step control can often be based on the estimated of the convergence quality of the corrector iteration. In [25] selects step size according to the number of corrector iterations. In general, the step length control scheme is problem dependent.

In practical situations, such as in power systems, saddle node bifurcation, to which out attention will mainly be given, are generic with the collapse type voltage problems. However, in some other situations, other bifurcations might occur more frequently and thus will be of greater interest. For instance, the type of bifurcation that connects equilibria with periodic motion, i.e., Hopf bifurcation, is also generic. Readers interested in problems, such as how to locate Hopf bifurcation point on the traced branch and the related topics are referred to reference [25].

2.3.4 Curve Tracing

Davidenko in his seminal paper [26] proposed that the solving equation Eq.2.12 is equivalent to solving the following differential equation.

$$F_x(x,\lambda)\,dx/d\lambda = -F_\lambda(x,\lambda) \tag{2.17a}$$

With the initial conditions $x(\lambda) = x_0$, where $F(x_0\ ,\ \lambda_0) = 0$. One can generate a sequence of solutions for changing λ by numerically solving the differential Eq.2.17b with an appropriate initial value.

$$\frac{dx}{d\lambda} = -[F_x]^{-1} F_\lambda \tag{2.17b}$$

However, here, the singularity of F_x creates numerical problems. Continuation methods can well alleviate this problem.

The continuation algorithm starts from a known solution and uses a predictor–corrector scheme to find subsequent solutions at different λ values.

The Eq.2.17a can be rearranged in the following form

$$F_x(x,\lambda)dx + F_\lambda(x,\lambda)d\lambda = 0 \tag{2.17c}$$

$$\begin{bmatrix} F_x(x,\lambda) & F_\lambda(x,\lambda) \end{bmatrix} \begin{bmatrix} dx \\ d\lambda \end{bmatrix} = 0$$

Let $T = [\ dx \quad d\lambda\ ]^T$, where T is a $n+1$ dimensional vector with $T_{n+1} = \lambda$. T is tangent to the solution branch of Eq.2.17c. Eq.2.17c consists of n equations and $(n+1)$ unknowns. To get a unique solution a normalization of T is needed. For this one can fix one of the elements of T at particular value. For example one can use $e_i^t\ T = T_i = 1.0$, where e_k is $(n+1)$ dimensional unit vector with k^{th} element equals to unity [25]
The tangent T is the solution of the linear system:

$$\begin{bmatrix} F_x F_\lambda \\ e_k^{tr} \end{bmatrix} z = e_{n+1} \tag{2.18}$$

Provided the full rank condition rank $(F_x, F_\lambda) = n$ holds along the whole branch, the above equation has a unique solution at any point on the branch (k may have to be changed to select a different continuation parameter at a particular step, especially at or near the turning point). Once the tangent vector has been found, the prediction can be easily made. If we define $Y = (x, \lambda)$, then:

$$(\overline{Y}_{j+1}) = (\overline{Y}_j) + \sigma_j T \tag{2.19a}$$

where σ_j designates the step size.

Parameterization and the corrector: Now that a prediction has been made, a method of correcting the approximate solution is needed. Actually the best way to present this corrector is to expand on parameterization, which is vital to the process. Various parameterization techniques are proposed in the mathematical literature. Local parameterization proposed by [27, 28] looks promising and is described here. In local parameterization, the local original set of equations is augmented by one equation that specifies the value of one of the state variables or λ. In local parameterization one can fix $\overline{Y_k} = \eta$ $(1 < k < n+1)$. Then we have to solve the following set of equations:

$$\begin{aligned} F(Y) &= 0 \\ \overline{Y}_k - \eta &= 0 \end{aligned} \tag{2.19b}$$

Selection of the continuation parameter corresponds to the variable that has the largest tangent vector component. Therefore $\overline{Y_k}$ at a particular step is the maximum of $(|T_1|, |T_2|, \cdots |T_{n+1}|)$. Now, once a suitable index k and value of η are chosen, a slightly modified Newton-Raphson iterative process can be used to solve the above set of equations. The general form of the iterative corrector process at the j^{th} step is:

$$\begin{bmatrix} F_Y(Y^j) \\ e_k \end{bmatrix} [\Delta Y^j] = \begin{bmatrix} F(Y^j) \\ 0 \end{bmatrix} \tag{2.20}$$

The corrector Jacobian can be seen to have the same form as the predicted Jacobian. Actually the index k used in the corrector is the same as that of used in the predictor and η will be equal to $\overline{Y_k}$, the predicted value Y_k. In the predictor it is made to have a non zero differential change ($d\overline{Y_k} = T_k = \pm 1$) and in the corrector its value is specified so that the values of other variables can be found.

