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FUNDAMENTAL ONE-DIMENSIONAL VARIABLES

A. Gaussian

The PDF, CDF, and CF of a Gaussian RV $\mathbf{X} \in N_n\left(\overline{\mathbf{X}}, \sigma^2\right)$ are given in (1.1), (1.2), and (1.5) respectively with σ_X replaced by σ. For $\overline{\mathbf{X}} = 0$, the even moments of the components of $\mathbf{X}$ are given by

$$E\left\{X_i^{2k}\right\} = \overline{X_i^{2k}} = \frac{(2k)!}{k!}\left(\frac{1}{2}\sigma^2\right)^k, k \text{ integer} \tag{2.1}$$

All odd moments are equal to zero.

B. Rayleigh

1. $n = 1$

Since the square root always yields a positive quantity, then by definition $R = \sqrt{X^2} = |X|$, i.e., a single-sided Gaussian RV with PDF and CDF given by

$$p_R(r) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right), r \geq 0 \tag{2.2}$$

$$P_R(r) = 1 - 2Q\left(\frac{r}{\sigma}\right), r \geq 0 \tag{2.3}$$

Also, the moments of R are given by

$$E\{R^k\} = \frac{(2\sigma^2)^{k/2}}{\sqrt{\pi}}\Gamma\left(\frac{k+1}{2}\right),\ k \text{ integer} \tag{2.4}$$

2. $n = 2$

Here R corresponds to a conventional Rayleigh RV with PDF and CDF

$$p_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right),\ r \geq 0 \tag{2.5}$$

$$P_R(r) = 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right),\ r \geq 0 \tag{2.6}$$

Also, the moments of R are given by

$$E\{R^k\} = (2\sigma^2)^{k/2}\Gamma\left(1 + \frac{k}{2}\right),\ k \text{ integer} \tag{2.7}$$

3. $n = 2m$

$$p_R(r) = \frac{2r^{2m-1}}{(2\sigma^2)^m (m-1)!}\exp\left(-\frac{r^2}{2\sigma^2}\right),\ r \geq 0 \tag{2.8}$$

$$P_R(r) = 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\sum_{i=0}^{m-1}\frac{1}{i!}\left(\frac{r^2}{2\sigma^2}\right)^i,\ r \geq 0 \tag{2.9}$$

$$E\{R^k\} = (2\sigma^2)^{k/2}\frac{\Gamma\left(m + \frac{k}{2}\right)}{(m-1)!},\ k \text{ integer} \tag{2.10}$$

4. $n = 2m + 1$

$$p_R(r) = \frac{2r^{2m}}{(2\sigma^2)^{m+1/2}\Gamma(m + 1/2)}\exp\left(-\frac{r^2}{2\sigma^2}\right),\ r \geq 0 \tag{2.11}$$

$$P_R(r) = ------ \tag{2.12}$$

$$E\{R^k\}=(2\sigma^2)^{k/2}\frac{\Gamma\left(m+\frac{k+1}{2}\right)}{\Gamma(m+1/2)},\ k\text{ integer} \tag{2.13}$$

C. Rician

1. $n=1$

$$p_R(r)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{r^2+a^2}{2\sigma^2}\right)\left[\exp\left(\frac{ar}{\sigma^2}\right)+\exp\left(-\frac{ar}{\sigma^2}\right)\right],\ r\geq 0 \tag{2.14}$$

$$P_R(r)=Q\left(\frac{a-r}{\sigma}\right)-Q\left(\frac{a+r}{\sigma}\right),\ r\geq 0 \tag{2.15}$$

$$E\{R^k\}=(2\sigma^2)^{k/2}\exp\left(-\frac{a^2}{2\sigma^2}\right)\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}}{}_1F_1\left(\frac{k+1}{2};\frac{1}{2};\frac{a^2}{2\sigma^2}\right),\ k\text{ integer} \tag{2.16}$$

where ${}_1F_1(\alpha;\beta;\gamma)$ is the confluent hypergeometric function [2] and $a=|\overline{X}|$.

2. $n=2$

Here R corresponds to a conventional Rician RV with parameter $a=\|\overline{\mathbf{X}}\|$.

$$p_R(r)=\frac{r}{\sigma^2}\exp\left(-\frac{r^2+a^2}{2\sigma^2}\right)I_0\left(\frac{ra}{\sigma^2}\right),\ r\geq 0 \tag{2.17}$$

$$P_R(r)=1-Q_1\left(\frac{a}{\sigma},\frac{r}{\sigma}\right),\ r\geq 0 \tag{2.18}$$

$$E\{R^k\}=(2\sigma^2)^{k/2}\exp\left(-\frac{a^2}{2\sigma^2}\right)\Gamma\left(1+\frac{k}{2}\right){}_1F_1\left(1+\frac{k}{2},1;\frac{a^2}{2\sigma^2}\right),\ k\text{ integer} \tag{2.19}$$

where

$$Q_1(\alpha,\beta)=\int_\beta^\infty x\exp\left(-\frac{x^2+\alpha^2}{2}\right)I_0(\alpha x)dx \tag{2.20}$$

is the first-order Marcum Q-function [8].[2]

