Geometrical Optics

1.1 INTRODUCTION

Geometrical optics uses light rays to describe image formation by spherical surfaces, lenses, mirrors, and optical instruments. Let us consider the real image of a real object, produced by a positive thin lens. Cones of light are assumed to diverge from each object point to the lens. There the cones of light are transformed into converging beams traveling to the corresponding real image points. We develop a very simple method for a geometrical construction of the image, using just two rays among the object, the image, and the lens. We decompose the object into object points and draw a line from each object point through the center of the lens. A formula is developed to give the distance of the image point, when the distance of the object point and the focal length of the lens are known. We assume that the line from object to image point makes only small angles with the axis of the system. This approximation is called the *paraxial theory*. Assuming that the object and image points are in a medium with refractive index 1 and that the lens has the focal length f, the simple mathematical formula

$$\frac{1}{-x_0} + \frac{1}{x_i} = \frac{1}{f} \tag{1.1}$$

gives the image position x_i when the object position x_0 and the focal length are known.

Formulas of this type can be developed for spherical surfaces, thin and thick lenses, and spherical mirrors, and one may call this approach the *thin lens model*.

For the description of the imaging process, we use the following laws.

1. Light propagates in straight lines.
2. The law of refraction,

$$n_1 \sin \theta_1 = n_2 \sin \theta_2. \tag{1.2}$$

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The light travels through the medium of refractive index n_1 and makes the angle θ_1 with the normal of the interface. After traversing the interface, the angle changes to θ_2, and the light travels in the medium with refractive index n_2.

3. The law of reflection

$$\theta_1 = \theta_2. \tag{1.3}$$

The law of reflection is the limiting case for the situation where both refraction indices are the same and one has a reflecting surface. The laws of refraction and reflection may be derived from Maxwell's theory of electromagnetic waves, but may also be derived from a "mechanical model" using Fermat's Principle.

The refractive index in a dielectric medium is defined as $n = c/v$, where v is the speed of light in the medium and c is the speed of light in a vacuum. The speed of light is no longer the ratio of the unit length of the length standard over the unit time of the time standard, but is now defined as 2.99792458×10^8 m/s for vacuum. For practical purposes one uses $c = 3 \times 10^8$ m/s, and assumes that in air the speed v of light is the same as c. In dielectric materials, the speed v is smaller than c and therefore, the refractive index is larger than 1.

Image formation by our eye also uses just one lens, but not a thin one of fixed focal length. The eye lens has a variable focal length and is capable of forming images of objects at various distances without changing the distance between the eye lens and the retina. Optical instruments, such as magnifiers, microscopes, and telescopes, when used with our eye for image formation, can be adjusted in such a way that we can use a fixed focal length of our eye. Image formation by our eye has an additional feature. Our brain inverts the image arriving on the retina, making us think that an inverted image is erect.

1.2 FERMAT'S PRINCIPLE AND THE LAW OF REFRACTION

In the seventeenth century philosophers contemplated the idea that nature always acts in an optimum fashion. Let us consider a medium made of different sections, with each having a different index of refraction. Light will move through each section with a different velocity and along a straight line. But since the sections have different refractive indices, the light does not move along a straight line from the point of incidence to the point of exit.

The mathematician Fermat formulated the calculation of the optimum path as an integral over the optical path

$$\int_{P_1}^{P_2} n \, ds. \tag{1.4}$$
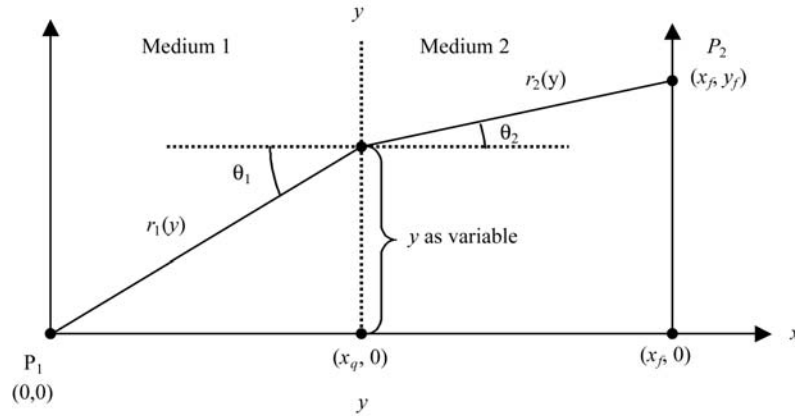
Path from P_1 to P_2 is $r1(y) + r2(y)$

Optical path from P_1 to P_2 is $n1r1(y) + n2r2(y)$

FIGURE 1.1   Coordinates for the travel of light from point P_1 in medium 1 to point P_2 in medium 2. The path in length units and the optical plath are listed.

The optical path is defined as the product of the geometrical path and the refractive index. In Figure 1.1 we show the length of the path from P_1 to P_2,

$$r_1(y) + r_2(y). \tag{1.5}$$

In comparison, the optical path is defined as

$$n_1 r_1(y) + n_2 r_2(y), \tag{1.6}$$

where n_1 is the refractive index in medium 1 and n_2 is the refractive index in medium 2.

The optimum value of the integral of Eq. (1.4) describes the shortest optical path from P_1 to P_2 through a medium in which it moves with two different velocities. It is important to compare only passes in the same neighborhood. In Figure 1.2 we show an example of what should not be compared.

In Figure 1.1, the light ray moves with v_1 in the first medium and is incident on the interface, making the angle θ_1 with the normal. After penetrating into the
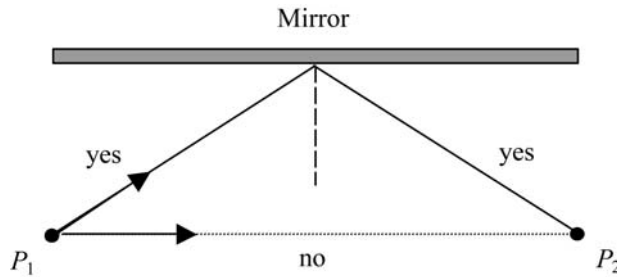

FIGURE 1.2   Application of Fermat's Principle to the reflection on a mirror. Only the path with the reflection on the mirror should be considered.

medium in which its speed is v_2, the angle with respect to the normal changes from θ_1 to θ_2.

Let us look at a popular example. A swimmer cries for help and a lifeguard starts running to help him. He runs on the sand with v_1, faster than he can swim in the water with v_2. To get to the swimmer in minimum time, he will not choose the straight line between his starting point and the swimmer in the water. He will run a much larger portion on the sand and then get into the water. Although the total length (in meter's) of this path is larger than the straight line, the total time is smaller. The problem is reduced to what the angles θ_1 and θ_2 are at the normal of the interface (Figure 1.1). We show that these two angles are determined by the law of refraction, assuming that the velocities are known.

In Figure 1.1 the light from point P_1 travels to point P_2 and passes the point Q at the boundary of the two media with indices n_1 and n_2. The velocity for travel from P_1 to Q is $v_1 = c/n_1$. The velocity for travel from Q to P_2 is $v_2 = c/n_2$. From Eq. (1.4) and Figure 1.1, the optical path is

$$n_1 r_1(y) + n_2 r_2(y), \tag{1.7}$$

where we have

$$r_1(y) = \sqrt{\{x_q^2 + y^2\}}$$
$$r_2(y) = \sqrt{\{(x_f - x_q)^2 + (y_f - y)^2\}} \tag{1.8}$$

and with $r_1(y) = v_1 t_1(y)$ and $r_2(y) = v_2 t_2(y)$ we get for the total time $T(y)$, to travel from P_1 to P_2,

$$T(y) = r_1(y)/v_1 + r_2(y)/v_2. \tag{1.9}$$

Only for the special case that $v_1 = v_2$, where the refractive indices are equal, will the light travel along a straight line. For different velocities, the total travel time through medium 1 and 2 will be a minimum. In FileFig 1.1 we show a graph of $T(y)$ and see the minimum for a specific value of y. In FileFig 1.2 we discuss the case where light is traveling through three media. To determine the optimum conditions we have to require that

$$dT(y)/dy = 0. \tag{1.10}$$

This may be done without a computer. We show it in FileFig 1.3 for two media. Using the expression for $r_1(y)$ and $r_2(y)$ of Figure 1.1, we have to differentiate

$$n_1 r_1(y) + n_2 r_2(y), \tag{1.11}$$

that is,

$$dT(y)/dy = d/dy\{(c/v_1)\sqrt{x_q^2 + y^2} + (c/v_2)\sqrt{(x_f - x_q)^2 + (y_f - y)^2}\} \tag{1.12}$$

and set it to zero. From FileFig 1.3 we get

$$y/(r_1(y)v_1) + (y - y_f)/(r_2(y)v_2) = 0. \tag{1.13}$$

With

$$\sin\theta_1 = y/r_1(y) \quad \text{and} \quad \sin\theta_2 = (y - y_f)/r_2(y) \tag{1.14}$$

we have

$$\sin\theta_1/v_1 = \sin\theta_2/v_2 \tag{1.15}$$

and after multiplication with c, the Law of Refraction,

$$n_1\sin\theta_1 = n_2\sin\theta_2. \tag{1.16}$$

FileFig 1.1   (G1FERMAT)

Graph of the total time for travel from P_1 to P_2, through medium 1, with velocities v_1, and medium 2, with v_2. For minimum travel time, the light does not travel along a straight line between P_1 and P_2. Changing the velocities will change the length of travel in each medium.

G1FERMAT

Fermat's Principle

Graph of total travel time: $t1$ is the time to go from the initial position $(0, 0)$ to point (xq, y) in medium with velocity $v1$. $t2$ is the time to go from point (xq, y) to the final position (xf, yf) in medium with velocity $v2$. There is a y value for minimum time. v_1 and v_2 are at the graph.

$$xq := 20 \quad xf := 40 \quad yf \equiv 40$$
$$y := 0, .1 \ldots 40.$$

Time in medium 1   *Time in medium 2*

$$t1(y) := \frac{1}{v1} \cdot \sqrt{(xq)^2 + y^2} \quad t2(y) := \frac{1}{v2} \cdot \sqrt{(xf - xq)^2 + (yf - y)^2}$$
$$T(y) := t1(y) + t2(y).$$

$$v1 \equiv 1 \qquad v2 \equiv 2.5.$$
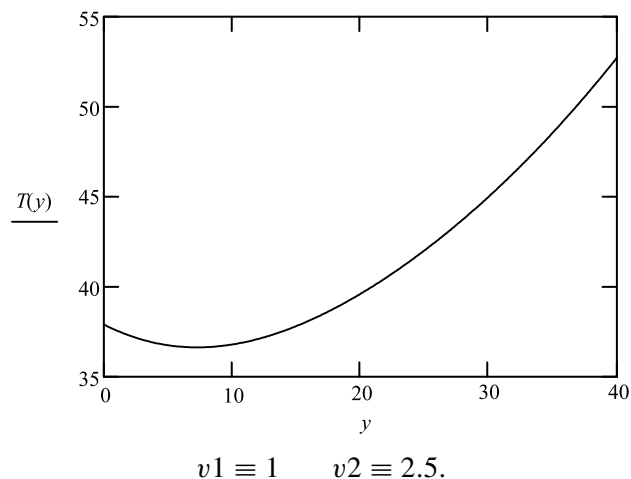
Changing the parameters $v1$ and $v2$ changes the minimum time for total travel.

Application 1.1.

1. *Compare the three choices*
   a. $v_1 < v_2$
   b. $v_1 = v_2$
   c. $v_1 > v_2$ *and how the minimum is changing.*

2. *To find the travel time t_1 in medium 1 and t_2 in medium 2 plot it on the graph and read the values at y for $T(y)$ at minimum.*

FileFig 1.2   (G2FERMAT)

Surface and contour graphs of total time for traversal through three media. Changing the velocities will change the minimum position.

G2FERMAT is only on the CD.

Application 1.2. *Change the velocities and observe the relocation of the minimum.*

FileFig 1.3   (G3FERREF)

Demonstration of the derivation of the law of refraction starting from Fermat's Principle. Differentiation of the total time of traversal. For optimum time, the expression is set to zero. Introducing c/n for the velocities.

G3FERREF is only on the CD.

1.3 PRISMS

A prism is known for the *dispersion* of light, that is, the decomposition of white light into its colors. The different colors of the incident light beam are deviated by different angles for different colors. This is called dispersion, and the angles depend on the refractive index of the prism material, which depends on the wavelength. Historically Newton used two prisms to prove his "Theory of Color." The first prism dispersed the light into its colors. The second prism, rotated by 90 degrees, was used to show that each color could not be decomposed any further. Dispersion is discussed in Chapter 8. Here we treat only the angle of deviation for a particular wavelength, depending on the value of the refractive index n.

1.3.1 Angle of Deviation

We now study the light path through a prism. In Figure 1.3 we show a cross-section of a prism with apex angle A and refractive index n. The incident ray makes an angle θ_1 with the normal, and the angle of deviation with respect to the incident light is call δ. We have from Figure 1.3 for the angles

$$\delta = \theta_1 - \theta_2 + \theta_4 - \theta_3 \qquad A = \theta_2 + \theta_3 \qquad (1.17)$$

and using the laws of refraction

$$\sin\theta_1 = n\sin\theta_2 \qquad n\sin\theta_3 = \sin\theta_4 \qquad (1.18)$$

we get for the angle of deviation, using asin for $\sin^{-1}$

$$\delta = \theta_1 + \text{asin}\left\{\left(\sqrt{n^2 - \sin^2(\theta_1)}\right)\sin(A) - \sin(\theta_1)\cos(A)\right\} - A. \qquad (1.19)$$

In FileFig 1.4 a graph is shown of δ (depending on the angle of incidence). A formula may be derived to calculate the minimum deviation δ_m of the prism, depending on n and A. From the Eq. (1.17) and (1.18) we have

$$\delta = \theta_1 - \theta_2 + \theta_4 - \theta_3, \qquad A = \theta_2 + \theta_3, \qquad (1.20)$$
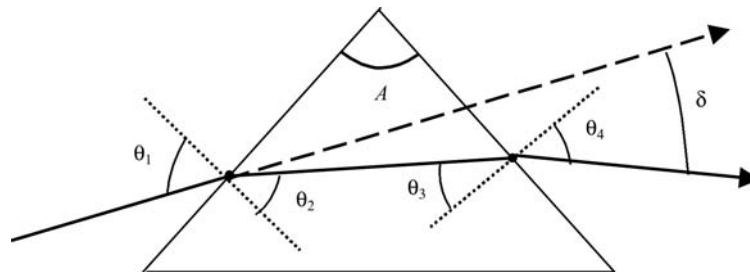

FIGURE 1.3 Angle of deviation δ of light incident at the angle θ_1 with respect to the normal. The apex angle of the prism is A.

and

$$\sin \theta_1 = n \sin \theta_2, \qquad n \sin \theta_3 = \sin \theta_4. \tag{1.21}$$

We can eliminate θ_2 and θ_4 and get two equations in θ_1 and θ_3,

$$\sin \theta_1 = n \sin(A - \theta_3) \tag{1.22}$$
$$n \sin \theta_3 = \sin(\delta + A - \theta_1). \tag{1.23}$$

The differentiations with respect to the angle of Eqs. (1.22) and (1.23) may be done using the "symbolic capabilities" of a computer (see FileFig 1.5). To calculate the optimum condition, the results of the differentiations have to be zero:

$$\cos \theta_1 d\theta_1 + n \cos(A - \theta_3)d\theta_3 = 0 \tag{1.24}$$
$$n \cos \theta_3 d\theta_3 + \cos(\delta + A - \theta_1)d\theta_1 = 0. \tag{1.25}$$

We consider these equations as two linear homogeneous equations of the unknown $d\theta_1$ and $d\theta_3$. In order to have a nontrivial solution of the system of the two linear equations, the determinant has to vanish. This is done in FileFig 1.5, and one gets

$$\cos \theta_1 \cos \theta_3 - \cos(A - \theta_3) \cos(\delta + A - \theta_1) = 0.$$

The minimum deviation δ_m, which depends only on n and A, may be calculated from

$$\delta_m = 2 \operatorname{asin} \{n \sin(A/2)\} - A, \tag{1.26}$$

where we use *asin* for $\sin^{-1}$. At the angle of minimum deviation, the light traverses the prism in a symmetric way. Equation (1.26) may be used to find the dependence of prism material on the refractive index n.

FileFig 1.4   (G4PRISM)

Graph of angle of deviation δ_1 as function of θ_1 for fixed values of apex angle A and refractive index n. For fixed A and n the angle of deviation δ has a minimum.
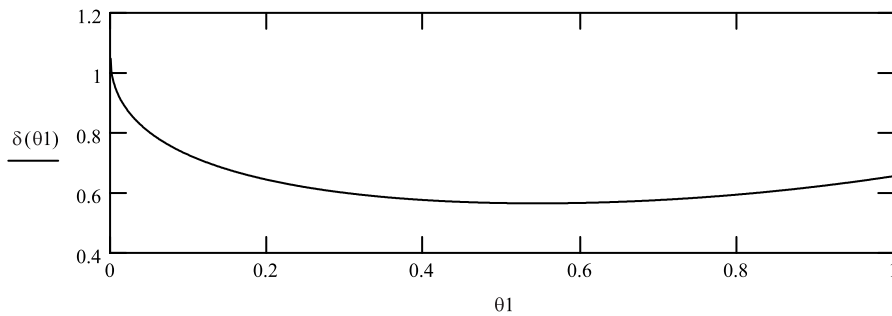
G4PRISM

Graph of the Angle of Deviation for Refraction on a Prism Depending on the Angle of Incidence

$\theta 1$ *is the angle of incidence with respect to the normal.* $\delta 1$ *is the angle of deviation. n is the refractive index and A is the apex angle.*

$$\theta 1 := 0, .001 \ldots 1 \qquad n := 2 \qquad A := \left(\frac{2 \cdot \pi}{360} \right) \cdot 30$$

$$\delta(\theta 1) := \theta 1 + \text{asin} \left(\sqrt{n^2 - \sin(\theta 1)^2} \cdot \sin(A) - \sin(\theta 1) \cdot \cos(A) \right) - A.$$

Application 1.4.

1. *Observe changes of the minimum depending on changing A and n.*

2. *Numerical determination of the angle of minimum deviation. Differentiate $\delta(\theta_1)$ and set the result to zero. Break the expression into two parts and plot them on the same graph. Read the value of the intersection point.*

FileFig 1.5   (G5PRISMIM)

Derivation of the formula for the refractive index determined by the angle of minimum deviation and apex angle A of prism.

G5PRISMIM is only on the CD.

1.4 CONVEX SPHERICAL SURFACES

Spherical surfaces may be used for image formation. All rays from an object point are refracted at the spherical surface and travel to an image point. The diverging light from the object point may converge or diverge after traversing the spherical surface. If it converges, we call the image point real; if it diverges we call the image point virtual.

1.4.1 Image Formation and Conjugate Points

We want to derive a formula to describe the imaging process on a convex spherical refracting surface between two media with refractive indices n_1 and n_2 (Figure 1.4). The light travels from left to right and a cone of light diverges from the object point P_1 to the convex spherical surface. Each ray of the cone is refracted at the spherical surface, and the diverging light from P_1 is converted to converging light, traveling to the image point P_2. The object point P_1 is assumed

to be in a medium with index n_1, the image point P_2 in the medium with index n_2. We assume that $n_2 > n_1$, and that the convex spherical surface has the radius of curvature $r > 0$.

For our derivation we assume that all angles are small; that is, we use the approximation of the paraxial theory. To find out what is small, one may look at a table of $y_1 = \sin\theta$ and compare it with $y_1 = \theta$. The angle should be in radians and then one may find angles for which y_1 and y_2 are equal to a desired accuracy.

We consider a cone of light emerging from point P_1. The outermost ray, making an angle α_1 with the axis of the system, is refracted at the spherical surface, and makes an angle α_2 with the axis at the image point P_2 (Figure 1.4). The refraction on the spherical surface takes place with the normal being an extension of the radius of curvature r, which has its center at C. We call the distance from P_1 to the spherical surface the object distance x_o, and the distance from the spherical surface to the image point P_2, the image distance x_i. In short, we may also use x_o for "object point" and x_i for "image point."

The incident ray with angle α_1 has the angle θ_1 at the normal, and penetrating in medium 2, we have the angle of refraction θ_2. Using the small angle approximation, we have for the law of refraction

$$\theta_2 = n_1\theta_1/n_2. \tag{1.27}$$

From Figure 1.4 we have the relations:

$$\alpha_1 + \beta = \theta_1 \quad \text{and} \quad \alpha_2 + \theta_2 = \beta. \tag{1.28}$$

For the ratio of the angles of refraction we obtain

$$\theta_1/\theta_2 = n_2/n_1 = (\alpha_1 + \beta)/(\beta - \alpha_2). \tag{1.29}$$

We rewrite the second part of the equation as

$$n_1\alpha_1 + n_2\alpha_2 = (n_2 - n_1)\beta. \tag{1.30}$$

The distance l in Figure 1.4 may be represented in three different ways.

$$\tan\alpha_1 = l/x_o, \qquad \tan\alpha_2 = l/x_i, \quad \text{and} \quad \tan\beta = l/r. \tag{1.31}$$

Using small angle approximation, we substitute Eq. (1.31) into Eq. (1.30) and get

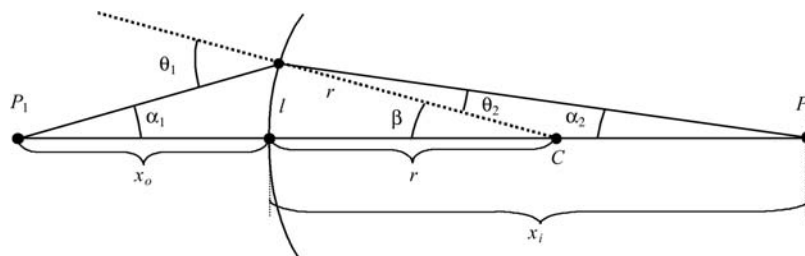$$n_1l/x_o + n_2l/x_i = (n_2 - n_1)l/r. \tag{1.32}$$

FIGURE 1.4   Coordinates for the derivation of the paraxial imaging equation.

The ls cancel out and we have obtained the image-forming equation for a spherical surface between media with refractive index n_1 and n_2, for all rays in a cone of light from P_1 to P_2:

$$n_1/x_0 + n_2/x_i = (n_2 - n_1)/r. \tag{1.33}$$

So far all quantities have been considered to be positive.

1.4.2 Sign Convention

In the following we distinguish between a convex and a concave spherical surface. The incident light is assumed to travel from left to right, and the object is to the left of the spherical surface. We place the spherical surface at the origin of a Cartesian coordinate system. For a convex spherical surface the radius of curvature r is positive; for a concave spherical surface r is negative. Similarly we have positive values for object distance x_0 and image distance x_i, when placed to the right of the spherical surface, and negative values when placed to the left.