The step length in Eq.2.19a can be determined by various approaches. The simplest one is by keeping the step length constant. However if we choose very small step length the number of steps needed may be vary large. On the other hand large step lengths may lead to convergence problems. [25] proposed a simple approach for step length selection. Based on this approach the new step length is given by:

$$(\sigma_j)_{new} = (\sigma_j)_{old} N_{opt} / N_j$$

where N_{opt} = optimal number of corrector iterations (this number is 6 for an error tolerance of 10^{-4}) and N_j = Number of iterations needed to approximate the previous continuation step.

With this the $\overline{Y_k}$ value η in Eq.2.19 can be calculated as:

$$\eta = \overline{Y}_k^j + (\sigma_j)_{new}$$

For most of the cases λ is the ideal parameter to choose for tracing. However this parameter creates problems near the fold points. Near the fold point the tangent is normal to the parameter axis. However with local parameterization, near the fold point one can choose the parameter other than λ to avoid these singularity problems. The identification of critical point can be realized by observing the sign change of $d\lambda$.

A one dimensional nonlinear problem is used here to show the basic steps involved in continuation (see Fig.2.4):

Numerical example 2: Consider the following simple example with a single unknown x

$$F(x,\lambda) = x^2 - 3x + \lambda = 0 \qquad (2.21a)$$

The Jacobian is

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial \lambda} \end{bmatrix} = [(2x-3) \quad 1]$$

Let the base solution (x_0, λ_0) be (3, 0). Then the series of solutions (x_1, λ_1), (x_2, λ_2), can be found using predictor-corrector continuation as below:

Continuation step 1:

Predictor

To start with, let λ be the continuation parameter. Calculate the tangent vector as below: (here the index k is equal to 2).

$$\begin{bmatrix} (2x_0 - 3) & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dx \\ d\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dx \\ d\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow dx = -\frac{1}{3} \text{ and } d\lambda = 1$$

Predict the next solution by solving:

$$\begin{bmatrix} \bar{x}_1 \\ \bar{\lambda}_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} + \sigma \begin{bmatrix} dx \\ d\lambda \end{bmatrix}$$

where σ is a scalar designating step size (say 0.5). Thus the predicted solution $(\bar{x}_1, \bar{\lambda}_1)$ becomes (2.8333, 0.5).

Continuation step 2:

Corrector

Correct the predicted solution by solving:

$$-\begin{bmatrix} (2\bar{x}_1 - 3) & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} f(\bar{x}_1, \bar{\lambda}_1) \\ 0 \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} 2.6666 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} 0.0277768 \\ 0 \end{bmatrix}$$

$$\Rightarrow \Delta x = -0.0104165 \text{ and } \Delta \lambda = 0$$

Repeat these correction iterations until reasonable accuracy is obtained (say ε=0.0001). Now, $\max\{\Delta x,\Delta\lambda\}>\varepsilon$. So update x_1 and λ_1 and repeat the corrector iteration.

$$\Rightarrow \bar{x}_{1,new} = \bar{x}_1 + \Delta x = 2.8229164$$

$$\bar{\lambda}_{1,new} = \bar{\lambda}_1 + \Delta\lambda = 0.5$$

Continuation step 3:
Corrector iteration:

$$-\begin{bmatrix} (2\bar{x}_{1,new} - 3) & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} F(\bar{x}_{1,new}, \bar{\lambda}_{1,new}) \\ 0 \end{bmatrix}$$

$$\Rightarrow \Delta x = -0.00004075 \text{ and } \Delta\lambda = 0$$

Now, $\max\{\Delta x,\Delta\lambda\}<\varepsilon$. So stop the corrector iterations. After the first continuation step, the point (x_1, λ_1) is equal to (2.8228757, 0.5). Repeat the entire process until we reach the critical point. For this example, the critical point it (1.5, 2.25).