3. $n=2m$

$$p_R(r)=\frac{r^m}{\sigma^2 a^{m-1}}\exp\left(-\frac{r^2+a^2}{2\sigma^2}\right)I_{m-1}\left(\frac{ra}{\sigma^2}\right),\ r\geq 0 \tag{2.21}$$

$$P_R(r)=1-Q_m\left(\frac{a}{\sigma},\frac{r}{\sigma}\right),\ r\geq 0 \tag{2.22}$$

$$E\{R^k\}=\left(2\sigma^2\right)^{k/2}\exp\left(-\frac{a^2}{2\sigma^2}\right)\frac{\Gamma\left(m+\frac{k}{2}\right)}{(m-1)!}\,{}_1F_1\left(m+\frac{k}{2};m;\frac{a^2}{2\sigma^2}\right),\ k\text{ integer} \tag{2.23}$$

where

$$Q_m(\alpha,\beta)=\frac{1}{\alpha^{m-1}}\int_\beta^\infty x^m\exp\left(-\frac{x^2+\alpha^2}{2}\right)I_m(\alpha x)dx \tag{2.24}$$

is the generalized (mth-order) Marcum Q-function [8].

4. $n=2m+1$

$$p_R(r)=\frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{r}{a}\right)^m\exp\left(-\frac{r^2+a^2}{2\sigma^2}\right)\left[\exp\left(\frac{ra}{\sigma^2}\right)\sum_{i=0}^{m-1}\frac{(-1)^i(m+i-1)!}{i!(m-i-1)!}\left(\frac{\sigma^2}{2ra}\right)^i\right.$$
$$\left.+(-1)^m\exp\left(-\frac{ra}{\sigma^2}\right)\sum_{i=0}^{m-1}\frac{(m+i-1)!}{i!(m-i-1)!}\left(\frac{\sigma^2}{2ra}\right)^i\right],\ r\geq 0 \tag{2.25}$$

$$P_R(r)=------ \tag{2.26}$$

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[2] More often than not in the literature, the subscript "1" identifying the order of the first-order Marcum Q-function is dropped from the notation. We shall maintain its identity in this text to avoid possible ambiguity with the two-dimensional Gaussian Q-function defined in Eq. (A.37) of Appendix A.

$$E\{R^k\} = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(m+\frac{k+1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} {}_1F_1\left(m+\frac{k+1}{2}; m+\frac{1}{2}; \frac{a^2}{2\sigma^2}\right), \quad (2.27)$$

k integer

D. Central Chi-Square

1. $n = 1$

$$p_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2 y}} \exp\left(-\frac{y}{2\sigma^2}\right), \; y \geq 0 \quad (2.28)$$

$$P_Y(y) = ------ \quad (2.29)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2}\right)^{1/2} \quad (2.30)$$

$$E\{Y^k\} = (2\sigma^2)^k \frac{\Gamma(k+1/2)}{\sqrt{\pi}}, \; k \text{ integer} \quad (2.31)$$

2. $n = 2m$

$$p_Y(y) = \frac{1}{2\sigma^2 \Gamma(m)} \left(\frac{y}{2\sigma^2}\right)^{m-1} \exp\left(-\frac{y}{2\sigma^2}\right), \; y \geq 0 \quad (2.32)$$

$$P_Y(y) = 1 - \exp\left(-\frac{y}{2\sigma^2}\right) \sum_{i=0}^{m-1} \frac{1}{i!} \left(\frac{y}{2\sigma^2}\right)^i, \; y \geq 0 \quad (2.33)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2}\right)^m \quad (2.34)$$

Since the kth moment of a central chi-square RV with $2m$ degrees of freedom is equal to the $2k$th moment of a Rayleigh RV of order $2m$, then it is straight-forward to obtain the moments of Y from (2.10) as

$$E\{Y^k\} = (2\sigma^2)^k \frac{\Gamma(m+k)}{(m-1)!}, k \text{ integer} \tag{2.35}$$

3. $n = 2m+1$

$$p_Y(y) = \frac{1}{2\sigma^2\Gamma(m+1/2)}\left(\frac{y}{2\sigma^2}\right)^{m-1/2} \exp\left(-\frac{y}{2\sigma^2}\right), y \geq 0 \tag{2.36}$$

$$P_Y(y) = ------ \tag{2.37}$$

$$\Psi_Y(s) = \left(\frac{1}{1-2js\sigma^2}\right)^{m+1/2} \tag{2.38}$$

$$E\{Y^k\} = (2\sigma^2)^k \frac{\Gamma\left(m+k+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}, k \text{ integer} \tag{2.39}$$