Using this sign convention, we write Eq. (1.33) with a minus sign, and have the equation of "spherical surface imaging" (observe the minus sign),

$$-\frac{n_1}{x_0} + \frac{n_2}{x_i} = \frac{n_2 - n_1}{r}. \tag{1.34}$$

The pair of object and image points are called conjugate points.

We may define $\zeta_o = x_o/n_1$, $\zeta_i = x_i/n_2$, and $\rho = r/(n_2 - n_1)$ and have from Eq. (1.34)

$$-1/\zeta_o + 1/\zeta_i = 1/\rho. \tag{1.35}$$

This simplification will be useful for other derivations of imaging equations.

1.4.3 Object and Image Distance, Object and Image Focus, Real and Virtual Objects, and Singularities

When the object point is placed to the left of the spherical surface, we call it a real object point. When it appears to the right of the spherical surface, we call it a virtual object point. A virtual object point is usually the image point produced by another system and serves as the object for the following imaging process. To get an idea, of how the positions of the image point depend on the positions of the object point, we use the equation of spherical surface imaging

$$-n_1/x_o + n_2/x_i = (n_2 - n_1)/r \tag{1.36}$$

or

$$x_i = n_2/[(n_2 - n_1)/r + n_1/x_o],$$

and plot a graph (FileFig 1.6). We choose an object point in air with $n_1 = 1$, a spherical convex surface of radius of curvature $r_1 = 10$, and refractive index $n_2 = 1.5$.

We do not add length units to the numbers. It is assumed that one uses the same length units for all numbers associated with quantities of the equations. When the object point is assumed to be at negative infinity, we have the image point at the image focus

$$x_{\text{if}} = n_2 r/(n_2 - n_1). \tag{1.37}$$

Similarly there is the object focus, when the image point is assumed to be at positive infinity

$$x_{\text{of}} = -n_1 r/(n_2 - n_1). \tag{1.38}$$

We see from the graph of FileFig 1.6 that there is a singularity at the object focus (at $x_o = -20$). To the left of the object focus all values of x_i are positive. To the right of the object focus the values of x_i are first negative, from the object focus to zero, and then positive to the right to infinity.

When $x_o = 0$ we have in Eq. (1.36) another singularity, and as a result we have $x_i = 0$. One may get around problems in plotting graphs around singularities t by using numerical values for x_o that never have values of the singular points.

In FileFig 1.7 we have calculated the image point for four specifically chosen object points, discussed below.

FileFig 1.6 (G6SINGCX)

Graph of image coordinate depending on object coordinate for convex spherical surface, for $r = 10$, $n_1 = 1$ and $n_2 = 1.5$. There are three sections. In the first and third sections, for a positive sign, the image is real. In the middle section, for a negative sign, the image is virtual.

G6SINGCX

Convex Single Refracting Surface

r is positive, light from left propagating from medium with n1 to medium with n2. xo on left of surface (negative).

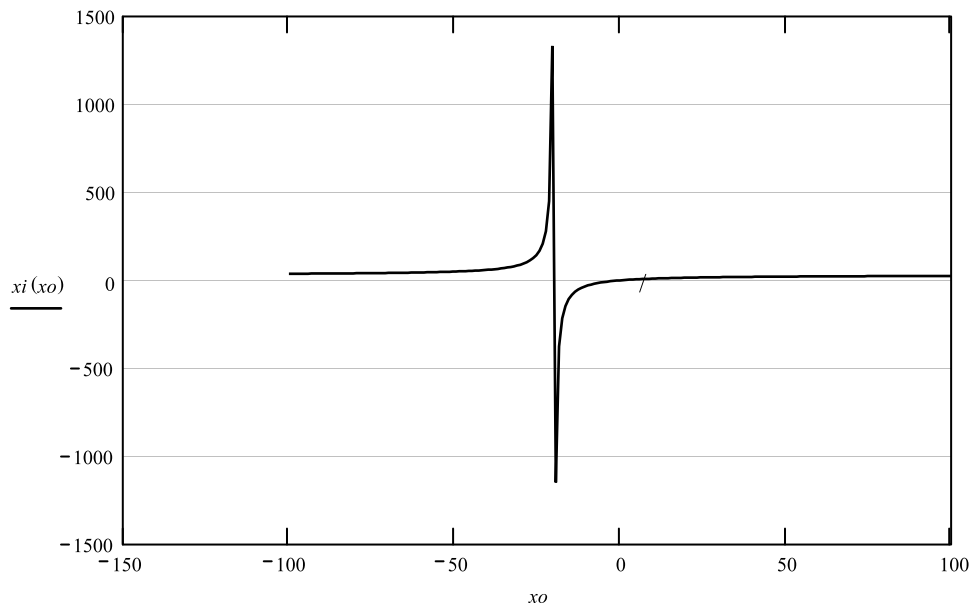*Calculation of Graph for xi as Function of xo over the Total Range of xo*

Graph for xi as function of xo over the range of xo to the left of xof. Graph for xi as function of xo over the range of xo to the right of xof.
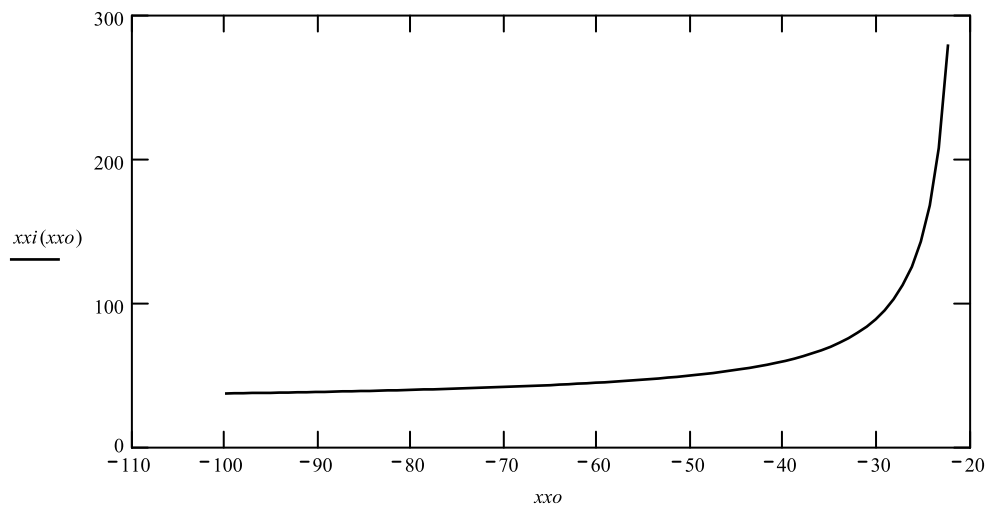
$$r \equiv 10 \qquad n1 := 1 \qquad n2 := 1.5.$$

Image focus *Object focus*

$$xif := n2 \cdot \frac{r}{n2 - n1} \qquad xif = 30 \qquad xof := n1 \cdot \frac{r}{n1 - n2} \qquad xof = -20$$

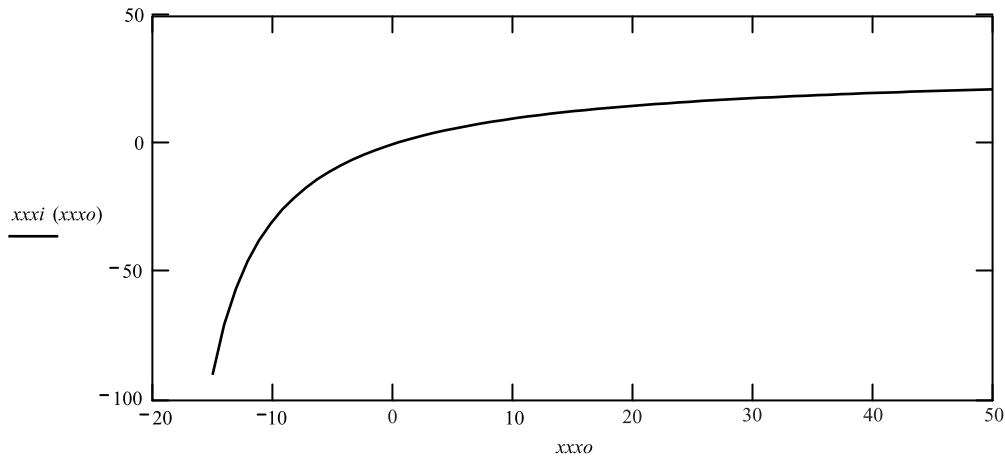$$xo := -100.001, -99.031 \ldots 100 \qquad xi(xo) := \frac{n2}{\left(\frac{n2-n1}{r}\right) + \frac{n1}{xo}} \cdot$$

$$xxo := -100.001, -99.031 \qquad xxi(xxo) := \frac{n2}{\left(\frac{n2-n1}{r}\right) + \frac{n1}{xxo}} \cdot$$

$$xxxo := -15.001, -14.031 \ldots 50$$

$$xxxi(xxxo) := \frac{n2}{\left(\frac{n2-n1}{r}\right) + \frac{n1}{xxxo}}.$$

Application 1.6.

1. *Change the refractive index and look at the separate graphs for the sections to the left and right of the object focus. To the left of the object focus, x_i is positive. To the right it is first negative until zero, and then positive. What are the changes?*
2. *Change the radius of curvature, and follow Application 1.*

FileFig 1.7   (G7SINGCX)

Convex spherical surface. Calculation of image and object foci. Calculation of image coordinate for four specifically chosen object coordinates.

G7SINGCX

Convex Single Refracting Surface

r is positive, light from left is propagating from medium with n1 to medium with n2. xo is on left of surface (negative).

Calculation for Four Positions for Real and Virtual Objects, to the Left and Right of the Objects Focus and Image Focus

Calculation of xi from given xo, refractive indices, and radius of curvature. Calculation of magnification

$$r \equiv 10 \qquad n1 := 1 \qquad n2 := 1.5.$$

Image focus   *Object focus*

$$xif := n2 \cdot \frac{r}{n2 - n1} \qquad xif = 30 \qquad xof := n1 \cdot \frac{r}{n1 - n2} \qquad xof = -20.$$

1. $x1o := -100$

$$x1i := \frac{n2}{\left(\frac{n2-n1}{r}\right) + \frac{n1}{x1o}} \qquad x1i = 37.5 \quad mm1 := x1i \cdot \frac{n1}{x1o \cdot n2} \quad mm1 = -0.25.$$

2. $x2o := -10$

$$x2i := \frac{n2}{\left(\frac{n2-n1}{r}\right) + \frac{n1}{x2o}} \qquad x2i = -30 \quad mm2 := x2i \cdot \frac{n1}{x2o \cdot n2} \quad mm2 = 2.$$

3. $x3o := -10$

$$x3i := \frac{n2}{\left(\frac{n2-n1}{r}\right) + \frac{n1}{x3o}} \qquad x3i = 15 \quad mm3 := x3i \cdot \frac{n1}{x3o \cdot n2} \quad mm3 = 0.5.$$

4. $x4o := 100$

$$x4i := \frac{n2}{\left(\frac{n2-n1}{r}\right) + \frac{n1}{x4o}} \qquad x4i = 25 \quad mm4 := x4i \cdot \frac{n1}{x4o \cdot n2} \quad mm4 = 0.167.$$

Application 1.7.

1. *Calculate Table 1.1 for refractive indices $n_1 = 1$ and $n_2 = 2.4$ (Diamond).*
2. *Calculate Table 1.1 for refractive indices $n_1 = 2.4$ and $n_2 = 1$.*

1.4.4 Real Objects, Geometrical Constructions, and Magnification

1.4.4.1 Geometrical Construction for Real Objects to the Left of the Object Focus

We consider an extended object consisting of many points. A conjugate point at the image corresponds to each point. When using a spherical surface for image formation, a cone of light emerges from each object point and converges to the conjugate image point. Let us present the object by an arrow, parallel to the positive y axis. The corresponding image will also appear at the image parallel to the y axis, but in the opposite direction (Figure 1.5).

The image position and size can then be determined by a simple geometrical construction. In Figure 1.5a we look at the ray connecting the top of the object arrow with the center of curvature of the spherical surface. We call the light ray corresponding to this line the *C-ray* (from center). A second ray, the *PF-ray*, starts at the top of the object arrow and is parallel to the axis along the distance to the spherical surface. It is refracted and travels to the image focal point F_i on the right side of the spherical surface (Figure 1.5c). The paraxial approximation
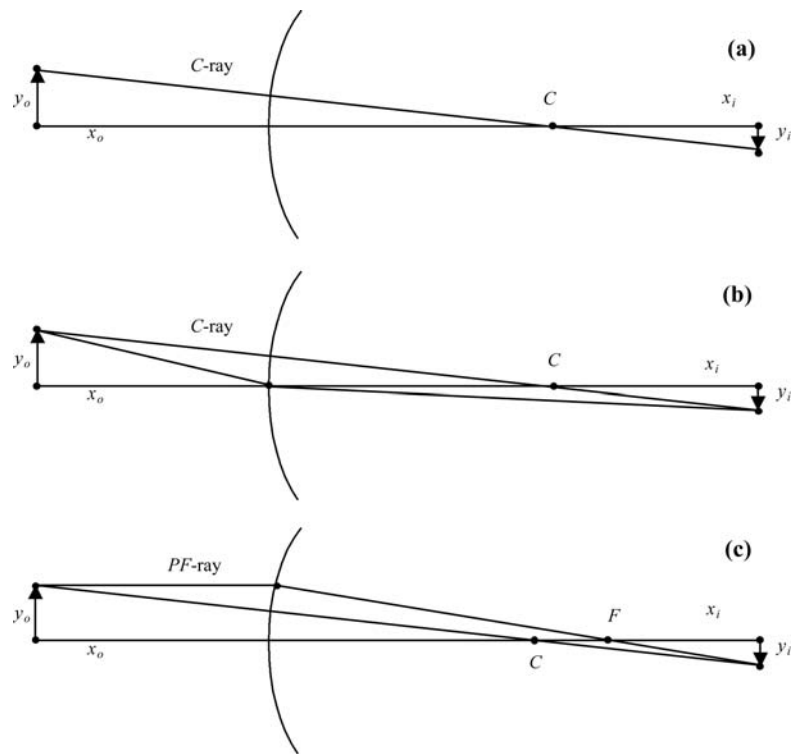
FIGURE 1.5   (a) The C-ray and conjugate points for extended image and object; (b) for the calculation of the lateral magnification we show the C-ray, and the ray from the top of y_0, refracted at the center of the spherical surface, connected to the top of y_i; (c) geometrical construction of image using the C-ray and the FP-ray.

requires that all C-rays and PF-rays have small angles with the axis of the system. The C-ray and the PF-ray meet at the top of the image arrow.

1.4.4.2 Geometrical Construction for Real Object to the Right of the Object Focus

We place the object arrow between the object focus and the spherical surface. From FileFig 1.7, with the input data we have used before, we find that the image position is at -30, when the object position is at -10. The geometrical construction is shown in Figure 1.7b. The C-ray and the PF-ray diverge in the forward direction to the right. However, if we trace both rays back they converge on the left side of the spherical surface. We find the top of an image arrow at the image position, at -30. We call the image, obtained by tracing the diverging rays back to a converging point, a *virtual image*. A virtual image may serve as a real object for a second imaging process.

We have listed in Table 1.1 the image positions for real object positions discussed so far and have indicated for images and objects if they are real or virtual.
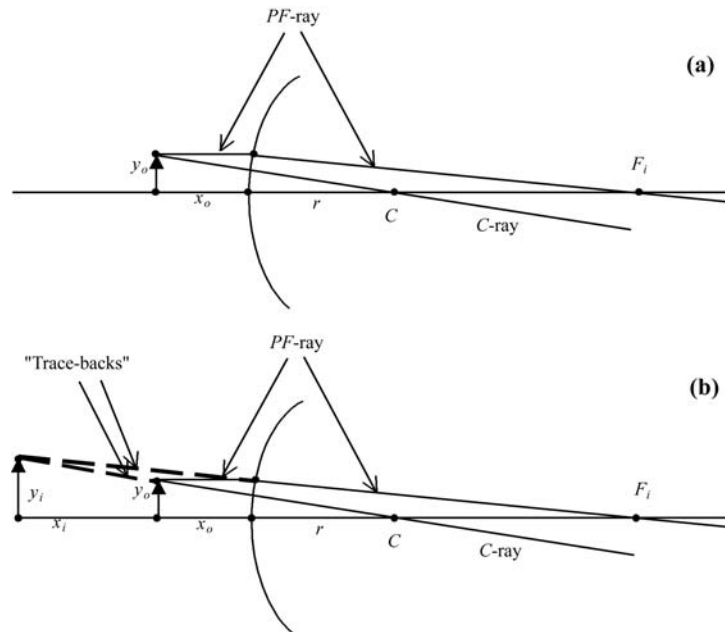
FIGURE 1.6 (a) The C-ray and the PF-ray diverge in the forward direction; (b) they are traced back to the virtual image.

1.4.4.3 Magnification

If we draw a C-ray from the top of the arrow representing the object, we find the top of the arrow presenting the image (Figure 1.5). The *lateral magnification m* is defined as

$$m = y_i/y_o. \tag{1.39}$$

It is obtained by using the proportionality of corresponding sides of right triangles, and taking care of the sign convention

$$-y_i/(x_i - r) = y_o/(-x_o + r). \tag{1.40}$$

For $m = y_i/y_o$ we have

$$m = -(x_i - r)/(-x_o + r). \tag{1.41}$$

Rewritten, eliminating the radius of curvature, one gets with Eq. (1.36),

$$m = y_i/y_o = (x_i/x_o)(n_1/n_2). \tag{1.42}$$

1.4.5 Virtual Objects, Geometrical Constructions, and Magnification

In Figure 1.7 we have made geometrical constructions of virtual objects to the left and right of the image focus. The objects are placed before and after the
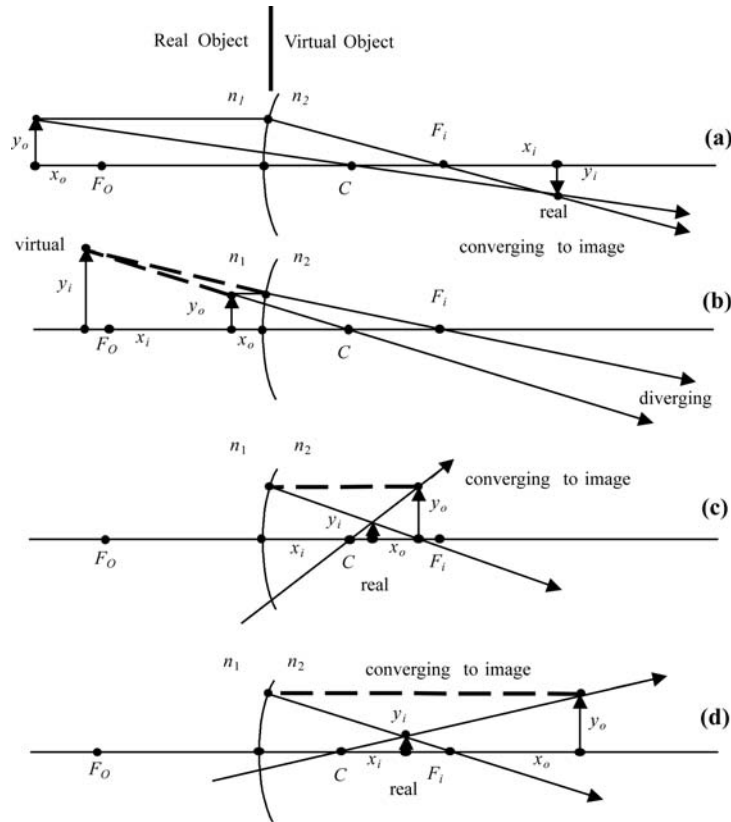
FIGURE 1.7   Geometrical construction of images for the convex spherical surface. The images of real objects are constructed in (a) and (b), for virtual objects in (c) and (d). The light converges to real images in (a), (c), (d). In (b) the light diverges and a virtual image is obtained by "trace back."

image focus. The magnification is obtained from Eq. (1.42) and the calculations are shown in FileFig 1.7.

In FileFig 1.7, we have calculated the four object positions listed in Table 1.1 and shown in Figure 1.7a to d.

1. Real object left of object focus

A real object is positioned to the left of the object focus. The construction uses the C-ray, PF-ray, and image focus. The rays converge to an image point, we have a real image.

2. Real object between object focus and spherical surface

We draw the C-ray and the PF-ray and use the image focus. The rays diverge in a forward direction. We trace both back to a point where they meet. The image is a virtual image.

3. and **4.** Virtual objects.

In Figures 1.7c and 1.7d we consider a virtual object to the right of the spherical surface, one to the left and another to the right of the image focus.

TABLE 1.1   Convex Surface. $r = 10$, $x_{\mathrm{if}} = 30$, $x_{\mathrm{of}} = -20^a$

x_o	x_i	m	Image	Object
-100	37.5	$-.25$	r	r
-10	-30	2	vi	r
20	15	05	r	vi
100	25	.0167	r	vi

[a] Calculations with G7SINGCX.

The C-ray is drawn through C in the "forward direction, but the PF-ray is now drawn first "backward" to the surface and then "forward" through the image focus. The C-ray and the PF-ray converge to real images for both positions of the virtual objects.

In Section 1.4 we discussed the case of Eq. (1.36) where $n_1 < n_2$ and r is positive. The case where $n_1 > n_2$ and r is negative will result in a very similar discussion and is considered as an application.

1.5 CONCAVE SPHERICAL SURFACES

The image-forming equation of a convex spherical surface (Eq. (1.34)), is changed for application to a concave spherical surface by changing the radius of curvature to a negative value. We show that this minor change makes image formation quite different.

Again we assume that the refractive index to the left of the surface is smaller than the refractive index on the right ($n_1 < n_2$). The formation of images of extended objects, their magnification, and geometrical construction are similar to the process discussed above for the convex spherical surface.

In FileFig 1.8 we have the graph for the dependence of x_i on x_o. In FileFig 1.9, we determine for four specific positions of x_o, for real and virtual objects, calculations of image positions and magnifications. Observe the difference in the position of object and image focus.

FileFig 1.8   (G8SINGCV)

Graph of image coordinate depending on object coordinate for concave spherical surface, for $r = -10$, $n_1 = 1$, and $n_2 = 1.5$. There are three sections. In the

first and third sections, for a negative sign, the image is virtual. In the middle section, for a positive sign, the image is real.

G8SINGCV is only on the CD.

Application 1.8.

