Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series *Texts in Applied Mathematics* (TAM).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (AMS) series, which will focus on advanced textbooks and research-level monographs.

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Preface

This book is based on a two-semester course in ordinary differential equations that I have taught to graduate students for two decades at the University of Missouri. The scope of the narrative evolved over time from an embryonic collection of supplementary notes, through many classroom tested revisions, to a treatment of the subject that is suitable for a year (or more) of graduate study.

If it is true that students of differential equations give away their point of view by the way they denote the derivative with respect to the independent variable, then the initiated reader can turn to Chapter 1, note that I write $\dot{x}$, not x', and thus correctly deduce that this book is written with an eye toward dynamical systems. Indeed, this book contains a thorough introduction to the basic properties of differential equations that are needed to approach the modern theory of (nonlinear) dynamical systems. But this is not the whole story. The book is also a product of my desire to demonstrate to my students that differential equations is the least insular of mathematical subjects, that it is strongly connected to almost all areas of mathematics, and it is an essential element of applied mathematics.

When I teach this course, I use the first part of the first semester to provide a rapid, student-friendly survey of the standard topics encountered in an introductory course of ordinary differential equations (ODE): existence theory, flows, invariant manifolds, linearization, omega limit sets, phase plane analysis, and stability. These topics, covered in Sections 1.1–1.8 of Chapter 1 of this book, are introduced, together with some of their important and *interesting* applications, so that the power and beauty of the subject is immediately apparent. This is followed by a discussion of linear systems theory and the proofs of the basic theorems on linearized stability in Chapter 2. Then, I conclude the first semester by presenting one or two realistic applications from Chapter 3. These applications provide a capstone for the course as well as an excellent opportunity to teach the mathematics graduate students some physics, while giving the engineering and physics students some exposure to applications from a mathematical perspective.

In the second semester, I introduce some advanced concepts related to existence theory, invariant manifolds, continuation of periodic orbits, forced oscillators, separatrix splitting, averaging, and bifurcation theory. Since there is not enough time in one semester to cover all of this material in depth, I usually choose just one or two of these topics for presentation in class. The material in the remaining chapters is assigned for private study according to the interests of my students.

My course is designed to be accessible to students who have only studied differential equations during one undergraduate semester. While I do assume some knowledge of linear algebra, advanced calculus, and analysis, only the most basic material from these subjects is required: eigenvalues and eigenvectors, compact sets, uniform convergence, the derivative of a function of several variables, and the definition of metric and Banach spaces. With regard to the last prerequisite, I find that some students are afraid to take the course because they are not comfortable with Banach space theory. These students are put at ease by mentioning that no deep properties of infinite dimensional spaces are used, only the basic definitions.

Exercises are an integral part of this book. As such, many of them are placed strategically within the text, rather than at the end of a section. These interruptions of the flow of the narrative are meant to provide an opportunity for the reader to absorb the preceding material and as a guide to further study. Some of the exercises are routine, while others are sections of the text written in "exercise form." For example, there are extended exercises on structural stability, Hamiltonian and gradient systems on manifolds, singular perturbations, and Lie groups. My students are strongly encouraged to work through the exercises. How is it possible to gain an understanding of a mathematical subject without doing some mathematics? Perhaps a mathematics book is like a musical score: by sight reading you can pick out the notes, but practice is required to hear the melody.

The placement of exercises is just one indication that this book is not written in axiomatic style. Many results are used before their proofs are provided, some ideas are discussed without formal proofs, and some advanced topics are introduced without being fully developed. The pure axiomatic approach forbids the use of such devices in favor of logical order. The other extreme would be a treatment that is intended to convey the ideas of the subject with no attempt to provide detailed proofs of basic results. While the narrative of an axiomatic approach can be as dry as dust, the excitement of an idea-oriented approach must be weighed against the fact that it might leave most beginning students unable to grasp the subtlety of the arguments required to justify the mathematics. I have tried to steer a middle course in which careful formulations and complete proofs are given for the basic theorems, while the ideas of the subject are discussed in depth and the path from the pure mathematics to the physical universe is clearly marked. I am reminded of an esteemed colleague who mentioned that a certain textbook "has lots of fruit, but no juice." Above all, I have tried to avoid this criticism.

Application of the implicit function theorem is a recurring theme in the book. For example, the implicit function theorem is used to prove the rectification theorem and the fundamental existence and uniqueness theorems for solutions of differential equations in Banach spaces. Also, the basic results of perturbation and bifurcation theory, including the continuation of subharmonics, the existence of periodic solutions via the averaging method, as well as the saddle node and Hopf bifurcations, are presented as applications of the implicit function theorem. Because of its central role, the implicit function theorem and the terrain surrounding this important result are discussed in detail. In particular, I present a review of calculus in a Banach space setting and use this theory to prove the contraction mapping theorem, the uniform contraction mapping theorem, and the implicit function theorem.

This book contains some material that is not encountered in most treatments of the subject. In particular, there are several sections with the title "Origins of ODE," where I give my answer to the question "What is this good for?" by providing an explanation for the appearance of differential equations in mathematics and the physical sciences. For example, I show how ordinary differential equations arise in classical physics from the fundamental laws of motion and force. This discussion includes a derivation of the Euler–Lagrange equation, some exercises in electrodynamics, and an extended treatment of the perturbed Kepler problem. Also, I have included some discussion of the origins of ordinary differential equations in the theory of partial differential equations. For instance, I explain the idea that a parabolic partial differential equation can be viewed as an ordinary differential equation in an infinite dimensional space. In addition, traveling wave solutions and the Galërkin approximation technique are discussed. In a later "origins" section, the basic models for fluid dynamics are introduced. I show how ordinary differential equations arise in boundary layer theory. Also, the ABC flows are defined as an idealized fluid model, and I demonstrate that this model has chaotic regimes. There is also a section on coupled oscillators, a section on the Fermi–Ulam–Pasta experiments, and one on the stability of the inverted pendulum where a proof of linearized stability under rapid oscillation is obtained using Floquet's method and some ideas from bifurcation theory. Finally, in conjunction with a treatment of the multiple Hopf bifurcation for planar systems, I present a short introduction to an algorithm for the computation of the Lyapunov quantities as an illustration of computer algebra methods in bifurcation theory.

Another special feature of the book is an introduction to the fiber contraction principle as a powerful tool for proving the smoothness of functions that are obtained as fixed points of contractions. This basic method is used first in a proof of the smoothness of the flow of a differential equation where its application is transparent. Later, the fiber contraction principle appears in the nontrivial proof of the smoothness of invariant manifolds at a rest point. In this regard, the proof for the existence and smoothness of stable and center manifolds at a rest point is obtained as a corollary of a more general existence theorem for invariant manifolds in the presence of a "spectral gap." These proofs can be extended to infinite dimensions. In particular, the applications of the fiber contraction principle and the Lyapunov–Perron method in this book provide an introduction to some of the basic tools of invariant manifold theory.

The theory of averaging is treated from a fresh perspective that is intended to introduce the modern approach to this classical subject. A complete proof of the averaging theorem is presented, but the main theme of the chapter is partial averaging at a resonance. In particular, the "pendulum with torque" is shown to be a universal model for the motion of a nonlinear oscillator near a resonance. This approach to the subject leads naturally to the phenomenon of "capture into resonance," and it also provides the necessary background for students who wish to read the literature on multifrequency averaging, Hamiltonian chaos, and Arnold diffusion.

I prove the basic results of one-parameter bifurcation theory—the saddle node and Hopf bifurcations—using the Lyapunov–Schmidt reduction. The fact that degeneracies in a family of differential equations might be unavoidable is explained together with a brief introduction to transversality theory and jet spaces. Also, the multiple Hopf bifurcation for planar vector fields is discussed. In particular, and the Lyapunov quantities for polynomial vector fields at a weak focus are defined and this subject matter is used to provide a link to some of the algebraic techniques that appear in normal form theory.

Since almost all of the topics in this book are covered elsewhere, there is no claim of originality on my part. I have merely organized the material in a manner that I believe to be most beneficial to my students. By reading this book, I hope that you will appreciate and be well prepared to use the wonderful subject of differential equations.

Columbia, Missouri Carmen Chicone
June 1999

Preface to the Second Edition

This edition contains new material, new exercises, rewritten sections, and corrections.

There are at least three nontrivial mathematical errors in the first edition: The proof of the Trotter product formula (Theorem 2.24) is valid only in case $e^{A+B} = e^A e^B$; the Floquet theorem (Theorem 2.47) on the existence of logarithms for matrices is valid only if the square of the real matrix in question has all positive eigenvalues; and the proof of the smoothness of invariant manifolds (Theorem 4.1) has a gap because the continuity of a certain fiber contraction with respect to its base space is assumed. The first two errors were pointed out by Mark Ashbaugh, the third by Mohamed ElBialy. These and many other less serious errors are corrected.

While much of the narrative has been revised, the most substantial additions and revisions not already mentioned are the following: the introductory Section 1.9.3 on contraction is rewritten to include a discussion of the continuity of fiber contractions and a more informative first application of the fiber contraction theorem, which is the proof of the smoothness of the solution of the functional equation $F \circ \phi - \phi = G$ (Theorem 1.234); Section 3.1 on the Euler-Lagrange equation is rewritten and expanded to include a more detailed discussion of Hamilton's theory, a presentation of Noether's Theorem, and several new exercises on the calculus of variations; Section 3.2 on classical mechanics has been revised by including more details; the application (in Section 3.5) of Floquet theory to the stability of the inverted pendulum is rewritten to incorporate a more elegant dimensionless model; a new Section 4.3.3 introduces the Lie derivative and applies it to prove the Hartman-Grobman theorem for flows; multidimensional continuation theory for periodic orbits in the presence of first integrals is discussed in the new Section 5.3.8, the basic result on the continuation of manifolds of periodic orbits in the presence of first integrals in involution is proved, and the Lie derivative is used again to characterize commuting flows; and the subject of dynamic bifurcation theory is introduced in a new Section 8.4 where the fundamental idea of delayed bifurcation is presented with applications to the pitchfork bifurcation and bursting.

Over 160 new exercises are included, most with multiple parts. While a few routine exercises are provided where I expect them to be helpful, most of the exercises are meant to challenge students on their understanding of the theory, stimulate interest, extend topics introduced in the narrative, and point the way to applications. Also, most exercises now have lettered parts for easy identification of portions of exercises for homework assignments.

As described in the Preface, the core first graduate course in ODE is contained in selections from the first three chapters. The instructor should budget class time so that all of the language and basic concepts of the subject (existence theory, flows, invariant manifolds, linearization, omega limit sets, phase plane analysis, and stability) are introduced and some applications are discussed in detail.

In my experience, sensitivity to the preparation of students is essential for a successful first graduate course in differential equations. Although the prerequisites are minimal, there are certainly some students who are unprepared for the challenges of a course based on this book if their exposure to differential equations is limited to no more than one undergraduate course where they studied only solution methods for linear second order equations. I have included some review (see Exercise 1.6) to serve as a bridge from their first course to this book. In addition, I often use some class time to review a few fundamental concepts (especially, the derivative as a linear transformation, compactness, connectedness, uniform convergence, linear spaces, eigenvalues, and Jordan canonical form) before they are encountered in context.

The second edition contains plenty of material for second semester courses, master's projects, and reading courses. Professionals might also find something of value.

I remain an enthusiastic teacher of the rich and important subject of differential equations. I hope that instructors will find this book a useful addition to their class design and preparation, and students will have a clear and faithful guide during their quest to learn the subject.

Columbia, Missouri Carmen Chicone
August 2005

2

Linear Systems and Stability of Nonlinear Systems

In this chapter we will study the differential equation

$$\dot{x} = A(t)x + f(x,t), \qquad x \in \mathbb{R}^n$$

where A is a smooth $n \times n$ matrix-valued function and f is a smooth function such that $f(0,t) = f_x(0,t) \equiv 0$. Note that if f has this form, then the associated *homogeneous linear system* $\dot{x} = A(t)x$ is the linearization of the differential equation along the *zero solution* $t \mapsto \phi(t) \equiv 0$.

One of the main objectives of the chapter is the proof of the basic results related to the principle of linearized stability. For example, we will prove that if the matrix A is constant and all of its eigenvalues have negative real parts, then the zero solution (also called the *trivial solution*) is asymptotically stable. Much of the chapter, however, is devoted to the general theory of homogeneous linear systems; that is, systems of the form $\dot{x} = A(t)x$. In particular, we will study the important special cases where A is a constant or periodic function.

In case $t \mapsto A(t)$ is a constant function, we will show how to reduce the solution of the system $\dot{x} = Ax$ to a problem in linear algebra. Also, by defining the matrix exponential, we will discuss the flow of this autonomous system as a one-parameter group with generator A.

Although the behavior of the general nonautonomous system $\dot{x} = A(t)x$ is not completely understood, the special case where $t \mapsto A(t)$ is a periodic matrix-valued function is reducible to the constant matrix case. We will develop a useful theory of periodic matrix systems, called Floquet theory, and use it to prove this basic result. The Floquet theory will appear again later when we discuss the stability of periodic nonhomogeneous systems. In particular, we will use Floquet theory in a stability analysis of the inverted pendulum (see Section 3.5).

Because linear systems theory is so well developed, it is used extensively in many areas of applied science. For example, linear systems theory is an essential tool for electromagnetics, circuit theory, and the theory of vibration. In addition, the results of this chapter are a fundamental component of control theory.

2.1 Homogeneous Linear Differential Equations

This section is devoted to a general discussion of the homogeneous linear system

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n$$

where $t \mapsto A(t)$ is a smooth function from some open interval $J \subseteq \mathbb{R}$ to the space of $n \times n$ matrices. Here, the continuity properties of matrix-valued functions are determined by viewing the space of $n \times n$ matrices as $\mathbb{R}^{n^2}$; that is, every matrix is viewed as an element in the Cartesian space by simply listing the rows of the matrix consecutively to form a row vector of length n^2. We will prove an important general inequality and then use it to show that solutions of linear systems cannot blow up in finite time. We will discuss the basic result that the set of solutions of a linear system is a vector space, and we will exploit this fact by showing how to construct the general solution of a linear homogeneous system with constant coefficients.

2.1.1 Gronwall's Inequality

The important theorem proved in this section does not belong to the theory of linear differential equations per se, but it is presented here because it will be used to prove the global existence of solutions of homogeneous linear systems.

Theorem 2.1 (Gronwall's Inequality). *Suppose that $a < b$ and let α, ϕ, and ψ be nonnegative continuous functions defined on the interval $[a, b]$. Moreover, suppose that α is differentiable on (a, b) with nonnegative continuous derivative $\dot{\alpha}$. If, for all $t \in [a, b]$,*

$$\phi(t) \le \alpha(t) + \int_a^t \psi(s)\phi(s)\, ds, \tag{2.1}$$

then

$$\phi(t) \le \alpha(t) e^{\int_a^t \psi(s)\, ds} \tag{2.2}$$

for all $t \in [a, b]$.

Proof. Assume for the moment that $\alpha(a) > 0$. In this case $\alpha(t) \geq \alpha(a) > 0$ on the interval $[a, b]$.

The function on the interval $[a, b]$ defined by $t \mapsto \alpha(t) + \int_a^t \psi(s)\phi(s)\,ds$ is positive and exceeds ϕ. Thus, we have that

$$\frac{\phi(t)}{\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds} \leq 1.$$

Multiply both sides of this inequality by $\psi(t)$, add and subtract $\dot{\alpha}(t)$ in the numerator of the resulting fraction, rearrange the inequality, and use the obvious estimate to obtain the inequality

$$\frac{\dot{\alpha}(t) + \psi(t)\phi(t)}{\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds} \leq \frac{\dot{\alpha}(t)}{\alpha(t)} + \psi(t),$$

which, when integrated over the interval $[a, t]$, yields the inequality

$$\ln\Big(\alpha(t) + \int_a^t \psi(s)\phi(s)\,ds\Big) - \ln(\alpha(a)) \leq \int_a^t \psi(s)\,ds + \ln(\alpha(t)) - \ln(\alpha(a)).$$

After we exponentiate both sides of this last inequality and use hypothesis (2.1), we find that, for each t in the interval $[a, b]$,

$$\phi(t) \leq \alpha(t)e^{\int_a^t \psi(s)\,ds} \leq \alpha(t)e^{\int_a^t \psi(s)\,ds}. \tag{2.3}$$

Finally, for the case $\alpha(a) = 0$, we have the inequality

$$\phi(t) \leq (\alpha(t) + \epsilon) + \int_a^t \psi(s)\phi(s)\,ds$$

for each $\epsilon > 0$. As a result of what we have just proved, we have the estimate

$$\phi(t) \leq (\alpha(t) + \epsilon)e^{\int_a^t \psi(s)\,ds}.$$

The desired inequality follows by passing to the limit (for each fixed $t \in [a, b]$) as $\epsilon \to 0$. □

Exercise 2.2. What can you say about a continuous function $f : \mathbb{R} \to [0, \infty)$ if

$$f(x) \leq \int_0^x f(t)\,dt?$$

Exercise 2.3. Prove the “specific Gronwall lemma” [198]: If, for $t \in [a, b]$,

$$\phi(t) \leq \delta_2(t - a) + \delta_1 \int_a^t \phi(s)\,ds + \delta_3,$$

where ϕ is a nonnegative continuous function on $[a, b]$, and $\delta_1 > 0$, $\delta_2 \geq 0$, and $\delta_3 \geq 0$ are constants, then

$$\phi(t) \leq \Big(\frac{\delta_2}{\delta_1} + \delta_3\Big)e^{\delta_1(t-a)} - \frac{\delta_2}{\delta_1}.$$

2.1.2 Homogeneous Linear Systems: General Theory

Consider the *homogeneous* linear system

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n. \tag{2.4}$$

By our general existence theory, the initial value problem

$$\dot{x} = A(t)x, \qquad x(t_0) = x_0 \tag{2.5}$$

has a unique solution that exists on some open interval containing t_0. The following theorem states that this open interval can be extended to the domain of A.

Theorem 2.4. *If $t \mapsto A(t)$ is continuous on the interval $\alpha < t < \beta$ and if $\alpha < t_0 < \beta$ (maybe $\alpha = -\infty$ or $\beta = \infty$), then the solution of the initial value problem (2.5) is defined on the open interval (α, β).*

Proof. Because the continuous function $t \mapsto A(t)$ is bounded on each compact subinterval of (α, β), it is easy to see that the function $(t, x) \mapsto A(t)x$ is locally Lipschitz with respect to its second argument. Consider the solution $t \mapsto \phi(t)$ of the initial value problem (2.5) given by the general existence theorem (Theorem 1.261) and let J_0 denote its maximal interval of existence. Suppose that J_0 does not contain (α, β). For example, suppose that the right hand end point b of J_0 is less than β. We will show that this assumption leads to a contradiction. The proof for the left hand end point is similar.

If $t \in J_0$, then we have

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} A(s)\phi(s)\, ds.$$

By the continuity of A and the compactness of $[t_0, b]$, there is some $M > 0$ such that $\|A(t)\| \le M$ for all $t \in [t_0, b]$. (The notation $\|\ \|$ is used for the matrix norm corresponding to some norm $|\ |$ on $\mathbb{R}^n$.) Thus, for $t \in J_0$, we have the following inequality:

$$\begin{aligned} |\phi(t)| &\le |x_0| + \int_{t_0}^{t} \|A(s)\||\phi(s)|\, ds \\ &\le |x_0| + \int_{t_0}^{t} M|\phi(s)|\, ds. \end{aligned}$$

In addition, by Gronwall's inequality, with $\psi(t) := M$, we have

$$|\phi(t)| \le |x_0| e^{M \int_{t_0}^{t} ds} = |x_0| e^{M(t - t_0)}.$$

Thus, $|\phi(t)|$ is uniformly bounded on $[t_0, b)$.