λ versus x curve for the example is shown in the Fig.2.4. For the predicted solutions at points (1) and (2), we can choose λ as the continuation parameter (i.e., fix λ at that particular value) and converge on to the curve with corrector iterations. But at (3), λ can not be a continuation parameter, as there is no solution for that value of λ. At this point (i.e. when we are close to the critical point), we use local parameterization technique and choose x as the continuation parameter and solve for the system. This can be clearly observed in the example we considered. We know the solution at the fold point as (1.5, 2.25). Consider the augmented Jacobian of the continuation process

$$J_{aug} = \begin{bmatrix} (2x - 3) & 1 \\ 0 & 1 \end{bmatrix}$$

At x = 1.5, $det\{J_{aug}\}$ =0. So the method diverges near the critical point, if λ is the continuation parameter. If we fix the value of x, instead of λ, then

$$J_{aug} = \begin{bmatrix} (2x - 3) & 1 \\ 1 & 0 \end{bmatrix}$$

$det\{J_{aug}\} \neq 0$. Then we can solve for the system.

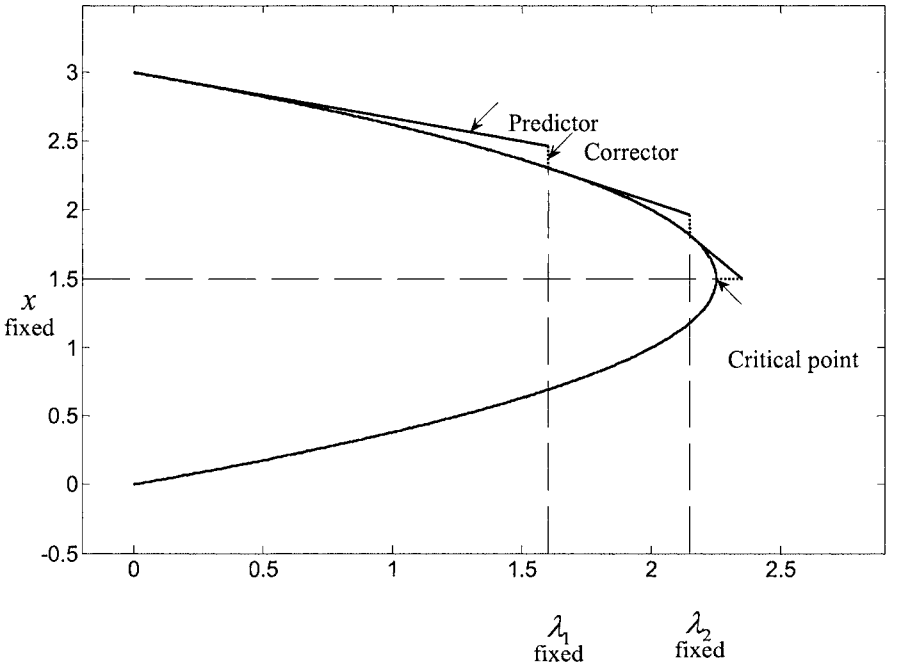


Fig.2.4 Illustration of predictor-corrector scheme

2.3.5 Direct method in computing the Saddle node bifurcation point: a one step continuation

In Section 2.3.4, discussion has been given to show that the tracing of a curve can be done via continuation. We've noticed that, on the traced curve, a particular point, namely, the critical point, or sometimes called the fold point (also related to saddle node bifurcation), is often of greater interest. If we are only interested in locating this point with respect to λ_c, or say, we are interested in the maxim allowable variation of λ where the corresponding linearization (Jacobian) is singular, we have yet another approach available, i.e., the direct method.