1. *Observe the singularity at the object focus, which is on the "other side" in comparison to the convex case.*
2. *Change the refractive index and look at the separate graphs for the sections to the left and right of the object focus. To the left of the object focus, x_i is negative to the left of zero, positive to the right. To the right of the object focus it is negative. What are the changes?*
3. *Change the radius of curvature, and follow Application 2.*

FileFig 1.9   (G9SINGCV)

Concave spherical surface. Calculation of the image and object foci, and image coordinate for four specifically chosen object coordinates.

G9SINGCV is only on the CD.

Application 1.9.

1. *Calculate Table 1.2 for refractive indices $n_1 = 1$ and $n_2 = 2.4$ (Diamond).*
2. *Calculate Table 1.2 for refractive indices $n_1 = 2.4$ and $n_2 = 1$ (Diamond).*

The results are listed in Table 1.2, together with the labeling of the real and virtual objects and image.

The geometrical constructions of the four cases calculated in FileFig 1.9 are shown in Figures 1.8a to 1.8d.

1. and **2.** *Real objects.*

A real object is positioned to the left of the spherical surface. The C-ray and PF-ray diverge in a forward direction. The PF-ray is traced back through the image focus (it is on the left). The C-ray and PF-ray meet at an image point. We have virtual images for both positions of the real object.

TABLE 1.2   Concave Surface. $r = -10$, $x_{if} = -30$, $x_{of} = 20^a$

x_o	x_i	m	Image	Object
−100	−25	.167	vi	r
−20	−15	.5	vi	r
10	30	.2	r	vi
100	−37.5	−.25	vi	vi

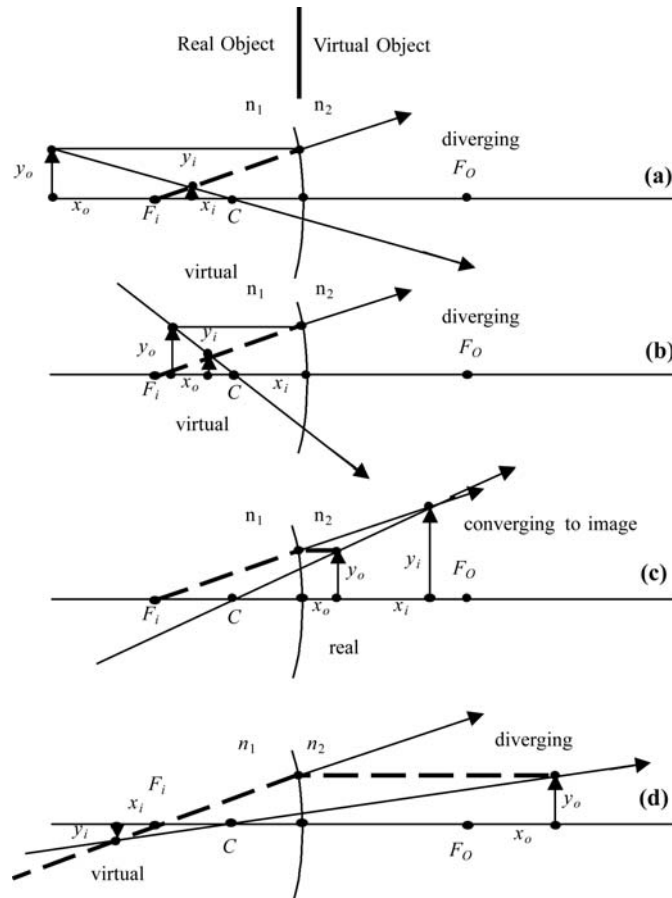[a] Calculations with C9SINGCV.

FIGURE 1.8 Geometrical construction of images for the concave spherical surface. The images of real objects are constructed in (a) and (b), for virtual objects in (c) and (d). The light converges to real images in (c). The light diverges in (a), (b), (d), and a virtual image is obtained by "trace back."

3. *Virtual object between spherical surface and object focus.*
We draw the C-ray and have to trace back the PF-ray to the surface and through the image focus. From there, we extend the ray in a forward direction. The rays converge in a forward direction and we have a real image.

4. *Virtual objects to the right of object focus.*
The C-ray is drawn through C in a forward direction. The PF-ray is traced back to the surface and then drawn backwards through the image focus. In the backward direction the two rays meet at a virtual image.

Comparing Figures 1.7 and 1.8, one finds that the regions of appearance of real and virtual images are dependent upon the singularities: one when the object distance is equal to the focal length, and the other when the object distance is zero. A virtual image is always found when the C-ray and PF-ray diverge in a

forward direction. If we could place a screen into the position of a virtual image, we could not detect it because the rays toward it are diverging.

The case where $n_1 > n_2$ and r is positive is very similar and is discussed as an application in FileFig 1.9.

Applications to Convex and Concave Spherical Surfaces

1. Single convex surface. A rod of material with refractive index $n2 = 1.5$ has on the side facing the incident light a convex spherical surface with radius of curvature $r = 50$ cm.

   a. What is the object distance in order to have the image at $+7$ cm?

   b. What is the object distance in order to have the image at -7 cm?

   c. Assume $r = 25$ cm; make a graph of x_i as a function of x_o for $n_1 = 1$, $n_2 = 1.33$, and do the graphical construction of the image (i) for real objects before and after the object focal point, and (ii) for virtual objects before and after the image focal point.

2. Rod sticks in water, calculation of image distance. A plastic rod of length 70 cm is stuck vertically in water. An object is positioned on the cross-section at the top of the rod, which sticks out of the water and faces the sun. On the other side in the water, the rod has a concave spherical surface, with respect to the incident light from the sun, with $r = -4$ cm. The refractive index of the rod is $n_1 = 1.5$ and of water $n_2 = 1.33$. Calculate the image distance of the object.

3. Single concave surface. A rod of material with refractive index $n_2 = 1.5$ has on one side a concave spherical surface with radius of curvature $r = -50$ cm.

   a. What is the object distance in order to have the image at $+5$ cm?

   b. What is the object distance in order to have the image at -5 cm?

   c. Assume $r = 25$ cm; make a graph of x_i as a function of x_o for $n_1 = 1$, $n_2 = 1.33$, and do the graphical construction of the image (i) for real objects before and after the image focal point, and (ii) for virtual objects before and after the object focal point.

4. Plastic film on water as spherical surface. A plastic film is mounted on a ring and placed on the surface of water. The film forms a spherical surface filled with water. The thickness of the film is neglected and therefore we have a convex surface of water of $n_2 = 1.33$. Sunlight is incident on the surface and the image is observed 100 cm deep in the water. Calculate the radius of curvature of the "spherical water surface."

1.6 THIN LENS EQUATION

1.6.1 Thin Lens Equation

A thin lens has two spherical surfaces with a short distance between them. The thin lens equation is a combination of the imaging equations applied to each of the two surfaces. In the derivation of the final equation, one ignores the distance between the spherical surfaces. The result is an imaging equation, which has the same absolute value for object and image focus. A positive lens has the object focus to the left and the image focus to the right. For the derivation, we assume that the lens has the refractive index n_2, real objects are in a medium with refractive index n_1, and virtual objects are in a medium with refractive index n_3.

To obtain the imaging equation of the thin lens we consider a convex and a concave spherical surface, separated by the distance a. The imaging equation for the first single spherical surface, as given in Eq. (1.35), is

$$-1/\zeta_o + 1/\zeta_i = 1/\rho_1, \tag{1.43}$$

where $\zeta_o = x_o/n_1$, $\zeta_i = x_i/n_2$, $\rho_1 = r_1/(n_2 - n_1)$, and all distances are measured from the center of the first surface. The imaging equation for the second spherical surface is described by

$$-1/\zeta_o' + 1/\zeta_i' = 1/\rho_2, \tag{1.44}$$

where $\zeta_o' = x_o'/n_2$, $\zeta_i' = x_i'/n_3$, $\rho_2 = r_2/(n_3 - n_2)$, and all distances are measured from the center of the second surface.

The two surfaces are positioned such that their distance in medium n_2 is "a" (Figure 1.9). To relate this distance to the image distance of the first surface and the object distance of the second surface, we place both at the same point (Figure 1.9). Measured from the first spherical surface the image is at $+\zeta_i$. Measured from the second spherical surface the object is at $-\zeta_o'$. Since ζ_o' and ζ_i are distances divided by the refractive index, we have to do the same with "a".
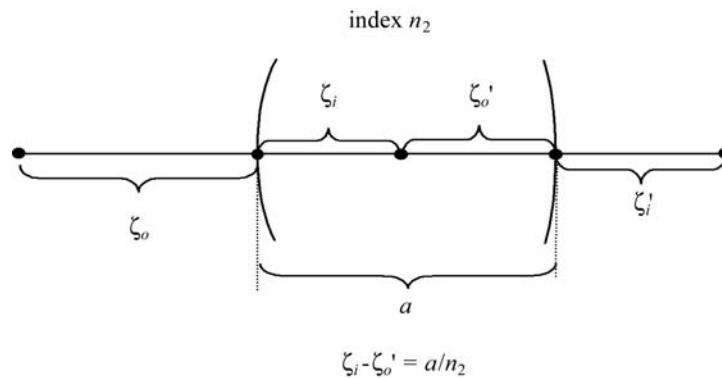

$$\zeta_i - \zeta_o' = a/n_2$$

FIGURE 1.9   Coordinates for the derivation of the thin lens equation.

To get the absolute value for a/n_2 we have

$$-\zeta_o' + \zeta_i = a/n_2. \tag{1.45}$$

The relation holds for the coordinates of each lens, and substitution into the equation for Surface 2 results in

$$-1/(-a/n_2 + \zeta_i) + 1/\zeta_i' = 1/\rho_2. \tag{1.46}$$

Adding Eq. (1.46) and the equation for Surface 1, that is, Eq. (1.43), we get

$$-1/\zeta_o + 1/\zeta_i - 1/(-a/n_2 + \zeta_i) + 1/\zeta_i' = 1/\rho_2 + 1/\rho_1. \tag{1.47}$$

The thickness a is now set to zero, two terms cancel each other out, and we obtain

$$-1/\zeta_o + 1/\zeta_i' = 1/\rho_1 + 1/\rho_2. \tag{1.48}$$

Rewriting Eq. (1.48) by using $\zeta_o = x_o/n_1$, $\zeta_i' = x_i'/n_3$, $\rho_1 = r_1/(n_2 - n_1)$, and $\rho_2 = r_2/(n_3 - n_2)$, and setting $x_i' = x_i$, we have

$$-n_1/x_0 + n_3/x_i = (n_2 - n_1)/r_1 + (n_3 - n_2)/r_2. \tag{1.49}$$

The focal length of the thin lens f is defined as

$$1/f = (n_2 - n_1)/r_1 + (n_3 - n_2)/r_2 \tag{1.50}$$

and depends on the refractive indices outside and inside the lens, and on the two radii of curvature. In most cases both sides of the lens have the same refractive index 1; that is, $n_3 = n_1 = 1$. Calling the refractive index of the lens n, we have

$$1/f = (n - 1)/r + (1 - n)/r'.$$

For a symmetric lens in air we obtain

$$1/f = 2(n - 1)/r.$$

Using $n_3 = n_1 = 1$ and the focal length of Eqs. (1.50), we have from Eq. (1.49) the *thin lens equation*,

$$-1/x_o + 1/x_i = 1/f. \tag{1.51}$$

There are positive and negative values for f, associated with positive and negative lenses. For example, a biconvex lens is a positive lens.

1.6.2 Object Focus and Image Focus

When f is positive, that is, for a positive lens, the object focus is on the left and has the coordinate $x_{of} = -f$, and the image focus is at $x_{if} = f$. When f is negative, that is, for a negative lens, the object focus is on the right and has the coordinate $x_{of} = |f|$, and the image focus is $x_{if} = -|f|$.
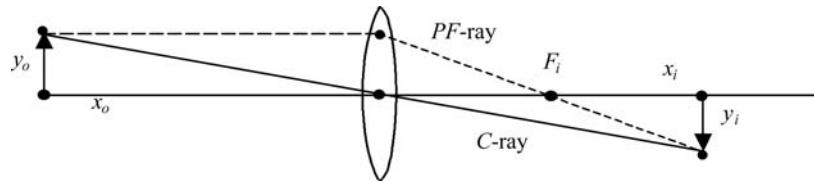
FIGURE 1.10   Graph of C-ray connecting object and image arrows. The length of the object arrow y_0 and image arrow y_i and their distances from the thin lens x_0 and x_i are also indicated.

1.6.3 Magnification

In Figure 1.10 we consider the case of a real object and real image and draw a line from the top of the object arrow through the center of the lens to the top of the arrow on the image arrow. The corresponding light ray is called the chief ray and is again referred to as the C-ray. It passes the lens at the center and therefore is not deviated by refraction. From the two "similar" triangles shown in Figure 1.10 we define the *magnification m* as

$$m = y_i/y_o = x_i/x_o. \tag{1.52}$$

1.6.4 Positive Lens, Graph, Calculations of Image Positions, and Graphical Constructions of Images

In FileFig 1.10 we show a graph of the thin lens equation. The image distance x_i is plotted as a function of x_o for positive f. There is a singularity at the object focus at $-f$. To the left of the object focus, x_i is positive. To the right between the object focus and lens, x_i is negative, and on the right of the lens it is positive. As a result, we have three sections. In the first and third sections, for a positive sign, the image is real. In the middle section, for a negative sign, the image is virtual.

In FileFig 1.11 we have chosen four specific values of object distances and calculate the corresponding image distances and magnifications.

FileFig 1.10   (G10TINPOS)

Graph of image coordinate x_i, depending on the object coordinate x_o for the thin lens equation with $f = 10$.

G10TINPOS

Positive Lens

Focal length f is positive, light from left propagating from medium with index 1 to lens of refractive index n. xo on left of surface (negative).

Calculation of Graph for xi as Function of xo over the Total Range of xo

Graph for xi as function of xo over the range of xo to the left of f. Graph for xi as function of xo over the range of xo to the right of f.
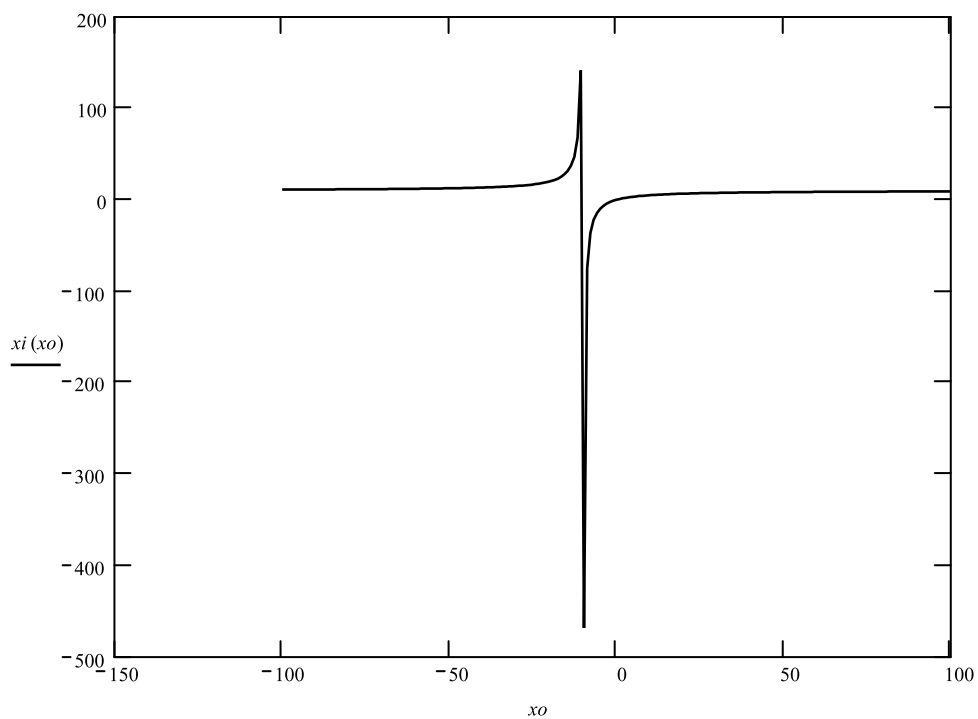
$$f \equiv 10$$

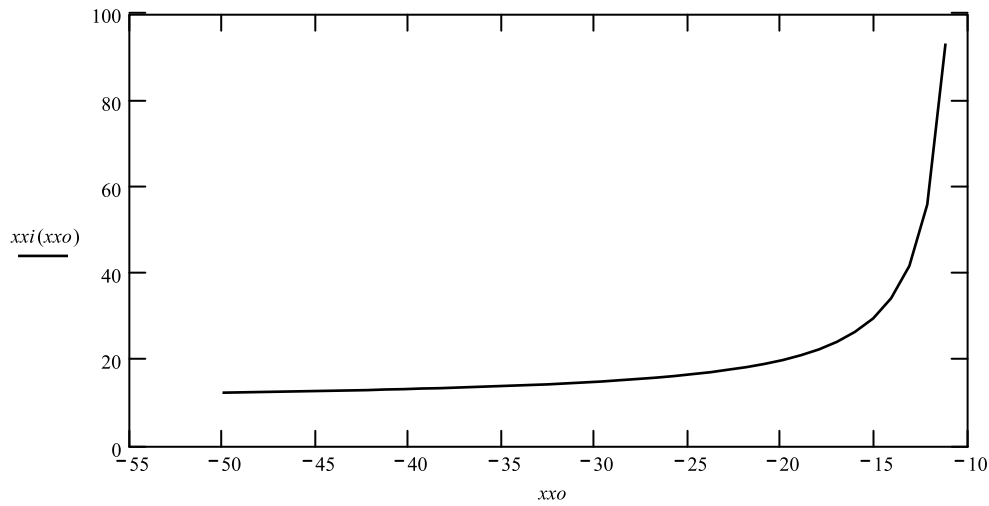*Image focus: f*   *Object focus: − f*

$$xo := -100.001, -99.031 \dots 100$$

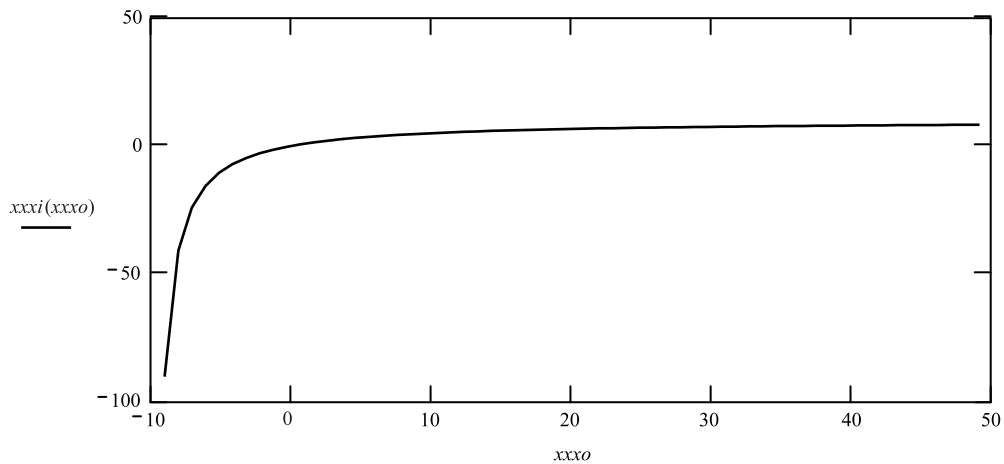$$xi(xo) := \frac{1}{\left(\frac{1}{f}\right) + \frac{1}{xo}}.$$

$$xxo := -50.001, -49.031 \dots - 11$$

$$xxi(xxo) := \frac{1}{\left(\frac{1}{f}\right) + \frac{1}{xxo}}.$$

$$xxxo := -9.001, -8.031 \ldots 50$$

$$xxxi(xxxo) := \frac{1}{\left(\frac{1}{f}\right) + \frac{1}{xxxo}}.$$

Application 1.10.

1. *Observe the singularity at the object focus, which has the same absolute value as the focal length but with a negative sign. The image focus has a positive sign. Note that they play different roles in the geometrical construction of the image.*

2. *Change the refractive index and describe what happens.*

3. *Change the focal length and describe what happens.*

FileFig 1.11   (G11TINPOS)

Calculation of image and object foci for $f = 10$. Calculation of image distances x_i and magnification for four specific values of object distance x_o

G11TINPOS

Positive Lens

Focal length f is positive, light from left propagating from medium with index 1 to lens of refractive index n. xo on left of lens (negative).

Calculation for Four Positions for Real and Virtual Objects, to the Left and Right of the Object Focus and Image Focus

Calculation of xi from given xo and focal length. Calculation of magnification.

$$f \equiv 10 \qquad n1 := 1 \qquad n2 : 1.5$$

Image focus: f $\hspace{6cm}$ *Object focus: $-f$*

1. $xo1 := -30$

$$xi1 := \frac{1}{\left(\frac{1}{f}\right) + \frac{1}{xo1}} \quad xi1 = 15 \quad mm1 := \frac{xi1}{xo1} \quad mm1 = -0.5.$$

2. $xo2 := -5$

$$xi2 := \frac{1}{\left(\frac{1}{f}\right) + \frac{1}{xo2}} \quad xi2 = -10 \quad mm2 := \frac{xi2}{xo2} \quad mm2 = 2.$$

3. $xo3 := 5$

$$xi3 := \frac{1}{\left(\frac{1}{f}\right) + \frac{1}{xo3}} \quad xi3 = 3.333 \quad mm3 := \frac{xi3}{xo3} \quad mm3 = 0.667.$$

4. $xo4 := 30$

$$xi4 := \frac{1}{\left(\frac{1}{f}\right) + \frac{1}{xo4}} \quad xi4 = 7.5 \quad mm1 := \frac{xi4}{xo4} \quad mm4 = 0.25.$$

Application 1.11. *The distance between the chosen object coordinate and the resulting image coordinate changes with the choice of the object coordinate.*

1. *Find analytically the condition for the shortest distance between image and object.*

2. *Make a graph of $y = -x_o + x_i$ depending on x_o and find the minimum.*
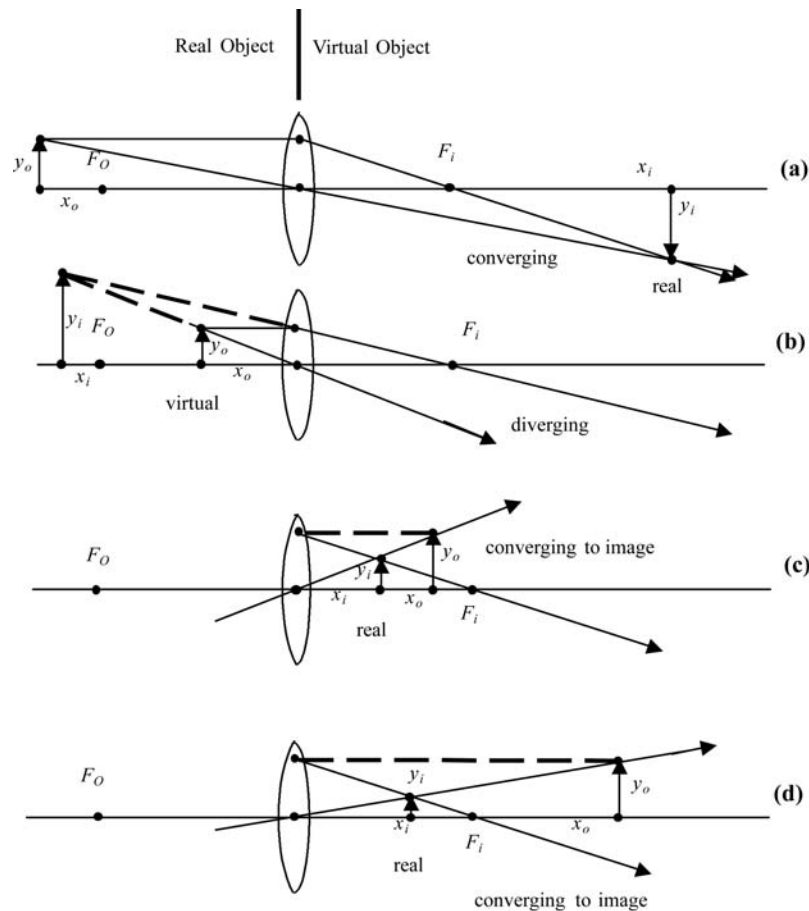
3. *Make a sketch.*

FIGURE 1.11   Geometrical construction of the images for a converging lens with positive f. Real objects for (a) and (b) and virtual objects for (c) and (d). The light converges to real images in (a), (c), (d). The light diverges in (b), and a virtual image is obtained by "trace back."