Because the boundary of $\mathbb{R}^n$ is empty, it follows from the extension theorem that $|\phi(t)| \to \infty$ as $t \to b^-$, in contradiction to the existence of the uniform bound. □

Exercise 2.5. Use Gronwall's inequality to prove the following important inequality: If $t \mapsto \beta(t)$ and $t \mapsto \gamma(t)$ are solutions of the smooth differential equation $\dot{x} = f(x)$ and both are defined on the time interval $[0, T]$, then there is a constant $L > 0$ such that

$$|\beta(t) - \alpha(t)| \leq |\beta(0) - \alpha(0)|e^{Lt}.$$

Thus, two solutions diverge from each other at most exponentially fast. Also, if the solutions have the same initial condition, then they coincide. Therefore, the result of this exercise provides an alternative proof of the general uniqueness theorem for differential equations.

Exercise 2.6. Prove that if A is a linear transformation of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a (smooth) function such that $|f(x)| \leq M|x| + N$ for positive constants M and N, then the differential equation $\dot{x} = Ax + f(x)$ has a complete flow.

Exercise 2.7. Suppose that $X(\cdot, \lambda)$ and $Y(\cdot, \lambda)$ are two vector fields with parameter $\lambda \in \mathbb{R}$, and the two vector fields agree to order N in λ; that is, $X(x, \lambda) = Y(x, \lambda) + O(\lambda^{N+1})$. If $x(t, \lambda)$ and $y(t, \lambda)$ are corresponding solutions defined on the interval $[0, T]$ with initial conditions at $t = 0$ that agree to order N in λ, prove that $x(T, \lambda)$ and $y(T, \lambda)$ agree to order N in λ. Hint: First prove the result for $N = 0$.

2.1.3 Principle of Superposition

The foundational result about linear homogeneous systems is the principle of superposition: The sum of two solutions is again a solution. A precise statement of this principle is the content of the next proposition.

Proposition 2.8. *If the homogeneous system (2.4) has two solutions $\phi_1(t)$ and $\phi_2(t)$, each defined on some interval (a, b), and if λ_1 and λ_2 are numbers, then $t \to \lambda_1\phi_1(t) + \lambda_2\phi_2(t)$ is also a solution defined on the same interval.*

Proof. To prove the proposition, we use the *linearity* of the differential equation. In fact, we have

$$\begin{aligned}\frac{d}{dt}(\lambda_1\phi_1(t) + \lambda_2\phi_2(t)) &= \lambda_1\dot{\phi}_1(t) + \lambda_2\dot{\phi}_2(t)\\ &= \lambda_1 A(t)\phi_1(t) + \lambda_2 A(t)\phi_2(t)\\ &= A(t)(\lambda_1\phi_1(t) + \lambda_2\phi_2(t)).\end{aligned}$$

□

As a natural extension of the principle of superposition, we will prove that the set of solutions of the homogeneous linear system (2.4) is a finite dimensional vector space of dimension n.

Definition 2.9. A set of n solutions of the homogeneous linear differential equation (2.4), all defined on the same open interval J, is called a *fundamental set* of solutions on J if the solutions are linearly independent functions on J.

Proposition 2.10. *If $t \to A(t)$ is defined on the interval (a, b), then the system (2.4) has a fundamental set of solutions defined on (a, b).*

Proof. If $c \in (a, b)$ and $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the usual basis vectors in $\mathbb{R}^n$, then there is a unique solution $t \mapsto \phi_i(t)$ such that $\phi_i(c) = \mathbf{e}_i$ for $i = 1, \dots, n$. Moreover, by Theorem 2.4, each function ϕ_i is defined on the interval (a, b). Let us assume that the set of *functions* $\{\phi_i : i = 1, \dots, n\}$ is linearly dependent and derive a contradiction. In fact, if there are scalars α_i, $i = 1, \dots, n$, not all zero, such that $\sum_{i=1}^n \alpha_i \phi_i(t) \equiv 0$, then $\sum_{i=1}^n \alpha_i \mathbf{e}_i \equiv 0$. In view of the linear independence of the usual basis, this is the desired contradiction. □

Proposition 2.11. *If $\mathcal{F}$ is a fundamental set of solutions of the linear system (2.4) on the interval (a, b), then every solution defined on (a, b) can be expressed as a linear combination of the elements of $\mathcal{F}$.*

Proof. Suppose that $\mathcal{F} = \{\phi_1, \dots, \phi_n\}$. Pick $c \in (a, b)$. If $t \mapsto \phi(t)$ is a solution defined on (a, b), then $\phi(c)$ and $\phi_i(c)$, for $i = 1, \dots, n$, are all vectors in $\mathbb{R}^n$. We will show that the set $B := \{\phi_i(c) : i = 1, \dots, n\}$ is a basis for $\mathbb{R}^n$. If not, then there are scalars α_i, $i = 1, \dots, n$, not all zero, such that $\sum_{i=1}^n \alpha_i \phi_i(c) = 0$. Thus, $y(t) := \sum_{i=1}^n \alpha_i \phi_i(t)$ is a solution with initial condition $y(c) = 0$. But the zero solution has the same initial condition. Thus, $y(t) \equiv 0$, and therefore $\sum_{i=1}^n \alpha_i \phi_i(t) \equiv 0$. This contradicts the hypothesis that $\mathcal{F}$ is a linearly independent set, as required.

Using the basis B, there are scalars $\beta_1, \dots, \beta_n \in \mathbb{R}$ such that $\phi(c) = \sum_{i=1}^n \beta_i \phi_i(c)$. It follows that both ϕ and $\sum_{i=1}^n \beta_i \phi_i$ are solutions with the same initial condition, and, by uniqueness, $\phi = \sum_{i=1}^n \beta_i \phi_i$. □

Definition 2.12. An $n \times n$ matrix function $t \mapsto \Psi(t)$, defined on an open interval J, is called a *matrix solution* of the homogeneous linear system (2.4) if each of its columns is a (vector) solution. A matrix solution is called a *fundamental matrix solution* if its columns form a fundamental set of solutions. In addition, a fundamental matrix solution $t \mapsto \Psi(t)$ is called the *principal fundamental matrix solution* at $t_0 \in J$ if $\Psi(t_0) = I$.

If $t \mapsto \Psi(t)$ is a matrix solution of the system (2.4) on the interval J, then $\dot{\Psi}(t) = A(t)\Psi(t)$ on J. By Proposition 2.10, there is a fundamental matrix solution. Moreover, if $t_0 \in J$ and $t \mapsto \Phi(t)$ is a fundamental matrix solution on J, then (by the linear independence of its columns) the matrix $\Phi(t_0)$ is invertible. It is easy to see that the matrix solution defined by $\Psi(t) := \Phi(t)\Phi^{-1}(t_0)$ is the principal fundamental matrix solution at t_0. Thus, system (2.4) has a principal fundamental matrix solution at each point in J.

Definition 2.13. The *state transition matrix* for the homogeneous linear system (2.4) on the open interval J is the family of fundamental matrix solutions $t \mapsto \Psi(t, \tau)$ parametrized by $\tau \in J$ such that $\Psi(\tau, \tau) = I$, where I denotes the $n \times n$ identity matrix.

Proposition 2.14. *If $t \mapsto \Phi(t)$ is a fundamental matrix solution for the system (2.4) on J, then $\Psi(t, \tau) := \Phi(t)\Phi^{-1}(\tau)$ is the state transition matrix. Also, the state transition matrix satisfies the Chapman–Kolmogorov identities*

$$\Psi(\tau, \tau) = I, \quad \Psi(t, s)\Psi(s, \tau) = \Psi(t, \tau)$$

and the identities

$$\Psi(t, s)^{-1} = \Psi(s, t), \qquad \frac{\partial \Psi}{\partial s}(t, s) = -\Psi(t, s)A(s).$$

Proof. See Exercise 2.15. □

A two-parameter family of operator-valued functions that satisfies the Chapman–Kolmogorov identities is called an *evolution family*.

In the case of constant coefficients, that is, in case $t \mapsto A(t)$ is a constant function, the corresponding homogeneous linear system is autonomous, and therefore its solutions define a flow. This result also follows from the Chapman–Kolmogorov identities.

To prove the flow properties, let us show first that if $t \mapsto A(t)$ is a constant function, then the state transition matrix $\Psi(t, t_0)$ depends only on the difference $t - t_0$. In fact, since $t \mapsto \Psi(t, t_0)$ and $t \mapsto \Psi(t+s, t_0+s)$ are both solutions satisfying the same initial condition at t_0, they are identical. In particular, with $s = -t_0$, we see that $\Psi(t, t_0) = \Psi(t - t_0, 0)$. If we define $\phi_t := \Psi(t, 0)$, then using the last identity together with the Chapman–Kolmogorov identities we find that

$$\Psi(t + s, 0) = \Psi(t, -s) = \Psi(t, 0)\Psi(0, -s) = \Psi(t, 0)\Psi(s, 0).$$

Thus, we recover the group property $\phi_{t+s} = \phi_t \phi_s$. Since, in addition, $\phi_0 = \Psi(0, 0) = I$, the family of operators ϕ_t defines a flow. In this context, ϕ_t is also called an *evolution group*.

If $t \mapsto \Phi(t)$ is a fundamental matrix solution for the linear system (2.4) and $v \in \mathbb{R}^n$, then $t \mapsto \Phi(t)v$ is a (vector) solution. Moreover, every solution is obtained in this way. In fact, if $t \mapsto \phi(t)$ is a solution, then there is some v such that $\Phi(t_0)v = \phi(t_0)$. (Why?) By uniqueness, we must have $\Phi(t)v = \phi(t)$. Also, note that $\Psi(t, t_0)v$ has the property that $\Psi(t_0, t_0)v = v$. In other words, Ψ "transfers" the initial state v to the final state $\Psi(t, t_0)v$. Hence, the name "state transition matrix."

Exercise 2.15. Prove Proposition 2.14.

Exercise 2.16. [Cocycles] A cocycle is a family of functions, each mapping from $\mathbb{R} \times \mathbb{R}^n$ to the set of linear transformations of $\mathbb{R}^n$ such that $\Phi(0, u) = I$ and $\Phi(t+s, u) = \Phi(t, \phi_s(u))\Phi(s, u)$. (To learn more about why cocycles are important, see [45].) Suppose $\dot{u} = f(u)$ is a differential equation on $\mathbb{R}^n$ with flow ϕ_t. Show that the family of principal fundamental matrix solutions $\Phi(t, u)$ of the family of variational equations $\dot{w} = Df(\phi_t(u))w$ is a cocycle over the flow ϕ_t.

The linear independence of a set of solutions of a homogeneous linear differential equation can be determined by checking the independence of a set of vectors obtained by evaluating the solutions at just one point. This useful fact is perhaps most clearly expressed by Liouville's formula, which has many other implications.

Proposition 2.17 (Liouville's Formula). *Suppose that $t \mapsto \Phi(t)$ is a matrix solution of the homogeneous linear system (2.4) on the open interval J. If $t_0 \in J$, then*

$$\det \Phi(t) = \det \Phi(t_0) e^{\int_{t_0}^{t} \operatorname{tr} A(s)\, ds}$$

where det *denotes determinant and* tr *denotes trace. In particular, $\Phi(t)$ is a fundamental matrix solution if and only if the columns of $\Phi(t_0)$ are linearly independent.*

Proof. The matrix solution $t \mapsto \Phi(t)$ is a differentiable function. Thus, we have that

$$\lim_{h \to 0} \frac{1}{h}[\Phi(t+h) - (I + hA(t))\Phi(t)] = 0.$$

In other words, using the "little oh" notation,

$$\Phi(t+h) = (I + hA(t))\Phi(t) + o(h). \tag{2.6}$$

(The little oh has the following meaning: $f(x) = g(x) + o(h(x))$ if

$$\lim_{x \to 0^+} \frac{|f(x) - g(x)|}{h(x)} = 0.$$

Thus, we should write $o(\pm h)$ in equation (2.6), but this technicality is not important in this proof.)

By the definition of the determinant of an $n \times n$ matrix, that is, if $B := (b_{ij})$, then

$$\det B = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)},$$

and the following result: The determinant of a product of matrices is the product of their determinants, we have that

$$\begin{aligned} \det \Phi(t+h) &= \det(I + hA(t)) \det \Phi(t) + o(h) \\ &= (1 + h \operatorname{tr} A(t)) \det \Phi(t) + o(h), \end{aligned}$$

and therefore

$$\frac{d}{dt}\det\Phi(t) = \operatorname{tr} A(t)\det\Phi(t).$$

Integration of this last differential equation gives the desired result. □

Exercise 2.18. Find a fundamental matrix solution of the system

$$\dot{x} = \begin{pmatrix} 1 & -1/t \\ 1+t & -1 \end{pmatrix} x, \qquad t > 0.$$

Hint: $x(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$ is a solution.

Exercise 2.19. Suppose that every solution of $\dot{x} = A(t)x$ is bounded for $t \geq 0$ and let $\Phi(t)$ be a fundamental matrix solution. Prove that $\Phi^{-1}(t)$ is bounded for $t \geq 0$ if and only if the function $t \mapsto \int_0^t \operatorname{tr} A(s)\,ds$ is bounded below. Hint: The inverse of a matrix is the adjugate of the matrix divided by its determinant.

Exercise 2.20. Suppose that the linear system $\dot{x} = A(t)x$ is defined on an open interval containing the origin whose right-hand end point is $\omega \leq \infty$ and the norm of every solution has a finite limit as $t \to \omega$. Show that there is a solution converging to zero as $t \to \omega$ if and only if $\int_0^\omega \operatorname{tr} A(s)\,ds = -\infty$. Hint: A matrix has a nontrivial kernel if and only if its determinant is zero (cf. [113]).

Exercise 2.21. [Transport Theorem] Let ϕ_t denote the flow of the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, and let Ω be a bounded region in $\mathbb{R}^n$. Define

$$V(t) = \int_{\phi_t(\Omega)} dx_1 dx_2 \cdots dx_n$$

and recall that the divergence of a vector field $f = (f_1, f_2, \ldots, f_n)$ on $\mathbb{R}^n$ with the usual Euclidean structure is

$$\operatorname{div} f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}.$$

(a) Use Liouville's theorem and the change of variables formula for multiple integrals to prove that

$$\dot{V}(t) = \int_{\phi_t(\Omega)} \operatorname{div} f(x) dx_1 dx_2 \cdots dx_n.$$

(b) Prove: The flow of a vector field whose divergence is everywhere negative contracts volume. (c) Suppose that $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and, for notational convenience, let $dx = dx_1 dx_2 \cdots dx_n$. Prove the transport theorem:

$$\frac{d}{dt}\int_{\phi_t(\Omega)} g(x,t)\,dx = \int_{\phi_t(\Omega)} g_t(x,t) + \operatorname{div}(gf)(x,t)\,dx.$$

(d) Suppose that the mass in every open set remains unchanged as it is moved by the flow (that is, mass is conserved) and let ρ denote the corresponding mass-density. Prove that the density satisfies the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho f) = 0.$$

(e) The flow of the system $\dot{x} = y$, $\dot{y} = x$ is area preserving. Show directly that the area of the unit disk is unchanged when it is moved forward two time units by the flow.

Exercise 2.22. Construct an alternate proof of Liouville's formula for the n-dimensional linear system $\dot{x} = A(t)x$ with fundamental matrix $\det \Phi(t)$ by differentiation of the function $t \mapsto \det \Phi(t)$ using the chain rule. Hint: Compute $\frac{d}{dt} \det \Phi(t)$ directly as a sum of n determinants of matrices whose components are the components of $\Phi(t)$ and their derivatives with respect to t. For this computation note that the determinant is a multilinear function of its rows (or columns). Use the multilinearity with respect to rows. Substitute for the derivatives of components of Φ using the differential equation and use elementary row operations to reduce each determinant in the sum to a diagonal component of $A(t)$ times $\det \Phi(t)$.

2.1.4 Linear Equations with Constant Coefficients

In this section we will consider the homogeneous linear system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n \tag{2.7}$$

where A is a real $n \times n$ (constant) matrix. We will show how to reduce the problem of constructing a fundamental set of solutions of system (2.7) to a problem in linear algebra. In addition, we will see that the principal fundamental matrix solution at $t = 0$ is given by the exponential of the matrix tA just as the fundamental scalar solution at $t = 0$ of the scalar differential equation $\dot{x} = ax$ is given by $t \mapsto e^{at}$.

Let us begin with the essential observation of the subject: The solutions of system (2.7) are intimately connected with the eigenvalues and eigenvectors of the matrix A. To make this statement precise, let us recall that a complex number λ is an eigenvalue of A if there is a complex *nonzero* vector v such that $Av = \lambda v$. In general, the vector v is called an *eigenvector associated with the eigenvalue* λ if $Av = \lambda v$. Moreover, the set of all eigenvectors associated with an eigenvalue forms a vector space. Because a real matrix can have complex eigenvalues, it is convenient to allow for complex solutions of the differential equation (2.7). Indeed, if $t \mapsto u(t)$ and $t \mapsto v(t)$ are real functions, and if $t \mapsto \phi(t)$ is defined by $\phi(t) := u(t) + iv(t)$, then ϕ is called a complex solution of system (2.7) provided that $\dot{u} + i\dot{v} = Au + iAv$. Of course, if ϕ is a complex solution, then we must have $\dot{u} = Au$ and $\dot{v} = Av$. Thus, it is clear that ϕ is a complex solution if and only if its real and imaginary parts are real solutions. This observation is used in the next proposition.

Proposition 2.23. *Let A be a real $n \times n$ matrix and consider the ordinary differential equation (2.7).*

(1) The function given by $t \mapsto e^{\lambda t}v$ is a real solution if and only if $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$, and $Av = \lambda v$.

(2) If $v \neq 0$ is an eigenvector for A with eigenvalue $\lambda = \alpha + i\beta$ such that $\beta \neq 0$, then the imaginary part of v is not zero. In this case, if $v = u + iw \in \mathbb{C}^n$, then there are two real solutions

$$t \to e^{\alpha t}[(\cos\beta t)u - (\sin\beta t)w],$$

$$t \to e^{\alpha t}[(\sin\beta t)u + (\cos\beta t)w].$$

Moreover, these solutions are linearly independent.

Proof. If $Av = \lambda v$, then

$$\frac{d}{dt}(e^{\lambda t}v) = \lambda e^{\lambda t}v = e^{\lambda t}Av = Ae^{\lambda t}v.$$

In particular, the function $t \to e^{\lambda t}v$ is a solution.

If $\lambda = \alpha + i\beta$ and $\beta \neq 0$, then, because A is real, v must be of the form $v = u + iw$ for some $u, w \in \mathbb{R}^n$ with $w \neq 0$. The real and imaginary parts of the corresponding solution

$$\begin{aligned} e^{\lambda t}v &= e^{(\alpha+i\beta)t}(u + iw) \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(u + iw) \\ &= e^{\alpha t}[(\cos\beta t)u - (\sin\beta t)w + i((\sin\beta t)u + (\cos\beta t)w)] \end{aligned}$$

are real solutions of the system (2.7). To show that these real solutions are linearly independent, suppose that some linear combination of them with coefficients c_1 and c_2 is identically zero. Evaluation at $t = 0$ and at $t = \pi/(2\beta)$ yields the equations

$$c_1u + c_2w = 0, \qquad c_2u - c_1w = 0.$$

By elimination of u we find that $(c_1^2 + c_2^2)w = 0$. Since $w \neq 0$, both coefficients must vanish. This proves (2).