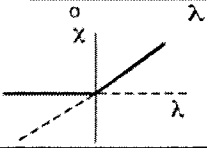
When the Jacobian becomes singular, $F(x, \lambda) = 0$ can not be solved by regular Newton–Raphson method in the present form. To avoid this singularity several methods have been published in the mathematical literature [29, 30, 31]. In these references the authors cleverly augmented the original system of equations in such a way that that for this enlarged system, the fold point becomes regular. If the fold point is mathematically character-

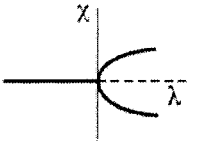
ized by the steady state Jacobian F_x having a simple and unique zero eigenvalue, with nonzero right eigenvector h and left eigenvector w, then

$$G(Y) = \begin{bmatrix} F(x,\lambda) \\ h_k - 1 \\ F_x(x,\lambda)h \end{bmatrix} = 0 \tag{2.21b}$$

where (x_c, λ_c) is a fold point of $F(x_c, \lambda) = 0$. This procedure basically augments the original equations of $F(x, \lambda) = 0$ by $F_x(x, \lambda)h = 0$, with $h_k = 1$. This augmentation makes the Jacobian G_x of enlarged system $G(Y)$ non-singular and guarantees a solution. The proof can be found in [32]. This approach has some drawbacks. The dimension of the nonlinear set of equations to be solved is twice that of the original number. The approach requires a good estimate for the vector h. However, convergence of the direct method is very fast if the initial operating point is close to the turning point. The enlarged system can be solved in such a way that it requires the solution of $n \times n$ (n is the dimension of the Jacobian $F_x(x, \lambda)$) linear systems, each with the same matrix. This method needs only one LU decomposition. At this turning point, rank $F_x(x, \lambda) = n-1$ and $F_\lambda (x_c, \lambda_c) \varepsilon$ *range* $F_x (x_c, \lambda_c)$, that is rank $F_x (x_c, \lambda_c) / F_\lambda (x_c, \lambda_c) = n$. These are called transversality conditions. Depending on the type of transversality condition, different types of static bifurcations can occur. Fold or saddle node is generic or the most commonly occurring static bifurcation. Table 2.1 summarizes the type of static bifurcation and corresponding transversality condition for a one-dimensional scalar system. Details can be found in Wiggins book [33]. The application of this method to power system voltage stability is reported by [34, 35].

Table 2.1 Static bifurcation types

Bifurcation Type	Transversality Condition	Prototype Equation	Bifurcation Diagram
Fold	$\frac{\partial F}{\partial \lambda} \neq 0; \frac{\partial^2 F}{\partial x^2} \neq 0$	$\lambda - x^2 = 0$	
Transcritical	$\frac{\partial^2 F}{\partial \lambda \partial x} \neq 0; \frac{\partial^2 F}{\partial x^2} \neq 0$	$\lambda x - x^2 = 0$	

Pitch Fork	$\frac{\partial^2 F}{\partial \lambda \partial x} \neq 0; \frac{\partial^2 F}{\partial x^2} \neq 0$	$\lambda x - x^3 = 0$	

Unstable mode Stable mode

Numerical example 3: In Eq.2.21b, the original system of equations is augmented in such a way that for the enlarged system, the turning point becomes regular. Solving for Eq.2.21b will yield the desired fold point. Related to numerical example 2 given above, the enlarged system of equations is:

$$\begin{cases} x^2 - 3x + \lambda = 0 \\ 2x - 3 = 0 \\ v = 1 \end{cases} \Rightarrow \begin{cases} x^2 - 3x + \lambda = 0 \\ 2x - 3 = 0 \end{cases} \tag{2.22}$$

Solving the above two equations, we directly get the critical point $(x_c, \lambda_c)^{tr} = (1.5, 2.25)^{tr}$.

Advantages:

The direct method can find the critical point where the Jacobian is singular by solving the enlarged system of power flow equations in one step. The left and right eigenvectors produced in the direct approach carry very important information. For instance, it was shown that, at saddle node bifurcations, the right eigenvector corresponding to the zero eigenvalue gives the trajectory of the system state variables [16]. The left eigenvector can be used to construct a normal vector [17, 36, 37, 38] at the bifurcations hypersurface.

