Preface

Historically, the idea of nonstandard analysis was to rigorously justify calculations with infinitesimal numbers. For example, formally, the chain rule of Leibniz' calculus for the function $F = f(g(x))$ can be written as

$$\frac{dF}{dx} = \frac{dF}{dg}\frac{dg}{dx},$$

and for a formal proof, one may just divide numerator and denominator by the "infinitesimal small number" dg.

Nowadays, nonstandard analysis has gone far beyond the realm of infinitesimals. In fact, it provides a machinery which enables one to describe "explicitly" mathematical concepts which by standard methods can only be described "implicitly" and in a cumbersome way. In the above example the "standard" notion of a limit is in a certain sense replaced by the "nonstandard" notion of an infinitesimal. If one applies a similar approach to other objects than the real numbers (like topological spaces or Banach spaces etc.), one has a tool which provides "explicit" definitions for objects which can in principle not be described explicitly by standard methods. Examples of such objects are sets which are not Lebesgue measurable, or functionals with certain properties like so-called Hahn–Banach limits. Since it is possible in nonstandard analysis to simply "calculate" with such objects, one can obtain results about them which are extremely hard to obtain by standard methods.

This book is an introduction to nonstandard analysis. In contrast to some other textbooks on this topic, it is not meant as an introduction to basic calculus by nonstandard analysis. Instead, the above mentioned applications in analysis (which are not easily accessible by standard methods) are our main motivation. The infinitesimals are only described as an elementary example for the provided machinery.

Consequently, the reader is supposed to be already familiar with (standard) basic calculus. For deeper understanding, also experience with (basic) topology and functional analysis is useful, although not mandatory: The book is self-contained (i.e., all required notions are introduced and all nontrivial facts are proved).

The machinery underlying nonstandard analysis is the so-called model theory. The reader is not assumed to have any knowledge in model theory. In this book, all required concepts of model theory are introduced. However, model theory is only developed to the purpose of nonstandard analysis. The reader who is interested in general model theory is referred to the literature. We concentrate here instead on the applications in analysis and topology.

The book contains also 84 exercises; most of them are of a reasonably mild difficulty (the exceptions are marked). The reader is strongly advised to solve the exercises, because this is the only way to get the routine required to apply the machinery of nonstandard analysis. To most exercises, a rather detailed solution is provided at the end of the book.