Finally, we will complete the proof of (1). Suppose that $\lambda = \alpha + i\beta$ and $v = u + iw$. If $e^{\lambda t}v$ is real, then $\beta = 0$ and $w = 0$. Thus, in fact, λ and v are real. On the other hand, if λ and v are real, then $e^{\lambda t}v$ is a real solution. In this case,

$$\lambda e^{\lambda t}v = Ae^{\lambda t}v,$$

and we have that $\lambda v = Av$. □

A fundamental matrix solution of system (2.7) can be constructed explicitly if the eigenvalues of A and their multiplicities are known. To illustrate the basic idea, let us suppose that $\mathbb{C}^n$ has a basis $\mathcal{B} := \{v_1, \dots, v_n\}$ consisting of eigenvectors of A, and let $\{\lambda_1, \dots, \lambda_n\}$ denote the corresponding eigenvalues. For example, if A has n distinct eigenvalues, then the set consisting of one eigenvector corresponding to each eigenvalue is a basis of $\mathbb{C}^n$. At any rate, if $\mathcal{B}$ is a basis of eigenvectors, then there are n corresponding solutions given by

$$t \mapsto e^{\lambda_i t} v_i, \quad i = 1, \dots, n,$$

and the matrix

$$\Phi(t) = [e^{\lambda_1 t} v_1, \dots, e^{\lambda_n t} v_n],$$

which is partitioned by columns, is a matrix solution. Because $\det \Phi(0) \neq 0$, this solution is a fundamental matrix solution, and moreover $\Psi(t) := \Phi(t)\Phi^{-1}(0)$ is the principal fundamental matrix solution of (2.7) at $t = 0$.

A principal fundamental matrix for a real system is necessarily real. To see this, let us suppose that $\Lambda(t)$ is the imaginary part of the principal fundamental matrix solution $\Psi(t)$ at $t = 0$. Since, $\Psi(0) = I$, we must have $\Lambda(0) = 0$. Also, $t \mapsto \Lambda(t)$ is a solution of the linear system. By the uniqueness of solutions of initial value problems, $\Lambda(t) \equiv 0$. Thus, even if some of the eigenvalues of A are complex, the principal fundamental matrix solution is real.

Continuing under the assumption that A has a basis $\mathcal{B}$ of eigenvectors, let us show that there is a change of coordinates that transforms the system $\dot{x} = Ax$, $x \in \mathbb{R}^n$, to a decoupled system of n scalar differential equations. To prove this result, let us first define the matrix $B := [v_1, \dots, v_n]$ whose columns are the eigenvectors in $\mathcal{B}$. The matrix B is invertible. Indeed, consider the action of B on the usual basis vectors and recall that the vector obtained by multiplication of a vector by a matrix is a linear combination of the columns of the matrix; that is, if $w = (w_1, \dots, w_n)$ is (the transpose of) a vector in $\mathbb{C}^n$, then the product Bw is equal to $\sum_{i=1}^n w_i v_i$. In particular, we have $B\mathbf{e}_i = v_i$, $i = 1, \dots, n$. This proves that B is invertible. In fact, B^{-1} is the unique linear map such that $B^{-1} v_i = \mathbf{e}_i$.

Using the same idea, let us compute

$$\begin{aligned} B^{-1}AB &= B^{-1}A[v_1, \dots, v_n] \\ &= B^{-1}[\lambda_1 v_1, \dots, \lambda_n v_n] \\ &= [\lambda_1 \mathbf{e}_1, \dots, \lambda_n \mathbf{e}_n] \\ &= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \end{aligned}$$

In other words, $D := B^{-1}AB$ is a diagonal matrix with the eigenvalues of A as its diagonal elements. The diffeomorphism of $\mathbb{C}^n$ given by the linear transformation $x = By$ transforms the system (2.7) to $\dot{y} = Dy$, as required. Or, using our language for general coordinate transformations, the push forward of the vector field with principal part $x \mapsto Ax$ by the diffeomorphism B^{-1} is the vector field with principal part $y \mapsto Dy$. In particular, the system $\dot{y} = Dy$ is given in components by

$$\dot{y}_1 = \lambda_1 y_1, \ \dots, \dot{y}_n = \lambda_n y_n.$$

Note that if we consider the original system in the new coordinates, then it is obvious that the functions

$$y_i(t) := e^{\lambda_i t}\mathbf{e}_i, \qquad i = 1, \dots, n$$

are a fundamental set of solutions for the differential equation $\dot{y} = Dy$. Moreover, by transforming back to the original coordinates, it is clear that the solutions

$$x_i(t) := e^{\lambda_i t}B\mathbf{e}_i = e^{\lambda_i t}v_i, \qquad i = 1, \dots, n$$

form a fundamental set of solutions for the original system (2.7). Thus, we have an alternative method to construct a fundamental matrix solution: Change coordinates to obtain a new differential equation, construct a fundamental set of solutions for the new differential equation, and then transform these new solutions back to the original coordinates. Even if A is not diagonalizable, a fundamental matrix solution of the associated differential equation can still be constructed using this procedure. Indeed, we can use a basic fact from linear algebra: If A is a real matrix, then there is a nonsingular matrix B such that $D := B^{-1}AB$ is in (real) Jordan canonical form (see [59], [121], and Exercise 2.37). Then, as before, the system (2.7) is transformed by the change of coordinates $x = By$ into the linear system $\dot{y} = Dy$.

We will eventually give a detailed description of the Jordan form and also show that the corresponding canonical system of differential equations can be solved explicitly. This solution can be transformed back to the original coordinates to construct a fundamental matrix solution of $\dot{x} = Ax$.

Exercise 2.24. (a) Find the principal fundamental matrix solutions at $t = 0$ for the matrix systems:

1. $\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x.$
2. $\dot{x} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} x.$
3. $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x.$

4. $\dot{x} = \begin{pmatrix} 7 & -8 \\ 4 & -5 \end{pmatrix} x.$

(b) Solve the initial value problem for system 2 with initial value $x(0) = (1, 0)$. (c) Find a change of coordinates (given by a matrix) that diagonalizes the system matrix of system 4. (d) Find the principal fundamental matrix solution at $t = 3$ for system 3.

Exercise 2.25. (a) Determine the flow of the first order system that is equivalent to the second order linear differential equation

$$\ddot{x} + \dot{x} + 4x = 0.$$

(b) Draw the phase portrait.

Exercise 2.26. [Euler's Equation] Euler's equation is the second order linear equation

$$t^2\ddot{x} + bt\dot{x} + cx = 0, \qquad t > 0$$

with the parameters b and c. (a) Show that there are three different solution types according to the sign of $(b-1)^2 - 4c$. Hint: Guess a solution of the form $x = r^t$ for some number r. (b) Discuss, for each of the cases in part (a), the behavior of the solution as $t \to 0^+$. (c) Write a time-dependent linear first order system that is equivalent to Euler's equation. (d) Determine the principal fundamental matrix solution for the first order system in part (c) in case $b = 1$ and $c = -1$.

Instead of writing out the explicit, perhaps complicated, formulas for the components of the fundamental matrix solution of an $n \times n$ linear system of differential equations, it is often more useful, at least for theoretical considerations, to treat the situation from a more abstract point of view. In fact, we will show that there is a natural generalization of the exponential function to a function defined on the set of square matrices. Using this matrix exponential function, the solution of a linear homogeneous system with constant coefficients is given in a form that is analogous to the solution $t \mapsto e^{ta}x_0$ of the scalar differential equation $\dot{x} = ax$.

Recall that the set of linear transformations $\mathcal{L}(\mathbb{R}^n)$ (respectively $\mathcal{L}(\mathbb{C}^n)$) on $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) is an n^2-dimensional Banach space with respect to the operator norm

$$\|A\| = \sup_{|v|=1} |Av|.$$

Most of the theory we will develop is equally valid for either of the vector spaces $\mathbb{R}^n$ or $\mathbb{C}^n$. When the space is not at issue, we will denote the Banach space of linear transformations by $\mathcal{L}(E)$ where E may be taken as either $\mathbb{R}^n$ or $\mathbb{C}^n$. The theory is also valid for the set of (operator norm) bounded linear transformations of an arbitrary Banach space.

Exercise 2.27. Prove: $\mathcal{L}(E)$ is a finite dimensional Banach space with respect to the operator norm.

Exercise 2.28. Prove: (a) If $A, B \in \mathcal{L}(E)$, then $\|AB\| \le \|A\|\|B\|$. (b) If $A \in \mathcal{L}(E)$ and k is a nonnegative integer, then $\|A^k\| \le \|A\|^k$.

Exercise 2.29. The space of $n \times n$ matrices is a topological space with respect to the operator topology. Prove that the set of matrices with n distinct eigenvalues is open and dense. A property that is defined on the countable intersection of open dense sets is called *generic* (see [121, p. 153–157]).

Proposition 2.30. *If $A \in \mathcal{L}(E)$, then the series $I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k$ is absolutely convergent.*

Proof. Define

$$S_N := 1 + \|A\| + \frac{1}{2!}\|A^2\| + \cdots + \frac{1}{N!}\|A^N\|$$

and note that $\{S_N\}_{N=1}^{\infty}$ is a monotone increasing sequence of real numbers. Since (by Exercise 2.28) $\|A^k\| \le \|A\|^k$ for every integer $k \ge 0$, it follows that $\{S_N\}_{N=1}^{\infty}$ is bounded above. In fact,

$$S_N \le e^{\|A\|}$$

for every $N \ge 1$. □

Define the *exponential map* $\exp : \mathcal{L}(E) \to \mathcal{L}(E)$ by

$$\exp(A) := I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k.$$

Also, let us use the notation $e^A := \exp(A)$.

The main properties of the exponential map are summarized in the following proposition.

Proposition 2.31. *Suppose that $A, B \in \mathcal{L}(E)$.*

(0) If $A \in \mathcal{L}(\mathbb{R}^n)$, then $e^A \in \mathcal{L}(\mathbb{R}^n)$.

(1) If B is nonsingular, then $B^{-1}e^AB = e^{B^{-1}AB}$.

(2) If $AB = BA$, then $e^{A+B} = e^Ae^B$.

(3) $e^{-A} = (e^A)^{-1}$. In particular, the image of exp is in the general linear group $GL(E)$ consisting of the invertible elements of $\mathcal{L}(E)$.

(4) $\frac{d}{dt}(e^{tA}) = Ae^{tA} = e^{tA}A$. In particular, $t \mapsto e^{tA}$ is the principal fundamental matrix solution of the system (2.7) at $t = 0$.

(5) $\|e^A\| \le e^{\|A\|}$.

Proof. The proof of (0) is obvious.

To prove (1), define

$$S_N := I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{N!}A^N,$$

and note that if B is nonsingular, then $B^{-1}A^nB = (B^{-1}AB)^n$. Thus, we have that

$$B^{-1}S_NB = I + B^{-1}AB + \frac{1}{2!}(B^{-1}AB)^2 + \cdots + \frac{1}{N!}(B^{-1}AB)^N,$$

and, by the definition of the exponential map,

$$\lim_{N\to\infty} B^{-1}S_NB = e^{B^{-1}AB}.$$

Using the continuity of the linear map on $\mathcal{L}(E)$ defined by $C \mapsto B^{-1}CB$, it follows that

$$\lim_{N\to\infty} B^{-1}S_NB = B^{-1}e^AB,$$

as required.

While the proof of (4) given here has the advantage of being self contained, there are conceptually simpler alternatives (see Exercises 2.32–2.33). As the first step in the proof of (4), consider the following proposition: If $s, t \in \mathbb{R}$, then $e^{(s+t)A} = e^{sA}e^{tA}$. To prove it, let us denote the partial sums for the series representation of e^{tA} by

$$\begin{aligned} S_N(t) &:= I + tA + \frac{1}{2!}(tA)^2 + \cdots + \frac{1}{N!}(tA)^N \\ &= I + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{N!}t^NA^N. \end{aligned}$$

We claim that

$$S_N(s)S_N(t) = S_N(s+t) + \sum_{k=N+1}^{2N} P_k(s,t)A^k \tag{2.8}$$

where $P_k(s,t)$ is a homogeneous polynomial of degree k such that

$$|P_k(s,t)| \le \frac{(|s|+|t|)^k}{k!}.$$

To obtain this identity, note that the kth order term of the product, at least for $0 \le k \le N$, is given by

$$\Big(\sum_{j=0}^{k} \frac{1}{(k-j)!j!}s^{k-j}t^j\Big)A^k = \Big(\frac{1}{k!}\sum_{j=0}^{k} \frac{k!}{(k-j)!j!}s^{k-j}t^j\Big)A^k = \frac{1}{k!}(s+t)^kA^k.$$

Also, for $N+1 \leq k \leq 2N$, the kth order term is essentially the same, only some of the summands are missing. In fact, these terms all have the form

$$\Big(\sum_{j=0}^{k} \frac{\delta(j)}{(k-j)!j!} s^{k-j} t^j\Big) A^k$$

where $\delta(j)$ has value zero or one. Each such term is the product of A^k and a homogeneous polynomial in two variables of degree k. Moreover, because $|\delta(j)| \leq 1$, we obtain the required estimate for the polynomial. This proves the claim.

Using equation (2.8), we have the following inequality

$$\begin{aligned} \|S_N(s)S_N(t) - S_N(s+t)\| &\leq \sum_{k=N+1}^{2N} |P_k(s,t)|\, \|A\|^k \\ &\leq \sum_{k=N+1}^{2N} \frac{(|s|+|t|)^k}{k!} \|A\|^k. \end{aligned}$$

Also, because the series

$$\sum_{k=0}^{\infty} \frac{(|s|+|t|)^k}{k!} \|A\|^k$$

is convergent, it follows that its partial sums, denoted Q_N, form a Cauchy sequence. In particular, if $\epsilon > 0$ is given, then for sufficiently large N we have

$$|Q_{2N} - Q_N| < \epsilon.$$

Moreover, since

$$Q_{2N} - Q_N = \sum_{k=N+1}^{2N} \frac{(|s|+|t|)^k}{k!} \|A\|^k,$$

it follows that

$$\lim_{N\to\infty} \|S_N(s)S_N(t) - S_N(s+t)\| = 0.$$

Using this fact and passing to the limit as $N \to \infty$ on both sides of the inequality

$$\begin{aligned} \|e^{sA}e^{tA} - e^{(s+t)A}\| \leq \|e^{sA}e^{tA} &- S_N(s)S_N(t)\| \\ &+ \|S_N(s)S_N(t) - S_N(s+t)\| \\ &+ \|S_N(s+t) - e^{(s+t)A}\|, \end{aligned}$$

we see that

$$e^{sA}e^{tA} = e^{(s+t)A}, \tag{2.9}$$

as required.

In view of the identity (2.9), the derivative of the function $t \mapsto e^{tA}$ is given by

$$\begin{aligned}\frac{d}{dt}e^{tA} &= \lim_{s\to 0}\frac{1}{s}(e^{(t+s)A} - e^{tA})\\ &= \lim_{s\to 0}\frac{1}{s}(e^{sA} - I)e^{tA}\\ &= \big(\lim_{s\to 0}\frac{1}{s}(e^{sA} - I)\big)e^{tA}\\ &= \big(\lim_{s\to 0}(A + R(s))\big)e^{tA}\end{aligned}$$

where

$$\|R(s)\| \le \frac{1}{|s|}\sum_{k=2}^{\infty}\frac{|s|^k}{k!}\|A\|^k \le |s|\sum_{k=2}^{\infty}\frac{|s|^{k-2}}{k!}\|A\|^k.$$

Moreover, if $|s| < 1$, then $\|R(s)\| \le |s|e^{\|A\|}$. In particular, $R(s) \to 0$ as $s \to 0$ and as a result,

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

Since $AS_N(t) = S_N(t)A$, it follows that $Ae^{tA} = e^{tA}A$. This proves the first statement of part (4). In particular $t \mapsto e^{tA}$ is a matrix solution of the system (2.7). Clearly, $e^0 = I$. Thus, the columns of e^0 are linearly independent. It follows that $t \mapsto e^{tA}$ is the principal fundamental matrix solution at $t = 0$, as required.

To prove (2), suppose that $AB = BA$ and consider the function $t \mapsto e^{t(A+B)}$. By (4), this function is a matrix solution of the initial value problem

$$\dot{x} = (A + B)x, \qquad x(0) = I.$$

The function $t \mapsto e^{tA}e^{tB}$ is a solution of the same initial value problem. To see this, use the product rule to compute the derivative

$$\frac{d}{dt}e^{tA}e^{tB} = Ae^{tA}e^{tB} + e^{tA}Be^{tB},$$

and use the identity $AB = BA$ to show that $e^{tA}B = Be^{tA}$. The desired result is obtained by inserting this last identity into the formula for the derivative. By the uniqueness of the solution of the initial value problem, the two solutions are identical.

To prove (3), we use (2) to obtain $I = e^{A-A} = e^A e^{-A}$ or, in other words, $(e^A)^{-1} = e^{-A}$.

The result (5) follows from the inequality

$$\|I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{N!}A^N\| \le \|I\| + \|A\| + \frac{1}{2!}\|A\|^2 + \cdots + \frac{1}{N!}\|A\|^N. \quad \square$$

We have defined the exponential of a matrix as an infinite series and used this definition to prove that the homogeneous linear system $\dot{x} = Ax$ has a fundamental matrix solution, namely, $t \mapsto e^{tA}$. This is a strong result because it does not use the existence theorem for differential equations. Granted, the uniqueness theorem is used. But it is an easy corollary of Gronwall's inequality (see Exercise 2.5). An alternative approach to the exponential map is to use the existence theorem and define the function $t \mapsto e^{tA}$ to be the principal fundamental matrix solution at $t = 0$. Proposition 2.31 can then be proved by using properties of the solutions of homogeneous linear differential equations.

Exercise 2.32. Show that the partial sums of the series representation of e^{tA} converge uniformly on compact subsets of $\mathbb{R}$. Use Theorem 1.248 to prove part (4) of Proposition 2.31.

Exercise 2.33. (a) Show that $\exp : \mathcal{L}(E) \to \mathcal{L}(E)$ is continuous. Hint: For $r > 0$, the sequence of partial sums of the series representation of $\exp(X)$ converges uniformly on $B_r(0) := \{X \in \mathcal{L}(E) : \|X\| < r\}$. (b) By Exercise 2.29, matrices with distinct eigenvalues are dense in $\mathcal{L}(E)$. Such matrices are diagonalizable (over the complex numbers). Show that if $A \in \mathcal{L}(E)$ is diagonalizable, then part (4) of Proposition 2.31 holds for A. (c) Use parts (a) and (b) to prove part (4) of Proposition 2.31. (d) Prove that $\exp : \mathcal{L}(E) \to \mathcal{L}(E)$ is differentiable and compute $D\exp(I)$.

Exercise 2.34. Define $\exp(A) = \Phi(1)$ where $\Phi(t)$ is the principal fundamental matrix at $t = 0$ for the system $\dot{x} = Ax$. (a) Prove that $\exp(tA) = \Phi(t)$. (b) Prove that $\exp(-A) = (\exp(A))^{-1}$.