The geometrical construction of the images for the values calculated in FileFig 1.11 are shown in Figures 1.11a to d.

1. *Real object and real image.*
   The object is presented by an arrow of length y_o, placed at the object point x_o. The image point and the length of the arrow presenting the image can be geometrically determined. The C-ray is drawn from the top of the object arrow through the center of the thin lens. The second ray, the PF-ray, is drawn from the object arrow parallel to the axis to the lens, and from there, through the image focus. The two rays meet at the position of the image arrow. In Figure 1.11a, we obtain for a real object a real image.

2. *Real object and virtual image.*
   In Figure 1.11b we place the real object between the object focus and the lens and draw the C-ray and PF-ray. These rays diverge in a forward direction

and both are traced back to the left. They meet at the virtual image. A virtual image is always found when the C-ray and the PF-ray diverge in a forward direction. If we could place a screen into the position of a virtual image, we could not see it, because the rays toward the virtual image are diverging.

3. and **4.** *Virtual object and real images.*

In Figures 1.11c and 1.11d we place the object to the right of the lens. We are considering virtual objects. A virtual object may be produced by the image formed by another optical imaging system. The virtual objects are placed between the lens and the image focus and to the right of the image focus. In both cases we draw the C-ray in a forward direction. The PF-1 ray is drawn first backward to the lens and then forward through the image focus. The C-ray and the PF-ray converge to real images for all positions of the virtual object.

The results of the calculations of the positive thin lens with $f = -10$ are listed in Table 1.3.

1.6.5 Negative Lens, Graph, Calculations of Image Positions, and Graphical Constructions of Images

In FileFig 1.12, we show graphs of the thin lens equation, plotting x_i as a function of x_o for negative f. We see the singularity is at the object focus f, which is now to the right of the lens. To the left of the lens, x_i is negative. Between the lens and the object focus, x_i is positive. To the right of the object focus, x_i is negative. As a result, we have three sections. In the first and third sections, for a negative sign, the image is virtual. In the middle section, for a positive sign, the image is real.

In FileFig 1.13, we have calculated for four specific values of object distance the corresponding image distances and the magnification. In Figure 1.12 we have the geometrical construction of the images for the values calculated in FileFig 1.13.

TABLE 1.3   Positive Lens. $f = 10$, Image Focus 10, Object Focus -10

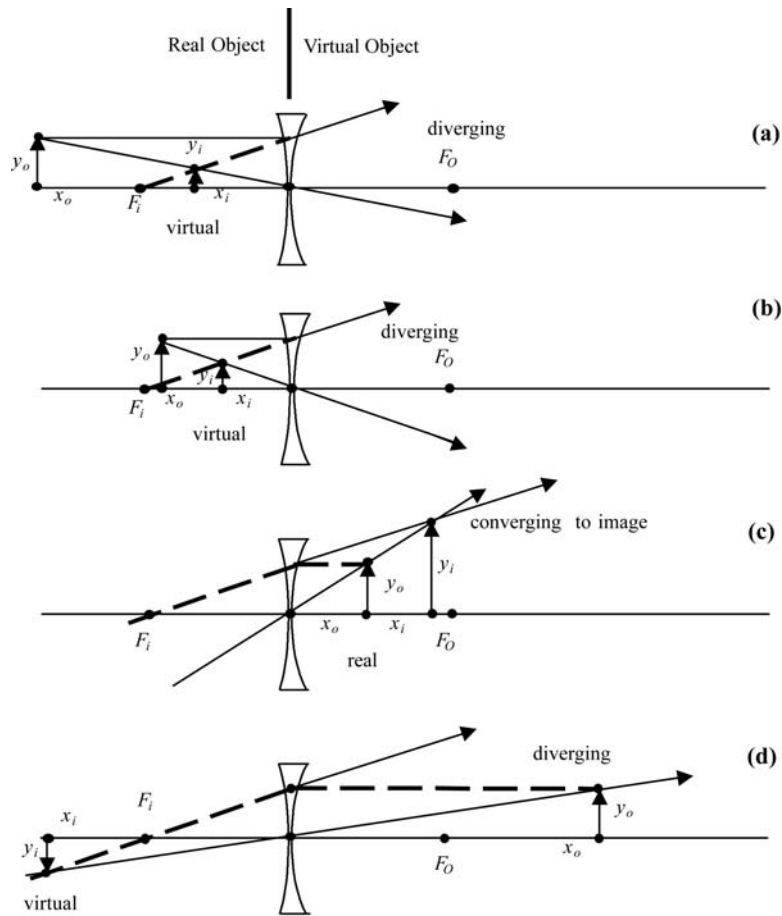| x_o	x_i	m	Image	Object
−30	15	−.5	r	r
−5	−10	2	vi	r
5	3.3	.67	r	vi
30	7.5	.25	r	vi

FIGURE 1.12   Geometrical construction of the images for a diverging lens with negative f. Real objects for (a) and (b) and virtual objects for (c) and (d). The light converges to real images in (c). The light diverges in (a), (b), (d), and a virtual image is obtained by "trace back."

FileFig 1.12   (G12TINNEG)

Graph of image coordinate x_i, depending on object coordinate x_o for the thin lens equation with $f = -10$.

G12TINNEG is only on the CD.

Application 1.12.

1. *Observe the singularity at object focus, which has the same absolute value as the focal length but with a positive sign. The image focus has a negative sign. Note that they play different roles in the geometrical construction of the image.*
2. *Change the refractive index and describe what happens.*
3. *Change the focal length and describe what happens.*

FileFig 1.13   (G13TINNEG)

Calculation of image focus and object focus for negative lens. Calculation of image distances x_i and magnification for four specific values of object distance x_o.

G13TINNEG is only on the CD.

Application 1.13. *The distance between the chosen object coordinate and resulting image coordinate changes with the choice of the object coordinate.*

1. *Modify the analytical calculation done in Application FF11 for the condition of the shortest distance between image and object.*
2. *Make a sketch.*

The geometrical construction of the images for the values calculated in FileFig 1.13 are shown in Figures 1.12a to d.

1. and 2. *Real object to the left of the lens and virtual image.*

   The object is presented by an arrow of length y_o, placed at the object point x_o to the left of the negative lens. The image point and the length of the arrow presenting the image can be geometrically determined using the C-ray and the PF-ray. The C-ray is drawn from the top of the object arrow through the center of the thin lens. The PF-ray is drawn from the object arrow parallel to the axis to the lens, and then diverges in a forward direction. It is traced back to the image focus. The two rays meet at the positions of the image arrow. In Figures 1.12a and 1.12b, we obtain for a real object a virtual image. A virtual image is obtained when the C-ray and the PF-ray diverge in a "forward" direction.

3. *Virtual object between lens and object focus.*

   In Figure 1.12c, we place the virtual object between the object focus and the lens and draw the C-ray. The PF-ray is first traced back to the lens, then connected to the image focus, and extended in the forward direction. The two rays meet in the forward direction at a real image.

4. *Virtual object on the right side of the object focus.*

   In Figure 1.12d, we place the virtual object to the right of the object focus and draw the C-ray. The PF-ray is first traced back to the lens, then connected to the image focus and extended further in the backward direction. The two rays meet in the backward direction for a virtual image.

The results of the calculations of the negative thin lens with $f = -10$ are listed in Table 1.4

For the geometrical construction, we note that the size of the lens does not matter. One uses a plane in the middle of the lens with sufficient extension in the y direction; see Figure 1.13a.

TABLE 1.4 Negative Lens. $f = -10$, Image Focus -10, Object focus 10

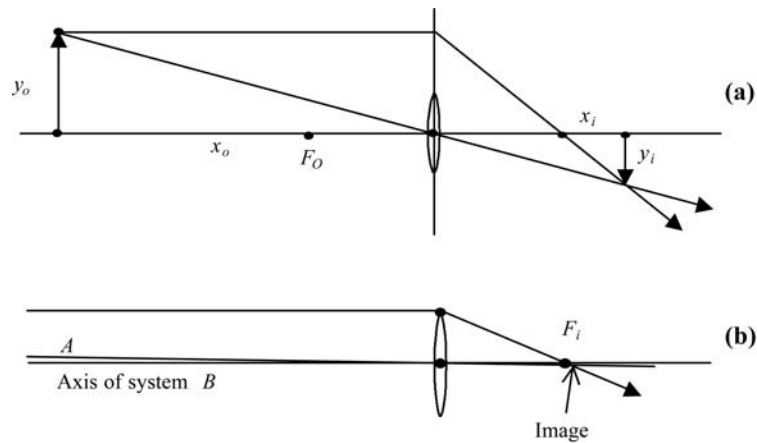| x_o	x_i	m	Image	Object
-30	-7.5	.25	vi	r
-5	-3.3	.67	r	r
5	10	2	vi	vi
30	-15	$-.5$	vi	vi

FIGURE 1.13   (a) Image formation of an object larger than the diameter of the lens. The extended plane of the lens is used; (b) image formation for an object at infinity. The axis B of the system is the ray from the center of the object through the center of the lens. the PF-ray is assumed to come from the top of an object at finite distance; the corresponding image is indicated.

If the object is at infinity, one uses for the object distance a finite number so that the image is not exactly at the focal point, where it would have a length equal to zero (Figure 1.13b).

1.6.6 Thin Lens and Two Different Media on the Outside

We go back to the thin lens equation and choose different indices of refraction at the two media on both sides of the lens. We start again from the definitions $\zeta_o = x_o/n_1$, $\zeta_i' = x_i/n_3$, $\rho_1 = r_1/(n_2 - n_1)$ and $\rho_2 = r_2/(n_3 - n_2)$ and have

$$-n_1/x_0 + n_3/x_i = (n_2 - n_1)/r_1 + (n_3 - n_2)/r_2. \tag{1.53}$$

We call the focal length of the thin lens f_n given by

$$1/f_n = (n_2 - n_1)/r_1 + (n_3 - n_2)/r_2 \tag{1.54}$$

and obtain the *thin lens equation*

$$-n_1/x_o + n_3/x_i = 1/f_n. \tag{1.55}$$

This equation is very similar to the spherical surface imaging equation discussed in Section 1.4a just as we found there, we have different values for the object focus and image focus.

For the object focus, when the image point is assumed to be at positive infinity, we have

$$x_{\text{of}} = -n_1 f_n \tag{1.56}$$

and for the image focus, obtained when the object point is assumed to be at negative infinity, we have

$$x_{\text{if}} = n_3 f_n. \tag{1.57}$$

The construction of the images for positive and negative lenses is similar to the procedure for the spherical surfaces and is not discussed further. The value of the focal length for different cases of the refractive indices may be calculated using FileFig 1.14.

FileFig 1.14   (G14TINFOC)

Calculation of the focal length and object and image focus of the thin lens for different combinations of the refractive indices.

G14TINFOC

Focal Length

1. *Calculation of focal length of thin lens of refractive index n2 in medium with refractive index n1.*
   First surface: $r1 := -5$. Second surface: $r2 := 5$. r is positive for convex surface, negative for concave surface. Refractive index of lens n2: $n2 := 1$. and Refractive index of medium n1: $n1 := 1.5$.

2. *Graph of focal length of thin lens with index n2 depending on refractive index of medium n1.*
   The range on n1 is divided into lower and higher ranges because of singularity. Refractive index of lens nn2: $nn2 := 1.5$. Lower range: $nn1 := 1, 1.1 \ldots nn2 - .00001$. Upper range: $nnn1 := nn2 + .1, nn2 + .2 \ldots 4$.

$$ff(nn1) := \frac{1}{\frac{nn2 - nn1}{r1} + \frac{nn1 - nn2}{r2}} \qquad fff(nnn1) := \frac{1}{\frac{nn2 - nnn1}{r1} + \frac{nnn1 - nn2}{r2}}.$$

$ff(nn1)$

$fff(nnn1)$

$nn1$

$nnn1$
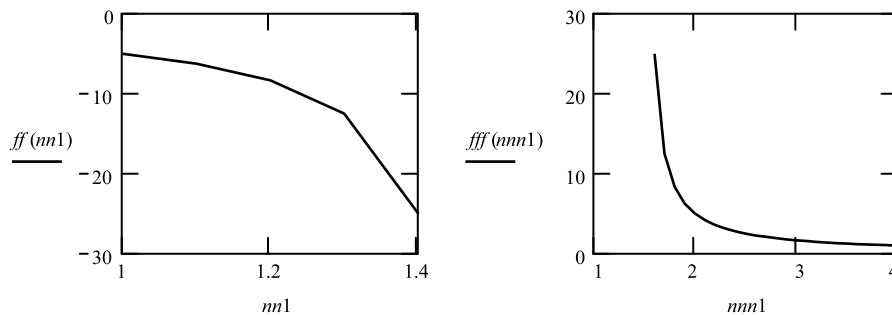
Application 1.14. *Consider the case $n_2 > n_1$. What is the result when interchanging n_1 and n_2?*

Applications for the Sections on Positive and Negative Lenses

1. Air lens in plastic. A plastic rod is flat on one side and has a spherical surface on the other side. The spherical surface is concave with respect to the incident light, which comes from the flat side. An identical second rod is taken and the two curved ends are put together, forming an air lens by the ends of the two rods. The cross-section of this lens has its thinnest point in the middle. Assume that the radii of curvatures of the spherical surfaces are $r = -r' = 10$ cm and the refractive index of the rod is $n = 1.5$. Sunlight is incident on an object on the face of the first rod at 20 cm from the air lens. Find the image distance.

2. Thin lens on water. A lens of refractive index $n = 1.5$ is put on water, one surface in air, the other in water. The lens is a symmetric biconvex lens and has a focal length of $f = 10$ cm in air. The refractive index of water is $n = 1.33$.
   a. Calculate the radii of curvature of the lens in air and the focal length to be used in the above position.
   b. Sunlight is shining on the lens; calculate the image distance in the water.

1.7   OPTICAL INSTRUMENTS

Optical instruments, such as magnifiers, and microscopes, enlarge tiny objects, making it possible to observe objects we can barely see with the naked eye. The magnifier gives us a modest magnification, in most cases less than ten times. The microscope makes it possible to observe objects of about 1 micron diameter, and the telescope enables us to see objects at a far distance in more detail. Our eye is a one lens system and may produce a real image of a real object, like a positive lens (Figure 1.11a). The real image of a real erect object of a positive lens is

inverted. However, our brain makes a "correction" (another inversion) and we "see" the object erect, as it is. In discussing optical instruments, we have to take this fact into account when making statements about image formation. For a microscope or astronomical telescope it does not matter much if the final image is erect or inverted. However, for the telescope of a sharpshooter it is important.

From Figures 1.11 and 1.12, we read a simple rule: If the image appears at the same side of the lens as an erect object, it is erect. If it appears on the other side of the lens, it is inverted.

1.7.1 Two Lens System

To obtain the final image distance of a two-lens system, one first applies the thin lens equation to the first lens and determines the image distance. The object distance for the second lens is calculated from the distance between the two lenses and the image distance of the first lens. The thin lens equation is then applied to the second lens and the final image distance for a two-lens system is obtained as the distance from the second lens. The formulas for this procedure are listed in FileFig 1.15.

For graphical constructions one proceeds in the same way. Using C- and PF-rays, one constructs the image of the first lens. The image is taken as an object for the second lens, and C- and PF-rays are used to construct the image formed by the second lens. The existence of the first lens is ignored when going through the second process.

The magnification of the system is the product m of the magnification of each of the two lenses. One has $m = m_1 m_2$ with $m_1 = x_{i1}/x_{o1}$ and $m_2 = x_{i2}/x_{o2}$, where m_1 is calculated with respect to the first lens and m_2 with respect to the second lens.

FileFig 1.15   (G15TINTOW)

Calculation of the final image distance of a two-lens system, for a given object distance of the first lens, focal length, and separation of the two lenses.

G15TINTOW

Two Thin Lenses, Distance Between Lenses: D

1. *First lens, $xo1$, $xi1$, $f1$*

$$xo1 := -5 \qquad f1 := 6$$

$$xi1 := \frac{1}{\left(\frac{1}{f1}\right) + \frac{1}{xo1}} \qquad xi1 = -30.$$

2. *Second Lens, xo2, xi2, f2, and Distance D (Positive Number)*

$$D := 10 \qquad f2 := 1.85.$$

The image distance of the first process is given with respect to the first lens. (Let us assume it is positive.) The object distance must be given with respect to the second lens, taking the distance D between the two lenses into account. (D is negative when counted from the second lens.) Therefore we have

$$xo2 := -D + xi1 \qquad xo2 = -40$$
$$xi2 := \frac{1}{\left(\frac{1}{f2}\right) + \frac{1}{xo2}} \qquad xi2 = 1.94.$$

3. *Magnification for each lens and product for the magnification of the system*

$$m1 := \frac{xi1}{xo1} \qquad m1 = 6$$
$$m2 := \frac{xi2}{xo2} \qquad m2 = -0.048$$

System

$$m1 \cdot m2 = -0.291.$$

Application 1.15.

1. *Distance between the lenses is larger than $2f$. Calculate the final image distance for two lenses at distance $D = 50$. Assume that the object distance from the first lens is -20. Give the magnification, and make a sketch of object and image, assuming that the object is erect. Consider the following cases.*
   a. *First lens $f_1 = 10$; second lens $f_2 = 10$.*
   b. *First lens $f_1 = 10$; second lens $f_2 = -10$.*
   c. *First lens $f_1 = -10$; second lens $f_2 = 10$.*
   d. *First lens $f_1 = -10$; second lens $f_2 = -10$.*
2. *Distance between the lenses is smaller than $2f$. Calculate the final image distance for two lenses at distance $D = 6$. Assume that the object distance from the first lens is -20. Give the magnification, and make a sketch of object and image, assuming that the object is erect. Consider the following cases.*
   a. *First lens $f_1 = 10$; second lens $f_2 = 10$.*
   b. *First lens $f_1 = 10$; second lens $f_2 = -10$.*
   c. *First lens $f_1 = -10$; second lens $f_2 = 10$.*
   d. *First lens $f_1 = -10$; second lens $f_2 = -10$.*

1.7.2 Magnifier and Object Positions

The size of an image on the retina increases when placed closer and closer to the eye. There is a shortest distance at which the object may be placed, called the near
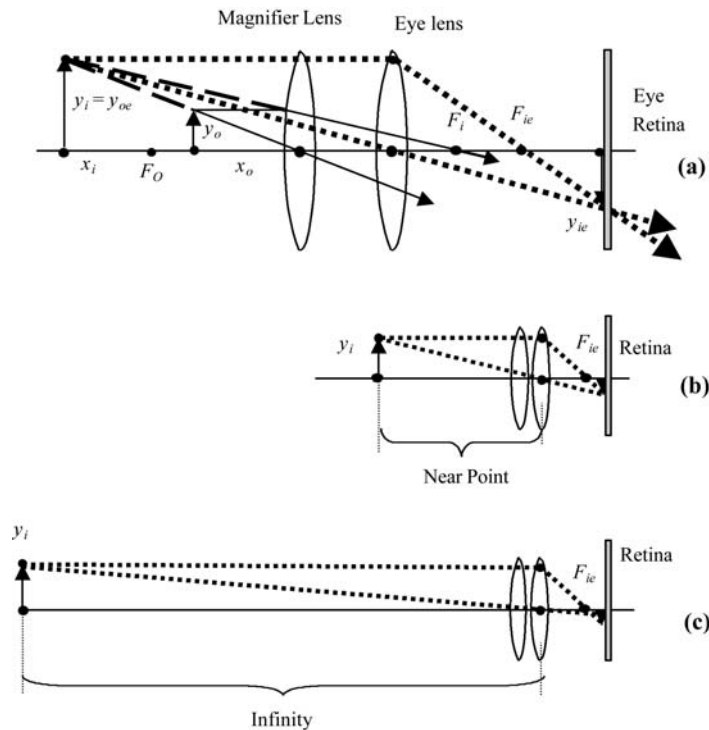
FIGURE 1.14   Two positive lenses in the magnifier configuration: (a) the virtual image y_i of the object y_0 serves as object Y_{oe} for the eye lens. The image y_{ie} (see bold dotted lines) appears on the retina upside down; we see it therefore erect; (b) the object of the eye in the near field configuration; (c) the object of the eye in the infinity configuration.

point at about 25 cm. For shorter distances the eye can no longer accommodate production of an image because the eye–retina distance is fixed. To increase the size of the object one may use a positive lens as a magnifier. In Figure 1.14, we show the magnifier and the eye as a two-thin-lens system. In FileFig 1.16 we show the calculation of the image distance for a two lens system. We assume that the positive lens and the eye are separated by a distance of $D = 1$ cm. Object distance and focal lengths of the lenses are both input data.