E. Noncentral Chi-Square

1. $n = 1$

$$p_Y(y) = \frac{1}{2\sqrt{2\pi\sigma^2 y}} \exp\left(-\frac{y+a^2}{2\sigma^2}\right)\left[\exp\left(\sqrt{\frac{a^2 y}{\sigma^4}}\right) + \exp\left(-\sqrt{\frac{a^2 y}{\sigma^4}}\right)\right], y \geq 0 \tag{2.40}$$

$$P_Y(y) = ------ \tag{2.41}$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2}\right)^{1/2} \exp\left(\frac{j\omega a^2}{1-2j\omega\sigma^2}\right) \tag{2.42}$$

$$E\{Y^k\} = (2\sigma^2)^k \exp\left(-\frac{a^2}{2\sigma^2}\right)\frac{\Gamma(k+1/2)}{\sqrt{\pi}}\,{}_1F_1\left(k+\frac{1}{2};\frac{1}{2};\frac{a^2}{2\sigma^2}\right), k \text{ integer} \tag{2.43}$$

2. $n = 2m$

$$p_Y(y) = \frac{1}{2\sigma^2}\left(\frac{y}{a^2}\right)^{(m-1)/2} \exp\left(-\frac{y+a^2}{2\sigma^2}\right) I_{m-1}\left(\sqrt{\frac{a^2 y}{\sigma^4}}\right), y \geq 0 \quad (2.44)$$

$$P_Y(y) = 1 - Q_m\left(\frac{a}{\sigma}, \frac{\sqrt{y}}{\sigma}\right), y \geq 0 \quad (2.45)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2}\right)^m \exp\left(\frac{j\omega a^2}{1-2j\omega\sigma^2}\right) \quad (2.46)$$

Since the kth moment of a noncentral chi-square RV with $2m$ degrees of freedom is equal to the $2k$th moment of a Rician RV of order $2m$, then it is straightforward to obtain the moments of Y from (2.23) as

$$E\{Y^k\} = (2\sigma^2)^k \exp\left(-\frac{a^2}{2\sigma^2}\right)\frac{(m+k-1)!}{(m-1)!} {}_1F_1\left(m+k; m; \frac{a^2}{2\sigma^2}\right), k \text{ integer} \quad (2.47)$$

3. $n = 2m+1$

$$p_Y(y) = \frac{1}{2\sigma^2}\left(\frac{y}{a^2}\right)^{(m-1/2)/2} \exp\left(-\frac{y+a^2}{2\sigma^2}\right) I_{m-1/2}\left(\sqrt{\frac{a^2 y}{\sigma^4}}\right), y \geq 0 \quad (2.48)$$

$$P_Y(y) = ------ \quad (2.49)$$

$$\Psi_Y(\omega) = \left(\frac{1}{1-2j\omega\sigma^2}\right)^{m+1/2} \exp\left(\frac{j\omega a^2}{1-2j\omega\sigma^2}\right) \quad (2.50)$$

$$E\{Y^k\} = (2\sigma^2)^k \exp\left(-\frac{a^2}{2\sigma^2}\right)\frac{\Gamma\left(m+k+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} {}_1F_1\left(m+k+\frac{1}{2}; m+\frac{1}{2}; \frac{a^2}{2\sigma^2}\right), \quad (2.51)$$

k integer

F. Log-Normal

Let $X \in N_1(\overline{X}, \sigma^2)$. Then the PDF of $\gamma = 10^{X/10}$ is given by

$$p_\gamma(\gamma) = \frac{\xi}{\sqrt{2\pi}\sigma\gamma}\exp\left[-\frac{\left(10\log_{10}\gamma - \overline{X}\right)^2}{2\sigma^2}\right], \gamma \ge 0 \tag{2.52}$$

$$P_\gamma(\gamma) = 1 - Q\left(\frac{10\log_{10}\gamma - \overline{X}}{\sigma}\right), \gamma \ge 0 \tag{2.53}$$

where $\xi = 10/\ln 10$ and $\overline{X}$ (dB) and σ^2 (dB) correspond to the mean and variance of $10\log_{10}\gamma$. The CF of γ is not obtainable in closed form but can be approximated by a Gauss-Hermite expansion as

$$\Psi_\gamma(\omega) \cong \frac{1}{\sqrt{\pi}}\sum_{n=1}^{N_p} H_{x_n} \exp\left(10^{\left(\sqrt{2}\sigma x_n + \overline{X}\right)/10} j\omega\right) \tag{2.54}$$

where x_n are the zeros and H_{x_n} are the weight factors of the N_p-order Hermite polynomial and can be found in Table 25.10 of [2]. In addition, the moments of γ are given by

$$E\{\gamma^k\} = \exp\left[\frac{k}{\xi}\overline{X} + \frac{1}{2}\left(\frac{k}{\xi}\right)^2\sigma^2\right], k \text{ integer} \tag{2.55}$$