Exercise 2.35. [Laplace Transform] (a) Prove that if A is an $n \times n$-matrix, then

$$e^{tA} - I = \int_0^t Ae^{\tau A}\, d\tau.$$

(b) Prove that if all eigenvalues of A have negative real parts, then

$$-A^{-1} = \int_0^\infty e^{\tau A}\, d\tau.$$

(c) Prove that if $s \in \mathbb{R}$ is sufficiently large, then

$$(sI - A)^{-1} = \int_0^\infty e^{-s\tau} e^{\tau A}\, d\tau;$$

that is, the Laplace transform of e^{tA} is $(sI - A)^{-1}$. (d) Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ using the method of the Laplace transform; that is, take the Laplace transform of both sides of the equation, solve the resulting algebraic equation, and then invert the transform to obtain the solution in the original variables. By definition, the Laplace transform of the (perhaps matrix valued) function f is

$$\mathcal{L}\{f\}s = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau.$$

To obtain a matrix representation for e^{tA}, let us recall that there is a real matrix B that transforms A to real Jordan canonical form. Of course, to construct the matrix B, we must at least be able to find the eigenvalues of A, a task that is equivalent to finding the roots of a polynomial of degree n. Thus, for $n \geq 5$, it is generally impossible to construct the matrix B explicitly. But if B is known, then by using part (1) of Proposition 2.31, we have that

$$B^{-1}e^{tA}B = e^{tB^{-1}AB}.$$

Thus, the problem of constructing a principal fundamental matrix is solved as soon as we find a matrix representation for $e^{tB^{-1}AB}$.

The Jordan canonical matrix $B^{-1}AB$ is block diagonal, where each block corresponding to a real eigenvalue has the form "diagonal + nilpotent," and, each block corresponding to a complex eigenvalue with nonzero imaginary part has the form "block diagonal + block nilpotent." In view of this block structure, it suffices to determine the matrix representation for e^{tJ} where J denotes a single Jordan block.

Consider a block of the form

$$J = \lambda I + N$$

where N is the nilpotent matrix with zero components except on the super diagonal, where each component is unity and note that $N^k = 0$. We have that

$$e^{tJ} = e^{t(\lambda I+N)} = e^{t\lambda I}e^{tN} = e^{t\lambda}(I + tN + \frac{t^2}{2!}N^2 + \cdots + \frac{t^{k-1}}{(k-1)!}N^{k-1})$$

where k is the dimension of the block.

If J is a Jordan block with diagonal 2×2 subblocks given by

$$R = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \tag{2.10}$$

with $\beta \neq 0$, then e^{tJ} is block diagonal with each block given by e^{tR}. To obtain an explicit matrix representation for e^{tR}, define

$$P := \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \qquad Q(t) := \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix},$$

and note that $t \mapsto e^{tP}$ and $t \mapsto Q(t)$ are both solutions of the initial value problem

$$\dot{x} = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} x, \qquad x(0) = I.$$

Thus, we have that $e^{tP} = Q(t)$ and

$$e^{tR} = e^{\alpha t}e^{tP} = e^{\alpha t}Q(t).$$

Finally, if the Jordan block J has the 2×2 block matrix R along its block diagonal and the 2×2 identity along its super block diagonal, then

$$e^{tJ} = e^{\alpha t} S(t) e^{tN} \tag{2.11}$$

where $S(t)$ is block diagonal with each block given by $Q(t)$, and N is the nilpotent matrix with 2×2 identity blocks on its super block diagonal. To prove this fact, note that J can be written as a sum $J = \alpha I + K$ where K has diagonal blocks given by P and super diagonal blocks given by the 2×2 identity matrix. Since the $n \times n$ matrix αI commutes with every matrix, we have that

$$e^{tJ} = e^{\alpha t} e^{tK}.$$

The proof is completed by observing that the matrix K can also be written as a sum of commuting matrices; namely, the block diagonal matrix with each diagonal block equal to P and the nilpotent matrix N.

We have outlined a procedure to find a matrix representation for e^{tA}. In addition, we have proved the following result.

Proposition 2.36. *If A is an $n \times n$ matrix, then e^{tA} is a matrix whose components are (finite) sums of terms of the form*

$$p(t)e^{\alpha t} \sin \beta t \text{ and } p(t)e^{\alpha t} \cos \beta t$$

where α and β are real numbers such that $\alpha + i\beta$ is an eigenvalue of A, and $p(t)$ is a polynomial of degree at most $n - 1$.

Exercise 2.37. [Jordan Form] Show that every real 2×2-matrix can be transformed to real Jordan canonical form and find the fundamental matrix solutions for the corresponding 2×2 real homogeneous linear systems of differential equations. Draw the phase portrait for each canonical system. Hint: For the case of a double eigenvalue suppose that $(A - \lambda I)V = 0$ and every eigenvector is parallel to V. Choose a vector W that is not parallel to V and note that $(A - \lambda I)W = Y \neq 0$. Since V and W are linearly independent, $Y = aV + bW$ for some real numbers a and b. Use this fact to argue that Y is parallel to V. Hence, there is a (nonzero) vector Z such that $(A - \lambda I)Z = V$. Define $B = [V, W]$ to be the indicated 2×2-matrix partitioned by columns and show that $B^{-1}AB$ is in Jordan form. To solve $\dot{x} = Ax$, use the change of variables $x = By$.

Exercise 2.38. Find the Jordan canonical form for the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

Exercise 2.39. Find the principal fundamental matrix solution at $t = 0$ for the linear differential equation whose system matrix is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & a & 2 & 0 \end{pmatrix},$$

where $a := 4-\omega^2$ and $0 \le \omega \le 1$, by changing variables so that the system matrix is in Jordan canonical form, computing the exponential, and changing back to the original variables.

Exercise 2.40. Suppose that $J = \lambda I + N$ is a $k \times k$-Jordan block and let B denote the diagonal matrix with main diagonal $1, \epsilon, \epsilon^2, \ldots, \epsilon^{k-1}$. (a) Show that $B^{-1}JB = \lambda I + \epsilon N$. (b) Prove: Given $\epsilon > 0$ and a matrix A, there is a diagonalizable matrix B such that $\|A - B\| < \epsilon$ (cf. Exercise 2.29). (c) Discuss the statement: A numerical algorithm for finding the Jordan canonical form will be ill conditioned.

Exercise 2.41. (a) Suppose that A is an $n \times n$-matrix such that $A^2 = I$. Find an explicit formula for e^{tA}. (b) Repeat part (a) in case $A^2 = -I$. (c) Solve the initial value problem

$$\dot{x} = \begin{pmatrix} 2 & -5 & 8 & -12 \\ 1 & -2 & 4 & -8 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(d) Specify the stable manifold for the rest point at the origin of the linear system

$$\dot{x} = \begin{pmatrix} 2 & -3 & 4 & -4 \\ 1 & -2 & 4 & -4 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{pmatrix} x.$$

Exercise 2.42. Prove that $\det e^A = e^{\operatorname{tr} A}$ for every $n \times n$ matrix A. Hint: Use Liouville's formula 2.17.

The scalar autonomous differential equation $\dot{x} = ax$ has the principal fundamental solution $t \mapsto e^{at}$ at $t = 0$. We have defined the exponential map on bounded linear operators and used this function to construct the analogous fundamental matrix solution $t \mapsto e^{tA}$ of the homogeneous autonomous system $\dot{x} = Ax$. The scalar nonautonomous homogeneous linear differential equation $\dot{x} = a(t)x$ has the principal fundamental solution

$$t \mapsto e^{\int_0^t a(s)\,ds}.$$

But, in the matrix case, the same formula with $a(s)$ replaced by $A(s)$ is not always a matrix solution of the linear system $\dot{x} = A(t)x$ (cf. [128] and see Exercise 2.49).

As an application of the methods developed in this section we will formulate and prove a special case of the Lie–Trotter product formula for the exponential of a sum of two $k \times k$-matrices when the matrices do not necessarily commute (see [218] for the general case).

Theorem 2.43. *If $\gamma : \mathbb{R} \to \mathcal{L}(E)$ is a C^1-function with $\gamma(0) = I$ and $\dot{\gamma}(0) = A$, then the sequence $\{\gamma^n(t/n)\}_{n=1}^{\infty}$ converges to $\exp(tA)$. In particular, if A and B are $k \times k$-matrices, then*

$$e^{t(A+B)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}A} e^{\frac{t}{n}B}\right)^n.$$

Proof. Fix $T > 0$ and assume that $|t| < T$. We will first prove the following proposition: There is a number $M > 0$ such that $\|\gamma^j(t/n)\| \le M$ whenever j and n are integers and $0 \le j \le n$. Using Taylor's theorem, we have the estimate

$$\|\gamma(t/n)\| \le 1 + \frac{T}{n}\|A\| + \frac{T}{n}\int_0^1 \|\dot{\gamma}(st/n) - A\|\, ds.$$

Since $\sigma \mapsto \|\dot{\gamma}(\sigma) - A\|$ is a continuous function on the compact set $S := \{\sigma \in \mathbb{R} : |\sigma| \le T\}$, we also have that $K := \sup\{\|\dot{\gamma}(\sigma) - A\| : \sigma \in S\} < \infty$, and therefore,

$$\|\gamma^j(t/n)\| \le \|\gamma(t/n)\|^j \le (1 + \frac{T}{n}(\|A\| + K))^n.$$

To finish the proof of the proposition, note that the sequence $\{(1+\frac{T}{n}(\|A\|+K))^n\}_{n=1}^{\infty}$ is bounded—it converges to $\exp(T(\|A\| + K))$.

Using the (telescoping) identity

$$\begin{aligned} e^{tA} - \gamma^n(t/n) &= \sum_{j=1}^{n} \big((e^{\frac{t}{n}A})^{n-j+1}\gamma^{j-1}(t/n) - (e^{\frac{t}{n}A})^{n-j}\gamma^j(t/n)\big) \\ &= \sum_{j=1}^{n} \big((e^{\frac{t}{n}A})^{n-j} e^{\frac{t}{n}A}\gamma^{j-1}(t/n) - (e^{\frac{t}{n}A})^{n-j}\gamma(t/n)\gamma^{j-1}(t/n)\big), \end{aligned}$$

we have the estimate

$$\begin{aligned} \|e^{tA} - \gamma^n(t/n)\| &\le \sum_{j=1}^{n} e^{\frac{n-j}{n}T\|A\|}\|e^{\frac{t}{n}A} - \gamma(t/n)\|\|\gamma^{j-1}(t/n)\| \\ &\le M\|e^{\frac{t}{n}A} - \gamma(t/n)\| \sum_{j=1}^{n} e^{(n-j)/nT\|A\|} \\ &\le Mne^{T\|A\|}\|e^{\frac{t}{n}A} - \gamma(t/n)\|. \end{aligned}$$

By Taylor's theorem (applied to each of the functions $\sigma \mapsto e^{\sigma A}$ and $\sigma \mapsto \gamma(\sigma)$), we obtain the inequality

$$\|e^{\frac{t}{n}A} - \gamma(t/n)\| \le \frac{T}{n}J(n)$$

where

$$J(n) := \int_0^1 \|\dot{\gamma}(st/n) - A\| \, ds + \int_0^1 \|A\| \|e^{\frac{st}{n}A} - I\| \, ds$$

is such that $\lim_{n\to\infty} J(n) = 0$. Since

$$\|e^{tA} - \gamma^n(t/n)\| \le MTe^{T\|A\|} J(n),$$

it follows that $\lim_{n\to\infty} \|e^{tA} - \gamma^n(t/n)\| = 0$, as required.

The second statement of the theorem follows from the first with A replaced by $A + B$ and $\gamma(t) := e^{tA}e^{tB}$. □

The product formula in Theorem 2.43 gives a method to compute the solution of the differential equation $\dot{x} = (A + B)x$ from the solutions of the equations $\dot{x} = Ax$ and $\dot{x} = Bx$. Of course, if A and B happen to commute (that is, $[A, B] := AB - BA = 0$), then the product formula reduces to $e^{t(A+B)} = e^{tA}e^{tB}$ by part (2) of Proposition 2.31. It turns out that $[A, B] = 0$ is also a necessary condition for this reduction. Indeed, let us note first that $t \mapsto e^{tA}e^{tB}$ is a solution of the initial value problem

$$\dot{W} = AW + WB, \qquad W(0) = I. \tag{2.12}$$

If $t \mapsto e^{t(A+B)}$ is also a solution, then by substitution and a rearrangement of the resulting equality, we have the identity

$$A = e^{-t(A+B)} A e^{t(A+B)}.$$

By computing the derivative with respect to t of both sides of this identity and simplifying the resulting equation, it follows that $[A, B] = 0$ (cf. Exercise 2.52).

What can we say about the product $e^{tA}e^{tB}$ in case $[A, B] \neq 0$? The answer is provided by (a special case of) the Baker-Campbell-Hausdorff formula

$$e^{tA}e^{tB} = e^{t(A+B)+(t^2/2)[A,B]+R(t,A,B)} \tag{2.13}$$

where $R(0, A, B) = R_t(0, A, B) = R_{tt}(0, A, B) = 0$ (see, for example, [222]).

To obtain formula (2.13), note that the curve $\gamma : \mathbb{R} \to \mathcal{L}(E)$ given by $t \mapsto e^{tA}e^{tB}$ is such that $\gamma(0) = I$. Also, the function $\exp : \mathcal{L}(E) \to \mathcal{L}(E)$ is such that $\exp(0) = I$ and $D\exp(0) = I$. Hence, by the inverse function theorem, there is a unique smooth curve $\Omega(t)$ in $\mathcal{L}(E)$ such that $\Omega(0) = 0$ and $e^{\Omega(t)} = e^{tA}e^{tB}$. Hence, the function $t \mapsto e^{\Omega(t)}$ is a solution of the initial value problem (2.12), that is,

$$D\exp(\Omega)\dot{\Omega} = Ae^{\Omega} + e^{\Omega}B. \tag{2.14}$$

By evaluation at $t = 0$, we have that $\dot{\Omega}(0) = A + B$. The equality $\ddot{\Omega}(0) = [A, B]$ is obtained by differentiating both sides of equation (2.14) with respect to t at $t = 0$. This computation requires the second derivative of exp at the origin in $\mathcal{L}(E)$. To determine this derivative, use the power series definition of exp to show that it suffices to compute the second derivative of the function $h : \mathcal{L}(E) \to \mathcal{L}(E)$ given by $h(X) = \frac{1}{2}X^2$. Since h is smooth, its derivatives can be determined by computing directional derivatives; in fact, we have that

$$Dh(X)Y = \frac{d}{dt}\frac{1}{2}(X + tY)^2\Big|_{t=0} = \frac{1}{2}(XY + YX),$$

$$D^2h(X)(Y, Z) = \frac{d}{dt}Dh(X + tZ)Y\Big|_{t=0} = \frac{1}{2}(YZ + ZY),$$

and $D^2 \exp(0)(Y, Z) = \frac{1}{2}(YZ + ZY)$. The proof of formula (2.13) is completed by applying Taylor's theorem to the function Ω.

Exercise 2.44. Compute the principal fundamental matrix solution at $t = 0$ for the system $\dot{x} = Ax$ where

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 2.45. Reduction to Jordan form is only one of many computational methods that can be used to determine the exponential of a matrix. Repeat Exercise 2.44 using the method presented in [110].

Exercise 2.46. Determine the phase portrait for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Make sure you distinguish the cases $\mu < -2$, $\mu > 2$, $\mu = 0$, $0 < \mu < 2$, and $-2 < \mu < 0$. For each case, find the principal fundamental matrix solution at $t = 0$.

Exercise 2.47. (a) Show that the general 2×2 linear system with constant coefficients decouples in polar coordinates, and the first-order differential equation for the angular coordinate θ can be viewed as a differential equation on the unit circle $\mathbb{T}^1$. (b) Consider the first-order differential equation

$$\dot{\theta} = \alpha \cos^2 \theta + \beta \cos\theta \sin\theta + \gamma \sin^2 \theta.$$

For $4\alpha\gamma - \beta^2 > 0$, prove that all orbits on the circle are periodic with period $4\pi(4\alpha\gamma - \beta^2)^{-1/2}$, and use this result to determine the period of the periodic orbits of the differential equation $\dot{\theta} = \eta + \cos\theta \sin\theta$ as a function of the parameter $\eta > 1$. Describe the behavior of this function as $\eta \to 1^+$ and give a qualitative explanation of the behavior. (c) Repeat the last part of the exercise for the differential equation $\dot{\theta} = \eta - \sin\theta$ where $\eta > 1$. (d) Show that an n-dimensional homogeneous linear differential equation induces a differential equation on the real projective space of dimension $n-1$. (e) There is an intimate connection between the linear second-order differential equation

$$\ddot{y} - (q(t) + \dot{p}(t)/p(t))\dot{y} + r(t)p(t)y = 0$$

and the Riccati equation

$$\dot{x} = p(t)x^2 + q(t)x + r(t).$$

In fact, these equations are related by $x = -\dot{y}/(p(t)y)$. For example $\ddot{y} + y = 0$ is related to the Riccati equation $\dot{u} = -1 - u^2$, where in this case the change of variables is $x = \dot{y}/y$. Note that the unit circle in $\mathbb{R}^2$, with coordinates $(y, \dot{y})$, has coordinate charts given by $(y, \dot{y}) \mapsto \dot{y}/y$ and $(y, \dot{y}) \mapsto y/\dot{y}$. Thus, the transformation from the linear second-order equation to the Riccati equation is a local coordinate representation of the differential equation induced by the second-order linear differential equation on the circle. Explore and explain the relation between this coordinate representation and the polar coordinate representation of the first-order linear system. (f) Prove the cross-ratio property for Riccati equations: If x_i, $i = 1, 2, 3, 4$, are four linearly independent solutions of a Riccati equation, then the quantity

$$\frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$

is constant. (g) Show that if one solution $t \mapsto z(t)$ of the Riccati equation is known, then the general solution can always be found by solving a linear equation after the substitution $x = z + 1/u$. (h) Solve the initial value problem

$$\dot{x} + x^2 + (2t+1)x + t^2 + t + 1 = 0, \qquad x(1) = 1.$$

(see [197, p. 30] for this equation, and [70] for more properties of Riccati equations).

Exercise 2.48. The linearized Hill's equations for the relative motion of two satellites with respect to a circular reference orbit about the earth are given by

$$\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2x &= 0, \\ \ddot{y} + 2n\dot{x} &= 0, \\ \ddot{z} + n^2z &= 0 \end{aligned}$$

where n is a constant related to the radius of the reference orbit and the gravitational constant. There is a five-dimensional manifold in the phase space corresponding to periodic orbits. An orbit with an initial condition not on this manifold contains a secular drift term. Determine the manifold of periodic orbits and explain what is meant by a secular drift term. Answer: The manifold of periodic orbits is the hyperplane given by $\dot{y} + 2nx = 0$.

Exercise 2.49. Find a matrix function $t \mapsto A(t)$ such that

$$t \mapsto \exp\Big(\int_0^t A(s)\,ds\Big)$$

is *not* a matrix solution of the system $\dot{x} = A(t)x$. Show that the given exponential formula is a solution in the scalar case. When is it a solution for the matrix case?

Exercise 2.50. In the Baker-Campbell-Hausdorff formula (2.13), the second-order correction term is $(t^2/2)[A, B]$. Prove that the third-order correction is $(t^3/12)([A, [A, B]] - [B, [A, B]])$.