From Figure 1.14, we see that the first lens produces a virtual erect image of a real erect object. The second lens (eye) treats the virtual erect image as a real erect object and produces a real inverted image on the retina. The final image on the retina is inverted. However, we "see" it upright because our brain does the conversion. The virtual image of the magnifier lens is the object of the eye lens. The object producing this virtual image may only be positioned with respect to the magnifier in such a way that the virtual image is not closer than the near point, but may have a distance as large as negative infinity. We therefore discuss the two cases: the virtual image is at the near point; and the virtual image is at infinity.

FileFig 1.16   (G16MAG2L)

Calculation of the image distance for a two-lens system consisting of a positive lens and the eye lens. Magnification for each lens and the system.

G16MAG2L is only on the CD.

Application 1.16. *Object distance at $x_{o1} = -5$, focal length of first lens $f_1 = 6$, distance D between lens and eye is $D = 0$, focal length of eye $f_2 = 1.85$. Study different resulting magnifications for changes of x_{o1} and f_1.*

1.7.2.1 Virtual Image at Near Point

The virtual image produced by the first lens is the real erect object for the second lens (eye), and is assumed to be at the near point (-25 cm). In the first step, we calculate the object distance for the first lens when the image is at -25 cm from the second lens (eye). In the second step we consider the eye. The calculation is shown in FileFig 1.17 where the magnification of the magnifier is given as

$$m_1 = x_{i1}/x_{o1} \tag{1.58}$$

and of the eye as

$$m_2 = x_{i2}/x_{o2}. \tag{1.59}$$

Considering only the magnification m1 of the magnifier, one may use the thin lens equation in order to express m1 in known quantities; that is, f_1 and $x_{i1} = -25$. We have

$$m_1 = x_{i1}/x_{o1} = x_{i1}(1/x_{o1}) = x_{i1}(-1)(1/f - 1/x_{i1}) = (1 - x_{i1}/f_1). \tag{1.60}$$

Neglecting the distance D between magnifier and eye lens, and setting $x_{i1} = -25$, we obtain for the magnification,

$$m_1 = 1 + 25/f_1. \tag{1.61}$$

1.7.2.2 Virtual Image at Infinity

The virtual image produced by the first lens is assumed to be at negative infinity ($-\infty$). It is the real erect object for the second lens (eye). The calculation is shown in FileFig 1.18 for $f_1 = 12$, and taking for x_{i1} the numerical value of -10^{10}. For the magnification of the magnifier we get, after using the thin-lens equation, similarly done as in Eq. (1.60),

$$m_1 = x_{i1}/x_{o1} = 1 - x_{i1}/f_1 = 8.333 \cdot 10^8,$$

This is a meaningless number. In order to discuss the case where the virtual image is at infinity, we have to change our approach and consider angular magnification.
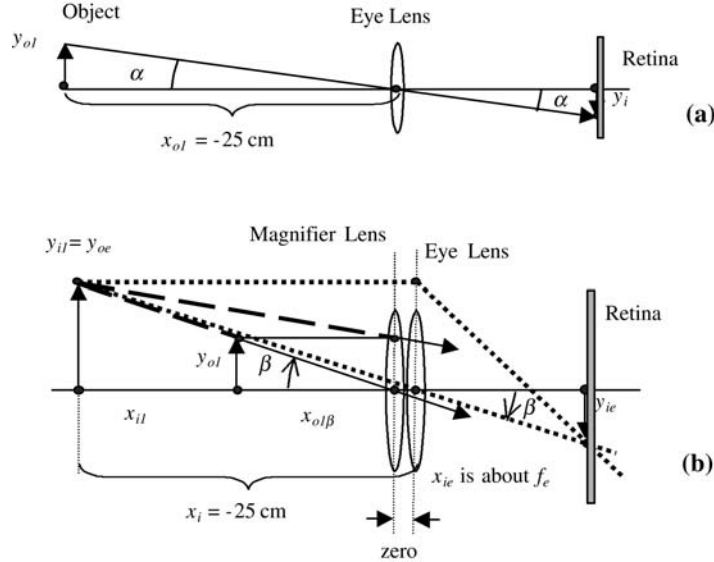
FIGURE 1.15 Angular magnification; (a) object at the near point, seen with the eye lens; object at the near point, seen with magnifier and eye lens.

1.7.2.3 Angular Magnification or Magnifying Power

To avoid the difficulties we encountered in Section 1.7.2.2, where we calculated meaningless numbers for the magnification, we take a different approach and use angular magnification. We compare the angles at the eye by looking at the object with and without a magnifier (Figure 1.15).

The object is positioned at the near point because that gives the largest magnification without a lens. First the eye looks at the object without a magnifier, (Figure 1.15a), where angle α is

$$\alpha = y_{o1}/x_{o1} = y_{o1}/(-25). \tag{1.62}$$

Then we introduce the magnifier and have for the angle β, as shown in Figure 1.15b,

$$\beta = y_{i1}/x_{i1} = y_{o1}/x_{o1\beta} = y_{o1}(1/x_{i1} - 1/f_1), \tag{1.63}$$

where $x_{o1\beta}$ is the object distance when calculating the angle β, and the thin-lens equation was used to eliminate $x_{o1\beta}$.

We define the angular magnification or magnifying power as

$$MP = \beta/\alpha = -25(1/x_{i1} - 1/f_1). \tag{1.64}$$

We now discuss the applications of angular magnification to the cases where the virtual image is at the near point and at infinity.

1. Near point.

The object is at the near point, and assuming $D = 0$ we have $x_{o1} = x_{i1} = -25$ and get

$$MP = 1 + 25/f_1. \tag{1.65}$$

This is the same expression we obtained in Section 1.7.2.1; for the case of the Near point, see Eq. (1.61).

FileFig 1.17   (G17MAGNP)

Calculations of the magnifier in the near point configuration. Assume $D = 0$. First step: Determination of object point for image point at -25 for first lens with $f_1 = 12$, result $x_{o1} = -8.108$. Second step: Determination of x_{i2} for $x_{o2} = -25$ and eye lens $f_2 = 1.85$, result $x_{i2} = 2$. Calculation of magnification.

G17MAGNP is only on the CD.

Application 1.17. *Find the resulting magnifications for three choices of f_1.*

2. Virtual image at infinity.

We consider the virtual image of lens 1 as the real object of lens 2. We have $x_{i1} = -\infty$, and have for the angular magnification

$$\begin{aligned} MP &= -25(1/x_{i1} - 1/f_1) \tag{1.66} \\ &= 25/f_1. \end{aligned}$$

This value is marked on magnifiers as MP times x. Example: for $f_1 = 5$ we would have $MP = 5x$.

In both cases the object is placed at the near point of the eye without a magnifier, and the resulting angular magnification depends on the focal length of the magnifier.

FileFig 1.18   (G18MAGIN)

Calculations of the magnifier for the "virtual image at infinity" configuration. Assume $D = 0$. First step: Determination of object point for image point $x_{i1} = -10^{10}$, that is, at $(-\infty)$ for the first lens with $f_1 = 12$, result is $x_{o1} = -12$. Second step: Determination of x_{i2} for $x_{o2} = (-\infty)$, for the eye lens $f_2 = 1.85$, result is $x_{i2} = 1.85$. Calculation of magnification.

G18MAGIN is only on the CD.

Application 1.18. *Study several resulting magnifications for three choices of f_1.*
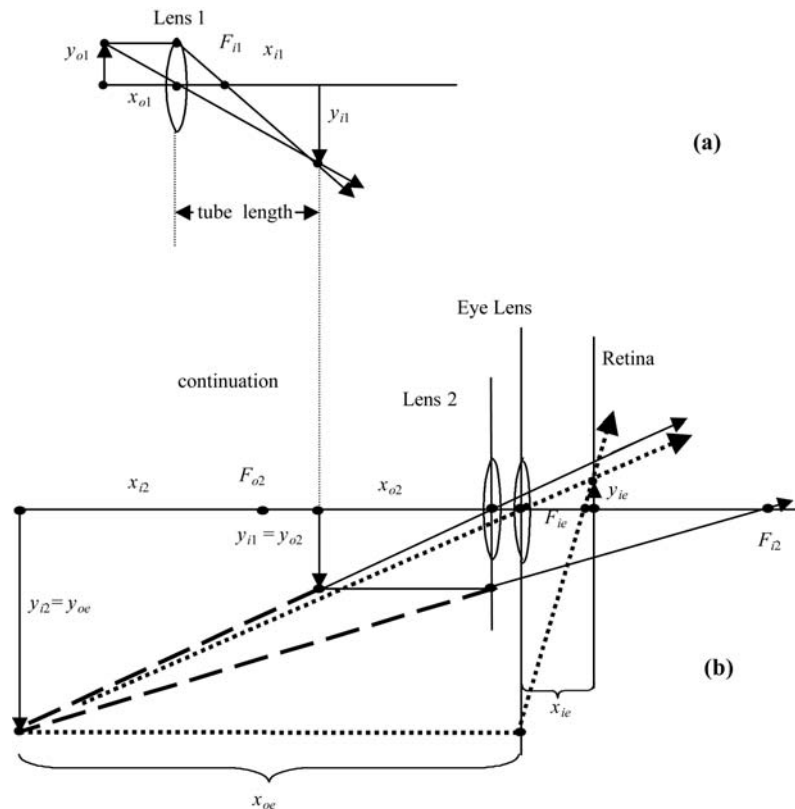
FIGURE 1.16 Microscope as three-lens system of objective, magnifier (ocular), and eye. The object is close to the focal length of the objective lens $L1$ and the image is y_{i1}. The magnifier $L2$ and eye lens act in the magnifier configuration on the image y_{02} produced by $L1$. The image y_{i2} is the object y_{0e} for the magnifier and the image y_{ie} appears on the retina erect; we see it therefore upside down.

1.7.3 Microscope

1.7.3.1 Microscope as Three-Lens System

In a compound microscope, the first lens $L1$ (objective lens) has a short focal length and forms a real inverted image of a real erect object. Then the magnifier configuration is applied, which is the second lens $L2$ (ocular lens) plus the eye lens. See Section 1.7.2 above and Figure 1.16. The final image on the retina is erect, but we see it upside down.

We ignore the eye lens and calculate the final image of a two lens system, using for the image distance x_{i1} the fixed value of tube length 16 cm plus F_{i1} (in cm), see Figure 1.16. The magnification is the product of m_1 of the objective lens, times m_2 of the ocular lens (magnifier). We discuss the following cases where the magnifier is used in (1) the near point configuration, and (2) the virtual image at infinity configuration.

1. Magnification, Near Point Configuration, Magnifying Power

In FileFig 1.19 we calculate the magnification, using $f_1 = 2$, $x_{i1} = 16 + f_1$, $f_2 = 6$, and $x_{i2} = -25$; we have for the magnification

$$m = m_1 m_2 = (x_{i1}/x_{o1})(x_{i2}/x_{o2}) = -41.34. \tag{1.67}$$

The magnifying power MP for the magnifier in the Near point configuration was obtained in Eq. (1.66), and was the same as the magnification $m_1 m_2$. Using the thin-lens equation to x_{o1} and x_{02} we have

$$MP = m_1 m_2 = x_{i1}(-1)(1/f_1 - 1/x_{i1})x_{i2}(-1)(1/f_2 - 1/x_{i2})$$
$$= (1 - [16 + f_1]/f_1)(1 + 25/f_2) \tag{1.68}$$

and as the result we get $m_1 m_2 = -41.34$. Neglecting f_1 with respect to 16 we have

$$MP \approx (1 - 16/f_1)(1 + 25/f_2) = -36.17. \tag{1.69}$$

The negative magnification indicates that we see the object upside down.

FileFig 1.19   (G19MICNP)

Calculations of the microscope in the near point configuration. The object is close to the focal point of lens 1. Lens 1: $f_1 = 2$ cm; $x_{i1} = +16 + 2$ cm, result $x_{o1} = -2.25$ cm. The magnifier lens L2 is in the near point configuration. Lens 2: $f_2 = 6$ cm, $x_{i2} = -25.008$ cm; $x_{o2} = -4.839$ cm. The angular magnification is also calculated.

G19MICNP is only on the CD.

Application 1.19. *Go through all the steps and study the resulting magnification by changing f_1 and f_2.*

2. Magnification, Virtual Image at Infinity, Magnifying Power

We assume that the virtual image is at infinity; that is, $x_{2i} = -\infty$. The calculations using the direct approach, which is $m = (x_{i1}/x_{o1})(x_{i2}/x_{o2})$, are shown in FileFig 1.20. Using $f_1 = 2$ cm, $x_{i1} = 16 + f_1$, $f_2 = 6$ cm, and $x_{i2} = -10^{10}$ cm, we obtain a meaningless number.

The magnifying power in the near point configuration of the magnifier was obtained in Eq. (1.68) as $MP = (1 - [16 + f_1]/f_1)(1 + 25/f_2)$. The second factor changes for the case where the "virtual image is at infinity," and the result is

$$MP = (1 - [16 + f_1]/f_1)(25/f_2) = -33.333. \tag{1.70}$$

Neglecting f_1 with respect to 16 one has

$$MP = -(16/f_1)(25/f_2) = -29.167. \tag{1.71}$$

One may also disregard the one in the first factor and have $MP = -(16/f_1)(25/f_2)$.

FileFig 1.20   (G20MICIN)

Calculations of the microscope in the "virtual image at infinity" configuration. The virtual image is at infinity; that is, $x_{2i} = -\infty$. Lens 1: $f_1 = 2$ cm; $x_{i1} = +16 + 2$ cm; result $x_{o1} = -2.25$ cm. Lens 2: $f_2 = 6$ cm; $x_{i2} = -10^{10}$ cm; result $x_{o2} = -6$ cm. The magnification is also calculated neglecting f_1.

G20MICIN is only on the CD.

Application 1.20. *Go through all the steps and study the resulting magnification by changing f_1 and f_2.*

1.7.3.2 Magnification of Commercial Microscopes

Commercial microscopes give the magnification of the objective and eye lens by a MPx value, similar to the one discussed above for the magnifier. For example, the magnifier power MP of the microscope was approximately $-(16/f_1)(25/f_2)$. Assuming $f_1 = 2$ and $f_2 = 6$, the objective would be marked 8x and the ocular 4x. The magnification of this microscope would be 32 times.

1.7.4 Telescope

1.7.4.1 Kepler Telescope

In a simple telescope, the first lens L1 forms an image of a far away object at a distance close to the focal point f1 of the objective lens (Figure 1.17). The object is considered real and erect, and the image is real inverted. The second lens is the magnifier lens and the eye and magnifier lens are used together in the virtual image at $-\infty$ configuration. In this setup the image of lens 1, which is the object of lens 2, is close to the focal point of f_2, and forms an inverted virtual image at infinity. When we look at this virtual image the final image on the retina is erect, but we see it upside down. The calculations are shown in FileFig 1.21. To find the approximate magnification of the telescope, we do not need to use the concept of magnifying power and can use the calculation of the magnification:

$$m = (x_{i1}/x_{o1})(x_{i2}/x_{o2}), \tag{1.72}$$

where $m_1 = x_{i1}/x_{o1}$ is about f_1/x_{o1}, because the image of lens 1 is close to the focal point. For $m_2 = x_{i2}/x_{o2}$ we have approximately $-x_{i2}/f_2$, because the object for f_2 is close to the focal point. Since x_{o1} and x_{i2} are both large numbers of the same order of magnitude, they cancel each other out and for the
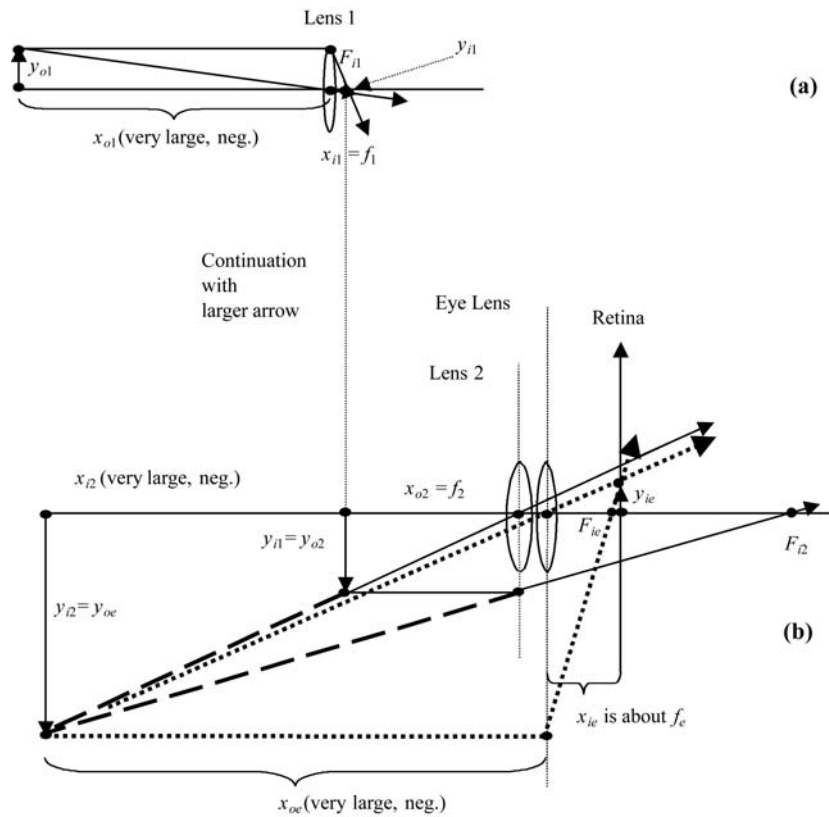
FIGURE 1.17 Optical diagram of a Kepler telescope: (a) the object is far away from the objective lens $L1$ and the image is y_{i1}, located close to $x_{i1} = f_1$; (b) the image y_{i1} is the object for the magnifier $L2$ and eye lens in the magnifier configuration and produces the virtual image $y_{i2} = y_{0e}$. The final image y_{ie} appears on the retina erect, we therefore see it upside down. The distance $f_1 + f_2$ is approximately the length of the telescope.

magnification we have

$$m = m_1 m_2 = -f_1/f_2. \tag{1.73}$$

Note that this is a negative number since f_1 and f_2 are both positive, and the object is "seen" inverted.

To get a large magnification, we need a large value of f_1 and a small one of f_2. The large value of the focal length of the first lens makes powerful telescopes "large."

FileFig 1.21   (G21TELK)

The Kepler telescope is treated as a two-lens system, assuming for x_{o1} and x_{i2} the same large negative numerical values. The magnification is calculated from

$m = (x_{i1}/x_{o1})(x_{i2}/x_{o2})$ *and results in* $m = m_1 m_2 = -f_1/f_2$. *Lens L1:* $f_1 = 30$; $x_{o1} = -10^{10}$; $x_{i2} = 30$. *Lens L2:* $f_2 = 6$; *distance* $a = f_1 + f_2$; $x_{i2} = -10^{10}$, $x_{o2} = -6$. *Calculation of magnification.*

G21TELK is only on the CD.

Application 1.21. *Study magnifications of 2 and 4 by changing* f_1 *and* f_2 *and make a sketch.*

1.7.4.2 Galilean Telescope

The Galilean telescope is the combination of a positive lens L1 and a negative lens L2. The positive lens forms a real inverted image of a far-away real erect object (Figure 1.18a). The negative lens replaces the magnifier. The image of lens 1 is the object for lens 2 and is virtual inverted, see Fig. 1.12(d). Lens 2 forms a virtual erect image of it, at negative infinity (Figure 1.18b). The eye looks at the virtual erect image of lens 2 as a real erect object and forms a real inverted image on the retina (we see it erect). The calculation is shown in FileFig 1.22.
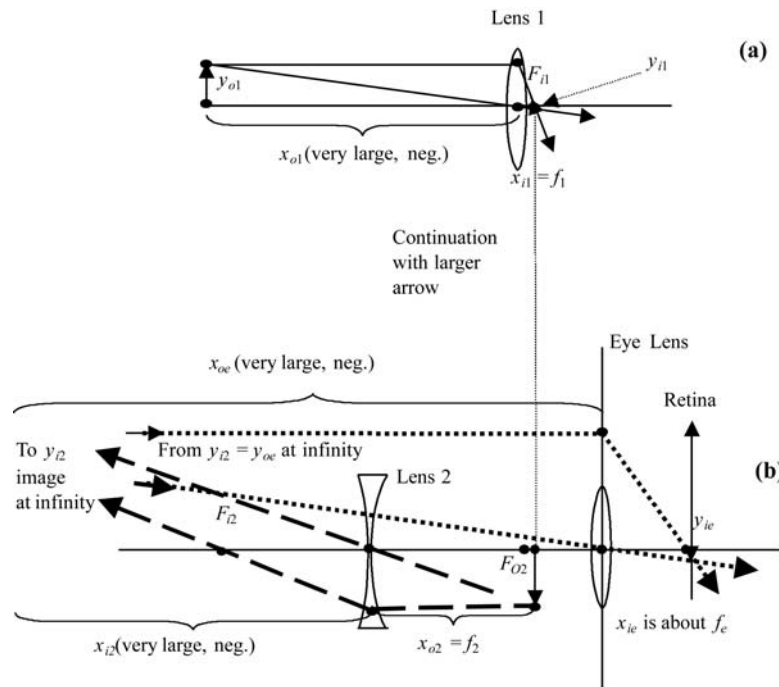
FIGURE 1.18   Optical diagram of a Galileo telescope: (a) the object is far away from the objective lens $L1$ and the image is y_{i1}, located close to $x_{i1} = f_1$; (b) the image of lens 1 is virtual inverted object for lens 2, and lens 2 forms a virtual erect image of it. This virtual erect image is the object of the eye lens and the image y_{ie} appears on the retina upside down, therefore we see it erect.

For the magnification one gets:

$$m = (x_{i1}/x_{o1})(x_{i2}/x_{o2}), \tag{1.74}$$

where $m_1 = x_{i1}/x_{o1}$ is approximately f_1/x_{o1} because the image of lens 1 is close to the focal point. The magnification of the second lens, $m_2 = x_{i2}/x_{o2}$, is approximately $-x_{i2}/f_2$, because the object of lens 2 is close to the focal point and to the right side of the lens, and f_2 is negative. Since x_{o1} and x_{i2} are both large numbers, of the same order of magnitude, they cancel each other out and we have for the magnification

$$m = m_1 m_2 = -f_1/f_2. \tag{1.75}$$

Note that this is a positive number since f_2 is a negative lens, and the object is seen erect. The Galilean telescope is used for many terrestrial applications in theaters and on ships.

FileFig 1.22   (G22TELG)

The Galilean telescope is treated as a two-lens system with the first lens having a positive focal length and the second lens a negative focal length. For x_{o1} and x_{i2} the same large negative numerical values are assumed. The magnification is calculated as $m = (x_{i1}/x_{o1})(x_{i2}/x_{o2})$ and results in $m = m_1 m_2 = -f_1/f_2$. (Note that the numerical value is positive.) Lens L1: $f_1 = 30$, $x_{o1} = -10^{10}$, $x_{i1} = 30$ Lens L2: $f_2 = -29.99$, $x_{i2} = -9 \cdot 10^4$; $x_{o2} = 30$.