Limitations:

In the direct approach, for a successful convergence, a good initial guess is needed. This method basically doubles the number of equations to be solved. However, some of these shortcomings can be overcome by following the approach proposed in reference [30]. In that paper, the authors explored the structure of equation Eq.2.21b. It's shown that the whole system can be resolved into four linear subsystems with the same coefficient matrix. Reference [34] applied this method to power system voltage stability studies.

When one is beset with the lack of good starting points for the Newton type iterative methods in solving nonlinear equations, or when one needs to lead a parameter study of nonlinear system equilibrium problems, it would probably be advisable to turn homotopy and continuation methods. Examples shown in this section manifests the applicability of the technique to engineering problems. This review does not present and exhaustive survey but a compact text on continuation methods. Readers interested in continuation, bifurcation, and related numerical methods may find the following references [25, 27, 39, 40] very helpful.

2.4 Hopf Bifurcation

2.4.1 Existence of Hopf bifurcation point

If (i) $F(x_c, \lambda_c) = 0$, (ii) the Jacobian matrix $(\partial F/\partial x)$ has a simple pair of purely imaginary eigenvalues, $\mu(\lambda_c) = \pm jw$, (iii) $\mathrm{d}(\mathrm{Re}(\mu(\lambda_c)))/\mathrm{d}\lambda \neq 0$. (Marsden & McCracken [41], Hassard *et al* [42].)

Then there is a birth or death of limit cycles at (x_c, λ_c) depending on the sign of derivative in (iii). λ_c is the value of the parameter at which Hopf bifurcation occurs. Requirement (iii) guarantees there is a transversal crossing of the imaginary axis by the pair of complex conjugate eigenvalues. Numerical determination of the Hopf bifurcation point involves estimation of the point (x_c, λ_c). A costly way of identifying the point is to evaluate all the eigenvalues of the Jacobian matrix. However, as in the static approach there are efficient ways of identifying the Hopf point by direct methods as well as by indirect methods.

2.4.1.1 Direct methods

Direct methods [40] calculate the Hopf point by solving one single suitably chosen equation. At the Hopf point, one pair of complex eigenvalues crosses the imaginary axis. Let this pair be:

$$\mu(\lambda) = \alpha(\lambda) - j\beta(\lambda)$$

with

$$\alpha(\lambda_c) = 0;\ \beta(\lambda_c) = 0;\ d\alpha(\lambda_c)/d\lambda = 0$$

For an eigenvalue μ of the Jacobian matrix $F_x[=\partial F/\partial x]$, the following equation is valid

$$F_x W = \mu W \tag{2.23}$$

where $W = u + jv$ is an eigenvector corresponding to the eigenvalue μ. Since $\alpha(X_c) = 0$, Eq.2.23 can be written as

$$F_x(u + jv) = (+j\beta)(u + jv)$$

$$F_x u + jF_x v = -\beta v + j\beta u$$

$$F_x v + \beta u = 0 \tag{2.24}$$

$$F_x v - \beta u = 0 \tag{2.25}$$

where u and v are vectors of dimension n. We have in fact $3n$ nonlinear algebraic Eqs.2.24 and 2.25 and $F(x,\lambda)=0$ with $3n+2$ unknowns ($x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, \lambda, \beta$). However the other two unknowns can be obtained by putting two normalizing conditions that force W to be non-zero. This means that practically we can choose two components of the vectors u and v arbitrarily. The Newton iterations method can be effectively used to solve this $3n$ by the $3n$ system to get the Hopf point. An efficient algorithm based on the direct approach is provided by [43]. The application of the boundary value problem for direct computation of the Hopf points was proposed by Seydel [25].

2.4.1.2 Indirect methods

The Hopf bifurcation point (x_c, λ_c) can also be located by an indirect approach. This can be achieved by obtaining the information collected during any continuation method described before, i.e., an iteration technique is used to solve the algebraic equation $\mathrm{Re}(\mu(\lambda)) = 0$ by means of the secant method. A change of sign of the real part $\alpha(\lambda)$ indicates that λ_c has

been passed. Therefore the check $\alpha(\lambda j)\ \alpha(\lambda j-1)<0$ should be performed after each continuation step $\lambda_{j-1} \rightarrow \lambda_j$.