Exercise 2.51. Show that the commutator relations

$$[A, [A, B]] = 0, \qquad [B, [A, B]] = 0$$

imply the identity

$$e^{tA}e^{tB} = e^{\Omega(t)} \tag{2.15}$$

where $\Omega(t) := t(A+B)+(t^2/2)[A, B]$. Is the converse statement true? Find (3×3) matrices A and B such that $[A, B] \neq 0$, $[A, [A, B]] = 0$, and $[B, [A, B]] = 0$. Verify identity (2.15) for your A and B. Hint: Suppose that $[A, [A, B]] = 0$ and $[B, [A, B]] = 0$. Use these relations to prove in turn $[\Omega(t), \dot{\Omega}(t)] = 0$, $D\exp(\Omega)\dot{\Omega} = \exp(\Omega)\dot{\Omega}$, and $[\exp(\Omega), \dot{\Omega}] = 0$. To prove the identity (2.15), it suffices to show that $t \mapsto \exp(\Omega(t))$ is a solution of the initial value problem (2.12). By substitution into the differential equation and some manipulation, prove that this function is a solution if and only if

$$\frac{d}{dt}(e^{-\Omega(t)}Ae^{\Omega(t)} - t[A, B]) = 0.$$

Compute the indicated derivative and use the hypotheses to show that it vanishes.

Exercise 2.52. Find $n\times n$ matrices A and B such that $[A, B] \neq 0$ and $e^Ae^B = e^{A+B}$ (see Problem 88-1 in *SIAM Review*, **31**(1), (1989), 125–126).

Exercise 2.53. Let A be an $n\times n$ matrix with components $\{a_{ij}\}$. Prove: Every component of e^A is nonnegative if and only if the off diagonal components of A are all nonnegative (that is, $a_{ij} \geq 0$ whenever $i \neq j$). Hint: The 'if' direction is an easy corollary of the Trotter product formula. But, this is not the best proof. To prove both directions, consider the positive invariance of the positive orthant in n-dimensional space under the flow of the system $\dot{x} = Ax$.

Exercise 2.54. [Lie Groups and Lax Pairs] Is the map

$$\exp : \mathcal{L}(E) \to GL(E)$$

injective? Is this map surjective? Do the answers to these questions depend on the choice of E as $\mathbb{R}^n$ or $\mathbb{C}^n$? Prove that the general linear group is a submanifold of $\mathbb{R}^N$ with $N = n^2$ in case $E = \mathbb{R}^n$, and $N = 2n^2$ in case $E = \mathbb{C}^n$. Show that the general linear group is a Lie group; that is, the group operation (matrix product), is a differentiable map from $GL(E) \times GL(E) \to GL(E)$. Consider the tangent space at the identity element of $GL(E)$. Note that, for each $A \in \mathcal{L}(E)$, the map $t \mapsto \exp(tA)$ is a curve in $GL(E)$ passing through the origin at time $t = 0$. Use this fact to prove that the tangent space can be identified with $\mathcal{L}(E)$. It turns out that $\mathcal{L}(E)$ is a Lie algebra. More generally, a vector space is called a Lie algebra if for each pair of vectors A and B, a product, denoted by $[A, B]$, is defined on the vector space such that the product is bilinear and also satisfies the following algebraic identities: (skew-symmetry) $[A, B] = -[B, A]$, and (the Jacobi identity)

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

Show that $\mathcal{L}(E)$ is a Lie algebra with respect to the product $[A, B] := AB - BA$. For an elementary introduction to the properties of these structures, see [117].

The delicate interplay between Lie groups and Lie algebras leads to a far-reaching theory. To give a flavor of the relationship between these structures, consider the map $\text{Ad} : GL(E) \to \mathcal{L}(\mathcal{L}(E))$ defined by $\text{Ad}(A)(B) = ABA^{-1}$. This map defines the adjoint representation of the Lie group into the automorphisms of the Lie algebra. Prove this. Also, Ad is a homomorphism of groups: $\text{Ad}(AB) = \text{Ad}(A)\,\text{Ad}(B)$. Note that we may as well denote the automorphism group of $\mathcal{L}(E)$ by $GL(\mathcal{L}(E))$. Also, define $\text{ad} : \mathcal{L}(E) \to \mathcal{L}(\mathcal{L}(E))$ by $\text{ad}(A)(B) = [A, B]$. The map ad is a homomorphism of Lie algebras. Now, $\varphi_t := \text{Ad}(e^{tA})$ defines a flow in $\mathcal{L}(E)$. The associated differential equation is obtained by differentiation. Show that φ_t is the flow of the differential equation

$$\dot{x} = Ax - xA = \text{ad}(A)x. \tag{2.16}$$

This differential equation is linear; thus, it has the solution $t \mapsto e^{t\,\text{ad}(A)}$. By the usual argument it now follows that $e^{t\,\text{ad}(A)} = \text{Ad}(e^{tA})$. In particular, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{L}(E) & \xrightarrow{\text{ad}} & \mathcal{L}(\mathcal{L}(E)) \\ \Big\downarrow \exp & & \Big\downarrow \exp \\ GL(E) & \xrightarrow{\text{Ad}} & GL(\mathcal{L}(E)). \end{array}$$

The adjoint representation of $GL(E)$ is useful in the study of the subgroups of $GL(E)$, and it is also used to identify the Lie group that is associated with a given Lie algebra. But consider instead the following application to spectral theory. A curve $t \mapsto L(t)$ in $\mathcal{L}(E)$ is called *isospectral* if the spectrum of $L(t)$ is the same as the spectrum of $L(0)$ for all $t \in \mathbb{R}$. We have the following proposition: Suppose that $A \in \mathcal{L}(E)$. If $t \mapsto L(t)$ is a solution of the differential equation (2.16), then the solution is isospectral. The proof is just a restatement of the content of the commutative diagram. In fact, $L(t)$ is similar to $L(0)$ because

$$L(t) = \text{Ad}(e^{tA})L(0) = e^{tA}L(0)e^{-tA}.$$

A pair of curves $t \mapsto L(t)$ and $t \mapsto M(t)$ is called a *Lax pair* if

$$\dot{L} = LM - ML.$$

The sign convention aside, the above proposition shows that if (L, M) is a Lax pair and if M is constant, then L is isospectral. Prove the more general result: If (L, M) is a Lax pair, then L is isospectral.

Finally, prove that

$$\frac{d}{dt}\big(e^{tA}e^{tB}e^{-tA}e^{-tB}\big)\Big|_{t=0} = 0$$

and

$$\frac{d}{dt}\big(e^{\sqrt{t}A}e^{\sqrt{t}B}e^{-\sqrt{t}A}e^{-\sqrt{t}B}\big)\Big|_{t=0} = AB - BA. \tag{2.17}$$

As mentioned above, $[A, B]$ is in the tangent space at the identity of $GL(E)$. Thus, there is a curve $\gamma(t)$ in $GL(E)$ such that $\gamma(0) = I$ and $\dot{\gamma}(0) = [A, B]$. One such curve is $t \mapsto e^{t[A,B]}$. Since the Lie bracket $[A, B]$ is an algebraic object computed from the tangent vectors A and B, it is satisfying that there is another such curve formed from the curves $t \mapsto e^{tA}$ and $t \mapsto e^{tB}$ whose respective tangent vectors at $t = 0$ are A and B.

Exercise 2.55. Prove that if α is a real number and A is an $n \times n$ real matrix such that $\langle Av, v\rangle \le \alpha|v|^2$ for all $v \in \mathbb{R}^n$, then $\|e^{tA}\| \le e^{\alpha t}$ for all $t \ge 0$. Hint: Consider the differential equation $\dot{x} = Ax$ and the inner product $\langle \dot{x}, x\rangle$. Prove the following more general result suggested by Weishi Liu. Suppose that $t \mapsto A(t)$ and $t \mapsto B(t)$ are smooth $n \times n$ matrix valued functions defined on $\mathbb{R}$ such that $\langle A(t)v, v\rangle \le \alpha(t)|v|^2$ and $\langle B(t)v, v\rangle \le 0$ for all $t \ge 0$ and all $v \in \mathbb{R}^n$. If $t \mapsto x(t)$ is a solution of the differential equation $\dot{x} = A(t)x + B(t)x$, then

$$|x(t)| \le e^{\int_0^t \alpha(s)\,ds}|x(0)|$$

for all $t \ge 0$.

Exercise 2.56. Let $v \in \mathbb{R}^3$, assume $v \ne 0$, and consider the differential equation

$$\dot{x} = v \times x, \quad x(0) = x_0$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. Show that the solution of the differential equation is a rigid rotation of the initial vector x_0 about the direction v. If the differential equation is written as a matrix system

$$\dot{x} = Sx$$

where S is a 3×3 matrix, show that S is skew symmetric and that the flow $\phi_t(x) = e^{tS}x$ of the system is a group of orthogonal transformations. Show that every solution of the system is periodic and relate the period to the length of v.

Exercise 2.57. Consider the linear system $\dot{x} = A(t)x$ where $A(t)$ is a skew-symmetric $n \times n$-matrix for each $t \in \mathbb{R}$ with respect to some inner product on $\mathbb{R}^n$, and let $|\ |$ denote the corresponding norm. Show that $|\phi(t)| = |\phi(0)|$ for every solution $t \mapsto \phi(t)$.

Exercise 2.58. [An Infinite Dimensional ODE] Let E denote the Banach space $C([0, 1])$ given by the set of all continuous functions $f : [0, 1] \to \mathbb{R}$ with the supremum norm

$$\|f\| = \sup_{s\in[0,1]} |f(s)|$$

and consider the operator $U : E \to E$ given by $(Uf)(s) = f(as)$ where $0 \le a \le 1$. Also, let $g \in E$ denote the function given by $s \to bs$ where b is a fixed real number. Find the solution of the initial value problem

$$\dot{x} = Ux, \qquad x(0) = g.$$

This is a simple example of an ordinary differential equation on an infinite dimensional Banach space (see Section 3.6).

Exercise 2.59. Write a report on the application of the Lie-Trotter formula to obtain numerical approximations of the solution of the initial value problem $\dot{x} = (A + B)x$, $x(0) = v$ with expressions of the form

$$T(t, n)v = (e^{(t/n)A}e^{(t/n)B})^n v.$$

For example, approximate $x(1)$ for such systems where

$$A := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad B := \begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Compare the results of numerical experiments using your implementation(s) of the "Lie-Trotter method" and your favorite choice of alternative method(s) to compute $x(1)$. Note that e^{tA} and e^{tB} can be input explicitly for the suggested example. Can you estimate the error $|T(1, n)v - e^{A+B}v|$? Generalizations of this scheme are sometimes used to approximate differential equations where the "vector field" can be split into two easily solved summands. Try the same idea to solve nonlinear ODE of the form $\dot{x} = f(x) + g(x)$ where e^{tA} is replaced by the flow of $\dot{x} = f(x)$ and e^{tB} is replaced by the flow of $\dot{x} = g(x)$.

2.2 Stability of Linear Systems

A linear homogeneous differential equation has a rest point at the origin. We will use our results about the solutions of constant coefficient homogeneous linear differential equations to study the stability of this rest point. The next result is fundamental.

Theorem 2.60. *Suppose that A is an $n \times n$ (real) matrix. The following statements are equivalent:*

(1) There is a norm $|\ \ |_a$ on $\mathbb{R}^n$ and a real number $\lambda > 0$ such that for all $v \in \mathbb{R}^n$ and all $t \geq 0$,

$$|e^{tA}v|_a \leq e^{-\lambda t}|v|_a.$$

(2) If $|\ \ |_g$ is an arbitrary norm on $\mathbb{R}^n$, then there is a constant $C > 0$ and a real number $\lambda > 0$ such that for all $v \in \mathbb{R}^n$ and all $t \geq 0$,

$$|e^{tA}v|_g \leq Ce^{-\lambda t}|v|_g.$$

(3) Every eigenvalue of A has negative real part.

Moreover, if $-\lambda$ exceeds the largest of all the real parts of the eigenvalues of A, then λ can be taken to be the decay constant in (1) or (2).

Corollary 2.61. *If every eigenvalue of A has negative real part, then the zero solution of $\dot{x} = Ax$ is asymptotically stable.*

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

To show (1) $\Rightarrow$ (2), let $| \quad |_a$ be the norm in statement (1) and $| \quad |_g$ the norm in statement (2). Because these norms are defined on the finite dimensional vector space $\mathbb{R}^n$, they are equivalent; that is, there are constants $K_1 > 0$ and $K_2 > 0$ such that for all $x \in \mathbb{R}^n$ we have

$$K_1|x|_g \leq |x|_a \leq K_2|x|_g.$$

(Prove this!) Hence, if $t \geq 0$ and $v \in \mathbb{R}^n$, then

$$|e^{tA}v|_g \leq \frac{1}{K_1}|e^{tA}v|_a \leq \frac{1}{K_1}e^{-\lambda t}|v|_a \leq \frac{K_2}{K_1}e^{-\lambda t}|v|_g.$$

To show (2) $\Rightarrow$ (3), suppose that statement (2) holds but statement (3) does not. In particular, A has an eigenvalue $\mu \in \mathbb{C}$, say $\mu = \alpha + i\beta$ with $\alpha \geq 0$. Moreover, there is at least one eigenvector $v \neq 0$ corresponding to this eigenvalue. By Proposition 2.23, the system $\dot{x} = Ax$ has a solution $t \mapsto \gamma(t)$ of the form $t \to e^{\alpha t}((\cos \beta t)u - (\sin \beta t)w)$ where $v = u + iw$, $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$. By inspection, $\lim_{t\to\infty} \gamma(t) \neq 0$. But if statement (2) holds, then $\lim_{t\to\infty} \gamma(t) = 0$, in contradiction.

To finish the proof we will show (3) $\Rightarrow$ (1). Let us assume that statement (3) holds. Since A has a finite set of eigenvalues and each of its eigenvalues has negative real part, there is a number $\lambda > 0$ such that the real part of each eigenvalue of A is less than $-\lambda$.

By Proposition 2.36, the components of e^{tA} are finite sums of terms of the form $p(t)e^{\alpha t} \sin \beta t$ or $p(t)e^{\alpha t} \cos \beta t$ where α is the real part of an eigenvalue of A and $p(t)$ is a polynomial of degree at most $n - 1$. In particular, if the matrix e^{tA}, partitioned by columns, is given by $[c_1(t), \ldots, c_n(t)]$, then each component of each vector $c_i(t)$ is a sum of such terms.

Let us denote the usual norm of a vector $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ by $|v|$. Also, $|v_i|$ is the absolute value of the real number v_i, or (if you like) the norm of the vector $v_i \in \mathbb{R}$. With this notation we have

$$|e^{tA}v| \leq \sum_{i=1}^{n} |c_i(t)||v_i|.$$

Because

$$|v_i| \leq \Big(\sum_{j=1}^{n} |v_j|^2\Big)^{1/2} = |v|,$$

it follows that

$$|e^{tA}v| \leq |v| \sum_{i=1}^{n} |c_i(t)|.$$

If $\beta_1, \ldots, \beta_\ell$ are the nonzero imaginary parts of the eigenvalues of A and if α denotes the largest real part of an eigenvalue of A, then using the structure of the components of the vector $c_i(t)$ it follows that

$$|c_i(t)|^2 \leq e^{2\alpha t} \sum_{k=0}^{2n-2} |d_{ki}(t)||t|^k$$

where each coefficient $d_{ki}(t)$ is a quadratic form in

$$\sin \beta_1 t, \ldots, \sin \beta_\ell t, \cos \beta_1 t, \ldots, \cos \beta_\ell t.$$

There is a constant $M > 0$ that does not depend on i or k such that the supremum of $|d_{ki}(t)|$ for $t \in \mathbb{R}$ does not exceed M^2. In particular, for each $i = 1, \ldots, n$, we have

$$|c_i(t)|^2 \leq e^{2\alpha t} M^2 \sum_{k=0}^{2n-2} |t|^k,$$

and as a result

$$|e^{tA}v| \leq |v| \sum_{i=1}^{n} |c_i(t)| \leq e^{\alpha t} nM|v| \Big(\sum_{k=0}^{2n-2} |t|^k \Big)^{1/2}.$$

Because $\alpha < -\lambda < 0$, there is some $\tau > 0$ such that for $t \geq \tau$, we have the inequality

$$e^{(\lambda+\alpha)t} nM \Big(\sum_{k=0}^{2n-2} |t|^k \Big)^{1/2} \leq 1,$$

or equivalently

$$e^{\alpha t} nM \Big(\sum_{k=0}^{2n-2} |t|^k \Big)^{1/2} \leq e^{-\lambda t}.$$

In particular, if $t \geq \tau$, then for each $v \in \mathbb{R}^n$ we have

$$|e^{tA}v| \leq e^{-\lambda t}|v|. \tag{2.18}$$

To finish the proof, we will construct a new norm for which the same inequality is valid for all $t \geq 0$. In fact, we will prove that

$$|v|_a := \int_0^\tau e^{\lambda s} |e^{sA} v| \, ds$$

is the required norm.

The easy proof required to show that $|\ \ |_a$ is a norm on $\mathbb{R}^n$ is left to the reader. To obtain the norm estimate, note that for each $t \geq 0$ there is a nonnegative integer m and a number T such that $0 \le T < \tau$ and $t = m\tau + T$. Using this decomposition of t, we find that

$$\begin{aligned}
|e^{tA}v|_a &= \int_0^\tau e^{\lambda s}|e^{sA}e^{tA}v|\,ds \\
&= \int_0^{\tau-T} e^{\lambda s}|e^{(s+t)A}v|\,ds + \int_{\tau-T}^{\tau} e^{\lambda s}|e^{(s+t)A}|\,ds \\
&= \int_0^{\tau-T} e^{\lambda s}|e^{m\tau A}e^{(s+T)A}v|\,ds \\
&\qquad + \int_{\tau-T}^{\tau} e^{\lambda s}|e^{(m+1)\tau A}e^{(T-\tau+s)A}v|\,ds.
\end{aligned}$$

Let $u = T + s$ in the first integral, let $u = T - \tau + s$ in the second integral, use the inequality (2.18), and, for $m = 0$, use the inequality $|e^{m\tau A}e^{uA}v| \le e^{-\lambda m\tau}|v|$, to obtain the estimates

$$\begin{aligned}
|e^{tA}v|_a &= \int_T^\tau e^{\lambda(u-T)}|e^{(m\tau+u)A}v|\,du + \int_0^T e^{\lambda(u+\tau-T)}|e^{((m+1)\tau+u)A}v|\,du \\
&\le \int_T^\tau e^{\lambda(u-T)}e^{-\lambda(m\tau)}|e^{uA}v|\,du \\
&\qquad + \int_0^T e^{\lambda(u+\tau-T)}e^{-\lambda(m+1)\tau}|e^{uA}v|\,du \\
&\le \int_0^\tau e^{\lambda u}e^{-\lambda(m\tau+T)}|e^{uA}v|\,du \\
&= e^{-\lambda t}\int_0^\tau e^{\lambda u}|e^{uA}v|\,du \\
&\le e^{-\lambda t}|v|_a,
\end{aligned}$$

as required. □

Recall that a matrix is *infinitesimally hyperbolic* if all of its eigenvalues have nonzero real parts. The following corollary of Theorem 2.60 is the basic result about the dynamics of hyperbolic linear systems.