G22TELG is only on the CD.

Application 1.22. *Go through all the stages and study magnifications by changing f_1 and f_2.*

Applications to Two- and Three-Lens Systems

1. Magnifier. A magnifier lens of $f_1 = 12$cm is placed 8 cm from the eye.
   a. Find the position of x_{o1} for
      i. the near point configuration; and
      ii. the infinite configuration.
   b. Give the magnification and the angular magnification.
2. Microscope. A microscope has a first lens (objective) with focal length .31 cm, a magnifier (ocular) lens of 1.79 cm, and the eye lens is assumed to be $f_e = 2$ cm. The focal length of the objective lens has been chosen so that the image is at about 16 cm. The distance between the lenses is 18 cm and we assume that the eye is in near point configuration. Calculate the magnification of the

first and of the second lenses, and compare the product with the magnifying power, as derived, and its approximation.

3. Microscope (near point). A microscope has a first lens (objective) with focal length 1.31 cm and a magnifier (ocular) lens of 1.79 cm. We assume that the image of the first lens is at 16 cm and the eye is in the near point configuration.

   a. Find the object distance for the objective lens.

   b. Find the distance from the first image and the magnifier lens.

   c. Find the distance between the lenses (length of microscope).

   d. Find the magnification.

4. Microscope $(-\infty)$. A microscope has a first lens (objective) with focal length 1.31 cm and a magnifier (ocular) lens of 1.79 cm. We assume that the image of the first lens is at 16 cm and the eye is relaxed, looking at $-\infty$.

   a. Find the objective distance for the objective lens.

   b. Find the distance from the first image and the magnifier lens.

   c. Find the distance between the lenses (length of microscope).

   d. Find the magnification.

5. Kepler telescope. Make a suggestion for construction of a Kepler telescope with magnifications of 4 and 10. At what higher number does the construction become unrealistic? Why?

6. Galilean telescope. A Galilean telescope has for the first lens $f_1 = 30$ cm and for the negative lens $f_2 = -9.9$ cm. If x_{o1} is large and the distance a between the two lenses is 20 cm, calculate x_{i2}, the image distance with respect to the negative lens. Calculate the magnification and show that for the object at infinity, one again has $M = -f_1/f_2$. The distance between the two lenses is then $f_1 + f_2$.

7. Laser beam expander. A laser beam of diameter of 2 mm should be expanded to a beam of 20 mm.

   a. A biconvex and a biconcave lens should be used. The beam first passes the biconcave lens of focal length -5 mm. Where should one place the biconvex lens of diameter of 30 mm and focal length of 50 mm?

   b. Two biconvex lenses should be used, one with $f = 5$ mm, the other with $f = 50$ mm. Make a sketch and give approximate values for the diameter of the lenses.

1.8 MATRIX FORMULATION FOR THICK LENSES

1.8.1 Refraction and Translation Matrices

A thick lens has two spherical surfaces separated by a dielectric material of a certain thickness. Previously we ignored the distance between the two surfaces
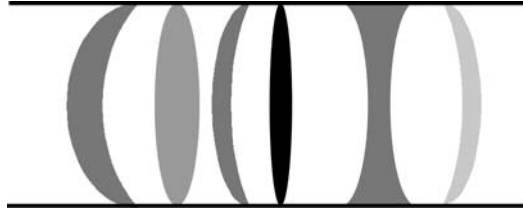
FIGURE 1.19 Multiple lens system. The lenses may have different radii of curvature and different refractive indices.

but now take it into account. One may calculate the image formation of the thick lens by first finding the image produced by the first surface. Then one uses this image as an object in the second imaging process and finds the image produced by the second surface. One could also use this procedure for lens systems with many lenses (Figure 1.19). However, one can develop a mathematical formalism to describe the image formation of a system of lenses by using the thin-lens equation. But one now has to measure the object and image distance from newly determined "principal planes," and not from the center of the thick lens. To do this, we first consider the case of refraction on a spherical surface (Figure 1.20). We want to represent the first surface by an operation which transforms the set of coordinates of the object into the set of coordinates of the first image. We show that this operation can be represented by a transformation matrix, which we call refraction matrix. Then we make a translation to get to the second surface, accomplished by a translation matrix, and the next operation on the second surface is again associated with a refraction matrix. This method is applicable to many different curved surfaces and their separations, having different thickness and refraction indices. The mathematical operation representing the processes of refraction at one and translation between two surfaces is a two-by-two matrix. The matrices are derived by using the paraxial theory, taking as the coordinates the distance from the axis of the point of the ray at the surface and the angle the ray makes with the axes (Figure 1.20a).

We now construct matrices to represent the refraction and translation operations. The matrices act on sets of two coordinates, written in the form of a vector. The initial coordinates (index1) in the plane of the object are acted on, and the result is the set of coordinates (index 2) in the plane of the image. We start from the equation for refraction on a single surface

$$-n_1/x_o + n_2/x_i = (n_2 - n_1)/r \tag{1.76}$$

and rewrite it, using α_1 and l_1 (see Figure 1.20a), as

$$n_1(\alpha_1/l_1) + n_2(-\alpha_2/l_2) = (n_2 - n_1)/r. \tag{1.77}$$

In addition we have for the second coordinate
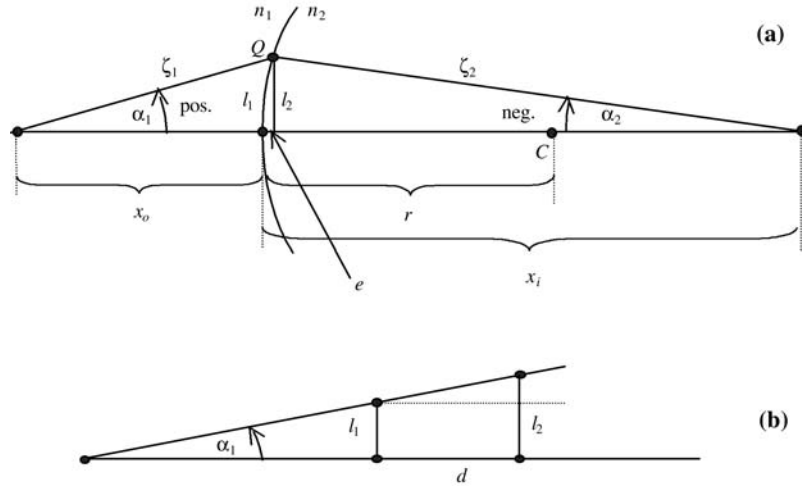
$$l_1 = l_2. \tag{1.78}$$

FIGURE 1.20   Coordinates for vector and matrix formulation: (a) the coordinates 1_1 and α_1 are used to form the vectors $I_1 = (l_1, \alpha_1)$, and the coordinates l_2 and α_2 are used to form the vectors $I_2 = (l_2, \alpha_2)$; (b) translation, the dependence of d on α_1, l_1, and l_2.

We define the vectors I_1 of object coordinates and I_2 of image coordinates using for I_1 the coordinates l_1 and α_1, and for I_2 we using l_2 and α_2,

$$I_1 = \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix} \qquad I_2 = \begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix}. \tag{1.79}$$

The two equations (1.77) and (1.78) may be written in matrix notation as

$$\begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(1/r)(n_2 - n_1)/n_2 & n_1/n_2 \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}. \tag{1.80}$$

For a proof, we may multiply the matrix with the vector and arrive back at Eqs. (1.76) to (1.78). In short notation we may also write

$$I_2 = R_{12} I_1.$$

The matrix R_{12} is called the refraction matrix of a single spherical surface

$$R_{12} = \begin{pmatrix} 1 & 0 \\ -(1/r)(n_2 - n_1)/n_2 & n_1/n_2 \end{pmatrix}. \tag{1.81}$$

For a plane surface, that is, for an infinite large radius of curvature, the matrix of Eq. (1.81) reduces to the refraction matrix of a plane surface

$$R = \begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}. \tag{1.82}$$

We get the translation matrix T, that is, the translation from one vertical plane to the next over the distance d, by taking into account that $l_2 = l_1 + \alpha_1 d$; see

Figure 1.20b.

$$T = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}. \tag{1.83}$$

1.8.2 Two Spherical Surfaces at Distance d and Principal Planes

1.8.2.1 The Matrix

For a thick lens we use the refraction and translation matrices. We apply the refraction matrix corresponding to the first spherical surface, the translation matrix corresponding to the thickness of the lens, and the refraction matrix corresponding to the second spherical surface. We again assume that the light comes from the left, and realize that the sequence of the matrices is the sequence of action on I_1. In other words, the first surface is represented by the matrix on the far right.

First operation:	Refraction on first surface:	Matrix on the right
Second operation:	Translation between the surfaces:	Matrix in the middle
Third operation:	Refraction on the second surface:	Matrix on the left.

For the refraction matrix of a thick lens of thickness d and two different spherical surfaces, we obtain

$$\begin{pmatrix} 1 & 0 \\ -(1/r_2)(n_3 - n_2)/n_3 & n_2/n_3 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(1/r_1)(n_2 - n_1)/n_2 & n_1/n_2 \end{pmatrix}. \tag{1.84}$$

Multiplication of the three matrices will give us one matrix representing the total action of the thick lens. To do this we define some abbreviations, called refracting powers P_{12}, P_{23}, and P, where P is related to the focal length of the thick lens.

$$P_{12} = -(1/r_1)(n_2 - n_1)/n_2 \tag{1.85}$$

$$P_{23} = -(1/r_2)(n_3 - n_2)/n_3, \quad \text{and} \tag{1.86}$$

$$P = -1/f = P_{23} + d P_{12} P_{23} + (n_2/n_3)P_{12}. \tag{1.87}$$

(From the 2,1 element P we get the focal length of the system.) We obtain the thick-lens matrix as

$$\begin{pmatrix} 1 + d P_{12} & d(n_1/n_2) \\ P & d(n_1/n_2)P_{23} + (n_1/n_3) \end{pmatrix}. \tag{1.88}$$

FileFig 1.23 (G23SYMB3M)

Symbolic calculation of the product of three matrices corresponding to a thick lens of refractive index n_2 and thickness d. The light is incident from a medium

with refractive index n_1 and transmitted into a medium with refractive index n_3. The case of the thin lens is derived by setting $d = 0$ and $n_1 = n_3$; one obtains the thin-lens matrix.

G23SYMB3M

Thin-Lens Matrix

Special case of the thin-lens matrix. We start with the symbolic calculation of two surfaces at distance d

$$P12 = (-1/r1)(n2 - n1)/n2 \qquad P23 = (-1/r2)(n3 - n2)/n3$$

$$
\begin{bmatrix} 1 & 0 \\ P23 & \dfrac{n2}{n3} \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ P12 & \dfrac{n1}{n2} \end{bmatrix}
$$

$$
\begin{bmatrix} 1 + d \cdot P12 & d \cdot \dfrac{n1}{n2} \\ \dfrac{(P23 \cdot n3 + P12 \cdot P23 \cdot d \cdot n3 + P12 \cdot n2)}{n3} & \dfrac{(P23 \cdot d \cdot n3 + n2)}{n3} \cdot \dfrac{n1}{n2} \end{bmatrix}
$$

$$P = P23 + d P12 P23 + (n2/n3)P12.$$

We go to the thin lens and set $d = 0$

$$
\begin{bmatrix} 1 & 0 \\ P23 & \dfrac{n2}{n3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ P12 & \dfrac{n1}{n2} \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 \\ \dfrac{(P23 \cdot n3 + P12 \cdot n2)}{n3} & \dfrac{1}{n3} \cdot n1 \end{bmatrix}.
$$

Since $n3$ and $n1$ are set to 1 we have

$$
\begin{bmatrix} 1 & 0 \\ (P23 + P12 \cdot n2) & 1 \end{bmatrix}.
$$

We set

$$P = (P23 + P12 \cdot n2)$$

and

$$P = \dfrac{-1}{f}; \quad f \text{ is the focal length of the lens.}$$

With

$$P12 = (-1/r1)(n2 - n1)/n2 \qquad P23 = (-1/r2)(n3 - n2)/n3$$

we obtain for $1/f = -((-1/r2)(1 - n2) + (-1/r1)(n2 - 1))$ and have finally for the thin-lens matrix,

$$\begin{bmatrix} 1 & 0 \\ \dfrac{-1}{f} & 1 \end{bmatrix}.$$

1.8.2.2 Application to the Thin Lens

We demonstrate more about the meaning and significance of the four matrix elements when reducing the matrix to the one corresponding to a thin lens. We use two surfaces close together; that is, we set $d = 0$ (Figure 1.21). The product matrix of Eq. (1.88) reduces to

$$\begin{pmatrix} 1 & 0 \\ P_{23} + (n_2/n_1)P_{12} & 1 \end{pmatrix}. \tag{1.89}$$

Assuming $n_1 = n_3 = 1$, we have for $P_{23} + (n_1/n_3)P_{12} = -(1 - n_2)/r_2 - (n_2 - 1)/r_1 = -1/f$, where f is the focal length of the thin lens. If we introduce these expressions into Eq. (1.89) and write the matrix with the coordinate vectors as in Eq. (1.80), we get

$$\begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(1/f) & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}. \tag{1.90}$$

We label the matrix elements $M_{0,0}$, $M_{0,1}$, $M_{1,0}$, and $M_{1,1}$.

By using the coordinates as done in Eq. (1.77) and (1.78), we want to show that Eq. (1.90) is equivalent to the thin-lens equation. Multiplication yields

$$\begin{aligned} l_2 &= l_1 \\ \alpha_2 &= -l_1/f + \alpha_1. \end{aligned} \tag{1.91}$$

From Figure 1.21 we have $\alpha_2 = -l_1/x_i$, and $\alpha_1 = -l_1/x_o$, and have

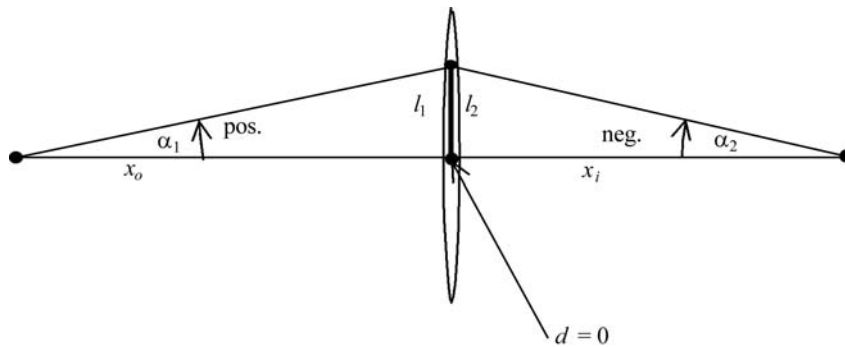$$l_1/(-x_o) + l_1/x_i = l_1/f. \tag{1.92}$$

FIGURE 1.21   Coordinates for the thin lens.

We see that if the 0,0 and 1,1 elements are 1 and the 0,1 element is zero, we may obtain the focal length of the thin lens from the 1,0 element; that is, $-1/f = P_{23} + (n_2/n_1)P_{12}$.

We have gone through this example of the thin lens to show how the procedure with the refraction matrix works to get to the object–image relation. We measure x_o and x_i from the surface of the thin lens, and apply in the usual way the thin-lens equation, and take the focal length from the 1,0 element.

1.8.2.3 Thick Lens

For a thick lens, the matrix elements 0,0 and 1,1 of Eq. (1.88) are not 1, and the 0,1 element is not zero. To apply a similar procedure to that discussed for the thin lens, we introduce a transformation in order to get the 0,0 and 1,1 element to 1 and the 0,1 element to 0. These three requirements may be obtained by application of a translation. We first translate by $-h$ the plane of the object and at the end we go back by a translation of hh. The introduction of these two new parameters corresponds to the displacements of the points from which we have to count x_o and x_i. We apply these two translations to the thick-lens matrix of Eq. (1.88) and have to calculate

$$\begin{pmatrix} 1 & hh \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + d\,P_{12} & d\,(n_1/n_2) \\ P & d(n_1/n_2)P_{23} + n_1/n_3 \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}. \quad (1.93)$$

We rewrite the thick-lens matrix, using the following abbreviations,

$$\begin{pmatrix} 1 & hh \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}. \quad (1.94)$$

The multiplication is done in FileFig 1.24, and we get as the result

$$\begin{pmatrix} M_{0,0} + hhM_{1,0} & -M_{0,0}h + M_{0,1} + hh(-M_{1,0}h + M_{1,1}) \\ M_{1,0} & -M_{1,0}h + M_{1,1} \end{pmatrix}. \quad (1.95)$$

There are three requirements to be fulfilled, and only two new parameters. We set $M_{0,0} + hhM_{1,0} = 1$ and $-M_{1,0}h + M_{1,1} = 1$, and calculate h and hh. In order to be successful, the introduction of the calculated values of h and hh from these two equations must make the 0,1 element zero. It can be shown analytically that $[-M_{0,0}h + M_{0,1} + hh(-M_{1,0}h + M_{1,1})] = 0$, and numerically as seen in FileFig 1.24.

We have the same form of the matrix as in Eq. (1.89) and find that the (1,0) element has not been changed by the transformation. We have $P = -1/f = M_{1,0}$. As a result of our transformation we have for the parameters h, hh, and the focal length

$$hh = (1 - M_{0,0})/M_{1,0} \qquad (1.96)$$

$$-h = (1 - M_{11})/M_{1,0} \qquad (1.97)$$

$$P = -1/f = M_{1,0}. \qquad (1.98)$$

FileFig 1.24   (G24SYMBH)

Symbolic calculations of the general transformation for a thick lens. Calculation of the two-spherical-surface matrix and displacement matrix with parameters $-h$ *and* hh*. A numerical example is presented for* $n_1 = 1$, $n_2 = 1.5$, $n_3 = 1$, $r_1 = 10$, $r_2 = -10$, *and* $d = 20$.

G24SYMBH

Symbolic Calculations of the Product of Three Matrices Corresponding to a General Thick Lens

1. Symbolic calculation of the matrix for the thick lens

$$
\begin{bmatrix} 1 & 0 \\ P23 & \dfrac{n2}{n3} \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ P12 & \dfrac{n1}{n2} \end{bmatrix}
$$

$$
P12 = (-1/r1)((n2 - n1)/n2)
$$
$$
P23 = (-1/r2)((n3 - n2)/n3)
$$

$$
\begin{bmatrix} 1 + d \cdot P12 & d \cdot \dfrac{n1}{n2} \\ \dfrac{(P23 \cdot n3 + P12 \cdot P23 \cdot d \cdot n3 + P12 \cdot n2)}{n3} & \dfrac{(P23 \cdot d \cdot n3 + n2)}{n3} \cdot \dfrac{n1}{n2} \end{bmatrix}.
$$

2. Determination of h and hh. For simpler calculation we define the matrix

$$
\begin{bmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{bmatrix}
$$

$$
M_{0,0} = 1 + d \cdot P12 \qquad\qquad M_{0,1} = d \cdot \dfrac{n1}{n2}
$$

$$
M_{1,0} = \dfrac{(P23 \cdot n3 + P12 \cdot P23 \cdot d \cdot n3 + P12 \cdot n2)}{n3} \qquad M_{1,1} = \dfrac{(P23 \cdot d \cdot n3 + n2)}{n3} \cdot \dfrac{n1}{n2}
$$

and determine h and hh,

$$
\begin{bmatrix} 1 & hh \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} M_{0,0} + hh \cdot M_{1,0} & -h \cdot M_{0,0} - h \cdot hh \cdot M_{1,0} + M_{0,1} + hh \cdot M_{1,1} \\ M_{1,0} & -M_{1,0} \cdot h + M_{1,1} \end{bmatrix}.
$$

3. The results for h, hh, and f are

$$
hh = \dfrac{1 - M_{0,0}}{M_{1,0}} \qquad h = \dfrac{-(1 - M_{1,1})}{M_{1,0}} \qquad f = \dfrac{-1}{M_{1,0}}.
$$

4. Numerical calculation

$$P12 := \frac{-1}{r1} \cdot \frac{n2 - n1}{n2} \qquad P23 := \frac{1}{r2} \cdot \frac{n3 - n2}{n3}$$
$$P12 = -3.333 \cdot 10^{-11} \qquad P23 = -0.05$$

$$M_{0,0} := 1 + d \cdot P12 \qquad M_{0,1} = d \cdot \frac{n1}{n2}$$
$$M_{0,0} = 1 \qquad M_{0,1} = 6.667$$
$$M_{1,0} := \frac{(P23 \cdot n3 + P12 \cdot P23 \cdot d \cdot n3 + P12 \cdot n2)}{n3}$$
$$M_{1,1} := \frac{(P23 \cdot d \cdot n3 + n2)}{n3} \cdot \frac{n1}{n2}$$
$$M_{1,0} = -0.05 \quad M_{1,1} = 0.667.$$

5. The result for h, hh, and f

$$hh := \frac{1 - M_{0,0}}{M_{1,0}} \qquad h := \frac{-(1 - M_{1,1})}{M_{1,0}} \qquad f := \frac{-1}{M_{1,0}}$$
$$hh = -6.667 \cdot 10^{-9} \qquad h = 6.667 \qquad f = 20.$$

6. The input values are globally defined

$$n1 \equiv 1 \qquad n2 \equiv 1.5 \qquad n3 \equiv 1 \qquad r1 \equiv 10^{10} \qquad r2 \equiv -10 \qquad d \equiv 10.$$

The transformation using the two matrices

$$\begin{pmatrix} 1 & hh \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \qquad\qquad (1.99)$$

has the effect that we have to count xo from the point on the axis determined by h, and xi from the point on the axis determined by hh. We do not count from the vertex of the spherical surfaces. If we call the vertex of the first surface V_1 and the vertex of the second surface V_2, we have a similar sign convention as we have used before:

1. if $h > 0$, the point to start calculating x_o is to the right of V_1; otherwise to the left; and

2. if $hh > 0$, the point to start calculating x_i is to the right of V_2; otherwise to the left.

The calculation is shown in FileFig 1.25. The planes perpendicular to the axis at h and hh are called *principal planes*. As a check one finds, that in the approximation of the thin lens, the difference $hh - h = 0$.

We state the general procedure for using the thin-lens equation with the matrix method: One calculates $hh = (1 - M_{0,0})/M_{1,0}$ and $-h = (1 - M_{11})/M_{1,0}$. The

focal length f is obtained from $P = -1/f = M_{1,0}$. One measures x_o from h and x_i from hh and applies the thin-lens equation.

