Preface

In 1879, Picard established the well-known and beautiful result that a transcendental entire function assumes all values infinitely often with one exception. Since then Hadamard (1893), Borel (1897) and Blumenthal (1910) had tried to give Picard's result a quantitative description and extend it to meromorphic functions. It was R. Nevanlinna, who achieved such an attempt in (1925) by establishing the so-called value distribution theory of meromorphic functions which was praised by H. Weyl (1943) as "One of the few great mathematical events of our century". Moreover, part of the significance of Nevanlinna's approach is that the concept of exceptional values can be given a geometric interpretation in terms of geometric objects like curves and mappings of subspaces of holomorphic curves from a complex plane $\mathbb{C}$ to a projective space $\mathbb{P}^n$. In the years since these results were achieved, mathematicians of comparable stature have made efforts to derive an analogous theory for meromorphic mappings and p-adic meromorphic functions. Besides the value distribution, the theory has had many applications to the analyticity, growth, existence, and unicity properties of meromorphic solutions to differential or functional equations. More recently, it has been found that there is a profound relation between Nevanlinna theory and number theory. C.F. Osgood [310], [311] first noticed a similarity between the 2 in Nevanlinna's defect relation and the 2 in Roth's theorem. S. Lang [230] pointed to the existence of a structure to the error term in Nevanlinna's second main theorem, conjectured what could be essentially the best possible form of this error term in general (based on his conjecture on the error term in Roth's theorem), and gave a quite detailed discussion in [235]. P.M. Wong [433] used a method of Ahlfors to prove Lang's conjecture in the one-dimensional case. As for higher dimensions, this problem was studied by S. Lang and W. Cherry [235], A. Hinkkanen [159], and was finally completed by Z. Ye [443]. The best possible form of error terms has been used in our present work to produce some sharp results.

In 1987, P. Vojta [415] gave a much deeper analysis of the situation, and compared the theory of heights in number theory with the characteristic functions of Nevanlinna theory. In his dictionary, the second main theorem, due to H. Cartan, corresponds to Schmidt's subspace theorem. Further, he proposed the general conjecture in number theory by comparing the second main theorem in Carlson-Griffiths-King's theory, or particularly influenced by Griffiths' conjecture, which also can be translated into a problem of non-Archimedean holomorphic curves posed by Hu and Yang [176]. Along this route, Shiffman's conjecture on hypersurface targets in value distribution theory corresponds to a subspace theorem for homogeneous polynomial forms in Diophantine approximation. Vojta's $(1,1)$-form conjecture is an analogue of an inequality of characteristic functions of holomorphic curves for line bundles. Being influenced by Mason's theorem, Oesterlé and Masser formulated the *abc*-conjecture. The generalized *abc*-conjectures for integers are counterparts of Nevanlinna's third main theorem and its variations in value distribution theory, and so on.

Roughly speaking, a significant analogy between Nevanlinna theory and Diophantine approximation seems to be that the sets $X(\kappa)$ of κ-rational points of a projective variety X defined over number fields κ are finite if and only if there are no non-constant holomorphic curves into X. Mordell's conjecture (Faltings' theorem) and Picard's theorem are classic examples in this direction. In higher-dimensional spaces, this corresponds to a conjecture due to S. Lang, that is, Kobayashi hyperbolic manifolds (which do not contain non-constant holomorphic curves) are Mordellic. Bloch-Green-Griffiths' conjecture on degeneracy of holomorphic curves into pseudo-canonical projective varieties is an analogue of the Bombieri-Lang conjecture on pseudo canonical varieties. We have introduced these problems and the related developments in this book. Generally, topics or problems in number theory are briefly introduced and translated as analogues of topics in value distribution theory. We have omitted the proofs of theorems in number theory. However, we have discussed the problems of value distribution in detail. In this book, we will not discuss value distribution theory of moving targets, say, K. Yamanoi's work [437], and their counterparts in number theory.

When a holomorphic curve f into X is not constant, we have to distinguish whether it is degenerate in Nevanlinna theory, that is, whether its image is contained or not in a proper subvariety. If it is degenerate, usually it is difficult to deal with it in value distribution theory. If f is non-degenerate, we can study its value distributions and measure its growth well by a characteristic function $T_f(r)$. Similarly, we should distinguish whether or not certain rational points are degenerate. Related to the degeneracy, it seems that for each number field κ, $X(\kappa)$ is contained in a proper Zariski closed subset if and only if there are no algebraically non-degenerate holomorphic curves into X. To compare with Nevanlinna theory, therefore, we need to rule out degenerate κ-rational points that are contained in a subspace of lower dimension, and give a proper measure for non-degenerate κ-rational points. By integrating heights over non-degenerate κ-rational points, we can obtain quantitative measurements $T_\kappa(r)$.

They have the following basic properties:

(i) f is constant if and only if $T_f(r)$ is bounded; there are no non-degenerate κ-rational points if and only if $T_\kappa(r)$ is bounded .

(ii) f is rational if and only if $T_f(r) = O(\log r)$; there are only finitely many non-degenerate κ-rational points if and only if $T_\kappa(r) = O(\log r)$.

It has been observed that there exist non-constant holomorphic curves into elliptic curves such that they must be surjective. Thus it is possible that there are infinitely many rational points on some elliptic curves. However, since non-constant holomorphic curves into elliptic curves have normal properties, say, they are surjective, then distribution of rational points on elliptic curves should be "normal". Really, elliptic curves are modular according to the Shimura-Taniyama-Weil conjecture, which was proved by Breuil, Conrad, Diamond, and Taylor [37] by extending work of Wiles [431], Taylor and Wiles [390]. Moreover, as a result of studies of the analogous results between Nevanlinna's value distribution theory and Diophantine approximation, some novel ideas and generalizations have been developed or derived in the two topics, with many open problems posed for further investigations.

The book consists of seven chapters: In Chapter 1, we introduce some basic notation and terminology on fields and algebraic geometry which are mainly used to explain clearly the topics in Chapter 3 related to number theory. Chapter 2 is a foundation of value distribution theory which is used in Chapter 4, Chapter 6 and Chapter 7 to introduce the analogues related to number theory in Nevanlinna theory, say, *abc*-problems, meromorphic solutions of Fermat's equations and the Waring problem, Green-Griffiths' conjecture, Griffiths' and Lang's conjectures, Riemann's ζ-function, and so on. Chapter 5 contains value distribution theory over non-Archimedean fields and some applications related to topics in number theory. Moreover, a few equidistribution formulae illustrating the differences with the classical Nevanlinna theory have been exhibited. Each chapter of this book is self-contained and this book is appended with a comprehensive and up-dated list of references. The book will provide not just some new research results and directions but challenging open problems in studying Diophantine approximation and Nevanlinna theory. One of the aims of this book is to make timely surveys on these new results and their related developments; some of which are newly obtained by the authors and have not been published yet. It is hoped that the publication of this book will stimulate, among our peers, further researches on Nevanlinna's value distribution theory, Diophantine approximation and their applications.

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