Corollary 2.62. *If A is an $n \times n$ (real) infinitesimally hyperbolic matrix, then there are two A-invariant subspaces E^s and E^u of $\mathbb{R}^n$ such that $\mathbb{R}^n = E^s \oplus E^u$. Moreover, if $|\ \ |_g$ is a norm on $\mathbb{R}^n$, then there are constants $\lambda > 0$, $\mu > 0$, $C > 0$, and $K > 0$ such that for all $v \in E^s$ and all $t \ge 0$*

$$|e^{tA}v|_g \le Ce^{-\lambda t}|v|_g,$$

and for all $v \in E^u$ and all $t \le 0$

$$|e^{tA}v|_g \le Ke^{\mu t}|v|_g.$$

Also, there exists a norm on $\mathbb{R}^n$ such that the above inequalities hold for $C = K = 1$ and $\lambda = \mu$.

Proof. The details of the proof are left as an exercise. But let us note that if A is infinitesimally hyperbolic, then we can arrange for the Jordan form J of A to be a block matrix

$$J = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}$$

where the eigenvalues of A_s all have negative real parts and the eigenvalues of A_u have positive real parts. Thus, there is an obvious J-invariant splitting of the vector space $\mathbb{R}^n$ into a stable space and an unstable space. By changing back to the original coordinates, it follows that there is a corresponding A-invariant splitting. The hyperbolic estimate on the stable space follows from Theorem 2.60 applied to the restriction of A to its stable subspace; the estimate on the unstable space follows from Theorem 2.60 applied to the restriction of $-A$ to the unstable subspace of A. Finally, an adapted norm on the entire space is obtained as follows:

$$|(v_s, v_u)|_a^2 = |v_s|_a^2 + |v_u|_a^2. \qquad \square$$

The basic result of this section—if all eigenvalues of the matrix A are in the left half plane, then the zero solution of the corresponding homogeneous system is asymptotically stable—is a special case of the principle of linearized stability. This result provides a method to determine the stability of the zero solution that does not require knowing other solutions of the system. As we will see, the same idea works in more general contexts. But, additional hypotheses are required for most generalizations.

Exercise 2.63. Find E^s, E^u, C, K, λ, and μ as in Corollary 2.62 (relative to the usual norm) for the matrix

$$A := \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}.$$

Exercise 2.64. As a continuation of Exercise 2.55, suppose that A is an $n \times n$ matrix and that there is a number $\lambda > 0$ such that every eigenvalue of A has real part less than $-\lambda$. Prove that there is an inner product and associated norm such that $\langle Ax, x\rangle \le -\lambda|x|^2$ for all $x \in \mathbb{R}^n$ and conclude that $|e^{tA}x| \le e^{-\lambda t}|x|$. This gives an alternative method of constructing an adapted norm (see [121, p. 146]). Show that there is a constant $C > 0$ such that $|e^{tA}x| \le Ce^{-\lambda t}|x|$ with respect to the usual norm. Moreover, show that there is a constant $k > 0$ such that if B is an $n \times n$ matrix, then $|e^{tB}x| \le Ce^{k\|B-A\|-\lambda t}|x|$. In particular, if $\|B - A\|$ is sufficiency small, then there is some $\mu > 0$ such that $|e^{tB}x| \le Ce^{-\mu t}|x|$.

Exercise 2.65. Suppose that A and B are $n \times n$-matrices and all the eigenvalues of B are positive real numbers. Also, let B^T denote the transpose of B. Show that there is a value μ_* of the parameter μ such that the rest point at the origin of the system

$$\dot{X} = AX - \mu X B^T$$

is asymptotically stable whenever $\mu > \mu_*$. Hint: X is a matrix valued variable. Show that the eigenvalues of the linear operator $X \mapsto AX - XB^T$ are given by differences of the eigenvalues of A and B. Prove this first in case A and B are diagonalizable and then use the density of the diagonalizable matrices (cf. [204, p. 331]).

2.3 Stability of Nonlinear Systems

Theorem 2.60 states that the zero solution of a constant coefficient homogeneous linear system is asymptotically stable if the spectrum of the coefficient matrix lies in the left half of the complex plane. The principle of linearized stability states that the same result is true for steady state solutions of nonlinear equations provided that the system matrix of the linearized system along the steady state solution has its spectrum in the left half plane. As stated, this principle is not a theorem. In this section, however, we will formulate and prove a theorem on linearized stability which is strong enough for most applications. In particular, we will prove that a rest point of an autonomous differential equation $\dot{x} = f(x)$ in $\mathbb{R}^n$ is asymptotically stable if all eigenvalues of the Jacobian matrix at the rest point have negative real parts. Our stability result is also valid for some nonhomogeneous nonautonomous differential equations of the form

$$\dot{x} = A(t)x + g(x,t), \qquad x \in \mathbb{R}^n \tag{2.19}$$

where $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth function.

A fundamental tool used in our stability analysis is the formula, called the *variation of parameters formula,* given in the next proposition.

Proposition 2.66 (Variation of Parameters Formula). *Consider the initial value problem*

$$\dot{x} = A(t)x + g(x,t), \qquad x(t_0) = x_0 \tag{2.20}$$

and let $t \mapsto \Phi(t)$ be a fundamental matrix solution for the homogeneous system $\dot{x} = A(t)x$ that is defined on some interval J_0 containing t_0. If $t \mapsto \phi(t)$ is the solution of the initial value problem defined on some subinterval of J_0, then

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(\phi(s),s)\,ds. \tag{2.21}$$

Proof. Define a new function z by $z(t) = \Phi^{-1}(t)\phi(t)$. We have

$$\dot{\phi}(t) = A(t)\Phi(t)z(t) + \Phi(t)\dot{z}(t).$$

Thus,

$$A(t)\phi(t) + g(\phi(t),t) = A(t)\phi(t) + \Phi(t)\dot{z}(t)$$

and

$$\dot{z}(t) = \Phi^{-1}(t)g(\phi(t), t).$$

Also note that $z(t_0) = \Phi^{-1}(t_0)x_0$.

By integration,

$$z(t) - z(t_0) = \int_{t_0}^{t} \Phi^{-1}(s)g(\phi(s), s)\, ds,$$

or, in other words,

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(\phi(s), s)\, ds. \qquad \square$$

Let us note that in the special case where the function g in the differential equation (2.20) is a constant with respect to its first variable, the variation of parameters formula solves the initial value problem once a fundamental matrix solution of the associated homogeneous system is determined.

Exercise 2.67. Consider the linear system

$$\dot{u} = -\delta^2 u + v + \delta w, \quad \dot{v} = -u - \delta^2 v + \delta w, \quad \dot{w} = -\delta w$$

where δ is a parameter. Find the general solution of this system using matrix algebra and also by using the substitution $z = u + iv$. Describe the phase portrait for the system for each value of δ. Find an invariant line and determine the rate of change with respect to δ of the angle this line makes with the positive w-axis. Also, find the angular velocity of the "twist" around the invariant line.

Exercise 2.68. (a) Use variation of parameters to solve the system

$$\dot{x} = x - y + e^{-t}, \qquad \dot{y} = x + y + e^{-t}.$$

(b) Find the set of initial conditions at $t = 0$ so that $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$ whenever $t \mapsto (x(t), y(t))$ satisfies one of these initial conditions.

Exercise 2.69. Suppose that $g : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and consider the family of solutions $t \mapsto \phi(t, \xi, \epsilon)$ of the family of differential equations

$$\dot{x} = Ax + \epsilon x + \epsilon^2 g(x)$$

with parameter ϵ such that $\phi(0, \xi, \epsilon) = \xi$. Compute the derivative $\phi_\epsilon(1, \xi, 0)$. Hint: Solve an appropriate variational equation using variation of parameters.

Exercise 2.70. The product $\Phi(t)\Phi^{-1}(s)$ appears in the variation of parameters formula where $\Phi(t)$ is the principal fundamental matrix for the system $\dot{x} = A(t)x$. Show that if A is a constant matrix or A is 1×1, then $\Phi(t)\Phi^{-1}(s) = \Phi(t - s)$. Prove that this formula does *not* hold in general for homogeneous linear systems.

Exercise 2.71. Give an alternative proof of Proposition 2.66 by verifying directly that the variation of parameters formula (2.21) is a solution of the initial value problem (2.20)

Exercise 2.72. Suppose that A is an $n\times n$-matrix all of whose eigenvalues have negative real parts. (a) Find a (smooth) function $f:\mathbb{R}\to\mathbb{R}$ so that a solution of the scalar equation $\dot{x}=-x+f(t)$ is not bounded for $t\geq 0$. (b) Show that there is a (smooth) function $f:\mathbb{R}\to\mathbb{R}^n$ so that a solution of the system $\dot{x}=Ax+f(t)$ is not bounded for $t\geq 0$. (c) Show that if the system $\dot{x}=Ax+f(t)$ does have a bounded solution, then all solutions are bounded.

Exercise 2.73. [Nonlinear Variation of Parameters] Consider the differential equations $\dot{y}=F(y,t)$ and $\dot{x}=f(t,x)$ and let $t\mapsto y(t,\tau,\xi)$ and $t\mapsto x(t,\tau,\xi)$ be the corresponding solutions such that $y(\tau,\tau,\xi)=\xi$ and $x(\tau,\tau,\xi)=\xi$. (a) Prove the nonlinear variation of parameters formula

$$x(t,\tau,\xi)=y(t,\tau,\xi)+\int_\tau^t [y_\tau(t,s,x(s,\tau,\xi))+y_\xi(t,s,x(s,\tau,\xi))f(s,x(s,\tau,\xi))]\,ds.$$

Hint: Define $z(s)=y(t,s,x(s,\tau,\xi))$, differentiate z with respect to s, integrate the resulting formula over the interval $[\tau,t]$, and note that $z(t)=x(t,\tau,\xi)$ and $z(\tau)=y(t,\tau,\xi)$. (b) Derive the variation of parameters formula from the nonlinear variation of parameters formula. Hint: Consider $\dot{y}=A(t)y$ and $\dot{x}=A(t)x+h(t,x)$. Also, let $\Phi(t)$ denote a fundamental matrix for $\dot{y}=A(t)y$ and note that $d/dt\Phi^{-1}(t)=-\Phi^{-1}(t)A(t)$. (c) Consider the differential equation $\dot{x}=-x^3$ and prove that $x(t,\xi)$ (the solution such that $x(0,\xi)=\xi$) is $O(1/\sqrt{t})$ as $t\to\infty$; that is, there is a constant $C>0$ such that $|x(t,\xi)|\leq C/\sqrt{t}$ as $t\to\infty$. Next suppose that M and δ are positive constants and $g:\mathbb{R}\to\mathbb{R}$ is such that $|g(x)|\leq Mx^4$ whenever $|x|<\delta$. Prove that if $t\mapsto x(t,\xi)$ is the solution of the differential equation $\dot{x}=-x^3+g(x)$ such that $x(0,\xi)=\xi$ and $|\xi|$ is sufficiently small, then $|x(t,\xi)|\leq C/\sqrt{t}$. Hint: First show that the origin is asymptotically stable using a Lyapunov function. Write out the nonlinear variation of parameters formula, make an estimate, and use Gronwall's inequality.

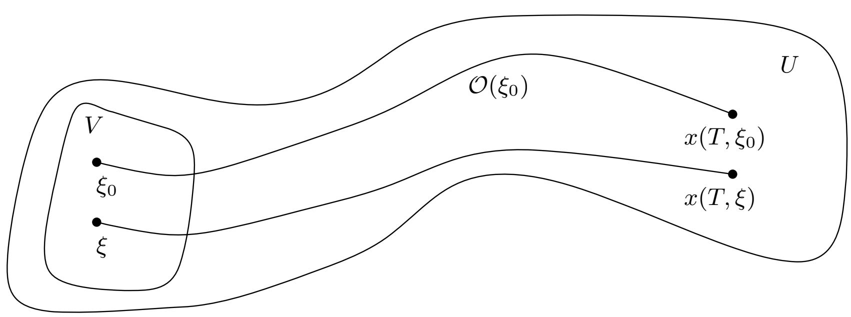

Figure 2.1: Local stability as in Proposition 2.75. For every open set U containing the orbit segment $\mathcal{O}(\xi_0)$, there is an open set V containing ξ_0 such that orbits starting in V stay in U on the time interval $0\leq t\leq T$.

The next proposition states an important continuity result for the solutions of nonautonomous systems with respect to initial conditions. To prove it, we will use the following lemma.

Lemma 2.74. *Consider a smooth function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. If $K \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}$ are compact sets, then there is a number $L > 0$ such that*

$$|f(x,t) - f(y,t)| \leq L|x - y|$$

for all $(x,t), (y,t) \in K \times A$.

Proof. The proof of the lemma uses compactness, continuity, and the mean value theorem. The details are left as an exercise. □

Recall that a function f as in the lemma is called Lipschitz with respect to its first argument on $K \times A$ with Lipschitz constant L.

Proposition 2.75. *Consider, for each $\xi \in \mathbb{R}^n$, the solution $t \mapsto \phi(t,\xi)$ of the differential equation $\dot{x} = f(x,t)$ such that $\phi(0,\xi) = \xi$. If $\xi_0 \in \mathbb{R}^n$ is such that the solution $t \mapsto \phi(t,\xi_0)$ is defined for $0 \leq t \leq T$, and if $U \subseteq \mathbb{R}^n$ is an open set containing the orbit segment $\mathcal{O}(\xi_0) = \{\phi(t,\xi_0) : 0 \leq t \leq T\}$, then there is an open set $V \subseteq U$, as in Figure 2.1, such that $\xi_0 \in V$ and $\{\phi(t,\xi) : \xi \in V,\ 0 \leq t \leq T\} \subseteq U$; that is, the solution starting at each $\xi \in V$ exists on the interval $[0,T]$, and its values on this interval are in U.*

Proof. Let $\xi \in \mathbb{R}^n$, and consider the two solutions of the differential equation given by $t \mapsto \phi(t,\xi_0)$ and $t \mapsto \phi(t,\xi)$. For t in the intersection of the intervals of existence of these solutions, we have that

$$\phi(t,\xi) - \phi(t,\xi_0) = \xi - \xi_0 + \int_0^t f(\phi(s,\xi),s) - f(\phi(s,\xi_0),s)\,ds$$

and

$$|\phi(t,\xi) - \phi(t,\xi_0)| \leq |\xi - \xi_0| + \int_0^t |f(\phi(s,\xi),s) - f(\phi(s,\xi_0),s)|\,ds.$$

We can assume without loss of generality that U is bounded, hence its closure is compact. It follows from the lemma that the smooth function f is Lipschitz on $U \times [0,T]$ with a Lipschitz constant $L > 0$. Thus, as long as $(\phi(t,\xi),t) \in U \times [0,T]$, we have

$$|\phi(t,\xi) - \phi(t,\xi_0)| \leq |\xi - \xi_0| + \int_0^t L|\phi(s,\xi) - \phi(s,\xi_0)|\,ds$$

and by Gronwall's inequality

$$|\phi(t,\xi) - \phi(t,\xi_0)| \leq |\xi - \xi_0|e^{Lt}.$$

Let $\delta > 0$ be such that δe^{LT} is less than the distance from $\mathcal{O}(\xi_0)$ to the boundary of U. Since, on the intersection J of the domain of definition of the solution $t \mapsto \phi(t,\xi)$ with $[0,T]$ we have

$$|\phi(t,\xi) - \phi(t,\xi_0)| \leq |\xi - \xi_0|e^{LT},$$

the vector $\phi(t,\xi)$ is in the bounded set U as long as $t \in J$ and $|\xi - \xi_0| < \delta$. By the extension theorem, the solution $t \mapsto \phi(t,\xi)$ is defined at least on the interval $[0,T]$. Thus, the desired set V is $\{\xi \in U : |\xi - \xi_0| < \delta\}$. □

We are now ready to formulate a theoretical foundation for Lyapunov's indirect method, that is, the method of linearization. The idea should be familiar: If the system has a rest point at the origin, the linearization of the system has an asymptotically stable rest point at the origin, and the nonlinear part is appropriately bounded, then the nonlinear system also has an asymptotically stable rest point at the origin.

Theorem 2.76. *Consider the initial value problem (2.20) for the case where $A := A(t)$ is a (real) matrix of constants. If all eigenvalues of A have negative real parts and there are positive constants $a > 0$ and $k > 0$ such that $|g(x,t)| \le k|x|^2$ whenever $|x| < a$, then there are positive constants C, b, and α that are independent of the choice of the initial time t_0 such that the solution $t \mapsto \phi(t)$ of the initial value problem satisfies*

$$|\phi(t)| \le C|x_0|e^{-\alpha(t-t_0)} \tag{2.22}$$

for $t \ge t_0$ whenever $|x_0| \le b$. In particular, the function $t \mapsto \phi(t)$ is defined for all $t \ge t_0$, and the zero solution (the solution with initial value $\phi(t_0) = 0$), is asymptotically stable.

Proof. By Theorem 2.60 and the hypothesis on the eigenvalues of A, there are constants $C > 1$ and $\lambda > 0$ such that

$$\|e^{tA}\| \le Ce^{-\lambda t} \tag{2.23}$$

for $t \ge 0$. Fix $\delta > 0$ such that $\delta < a$ and $Ck\delta - \lambda < 0$, define $\alpha := \lambda - Ck\delta$ and $b := \delta/C$, and note that $\alpha > 0$ and $0 < b < \delta < a$.

If $|x_0| < b$, then there is a maximal half-open interval $J = \{t \in \mathbb{R} : t_0 \le t < \tau\}$ such that the solution $t \to \phi(t)$ of the differential equation with initial condition $\phi(t_0) = x_0$ exists and satisfies the inequality

$$|\phi(t)| < \delta \tag{2.24}$$

on the interval J.

For $t \in J$, use the estimate

$$|g(\phi(t),t)| \le k\delta|\phi(t)|,$$

the estimate (2.23), and the variation of parameters formula

$$\phi(t) = e^{(t-t_0)A}x_0 + e^{tA}\int_{t_0}^{t} e^{-sA}g(\phi(s),s)\,ds$$

to obtain the inequality

$$|\phi(t)| \le Ce^{-\lambda(t-t_0)}|x_0| + \int_{t_0}^{t} Ce^{-\lambda(t-s)}k\delta|\phi(s)|\,ds.$$

Rearrange the inequality to the form

$$e^{\lambda(t-t_0)}|\phi(t)| \le C|x_0| + Ck\delta \int_{t_0}^{t} e^{\lambda(s-t_0)}|\phi(s)|\, ds$$

and apply Gronwall's inequality to obtain the estimate

$$e^{\lambda(t-t_0)}|\phi(t)| \le C|x_0|e^{Ck\delta(t-t_0)};$$

or equivalently

$$|\phi(t)| \le C|x_0|e^{(Ck\delta-\lambda)(t-t_0)} \le C|x_0|e^{-\alpha(t-t_0)}. \qquad (2.25)$$

Thus, if $|x_0| < b$ and $|\phi(t)| < \delta$ for $t \in J$, then the required inequality (2.22) is satisfied for $t \in J$.