FileFig 1.25 (G25SYMBGTH)

Calculation of the general transformation for a thin lens. Calculation of the product of the two-spherical-surface matrix, and the displacement matrix. Determination of the parameters $-h$ and hh. Specialization for the case of the thin lens. Numerical example for $n_1 = 1$, $n_2 = 1.5$, $n_3 = 1.3$, $r_1 = 120$, and $r_2 = -10$.

G25SYMBGTH is only on the CD.

1.8.2.4 Application to the Hemispherical Thick Lens

We consider a thick lens of hemispherical shape (see Figure 1.22). In FileFig 1.26 we present the calculations and for the choice of parameters: $n_2 = 1.5$, $n_1 = n_3 = 1$, $r_1 = 20$, and $r_2 = \infty$.

If we set $n_2 = n = 1.5$, $n_1 = n_3 = 1$, $r_1 = r = d$, and $r_2 = \infty$, we have the result that $P_{12} = -1/3r$, $P_{23} = 0$, $P = -1/2r$; that is, $f = 2r$, $h = 0$, and $hh = -2r/3$.
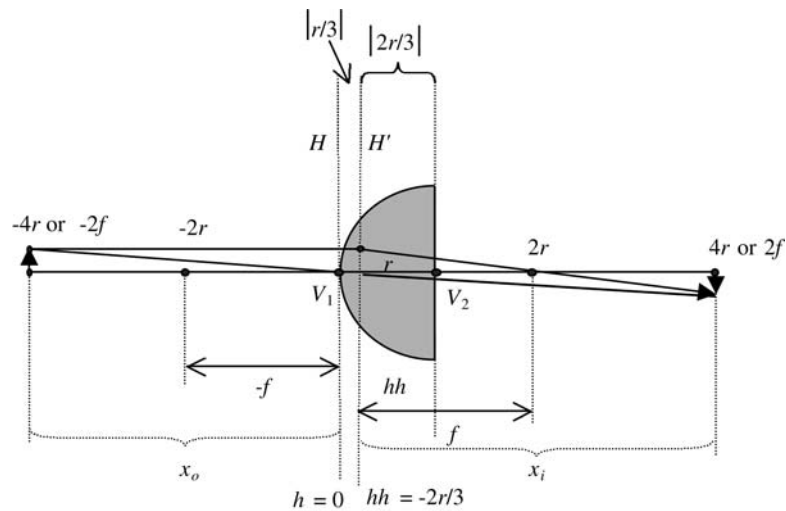
FIGURE 1.22 Coordinates for a hemispherical thick lens of index n. The principal planes are indicated as H and H'.
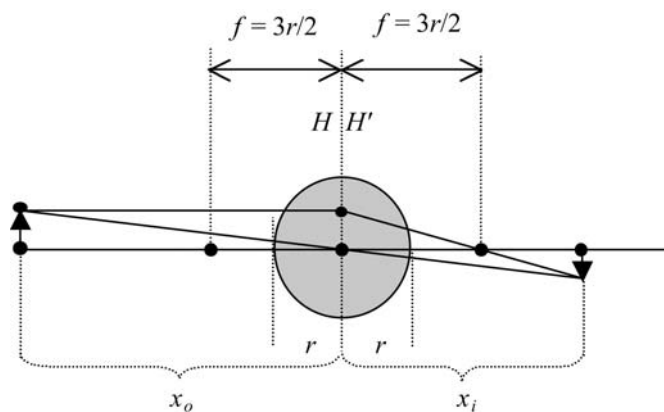
FIGURE 1.23   Coordinates for a spherical thick lens.

FileFig 1.26   (G26HEM)

Calculations of the hemispherical thick lens with curved surface to the left. For the numerical values we take $n_2 = 1.5$, $n_1 = n_3 = 1$, $r_1 = 10 = d$, and $r_2 = \infty$.

G26HEM is only on the CD.

Application 1.26. *Repeat the calculations for a hemispherical thick lens with curved surface to the right.*

1.8.2.5 Application to Glass Sphere

We consider a thick lens of spherical shape (Figure 1.23). In FileFig 1.27 we show the calculations for $n_2 = 1.5 = n$, $n_1 = n_3 = 1$, $r_1 = -r_2 = 10$, and $d = 2r_1 = 20$. The result is $P_{12} = -1/3r$, $P_{23} = -1/2r$, $P = -2/3r$; that is, $f = 3r/2$, $h = r$, and $hh = -r$.

From Figure 1.23 we see that the principal planes are at the center, as expected for a symmetric lens. We have to start at the center to measure x_o and x_i and apply the thin lens equation with focal length $f = 3r/2$. For the numerical calculations we use $r_1 = 10$ and have $h = 10$, and $hh = -10$.

FileFig 1.27   (G27SPH)

Calculation of the spherical thick lens. For the numerical values we have chosen $n_2 = 1.5$, $n_1 = n_3 = 1$, $r_1 = 10$, $r_2 = -10$, and $d = 20$.

G27SPH is only on the CD.

Application 1.27. *Go over the calculations for two different sets of parameters* n_2, n_1, n_3, r_1, d, *and* r_2.

1.8.3 System of Lenses

1.8.3.1 System of Two Thin Lenses in Air

We now study the application of matrices to the calculation of the final image produced by a system of two lenses. First we consider a system of two thin lenses of focal length f_1 and f_2 and at distance a between them

$$\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix}. \tag{1.100}$$

Multiplication yields

$$\begin{pmatrix} (f_1 - a)/f_1 & a \\ -(f_1 - a + f_2)/f_1 f_2 & -(a - f_2)/f_2 \end{pmatrix}. \tag{1.101}$$

Since the (0,0) and (1,1) elements are not zero and the (0,1) element is not 1 we have to apply the transformation to principal planes, as we did for the single thick lens. We have to evaluate (Figure 1.24)

$$\begin{pmatrix} 1 & hh \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (f_1 - a)/f_1 & a \\ -(f_1 - a + f_2)/f_1 f_2 & -(a - f_2)/f_2 \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}. \tag{1.102}$$

This is done in FileFig 1.28 and the result is

$$h = -a/Pf_2$$
$$hh = a/Pf_1$$
$$P = (-1/f_2)(1 - a/f_1) - 1/f_1.$$
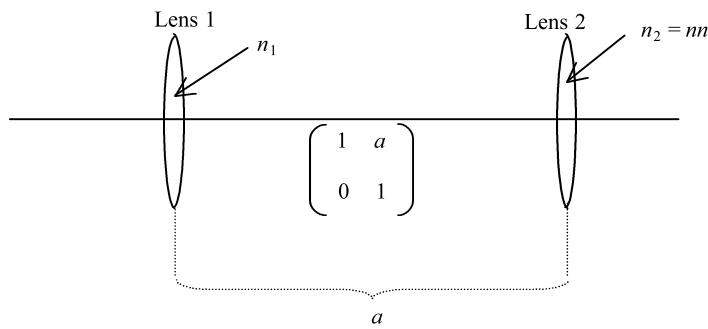
FIGURE 1.24 Coordinates for two lenses in air with corresponding matrices.

For the application to calculate the image from a given object point, focal lengths, and distance between the lenses, we measure x_o from h, x_i from hh, and get the focal length from $-1/f = P$.

FileFig 1.28   (G28SYST2LTI)

Calculation for a system of two thin lenses. For the numerical values we have chosen $f_1 = 10$, $f_2 = 10$, and $a = 100$.

G28SYST2LTI

Symbolic Calculation to Determine the Principal Planes for Two Thin Lenses at Distance a

The matrix (M) as the product of the two lenses and the displacement between them

$$
\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f1} & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} \dfrac{(f1-a)}{f1} & a \\ \dfrac{-(f1-a+f2)}{(f2 \cdot f1)} & \dfrac{-(a-f2)}{f2} \end{bmatrix}.
$$

Special case $a = 0$, two thin lenses in contact

$$
\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f2} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f1} & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 \\ \dfrac{-(f1+f2)}{(f2 \cdot f1)} & 1 \end{bmatrix}.
$$

Principal planes with h and hh, and $P = (-1/f2)(1 - a/f1) - 1/f1$

$$
\begin{bmatrix} 1 & hh \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{-(-f1+a)}{f1} & a \\ P & \dfrac{-(a-f2)}{f2} \end{bmatrix} \cdot \begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}
$$

$$\begin{bmatrix} \frac{(f1-a+hh\cdot P\cdot f1)}{f1} & \frac{(-h\cdot f2\cdot f1+h\cdot f2\cdot a-h\cdot f2\cdot hh\cdot P\cdot f1+f1\cdot a\cdot f2-f1\cdot hh\cdot a+f1\cdot hh\cdot f2)}{(f1\cdot f2)} \\ P & \frac{-P\cdot h\cdot f2+a-f2)}{f2} \end{bmatrix}.$$

If the $(0, 0)$ and $(1, 1)$ elements are one, we have for $hh = a/P \cdot f1$ and $h = a/P \cdot f2$, P is always $-1/f$

$$P = (-1/f2)(1 - a/f1) - 1/f1$$

$$P := \left(\frac{-1}{f2}\right) \cdot \left(1 - \frac{a}{f1}\right) - \frac{1}{f1} \qquad hh := \frac{a}{P \cdot f1} \qquad h := \frac{-a}{P \cdot f2}$$

$$M := \begin{bmatrix} \frac{(f1-a+hh\cdot P\cdot f1)}{f1} & \frac{(-h\cdot f2\cdot f1+h\cdot f2\cdot a-h\cdot f2\cdot hh\cdot P\cdot f1+f1\cdot a\cdot f2-f1\cdot hh\cdot a+f1\cdot hh\cdot f2)}{(f1\cdot f2)} \\ P & \frac{-(P\cdot h\cdot f2+a-f2)}{f2} \end{bmatrix}$$

$$f1 \equiv 10 \qquad f2 \equiv 10 \qquad a \equiv 100$$

$$M = \begin{bmatrix} 1 & 0 \\ 0.8 & 1 \end{bmatrix} \qquad f := \frac{-1}{P}$$

$$hh = 12.5 \qquad h = -12.5 \qquad f = -1.25.$$

Application 1.28. *Consider the case where $a = 0$, and compare the resulting focal length f with $1/(1/f_1 + 1/f_2)$.*

1.8.3.2 System of Two Thick Lenses

We consider two thick lenses and assume that lens 1 has the refractive index n lens 2 the index nn. We also assume that the radii of curvature of the four spherical surfaces are labeled r_1 to r_4 and that the distance between lens 1 and lens 2 is a. The matrix for the system is obtained from the sequence of three matrices (Figure 1.25).

We start on the right with the thick-lens matrix of the first lens, then the translation matrix, and then to the left the thick-lens matrix of the second lens. The calculation is shown in FileFig 1.29 and one obtains
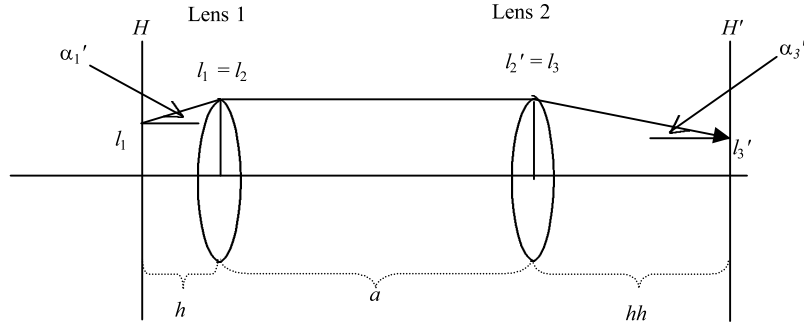
$$\begin{pmatrix} 1 + d_2 P_{34} & d_2/nn \\ P_2 & d_2(P_{45}/nn) + 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + d_1 P_{12} & d_1/n \\ P_1 & d_1(P_{23}/n) + 1 \end{pmatrix} \tag{1.103}$$

with

$$P_{12} = -(1/r_1)(n - 1)/n$$
$$P_{23} = -(1/r_2)(1 - n)$$
$$P_{34} = -(1/r_3)(nn - 1)/nn$$

$$\begin{pmatrix} 1 + d_2\,P_{34} & d_2/nn \\ P_2 & d_2(P_{45}/nn) + 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + d_1\,P_{12} & d_1/n \\ P_1 & d_1(P_{23}/n) + 1 \end{pmatrix}$$

FIGURE 1.25   Coordinates for two thick lenses in air with corresponding matrices.

$$P_{45} = -(1/r_4)(1 - nn)$$
$$P_1 = P_{23} + P_{12}P_{23}d_1 + P_{12}n$$
$$P_2 = P_{45} + P_{34}P_{45}d_2 + P_{34}nn.$$

To determine the principal planes of this system, we call M the product of the three matrices in Eq. (1.103), and have to calculate (see FileFig 1.29)

$$\begin{pmatrix} 1 & hh \\ 0 & 1 \end{pmatrix} \quad M \quad \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}. \tag{1.104}$$

We have to set in the product matrix the (0,0) and (1,1) elements equal to one, and it follows that the (0,1) element is 0. The result of the transformation is:

$$h = -(1 - M_{1,1})/M_{1,0}$$
$$hh = (1 - M_{0,0})/M_{1,0}$$
$$1/f = -M_{1,0}.$$

To calculate the image distance for a given object distance, we have to measure x_o from h, x_i from hh, and apply the thin-lens equation with focal length f calculated from $-1/f = M_{1,0}$.

For a specific example of a system of two thick lenses we choose a system of two hemispherical lenses. Each lens is one-half of a sphere, and we assume that the distance a is zero. The results are the same as we found in Section 1.8.2.4 for a sphere.

In Figure 1.26, we show the two hemispherical lenses with their refracting powers P_{12} and P_{45}, each of thickness d and a refractive index n with the corresponding matrices. The details of the calculation are shown in FileFig 1.29.
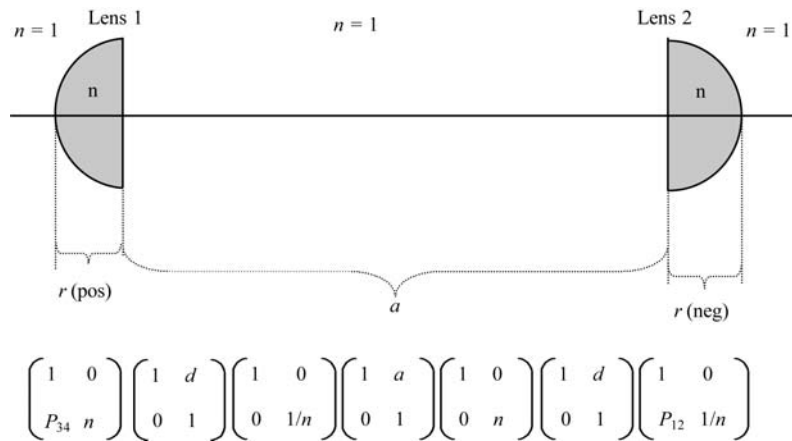
$$\begin{pmatrix} 1 & 0 \\ P_{34} & n \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ P_{12} & 1/n \end{pmatrix}$$

FIGURE 1.26 Two hemispherical lenses at distance a, and the corresponding matrices. The lenses have refractive index n, thickness $d = r$, P_{12} is the refracting power of the first spherical surface, and P_{45} of the last.

FileFig 1.29   (G29SYST2LTC)

Calculation for a system of two thick lenses with refractive indices n and nn at distance a. The choices of the numerical values are n = 1.5, nn = 1.5, d_1 = 10, d_2 = 10, a = 100, r_1 = 10, r_2 = −10, r_3 = 10, and r_4 = −10. See also (G27SPH).

G29SYST2LTC

Symbolic Calculation of the Principal Planes for Two Thick Lenses of Refractive Indices n and nn in Air.

Distance between lenses is a and the thickness of the first is $d1$, of the second $d2$. Radii of curature are $r1$ to $r4$. The matrix of the first lens is on the right.

$$\begin{pmatrix} 1 & 0 \\ P45 & \frac{nn}{1} \end{pmatrix} \cdot \begin{pmatrix} 1 & d2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ P34 & \frac{1}{nn} \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$$\cdot \left[\begin{pmatrix} 1 & 0 \\ P23 & \frac{n}{1} \end{pmatrix} \cdot \begin{pmatrix} 1 & d1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ P12 & \frac{1}{n} \end{pmatrix} \right]$$

$$P12 = -(1/r1)(n-1)/n \qquad P23 = -(1/r2)(n-1)/n$$

$$P34 = -(1/r3)(nn-1)/nn \qquad P45 = -(1/r4)(1-nn)$$

Matrix for the first lens

$$\left[\begin{pmatrix} 1 & 0 \\ P23 & \frac{n}{1} \end{pmatrix} \cdot \begin{pmatrix} 1 & d1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ P12 & \frac{1}{n} \end{pmatrix} \right]$$

$$\begin{bmatrix} 1 + d1 \cdot P12 & \frac{d1}{n} \\ P23 + P12 \cdot P23 \cdot d1 + P12 \cdot n & \frac{(P23 \cdot d1 + n)}{n} \end{bmatrix}$$

Matrix for the second lens

$$\begin{pmatrix} 1 & 0 \\ P45 & \frac{nn}{1} \end{pmatrix} \cdot \begin{pmatrix} 1 & d2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ P34 & \frac{1}{nn} \end{pmatrix}$$

$$\begin{bmatrix} 1 + d2 \cdot P34 & \frac{d2}{nn} \\ P45 + P34 \cdot P45 \cdot d2 + P34 \cdot nn & \frac{(P45 \cdot d2 + nn)}{nn} \end{bmatrix}$$

For the determination of h and hh

$$\begin{bmatrix} 1 & hh \\ 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 + d2 \cdot P34 & \frac{d2}{nn} \\ P45 + P34 \cdot P45 \cdot d2 + P34 \cdot nn & \frac{(P45 \cdot d2 + nn)}{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 + d1 \cdot P12 & \frac{d1}{n} \\ P23 + P12 \cdot P23 \cdot d1 + P12 \cdot n & \frac{(P23 \cdot d1 + n)}{n} \end{bmatrix} \cdot \begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}$$

Multiplication results in a very large expression, and we go right away to numerical calculations.

We have for the powers of refraction

$$P12 := -\frac{n-1}{r1 \cdot n} \qquad P23 := -\frac{1-n}{r2}$$

$$P34 := -\frac{nn-1}{r3 \cdot nn} \qquad P45 := -\frac{1-nn}{r4}.$$

The thick lens matrix is then

$$M := \begin{bmatrix} 1 + d2 \cdot P34 & \frac{d2}{nn} \\ P45 + P34 \cdot P45 \cdot d2 + P34 \cdot nn & \frac{(P45 \cdot d2 + nn)}{nn} \end{bmatrix} \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$$\cdot \begin{bmatrix} 1 + d1 \cdot P12 & \frac{d1}{n} \\ P23 + P12 \cdot P23 \cdot d1 + P12 \cdot n & \frac{(P23 \cdot d1 + n)}{n} \end{bmatrix}.$$

The result is

$$\begin{bmatrix} 0.333 & 13.333 \\ -0.667 & 0.333 \end{bmatrix}.$$

We define M as

$$\begin{bmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{bmatrix}.$$

For the determination of h and hh we multiply by the two translation matrices

$$\begin{bmatrix} 1 & hh \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} M_{0,0} + hh \cdot M_{1,0} & -h \cdot M_{0,0} - h \cdot hh \cdot M_{1,0} + M_{0,1} + hh \cdot M_{1,1} \\ M_{1,0} & -M_{1,0} \cdot h + M_{1,1} \end{bmatrix}$$

$$hh := \frac{1 - (M_{0,0})}{M_{1,0}} \qquad h := \frac{1 - (M_{1,1})}{(-M)_{1,0}} \qquad f := -\frac{1}{M_{1,0}}$$

$$hh = -10 \qquad h = 10 \qquad f = 15.$$

Input Data

$$n \equiv 1.5 \qquad nn \equiv 1.5 \qquad d1 \equiv 10 \qquad d2 \equiv 10 \qquad a \equiv 0$$

$$r1 \equiv 10 \qquad r2 \equiv 10^{10} \qquad r3 \equiv 10^{10} \qquad r4 \equiv -10.$$

Check the form of the final matrix product

$$MM := \begin{bmatrix} 1 & hh \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}$$

$$MM = \begin{bmatrix} 1 & -1.776 \cdot 10^{-15} \\ -0.667 & 1 \end{bmatrix}.$$

Applications to Matrix Method

1. An exercise for matrix multiplication. Draw two cartesian coordinate systems x, y and x', y, the second rotated by the angle θ with respect to the first. Identify the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with the rotation of x, y into x', y'.

   a. Is this a rotation in the mathematical positive or negative sense?

   b. The matrix for rotation in the opposite direction A^{-1} is obtained by substituting for θ the negative value $-\theta$.

   c. Show that $A \, A^{-1}$ is the unit matrix.

   d. The transposed matrix A^T is obtained from A by interchanging the 2,1 and 1,2 elements. In our case the A^T is equal to A^{-1} and $A A^T$ is the unit matrix.

  e. Show that $A\,A$ is the same matrix if we substitute into A the angle 2θ.

2. Noncommutation of matrices. In general two matrices A and B may not be commuted; that is, AB is not equal to BA. We show this in the following example for a different sequence of the same matrices. We consider two hemispherical thick lenses where light is coming from the left. The light hits the first lens $L1$ at a spherical surface of radius of curvature r, then traverses the thickness d, and emerges from a plane surface. The second, $L2$, has the reverse order; first the plane surface, then thickness d, and then the curved surface with the same radius of curvature r. The refractive indices of the lenses are n_2 and outside we assume $n_1 = n_3 = 1$. Make a sketch. See how the two lenses are different. The product matrices for lens 1 and lens 2 are different for the two cases. Compare the position of the principal planes. Compare for the case where $r = \infty$.

3. Calculate, using the matrix method, the position of the two principal planes for a system of two thin lenses, both of focal length f, and a distance f.

4. Consider a convex-concave lens. The first surface has a radius of curvature $r_1 = 20$ cm, the second, a radius of curvature $r_2 = -10$ cm with thickness of $d = 5$ cm.

   a. Calculate the principal planes and focal length and find the image of an object positioned at 5 cm to the left of the first surface.

   b. Find the same result by using twice the imaging equation of a single surface.

5. Thick concentration lens. A thick lens of radius of curvature $-r_1 = r_2 = -5$ mm and thickness of 4 mm is used to concentrate incident parallel light on a detector. Using the matrix method, find the position with respect to the detector plane.

6. Plane-convex and convex-plane lens. The radii of curvature for the convex surface is $r = 10$ cm and for the concave surface $r = -10$ cm and the thickness is 4 cm.

   a. Compare h, hh, and f for both lenses.

   b. An object is placed 100 cm to the left of the first surface. Find the image point for both lenses.