A good comparison of various methods of computing Hopf bifurcating points is given by [44]. Application of Hopf bifurcation to power system problems can be found in [12, 14, 22, 45].

2.5 Complex Bifurcation

Further variation of the parameter beyond the Hopf point may lead to other complex phenomena; basically one has to trace the monodromy matrix of a periodic orbit for different values of the parameter. The stability of periodic solution is determined by Floquet multipliers which are the eigenvalues of the monodromy matrix. For a particular value of λ, the monodromy matrix has n-Floquet multipliers. The magnitude of one of them is always equal to unity. The other n-1 Floquet multipliers determine (local) stability by the following rule [25, 46].

- $x(t)$ is stable if $|\mu_j|<1$, for $j=1,\ldots, n-1$;
- $x(t)$ is unstable if $|\mu_j|>1$, for some j.

On the stable periodic orbit, the n-1 multipliers are always inside the unit circle. The multipliers are the functions of the parameter under consideration. When we vary the parameter, some of the multipliers may cross the unit circle. The multiplier crossing the unit circle is called the critical multiplier. Different types of branching occur depending on where a critical multiplier or pair of complex conjugate multipliers leaves the unit circle. Three associated types of branching are (i) the critical multiplier goes outside the unit circle along the positive real axis, with $|\mu(p_c)|=1$, (ii) the multiplier goes outside the unit circle along the negative real axis with $|\mu(p_c)|=-1$ and (iii) a pair of complex conjugate multipliers crosses the unit circle with a non-zero imaginary part. All these types refer to a loss of stability when λ passes through λ_c. (On the other hand, if a critical multiplier enters the unit circle, the system gains stability.) In the case (i) typically, turning points of the periodic orbit occur with a gain or loss of stability. Transcritical or pitchfork type bifurcations in periodic orbits are also possible for this case. In the case (ii), the system oscillates with period two. In the case (iii), the phenomenon of bifurcation into a torus occurs,

which is also called secondary Hopf bifurcation, or generalized Hopf bifurcation. The period doubling bifurcation often occurs repeatedly which generally leads to chaos. Lyapunov exponents are generally used to identify the chaos [47]. The exponential serves as a measure for exponential divergence or contraction of nearby trajectories. Chaos is characterized by at least one positive Lyapunov exponent, which reflects a stretching into one or more directions. In general, chaos has the following ingredients [47]: (i) the underlying dynamics is deterministic, (ii) no external noise has been introduced, (iii) seemingly erratic behavior of individual trajectories depends sensitively on small changes of initial conditions; (iv) in contrast to a single trajectory, some global characteristics are obtained by averaging over many trajectories or over a long time (e.g., a positive Lyapunov exponent) that do not depend on initial conditions; (v) when a parameter is tuned, the erratic state is reached via a sequence of events, including the appearance of one or more sub-harmonics. In the last few years, a great number of conferences and workshops devoted to chaotic dynamics have been organized. In most of them, papers by researchers from various branches of science and engineering have been presented. Research in chaos is well documented by [47]. Numerical methods to identify chaos can be found in [48]. Observations of chaos in power systems are reported in [12, 13, 49]. Fig2.5 gives the overall possible bifurcation scenario.

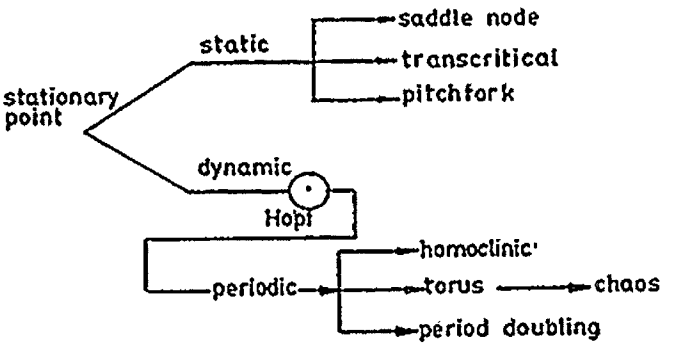


Fig. 2.5 List of possible bifurcations

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