If J is not the interval $[t_0, \infty)$, then $\tau < \infty$. Because $|x_0| < \delta/C$ and in view of the inequality (2.25), we have that

$$|\phi(t)| < \delta e^{-\alpha(t-t_0)} \qquad (2.26)$$

for $t_0 \le t < \tau$. In particular, the solution is bounded by δ on the interval $[t_0, \tau)$. Therefore, by the extension theorem there is some number $\epsilon > 0$ such that the solution is defined on the interval $K := [t_0, \tau + \epsilon)$. Using the continuity of the function $t \mapsto |\phi(t)|$ on K and the inequality (2.26), it follows that

$$|\phi\tau)| \le \delta e^{-\alpha(\tau-t_0)} < \delta.$$

By using this inequality and again using the continuity of the function $t \mapsto |\phi(t)|$ on K, there is a number $\eta > 0$ such that $t \mapsto \phi(t)$ is defined on the interval $[t_0, \tau + \eta)$, and, on this interval, $|\phi(t)| < \delta$. This contradicts the maximality of τ. □

Corollary 2.77. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth (at least class C^2), $f(\xi) = 0$, and all eigenvalues of $Df(\xi)$ have negative real parts, then the differential equation $\dot{x} = f(x)$ has an asymptotically stable rest point at ξ. Moreover, if $-\alpha$ is a number larger than every real part of an eigenvalue of $Df(\xi)$, and ϕ_t is the flow of the differential equation, then there is a neighborhood U of ξ and a constant $C > 0$ such that*

$$|\phi_t(x) - \xi| \le C|x - \xi|e^{-\alpha t}$$

whenever $x \in U$ and $t \ge 0$.

Proof. It suffices to prove the corollary for the case $\xi = 0$. By Taylor's theorem (Theorem 1.237), we can rewrite the differential equation in the form $\dot{x} = Df(0)x + g(x)$ where

$$g(x) := \int_0^1 (Df(sx) - Df(0))x\, ds.$$

The function $\xi \mapsto Df(\xi)$ is class C^1. Thus, by the mean value theorem (Theorem 1.53),

$$\begin{aligned}\|Df(sx) - Df(0)\| &\le |sx| \sup_{\tau\in[0,1]} \|D^2 f(\tau s x)\| \\ &\le |x| \sup_{\tau\in[0,1]} \|D^2 f(\tau x)\|.\end{aligned}$$

Again, by the smoothness of f, there is an open ball B centered at the origin and a constant $k > 0$ such that

$$\sup_{\tau\in[0,1]} \|D^2 f(\tau x)\| < k$$

for all $x \in B$. Moreover, by an application of Proposition 1.235 and the above estimates we have that

$$|g(x)| \le \sup_{s\in[0,1]} |x|\|Df(sx) - Df(0)\| \le k|x|^2$$

whenever $x \in B$. The desired result now follows directly from Theorem 2.76.

□

Exercise 2.78. Generalize the previous result to the Poincaré–Lyapunov Theorem: Let

$$\dot{x} = Ax + B(t)x + g(x,t), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n$$

be a smooth initial value problem. If

(1) A is a constant matrix with spectrum in the left half plane,

(2) $B(t)$ is the $n\times n$ matrix, continuously dependent on t such that $\|B(t)\| \to 0$ as $t \to \infty$,

(3) $g(x,t)$ is smooth and there are constants $a > 0$ and $k > 0$ such that

$$|g(x,t)| \le k|x|^2$$

for all $t \ge 0$ and $|x| < a$,

then there are constants $C > 1$, $\delta > 0$, $\lambda > 0$ such that

$$|x(t)| \le C|x_0|e^{-\lambda(t-t_0)}, \qquad t \ge t_0$$

whenever $|x_0| \le \delta/C$. In particular, the zero solution is asymptotically stable.

Exercise 2.79. This exercise gives an alternative proof of the principle of linearized stability for autonomous systems using Lyapunov's direct method. (a) Consider the system

$$\dot{x} = Ax + g(x), \qquad x \in \mathbb{R}^n$$

where A is a real $n \times n$ matrix and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Suppose that every eigenvalue of A has negative real part, and that for some $a > 0$, there is a constant $k > 0$ such that, using the usual norm on $\mathbb{R}^n$,

$$|g(x)| \leq k|x|^2$$

whenever $|x| < a$. Prove that the origin is an asymptotically stable rest point by constructing a quadratic Lyapunov function. Hint: Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^n$, and let A^* denote the transpose of the real matrix A. Suppose that there is a real symmetric positive definite $n \times n$ matrix that also satisfies *Lyapunov's equation*

$$A^*B + BA = -I$$

and define $V : \mathbb{R}^n \to \mathbb{R}$ by

$$V(x) = \langle x, Bx \rangle.$$

Show that the restriction of V to a sufficiently small neighborhood of the origin is a strict Lyapunov function. To do this, you will have to estimate a certain inner product using the Schwarz inequality. Finish the proof by showing that

$$B := \int_0^\infty e^{tA^*} e^{tA}\, dt$$

is a symmetric positive definite $n \times n$ matrix which satisfies Lyapunov's equation. To do this, prove that A^* and A have the same eigenvalues. Then use the exponential estimates for hyperbolic linear systems to prove that the integral converges. (b) Give an alternative method to compute solutions of Lyapunov's equation using the following outline: Show that Lyapunov's equation in the form $A^*B + BA = S$, where A is diagonal, S is symmetric and positive definite, and all pairs of eigenvalues of A have nonzero sums, has a symmetric positive-definite solution B. In particular, under these hypotheses, the operator $B \mapsto A^*B + BA$ is invertible. Show that the same result is true without the hypothesis that A is diagonal. Hint: Use the density of the diagonalizable matrices and the continuity of the eigenvalues of a matrix with respect to its components (see Exercises 2.65 and 8.1). (c) Prove that the origin is asymptotically stable for the system $\dot{x} = Ax + g(x)$ where

$$A := \begin{pmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \qquad g(u, v, w) := \begin{pmatrix} u^2 + uv + v^2 + wv^2 \\ w^2 + uvw \\ w^3 \end{pmatrix}$$

and construct the corresponding matrix B that solves Lyapunov's equation.

Exercise 2.80. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is *conservative;* that is, there is some function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = \operatorname{grad} g(x)$. Also, suppose that M and Λ are symmetric positive definite $n \times n$ matrices. Consider the differential equation

$$M\ddot{x} + \Lambda\dot{x} + f(x) = 0, \qquad x \in \mathbb{R}^n$$

and note that, in case M and Λ are diagonal, the differential equation can be viewed as a model of n particles each moving according to Newton's second law in a conservative force field with viscous damping. (a) Prove that the function $V : \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(x, y) := \frac{1}{2}\langle My, y\rangle + \int_0^1 \langle f(sx), x\rangle\, ds$$

decreases along orbits of the associated first-order system

$$\dot{x} = y, \qquad M\dot{y} = -\Lambda y - f(x);$$

in fact, $\dot{V} = -\langle \Lambda y, y\rangle$. Conclude that the system has no periodic orbits. (b) Prove that if $f(0) = 0$ and $Df(0)$ is positive definite, then the system has an asymptotically stable rest point at the origin. Prove this fact in two ways: using the function V and by the method of linearization. Hint: To use the function V see Exercise 1.171. To use the method of linearization, note that M is invertible, compute the system matrix for the linearization in block form, suppose there is an eigenvalue λ, and look for a corresponding eigenvector in block form, that is the transpose of a vector (x, y). This leads to two equations corresponding to the block components corresponding to x and y. Reduce to one equation for x and then take the inner product with respect to x.

2.4 Floquet Theory

We will study linear systems of the form

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n \tag{2.27}$$

where $t \to A(t)$ is a T-periodic continuous matrix-valued function. The main theorem in this section, Floquet's theorem, gives a canonical form for fundamental matrix solutions. This result will be used to show that there is a periodic time-dependent change of coordinates that transforms system (2.27) into a homogeneous linear system with constant coefficients.

Floquet's theorem is a corollary of the following result about the range of the exponential map.

Theorem 2.81. *If C is a nonsingular $n \times n$ matrix, then there is an $n \times n$ matrix B (which may be complex) such that $e^B = C$. If C is a nonsingular real $n \times n$ matrix, then there is a real $n \times n$ matrix B such that $e^B = C^2$.*

Proof. If S is a nonsingular $n \times n$ matrix such that $S^{-1}CS = J$ is in Jordan canonical form, and if $e^K = J$, then $Se^KS^{-1} = C$. As a result, $e^{SKS^{-1}} = C$ and $B = SKS^{-1}$ is the desired matrix. Thus, it suffices to consider the nonsingular matrix C or C^2 to be a Jordan block.

For the first statement of the theorem, assume that $C = \lambda I + N$ where N is nilpotent; that is, $N^m = 0$ for some integer m with $0 \le m < n$. Because C is nonsingular, $\lambda \neq 0$ and we can write $C = \lambda(I + (1/\lambda)N)$. A computation using the series representation of the function $t \mapsto \ln(1 + t)$

at $t = 0$ shows that, formally (that is, without regard to the convergence of the series), if $B = (\ln \lambda)I + M$ where

$$M = \sum_{j=1}^{m-1} \frac{(-1)^{j+1}}{j\lambda^j} N^j,$$

then $e^B = C$. But because N is nilpotent, the series are finite. Thus, the formal series identity is an identity. This proves the first statement of the theorem.

The Jordan blocks of C^2 correspond to the Jordan blocks of C. The blocks of C^2 corresponding to real eigenvalues of C are all of the type $rI + N$ where $r > 0$ and N is real nilpotent. For a real matrix C all the complex eigenvalues with nonzero imaginary parts occur in complex conjugate pairs; therefore, the corresponding real Jordan blocks of C^2 are block diagonal or "block diagonal plus block nilpotent" with 2×2 diagonal subblocks of the form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

as in equation (2.10). Some of the corresponding real Jordan blocks for the matrix C^2 might have real eigenvalues, but these blocks are again all block diagonal or "block diagonal plus block nilpotent" with 2×2 subblocks.

For the case where a block of C^2 is $rI + N$ where $r > 0$ and N is real nilpotent a real "logarithm" is obtained by the matrix formula given above. For block diagonal real Jordan block, write

$$R = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $r > 0$, and note that a real logarithm is given by

$$\ln rI + \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

Finally, for a "block diagonal plus block nilpotent" Jordan block, factor the Jordan block as follows:

$$\mathcal{R}(I + \mathcal{N})$$

where $\mathcal{R}$ is block diagonal with R along the diagonal and $\mathcal{N}$ has 2×2 blocks on its super diagonal all given by R^{-1}. Note that we have already obtained logarithms for each of these factors. Moreover, it is not difficult to check that the two logarithms commute. Thus, a real logarithm of the Jordan block is obtained as the sum of real logarithms of the factors. □

Theorem 2.81 can be proved without reference to the Jordan canonical form (see [5]).

Theorem 2.82 (Floquet's Theorem). *If $\Phi(t)$ is a fundamental matrix solution of the T-periodic system (2.27), then, for all $t \in \mathbb{R}$,*

$$\Phi(t+T) = \Phi(t)\Phi^{-1}(0)\Phi(T).$$

In addition, there is a matrix B (which may be complex) such that

$$e^{TB} = \Phi^{-1}(0)\Phi(T)$$

and a T-periodic matrix function $t \mapsto P(t)$ (which may be complex valued) such that $\Phi(t) = P(t)e^{tB}$ for all $t \in \mathbb{R}$. Also, there is a real matrix R and a real $2T$-periodic matrix function $t \to Q(t)$ such that $\Phi(t) = Q(t)e^{tR}$ for all $t \in \mathbb{R}$.

Proof. Since the function $t \mapsto A(t)$ is periodic, it is defined for all $t \in \mathbb{R}$. Thus, by Theorem 2.4, all solutions of the system are defined for $t \in \mathbb{R}$.

If $\Psi(t) := \Phi(t+T)$, then $\Psi(t)$ is a matrix solution. Indeed, we have that

$$\dot{\Psi}(t) = \dot{\Phi}(t+T) = A(t+T)\Phi(t+T) = A(t)\Psi(t),$$

as required.

Define

$$C := \Phi^{-1}(0)\Phi(T) = \Phi^{-1}(0)\Psi(0),$$

and note that C is nonsingular. The matrix function $t \mapsto \Phi(t)C$ is clearly a matrix solution of the linear system with initial value $\Phi(0)C = \Psi(0)$. By the uniqueness of solutions, $\Psi(t) = \Phi(t)C$ for all $t \in \mathbb{R}$. In particular, we have that

$$\begin{aligned}\Phi(t+T) &= \Phi(t)C = \Phi(t)\Phi^{-1}(0)\Phi(T),\\ \Phi(t+2T) &= \Phi((t+T)+T) = \Phi(t+T)C = \Phi(t)C^2.\end{aligned}$$

By Theorem 2.81, there is a matrix B, possibly complex, such that

$$e^{TB} = C.$$

Also, there is a real matrix R such that

$$e^{2TR} = C^2.$$

If $P(t) := \Phi(t)e^{-tB}$ and $Q(t) := \Phi(t)e^{-tR}$, then

$$\begin{aligned}P(t+T) &= \Phi(t+T)e^{-TB}e^{-tB} = \Phi(t)Ce^{-TB}e^{-tB} = \Phi(t)e^{-tB} = P(t),\\ Q(t+2T) &= \Phi(t+2T)e^{-2TR}e^{-tR} = \Phi(t)e^{-tR} = Q(t).\end{aligned}$$

Thus, we have $P(t+T) = P(t)$, $Q(t+2T) = Q(t)$, and

$$\Phi(t) = P(t)e^{tB} = Q(t)e^{tR},$$

as required. □

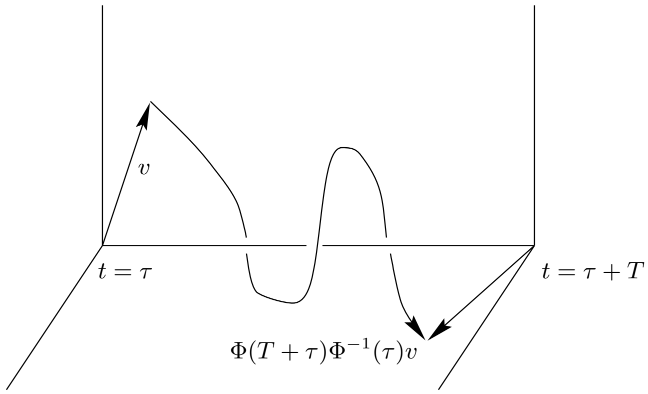

Figure 2.2: The figure depicts the geometry of the monodromy operator for the system $\dot{x} = A(t)x$ in the extended phase space. The vector v in $\mathbb{R}^n$ at $t = \tau$ is advanced to the vector $\Phi(T + \tau)\Phi^{-1}(\tau)v$ at $t = \tau + T$.

The representation $\Phi(t) = P(t)e^{tB}$ in Floquet's theorem is called a *Floquet normal form* for the fundamental matrix $\Phi(t)$. We will use this normal form to study the stability of the zero solution of periodic homogeneous linear systems.

Let us consider a fundamental matrix solution $\Phi(t)$ for the periodic system (2.27) and a vector $v \in \mathbb{R}^n$. The vector solution of the system starting at time $t = \tau$ with initial condition $x(\tau) = v$ is given by

$$t \mapsto \Phi(t)\Phi^{-1}(\tau)v.$$

If the initial vector is moved forward over one period of the system, then we again obtain a vector in $\mathbb{R}^n$ given by $\Phi(T + \tau)\Phi^{-1}(\tau)v$. The operator

$$v \mapsto \Phi(T + \tau)\Phi^{-1}(\tau)v$$

is called a *monodromy operator* (see Figure 2.2). Moreover, if we view the periodic differential equation (2.27) as the autonomous system

$$\dot{x} = A(\psi)x, \qquad \dot{\psi} = 1$$

on the phase cylinder $\mathbb{R}^n \times \mathbb{T}$ where ψ is an angular variable modulo T, then each monodromy operator is a (stroboscopic) Poincaré map for our periodic system. For example, if $\tau = 0$, then the Poincaré section is the fiber $\mathbb{R}^n$ on the cylinder at $\psi = 0$. Of course, each fiber $\mathbb{R}^n$ at $\psi = mT$ where m is an integer is identified with the fiber at $\psi = 0$, and the corresponding Poincaré map is given by

$$v \mapsto \Phi(T)\Phi^{-1}(0)v.$$

The eigenvalues of a monodromy operator are called *characteristic multipliers* of the corresponding time-periodic homogeneous system (2.27). The next proposition states that characteristic multipliers are nonzero complex numbers that are intrinsic to the periodic system—they do not depend on the choice of the fundamental matrix or the initial time.

Proposition 2.83. *The following statements are valid for the periodic linear homogeneous system (2.27).*

(1) Every monodromy operator is invertible. Equivalently, every characteristic multiplier is nonzero.

(2) All monodromy operators have the same eigenvalues. In particular, there are exactly n characteristic multipliers, counting multiplicities.

Proof. The first statement of the proposition is obvious from the definitions.

To prove statement (2), let us consider the principal fundamental matrix $\Phi(t)$ at $t = 0$. If $\Psi(t)$ is a fundamental matrix, then $\Psi(t) = \Phi(t)\Psi(0)$. Also, by Floquet's theorem,

$$\Phi(t + T) = \Phi(t)\Phi^{-1}(0)\Phi(T) = \Phi(t)\Phi(T).$$

Consider the monodromy operator $\mathcal{M}$ given by

$$v \mapsto \Psi(T + \tau)\Psi^{-1}(\tau)v$$

and note that

$$\begin{aligned}\Psi(T + \tau)\Psi^{-1}(\tau) &= \Phi(T + \tau)\Psi(0)\Psi^{-1}(0)\Phi^{-1}(\tau)\\ &= \Phi(T + \tau)\Phi^{-1}(\tau)\\ &= \Phi(\tau)\Phi(T)\Phi^{-1}(\tau).\end{aligned}$$

In particular, the eigenvalues of the operator $\Phi(T)$ are the same as the eigenvalues of the monodromy operator $\mathcal{M}$. Thus, all monodromy operators have the same eigenvalues. □

Because

$$\Phi(t + T) = \Phi(t)\Phi^{-1}(0)\Phi(T),$$

some authors define characteristic multipliers to be the eigenvalues of the matrices defined by $\Phi^{-1}(0)\Phi(T)$ where $\Phi(t)$ is a fundamental matrix. Of course, both definitions gives the same characteristic multipliers. To prove this fact, let us consider the Floquet normal form $\Phi(t) = P(t)e^{tB}$ and note that $\Phi(0) = P(0) = P(T)$. Thus, we have that

$$\Phi^{-1}(0)\Phi(T) = e^{TB}.$$

Also, by using the Floquet normal form,

$$\begin{aligned}\Phi(T)\Phi^{-1}(0) &= P(T)e^{TB}\Phi^{-1}(0)\\ &= \Phi(0)e^{TB}\Phi^{-1}(0)\\ &= \Phi(0)(\Phi^{-1}(0)\Phi(T))\Phi^{-1}(0),\end{aligned}$$

and therefore $\Phi^{-1}(0)\Phi(T)$ has the same eigenvalues as the monodromy operator given by

$$v \mapsto \Phi(T)\Phi^{-1}(0)v.$$

In particular, the traditional definition agrees with our geometrically motivated definition.

Returning to consideration of the Floquet normal form $P(t)e^{tB}$ for the fundamental matrix $\Phi(t)$ and the monodromy operator

$$v \mapsto \Phi(T+\tau)\Phi^{-1}(\tau)v,$$

note that $P(t)$ is invertible and

$$\Phi(T+\tau)\Phi^{-1}(\tau) = P(\tau)e^{TB}P^{-1}(\tau).$$

Thus, the characteristic multipliers of the system are the eigenvalues of e^{TB}. A complex number μ is called a *characteristic exponent* (or a *Floquet exponent*) of the system, if ρ is a characteristic multiplier and $e^{\mu T} = \rho$. Note that if $e^{\mu T} = \rho$, then $\mu + 2\pi ik/T$ is also a Floquet exponent for each integer k. Thus, while there are exactly n characteristic multipliers for the periodic linear system (2.27), there are infinitely many Floquet exponents.