See also on the CD

1.9 PLANE AND SPHERICAL MIRRORS

1.9.1 Plane Mirrors and Virtual Images

A two-dimensional object appears in a flat mirror as a virtual and left–right inverted image. First we look at one reflected ray (Figure 1.27a). We observe the law of reflection, which says the angle of incidence has the same absolute value as the angle of reflection.

In Figure 1.27b we show the reflection of a cone of light, emerging from a point source. The object point appears to us as it is on the "other side" of the mirror. Now we look at a three-dimensional object, represented by the arrows of a right-handed coordinate system. The virtual image produced by the flat mirror appears as a left-handed coordinate system. This may be seen by comparing the image of our left hand as it appears in a mirror, with our right hand placed before the mirror. Similarly one finds "Ambulance" written on the front of an ambulance truck written in letters from left to right. A driver in a car before the truck can read it "normally" in the rear view mirror.

1.9.2 Spherical Mirrors and Mirror Equation

Spherical concave mirrors of diameters of a few meters are used in astronomical telescopes, replacing the first lens, as discussed in Section 1.7 on optical instruments. A real inverted image is produced by a real erect object. Spherical convex
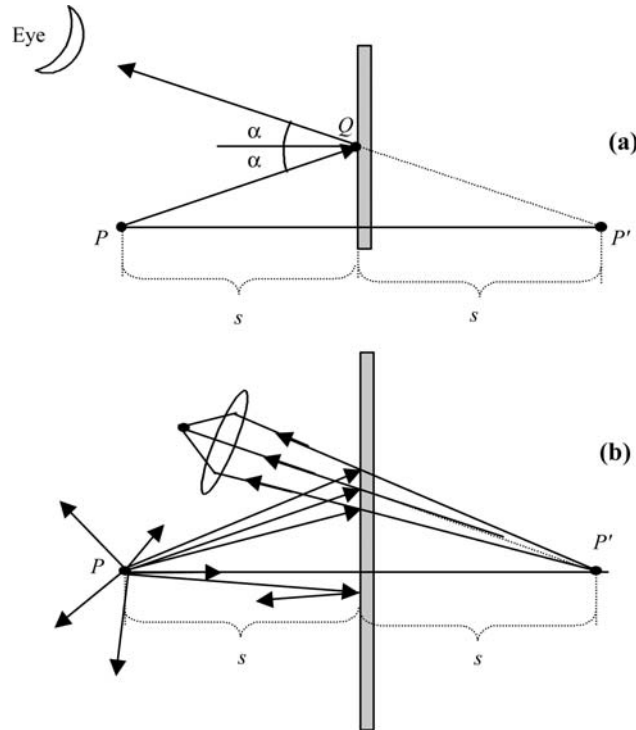
FIGURE 1.27   (a) Coordinates of the law of reflection; (b) virtual image of a real point source produced by a plane mirror. The image is observed by using a lens, which may be the eye lens.

mirrors with much smaller diameters are used for cosmetic applications, where an erect virtual image is formed from an erect object. Our eye uses a positive lens for the image formation on the retina, but "sees" the virtual image erect, as discussed in Section 1.7.

We derive the image-forming equation for spherical mirrors by looking at the image-forming equation of a single spherical surface

$$n_1/(-x_o) + n_2/x_i = (n_2 - n_1)/r. \tag{1.105}$$

By formally setting $n_1 = -n_2$ we get the imaging equation for a spherical mirror

$$n_1/(-x_o) + (-n_1)/x_i = (-n_1 - n_1)/r \tag{1.106}$$

$$n_1/(-x_o) + (-n_1)/x_i = (-2n_1)/r, \tag{1.107}$$

where r is the radius of curvature of the spherical surface.

Division by $-n_1$ results in the *spherical mirror equation*

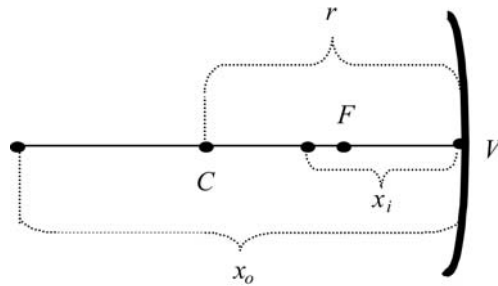$$\frac{1}{x_o} + \frac{1}{x_i} = \frac{2}{r}. \tag{1.108}$$

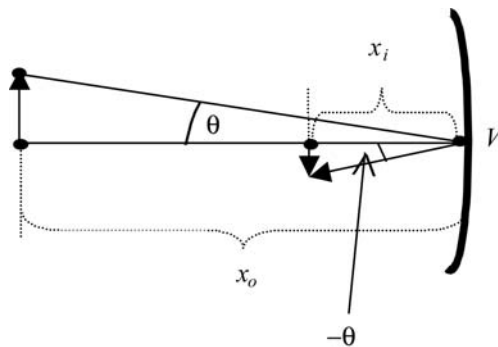**FIGURE 1.28** Coordinates for the production of a real image from a real object.

FIGURE 1.29 Magnification with respect to real object and real image.

1.9.3 Sign Convention

The light is assumed to be incident from the left. The object points x_o are to the left of the mirror, and x_0 is always negative. No positive values are considered. If x_i is negative we have a real image. If x_i is positive we have a virtual image (Figure 1.28).

For a convex spherical mirror, r is positive. For a concave spherical mirror, r is negative.

1.9.4 Magnification

For the magnification (Figure 1.29, we have

$$m = y_i/y_o = -x_i/x_o. \tag{1.109}$$

1.9.5 Graphical Method and Graphs of x_i Depending on x_o

1.9.5.1 Concave Spherical Mirror

Geometrical Construction

1. Choose y_o and draw the PF-ray to the mirror and then reflected back through the focus F, given by $r/2$.
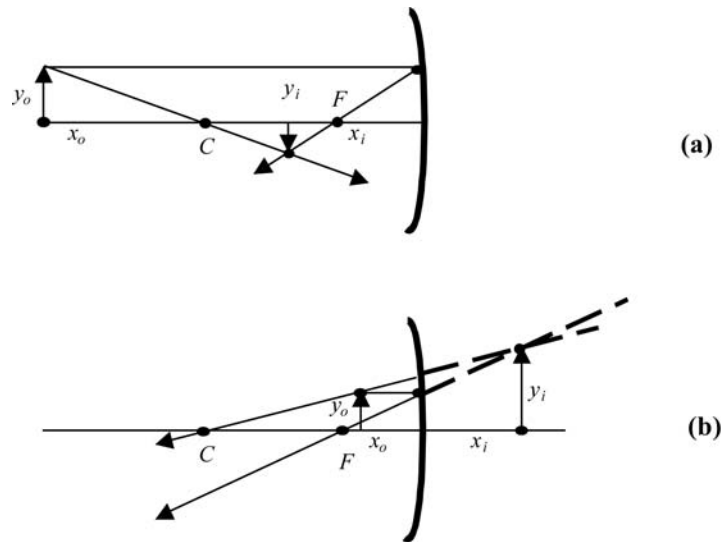
FIGURE 1.30   Geometrical construction of images for concave spherical mirror. The object is: (a) to the left of the focal point; (b) to the right of the focal point.

2. The ray incident through the top of the arrow and then going to the center of curvature C is reflected back onto itself (Figure 1.30).

Both may be extended to the other side of the mirror when x_o is between the focus and mirror.

Graph of x_i as Function of x_o

A concave spherical mirror has a negative radius of curvature. In FileFig 1.30 we calculate the image points for given object points and radius of curvature. The light comes from the left. For $x_{if} = -\infty$, the focus $x_{\text{of}} = r/2$, and since r is negative for a convex mirror, it is to the left of the mirror. This is the only focus we have for a concave mirror. The focus is also a singularity. We obtain real images for x_o to the left and virtual images for x_o to the right.

FileFig 1.30   (G30MIRCV)

Concave spherical mirror. Calculation of image positions from given object positions. Graph for image positions depending on object positions for radius of curvature $r = -50$, that is, $r/2 = -25$, and x_o from -100 to -0.1.
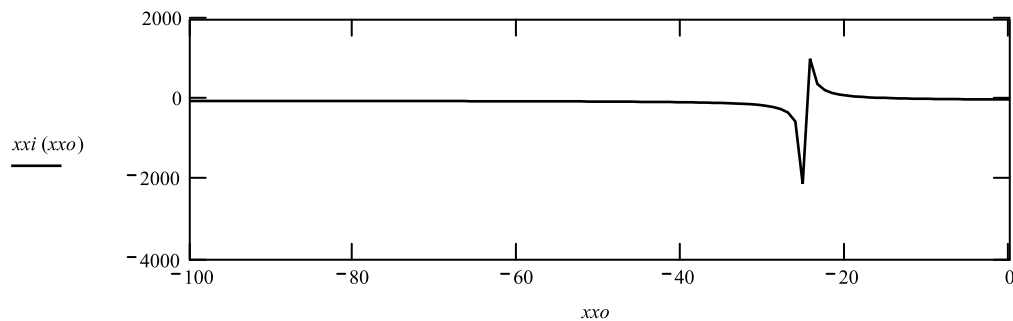
G30MIRCV

Concave Mirror

Raduis of curvature is negative; xo is on left, and is negative. To get around the singularity at $-xo = f$ one chooses the increments such that the value for the singularity does not appear.

$$r := -50$$

$$xo := -60$$

$$xi := \frac{1}{\left(\frac{1}{2}\right) - \frac{1}{xo}} \qquad xi = -42.857$$

$$m := \frac{-xi}{xo} \qquad m = -0.714.$$

Graph

$$xxo := -100, -99.1\ldots - .1$$

$$xxi(xxo) := \frac{1}{\left(\frac{1}{\frac{r}{2}}\right) - \frac{1}{xxo}}.$$

1.9.5.2 Convex Spherical Mirror

Geometrical Construction

1. Choose y_o and draw the PF-ray to the mirror and trace it forward to the focus F'.

2. The ray from the top of the arrow to the center of curvature C is reflected back onto itself (Figure 1.31).

Graph of x_i as Function of x_o

A convex spherical mirror has a positive radius of curvature. We show in FileFig 1.31 a graph of the image points as a function of the object points for x_o from
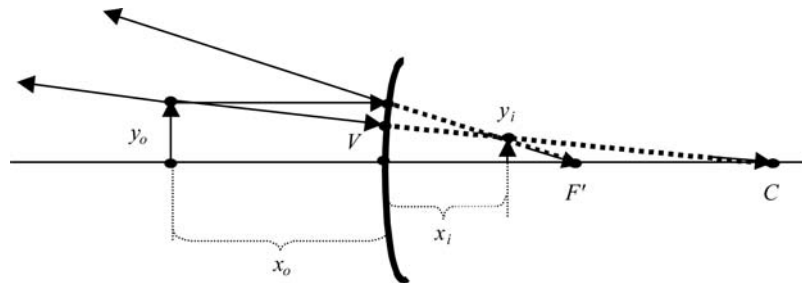
FIGURE 1.31 Geometrical construction of image for convex spherical mirror. The image y_i of object y_o is for any object distance to the right of the mirror (always virtual).

-100 to $-.1$. When the light comes from the left, there is no singularity at $r/2 = x_o$, and we obtain virtual images for all positions of x_o.

FileFig 1.31   (G31MIRCX)

Convex spherical mirror. Calculation of image position from given object position. Graph for image position depending on the object position coordinate for the radius of curvature $r = 50$, that is, $r/2 = 25$, and x_o from -100 to -0.1.

G31MIRCX is only on the CD.

A summary of the image formation and the dependence on the various parameters is given in Table 1.5.

Applications to Spherical Mirrors

1. A corner mirror is made of two flat mirrors, joined together at an angle of 90 degrees. Show that the light incident on one mirror is parallel to the light leaving the other mirror for any angle of incidence.

2. Do the geometrical construction of:

TABLE 1.5

|	Concave	Concave	Concave	Convex
Object x_o	Left of f	at f	Right of f	Any
Image x_i	Negative	- Infinity	Positive	Positive
Magnification	Negative		Positive	Positive
	Real		Virtual	Virtual
	Inverted		Upright	Upright

   a. convex spherical mirrors for (i) object at $-\infty$; (ii) object to the left of focus; and (iii) object to the right of focus.

   b. concave spherical mirror with same focal length, for the three positions of x_o about the same values as in **a**.

1.10 MATRICES FOR A REFLECTING CAVITY AND THE EIGENVALUE PROBLEM

The first Ne–He laser used a Fabry–Perot cavity with two flat mirrors at a separation of 1 m. It was very difficult to align this cavity, and the first alignment was done by accident. One of the researchers bumped into the table, causing the flat mirrors to vibrate, and laser action was observed. Later, spherical mirrors were used to construct easy to align cavities.

For our discussion of laser cavities, consisting of two reflecting spherical surfaces, we first look at a periodic lens line, equivalently representing the "round trips" of the light in a reflecting cavity. One section of the lens line is shown in Figure 1.32, where the forward and backward traveling light are shown separately. The next section has the same configuration and the light enters and leaves each section in the same way. The first and third lenses are shared by two sections. Therefore we have drawn them as half-lenses and assigned to them twice the focal length. The sequence of matrices for the lens line is

$$
\begin{pmatrix} 1 & 0 \\ -1/2f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/2f_1 & 1 \end{pmatrix}.
$$

$$(1.110)$$
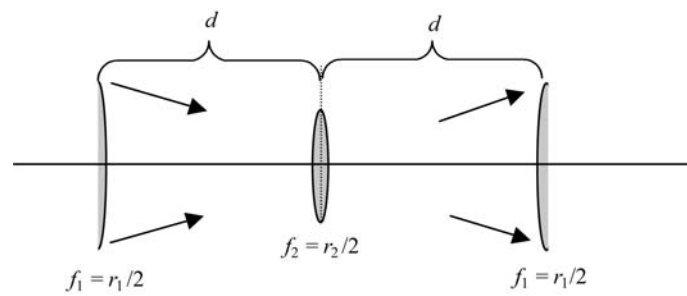
FIGURE 1.32 Unit cell of a lens system of periodic appearance. The light enters each cell in the same way. Such a periodic arrangement may be used to represent the reflection in a mirror cavity. The first and third lenses are only half-lenses and the focal lengths are twice as large. Rays of a possible light path are indicated.

We substitute for the focal length of the lenses one-half of the value of the radius of curvature of the corresponding mirrors. We use $f = r/2$ and obtain for the mirror cavity

$$\begin{pmatrix} 1 & 0 \\ -1/r_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2/r_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/r_1 & 1 \end{pmatrix}.$$

(1.111)

The first three matrices describe the travel of the light from the first to the second mirror. The last two matrices describe the travel from the second mirror back to the first.

We introduce the resonator parameters g_1 and g_2,

$$g_1 = 1 - d/r_1 \quad \text{and} \quad g_2 = 1 - d/r_2 \tag{1.112}$$

and calculate the product of the five matrices using FileFig 1.32.

FileFig 1.32   (G32RESGG)

Calculation of the product of the five matrices of the lens line corresponding to a cavity with two reflecting mirrors. Calculation of the eigenvalues of the cavity using g_1, g_2, and d. Graphs of the stability relation.

G32RESGG

Calculation of Resonator Using $g1$, $g2$, and d

$$\begin{pmatrix} 1 & 0 \\ \frac{g1-1}{d} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{2\cdot(g2-1)}{d} & 1 \end{bmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{g1-1}{d} & 1 \end{pmatrix}$$

$$\begin{bmatrix} -1 + 2\cdot g1 \cdot g2 & 2\cdot d \cdot g2 \\ 2\cdot g1 \cdot \frac{(-1+g1\cdot g2)}{d} & -1 + 2\cdot g1 \cdot g2 \end{bmatrix}$$

$$\text{eigenvals} \begin{bmatrix} \begin{bmatrix} -1 + 2\cdot g1 \cdot g2 & 2\cdot d \cdot g2 \\ 2\cdot g1 \cdot \frac{(-1+g1\cdot g2)}{d} & -1 + 2\cdot g1 \cdot g2 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} (1, 1, 1) = -1 + 2\cdot g1 \cdot g2 + 2\cdot \sqrt{-g1\cdot g2 + g1^2 \cdot g2^2} \\ (1, 1, 2) = -1 + 2\cdot g1 \cdot g2 - 2\cdot \sqrt{-g1\cdot g2 + g1^2 \cdot g2^2} \end{bmatrix}$$

$$r1 := 1 \qquad r2 := 1 \qquad d := 2$$

$$g1 := 1 - \frac{d}{r1} \qquad g2 := 1 - \frac{d}{r2}$$

$$\lambda 1 := -1 + 2\cdot g1 \cdot g2 + 2\sqrt{-g1\cdot g2 + g1^2 \cdot g2^2}$$

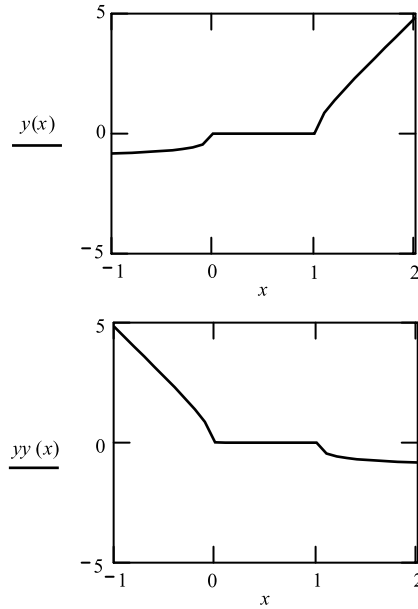$$\lambda 2 := -1 + 2 \cdot g1 \cdot g2 - 2\sqrt{-g1 \cdot g2 + g1^2 \cdot g2^2}$$
$$\lambda 1 = 1 \qquad \lambda 2 = 1.$$

We set the product $g1g2 = x$ and plot it over the range from -1 to 2.

$$x := -1, -9 \ldots 2$$
$$y(x) := |(2 \cdot x - 1) + \sqrt{(2 \cdot x - 1)^2 - 1}| - 1$$
$$yy(x) := |(2 \cdot x - 1) - \sqrt{(2 \cdot x - 1)^2 - 1}| - 1$$

We obtain from FileFig 1.32 the matrix product of the matrices of Eq. (1.111),

$$\begin{pmatrix} -1 + 2g_1g_2 & 2dg_2 \\ 2g_1(-1 + 2g_1g_2)/d & -1 + 2g_1g_2 \end{pmatrix}. \tag{1.113}$$

The round trip in the cavity must have the symmetry of the path of the rays at the beginning and end of a unit cell of the lens line. This corresponds to a mode of oscillation of the cavity. The eigenvalues of this oscillation are obtained from the eigenvalues of the product matrix and the calculation is shown in FileFig 1.32. First the product of the five matrices is calculated using the symbolic method. Then the eigenvalues are obtained

$$\lambda_1 = (2g_1g_2 - 1) + [(2g_1g_2 - 1)^2 - 1]^{1/2} \tag{1.114}$$
$$\lambda_2 = (2g_1g_2 - 1) - [(2g_1g_2 - 1)^2 - 1]^{1/2}. \tag{1.115}$$

The coordinates, used for setting up the matrices, may now be transformed into a new coordinate system. In this coordinate system the matrix describing the round trip in the cavity is a diagonal matrix. When the diagonal matrix is a unit

matrix, the light may pass through many round trips and no light will escape. One calls such a resonator stable, and the condition for stability is where the magnitudes of the eigenvalues are equal to 1.

$$|\lambda_1| = |\lambda_2| = 1. \tag{1.116}$$

We may write for Eq. (1.114),

$$\lambda_1 = (2g_1g_2 - 1) + [(2g_1g_2 - 1)^2 - 1]^{1/2} \tag{1.117}$$

or

$$\lambda_1 = (2g_1g_2 - 1) + i[1 - (2g_1g_2 - 1)^2]^{1/2}. \tag{1.118}$$

The real and imaginary parts of Eq. (1.118) must be on a circle of radius 1; that is,

$$|(2g_1g_2 - 1)| \leq 1, \quad \text{or} \quad 0 \leq g_1g_2 \leq 1. \tag{1.119}$$

in agreement with the imaginary part and plotted in FileFig 1.32.

In FileFig 1.33 we show a repetition of the calculations, starting from the five matrices of the cavity in Eq. (1.111), but now in terms of r_1, r_2, and d. In Figure 1.33, we show schematics of the Fabry–Perot, a focal, a confocal and a spherical cavity for values of the parameters r_1, r_2, and d, and also of g_1 and g_2. For both representations one finds that the absolute values of the eigenvalues λ_1 and λ_2 are always 1.

FileFig 1.33   (G33RESCY)

Calculation of the eigenvalues of the cavity with two reflecting mirrors using r_1, r_2, and d. Numerical calculation with $r_1 = 1$, $r_2 = 1$, and $d = 2$.

G33RESCY is only on the CD.

Application 1.33. *Use the values of the parameters r_1, r_2, and d for the Fabry–Perot, focal, confocal, and spherical cavities, and find that all are stable cavities.*
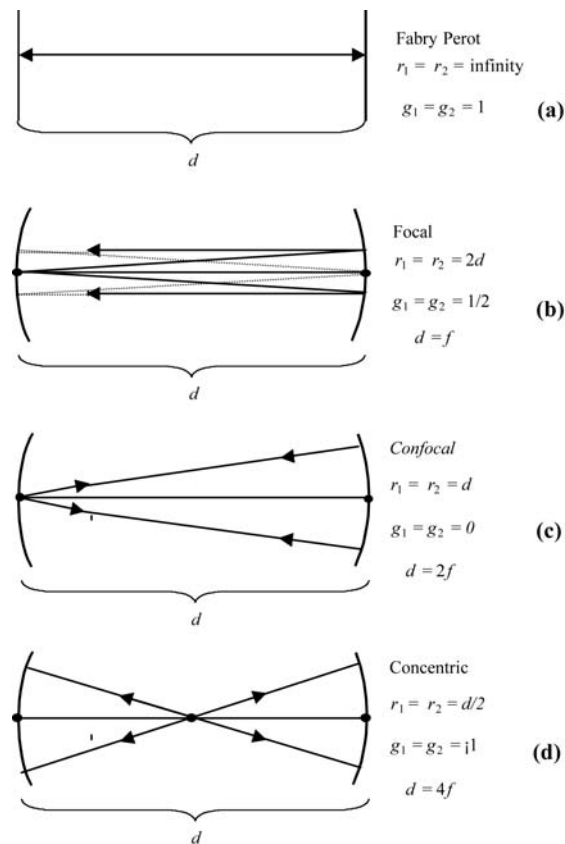
FIGURE 1.33 Schematic of light path for four cavities with different values of radii of curvature and length of cavity. The corresponding values of g_1 and g_2 are indicated: (a) Fabry–Perot; (b) focal; (c) confocal; (d) concentric.