Exercise 2.84. Suppose that $a : \mathbb{R} \to \mathbb{R}$ is a T-periodic function. Find the characteristic multiplier and a Floquet exponent of the T-periodic system $\dot{x} = a(t)x$. Also, find the Floquet normal form for the principal fundamental matrix solution of this system at $t = t_0$.

Exercise 2.85. For the autonomous linear system $\dot{x} = Ax$ a fundamental matrix solution $t \mapsto \Phi(t)$ satisfies the identity $\Phi(T-t) = \Phi(T)\Phi^{-1}(t)$. Show that, in general, this identity does not hold for nonautonomous homogeneous linear systems. Hint: Write down a Floquet normal form matrix $\Phi(t) = P(t)e^{tB}$ that does not satisfy the identity and then show that it is the solution of a (periodic) nonautonomous homogeneous linear system.

Exercise 2.86. Suppose as usual that $A(t)$ is T-periodic and the Floquet normal form of a fundamental matrix solution of the system $\dot{x} = A(t)x$ has the form $P(t)e^{tB}$. (a) Prove that

$$\operatorname{tr} B = \frac{1}{T}\int_0^T \operatorname{tr} A(t)\,dt.$$

Hint: Use Liouville's formula 2.17. (b) By (a), the sum of the characteristic exponents is given by the right-hand side of the formula for the trace of B. Prove that the product of the characteristic multipliers is given by $\exp(\int_0^T \operatorname{tr} A(t)\,dt)$.

Let us suppose that a fundamental matrix for the system (2.27) is represented in Floquet normal form by $P(t)e^{tB}$. We have seen that the characteristic multipliers of the system are the eigenvalues of e^{TB}, but the definition of the Floquet exponents does not mention the eigenvalues of B. Are the eigenvalues of B Floquet exponents? This question is answered affirmatively by the following general theorem about the exponential map.

Theorem 2.87. *If A is an $n \times n$ matrix and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A repeated according to their algebraic multiplicity, then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k and $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are the eigenvalues of e^A.*

Proof. We will prove the theorem by induction on the dimension n.

The theorem is clearly valid for 1×1 matrices. Suppose that it is true for all $(n-1) \times (n-1)$ matrices. Define $\lambda := \lambda_1$, and let $v \neq 0$ denote a corresponding eigenvector so that $Av = \lambda v$. Also, let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the usual basis of $\mathbb{C}^n$. There is a nonsingular $n \times n$ matrix S such that $Sv = \mathbf{e}_1$. (Why?) Thus,

$$SAS^{-1}\mathbf{e}_1 = \lambda \mathbf{e}_1,$$

and it follows that the matrix SAS^{-1} has the block form

$$SAS^{-1} = \begin{pmatrix} \lambda & * \\ 0 & \widetilde{A} \end{pmatrix}.$$

The matrix SA^kS^{-1} has the same block form, only with the block diagonal elements λ^k and $\widetilde{A}^k$. Clearly the eigenvalues of this block matrix are λ^k together with the eigenvalues of $\widetilde{A}^k$. By induction, the eigenvalues of $\widetilde{A}^k$ are the kth powers of the eigenvalues of $\widetilde{A}$. This proves the second statement of the theorem.

Using the power series definition of exp, we see that $e^{SAS^{-1}}$ has block form, with block diagonal elements e^λ and $e^{\widetilde{A}}$. Clearly, the eigenvalues of this block matrix are e^λ together with the eigenvalues of $e^{\widetilde{A}}$. Again using induction, it follows that the eigenvalues of $e^{\widetilde{A}}$ are $e^{\lambda_2}, \ldots, e^{\lambda_n}$. Thus, the eigenvalues of $e^{SAS^{-1}} = Se^AS^{-1}$ are $e^{\lambda_1}, \ldots, e^{\lambda_n}$. □

Theorem 2.87 is an example of a spectral mapping theorem. If we let $\sigma(A)$ denote the spectrum of the matrix A, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not invertible, then, for our finite dimensional matrix, $\sigma(A)$ coincides with the set of eigenvalues of A. Theorem 2.87 can be restated as follows: $e^{\sigma(A)} = \sigma(e^A)$.

The next result uses Floquet theory to show that the differential equation (2.27) is equivalent to a homogeneous linear system with constant coefficients. This result demonstrates that the stability of the zero solution can often be determined by the Floquet multipliers.

Theorem 2.88. *If the principal fundamental matrix solution of the T-periodic differential equation $\dot{x} = A(t)x$ (system (2.27)) at $t = 0$ is given by $Q(t)e^{tR}$ where Q and R are real, then the time-dependent change of coordinates $x = Q(t)y$ transforms this system to the (real) constant coefficient linear system $\dot{y} = Ry$. In particular, there is a time-dependent ($2T$-periodic) change of coordinates that transforms the T-periodic system to a (real) constant coefficient linear system.*

(1) If the characteristic multipliers of the periodic system (2.27) all have modulus less than one; equivalently, if all characteristic exponents have negative real part, then the zero solution is asymptotically stable.

(2) If the characteristic multipliers of the periodic system (2.27) all have modulus less than or equal to one; equivalently, if all characteristic exponents have nonpositive real part, and if the algebraic multiplicity equals the geometric multiplicity of each characteristic multiplier with modulus one; equivalently, if the algebraic multiplicity equals the geometric multiplicity of each characteristic exponent with real part zero, then the zero solution is Lyapunov stable.

(3) If at least one characteristic multiplier of the periodic system (2.27) has modulus greater than one; equivalently, if a characteristic exponent has positive real part, then the zero solution is unstable.

Proof. We will prove the first statement of the theorem and part (1). The proof of the remaining two parts is left as an exercise. For part (2), note that since the differential equation is linear, the Lyapunov stability may reasonably be determined from the eigenvalues of a linearization.

By Floquet's theorem, there is a real matrix R and a real $2T$-periodic matrix $Q(t)$ such that the *principal* fundamental matrix solution $\Phi(t)$ of the system at $t = 0$ is represented by

$$\Phi(t) = Q(t)e^{tR}.$$

Also, there is a matrix B and a T-periodic matrix P such that

$$\Phi(t) = P(t)e^{tB}.$$

The characteristic multipliers are the eigenvalues of e^{TB}. Because $\Phi(0)$ is the identity matrix, we have that

$$\Phi(2T) = e^{2TR} = e^{2TB},$$

and in particular

$$(e^{TB})^2 = e^{2TR}.$$

By Theorem 2.87, the eigenvalues of e^{2TR} are the squares of the characteristic multipliers. These all have modulus less than one. Thus, by another application of Theorem 2.87, all eigenvalues of the real matrix R have negative real parts.

Consider the change of variables $x = Q(t)y$. Because

$$x(t) = Q(t)e^{tR}x(0)$$

and $Q(t)$ is invertible, we have that $y(t) = e^{tR}x(0)$; and therefore,

$$\dot{y} = Ry.$$

By our previous result about linearization (Lyapunov's indirect method), the zero solution of $\dot{y} = Ry$ is asymptotically stable. In fact, by Theorem 2.60, there are numbers $\lambda > 0$ and $C > 0$ such that

$$|y(t)| \leq Ce^{-\lambda t}|y(0)|$$

for all $t \geq 0$ and all $y(0) \in \mathbb{R}^n$. Because Q is periodic, it is bounded. Thus, by the relation $x = Q(t)y$, the zero solution of $\dot{x} = A(t)x$ is also asymptotically stable. □

While the stability theorem just presented is very elegant, in applied problems it is usually impossible to compute the eigenvalues of e^{TB} explicitly. In fact, because $e^{TB} = \Phi(T)$, it is not at all clear that the eigenvalues can be found without solving the system, that is, without an explicit representation of a fundamental matrix. Note, however, that we only have to approximate *finitely* many numbers (the Floquet multipliers) to determine the stability of the system. This fact is important! For example, the stability can often be determined by applying a numerical method to approximate the Floquet multipliers.

Exercise 2.89. If the planar system $\dot{u} = f(u)$ has a limit cycle, then it is possible to find coordinates in a neighborhood of the limit cycle so that the differential equation has the form

$$\dot{\rho} = h(\rho, \varphi)\rho, \qquad \dot{\varphi} = \omega$$

where ω is a constant and for each ρ the function $\varphi \mapsto h(\rho, \varphi)$ is $2\pi/\omega$-periodic. Prove: If the partial derivative of h with respect to ρ is identically zero, then there is a coordinate system such that the differential equation in the new coordinates has the form

$$\dot{r} = cr, \qquad \dot{\phi} = \omega.$$

Hint: Use Exercise 2.84 and Theorem 2.88.

Exercise 2.90. View the damped periodically-forced Duffing equation $\ddot{x}+\dot{x}-x+x^3=\epsilon\sin\omega t$ on the phase cylinder. The unperturbed system ($\epsilon=0$) has a periodic orbit on the phase cylinder with period $2\pi/\omega$ corresponding to its rest point at the origin of the phase plane. Determine the Floquet multipliers associated with this periodic orbit of the unperturbed system; that is, the Floquet multipliers of the linearized system along the periodic orbit.

Exercise 2.91. Consider the system of two coupled oscillators with periodic parametric excitation

$$\ddot{x}+(1+a\cos\omega t)x=y-x,\quad \ddot{y}+(1+a\cos\omega t)y=x-y$$

where a and ω are nonnegative parameters. (See Section 3.3 for a derivation of the coupled oscillator model.) (a) Prove that if $a=0$, then the zero solution is Lyapunov stable. (b) Using a numerical method (or otherwise), determine the Lyapunov stability of the zero solution for fixed but arbitrary values of the parameters. (c) What happens if viscous damping is introduced into the system? Hint: A possible numerical experiment might be designed as follows. For each point in a region of (ω,a)-space, mark the point green if the corresponding system has a Lyapunov stable zero solution; otherwise, mark it red. To decide which regions of parameter space might contain interesting phenomena, recall from your experience with second-order scalar differential equations with constant coefficients (mathematical models of springs) that resonance is expected when the frequency of the periodic excitation is rationally related to the natural frequency of the system. Consider resonances between the frequency ω of the excitation and the frequency of periodic motions of the system with $a=0$, and explore the region of parameter space near these parameter values. Although interesting behavior does occur at resonances, this is not the whole story. Because the monodromy matrix is symplectic (see [11, Sec. 42]), the characteristic multipliers have two symmetries: If λ is a characteristic multiplier, then so is its complex conjugate and its reciprocal. It follows that on the boundary between the stable and unstable regions a pair of characteristic exponents coalesce on the unit circle. Thus, it is instructive to determine the values of ω, with $a=0$, for those characteristic multipliers that coalesce. These values of ω determine the points where unstable regions have boundary points on the ω-axis.

Is there a method to determine the characteristic exponents without finding the solutions of the differential equation (2.27) explicitly? An example of Lawrence Marcus and Hidehiko Yamabe shows no such method can be constructed in any obvious way from the eigenvalues of $A(t)$. Consider the π-periodic system $\dot{x}=A(t)x$ where

$$A(t)=\begin{pmatrix}-1+\frac{3}{2}\cos^2 t & 1-\frac{3}{2}\sin t\cos t\\ -1-\frac{3}{2}\sin t\cos t & -1+\frac{3}{2}\sin^2 t\end{pmatrix}. \tag{2.28}$$

It turns out that $A(t)$ has the (time independent) eigenvalues $\frac{1}{4}(-1\pm\sqrt{7}\,i)$. In particular, the real part of each eigenvalue is negative. On the other hand,

$$x(t)=e^{t/2}\begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}$$

is a solution, and therefore the zero solution is unstable!

The situation is not hopeless. An important example (Hill's equation) where the stability of the zero solution of the differential equation (2.27) can be determined in some cases is discussed in the next section.

Exercise 2.92. (a) Find the principal fundamental matrix solution $\Phi(t)$ at $t = 0$ for the Marcus–Yamabe system; its system matrix $A(t)$ is given in display (2.28). (b) Find the Floquet normal form for $\Phi(t)$ and its "real" Floquet normal form. (c) Determine the characteristic multipliers for the system. (d) The matrix function $t \mapsto A(t)$ is isospectral. Find a matrix function $t \mapsto M(t)$ such that $(A(t), M(t))$ is a Lax pair (see Exercise 2.54). Is every isospectral matrix function the first component of a Lax pair?

The Floquet normal form can be used to obtain detailed information about the solutions of the differential equation (2.27). For example, if we use the fact that the Floquet normal form decomposes a fundamental matrix into a periodic part and an exponential part, then it should be clear that for some systems there are periodic solutions and for others there are no nontrivial periodic solutions. It is also possible to have "quasi-periodic" solutions. The next lemma will be used to prove these facts.

Lemma 2.93. *If μ is a characteristic exponent for the homogeneous linear T-periodic differential equation (2.27) and $\Phi(t)$ is the principal fundamental matrix solution at $t = 0$, then $\Phi(t)$ has a Floquet normal form $P(t)e^{tB}$ such that μ is an eigenvalue of B.*

Proof. Let $\mathcal{P}(t)e^{t\mathcal{B}}$ be a Floquet normal form for $\Phi(t)$. By the definition of characteristic exponents, there is a characteristic multiplier λ such that $\lambda = e^{\mu T}$, and, by Theorem 2.87, there is an eigenvalue ν of $\mathcal{B}$ such that $e^{\nu T} = \lambda$. Also, there is some integer $k \neq 0$ such that $\nu = \mu + 2\pi i k/T$.

Define $B := \mathcal{B} - (2\pi i k/T)I$ and $P(t) = \mathcal{P}(t)e^{(2\pi i k t/T)I}$. Note that μ is an eigenvalue of B, the function P is T-periodic, and

$$P(t)e^{tB} = \mathcal{P}(t)e^{t\mathcal{B}}.$$

It follows that $\Phi(t) = P(t)e^{tB}$ is a representation in Floquet normal form where μ is an eigenvalue of B. □

A basic result that is used to classify the possible types of solutions that can arise is the content of the following theorem.

Theorem 2.94. *If λ is a characteristic multiplier of the homogeneous linear T-periodic differential equation (2.27) and $e^{T\mu} = \lambda$, then there is a (possibly complex) nontrivial solution of the form*

$$x(t) = e^{\mu t}p(t)$$

where p is a T-periodic function. Moreover, for this solution $x(t+T) = \lambda x(t)$.

Proof. Consider the principal fundamental matrix solution $\Phi(t)$ at $t = 0$. By Lemma 2.93, there is a Floquet normal form representation $\Phi(t) = P(t)e^{tB}$ such that μ is an eigenvalue of B. Hence, there is a vector $v \neq 0$ such that $Bv = \mu v$. Clearly, it follows that $e^{tB}v = e^{\mu t}v$, and therefore the solution $x(t) := \Phi(t)v$ is also represented in the form

$$x(t) = P(t)e^{tB}v = e^{\mu t}P(t)v.$$

The solution required by the first statement of the theorem is obtained by defining $p(t) := P(t)v$. The second statement of the theorem is proved as follows:

$$x(t+T) = e^{\mu(t+T)}p(t+T) = e^{\mu T}e^{\mu t}p(t) = \lambda x(t). \qquad \square$$

Theorem 2.95. *Suppose that λ_1 and λ_2 are characteristic multipliers of the homogeneous linear T-periodic differential equation (2.27) and μ_1 and μ_2 are characteristic exponents such that $e^{T\mu_1} = \lambda_1$ and $e^{T\mu_2} = \lambda_2$. If $\lambda_1 \neq \lambda_2$, then there are T-periodic functions p_1 and p_2 such that*

$$x_1(t) = e^{\mu_1 t}p_1(t) \quad \text{and} \quad x_2(t) = e^{\mu_2 t}p_2(t)$$

are linearly independent solutions.

Proof. Let $\Phi(t) = P(t)e^{tB}$ (as in Lemma 2.93) be such that μ_1 is an eigenvalue of B. Also, let v_1 be a nonzero eigenvector corresponding to the eigenvalue μ_1. Since λ_2 is an eigenvalue of the monodromy matrix $\Phi(T)$, by Theorem 2.87 there is an eigenvalue μ of B such that $e^{T\mu} = \lambda_2 = e^{T\mu_2}$. It follows that there is an integer k such that $\mu_2 = \mu + 2\pi i k/T$. Also, because $\lambda_1 \neq \lambda_2$, we have that $\mu \neq \mu_1$. Hence, if v_2 is a nonzero eigenvector of B corresponding to the eigenvalue μ, then the eigenvectors v_1 and v_2 are linearly independent.

As in the proof of Theorem 2.94, there are solutions of the form

$$x_1(t) = e^{\mu_1 t}P(t)v_1, \qquad x_2(t) = e^{\mu t}P(t)v_2.$$

Moreover, because $x_1(0) = v_1$ and $x_2(0) = v_2$, these solutions are linearly independent. Finally, let us note that x_2 can be written in the required form

$$x_2(t) = \big(e^{\mu t}e^{2\pi k i/T}\big)\big(e^{-2\pi k i/T}P(t)v_2\big). \qquad \square$$

The T-periodic system (2.27) has the Floquet normal form

$$t \mapsto Q(t)e^{tR}$$

where Q is a real $2T$-periodic function and R is real matrix. By Theorem 2.36 and 2.88, all solutions of the system are represented as finite sums of real solutions of the two types

$$q(t)r(t)e^{\alpha t}\sin\beta t \quad \text{and} \quad q(t)r(t)e^{\alpha t}\cos\beta t,$$

where q is $2T$-periodic, r is a polynomial of degree at most $n-1$, and $\alpha+i\beta$ is an eigenvalue of R. We will use Theorem 2.94 to give a more detailed description of the nature of these real solutions.

If the characteristic multiplier λ is a positive real number, then there is a corresponding real characteristic exponent μ. In this case, if the periodic function p in Theorem 2.94 is complex, then it can be represented as $p = r + is$ where both r and s are real T-periodic functions. Because our T-periodic system is real, both the real and the imaginary parts of a solution are themselves solutions. Hence, there is a real nontrivial solution of the form $x(t) = e^{\mu t}r(t)$ or $x(t) = e^{\mu t}s(t)$. Such a solution is periodic if and only if $\lambda = 1$ or equivalently if $\mu = 0$. On the other hand, if $\lambda \neq 1$ or $\mu \neq 0$, then the solution is unbounded either as $t \to \infty$ or as $t \to -\infty$. If $\lambda < 1$ (equivalently, $\mu < 0$), then the solution is asymptotic to the zero solution as $t \to \infty$. On the other hand, if $\lambda > 1$ (equivalently, $\mu > 0$), then the solution is unbounded as $t \to \infty$.

If the characteristic multiplier λ is a negative real number, then μ can be chosen to have the form $\nu + \pi i/T$ where ν is real and $e^{T\mu} = \lambda$. Hence, if we again take $p = r + is$, then we have the solution

$$e^{\mu t}p(t) = e^{\nu t}e^{\pi i t/T}(r(t) + is(t))$$

from which real nontrivial solutions are easily constructed. For example, if the real part of the complex solution is nonzero, then the real solution has the form

$$x(t) = e^{\nu t}(r(t)\cos(\pi t/T) - s(t)\sin(\pi t/T)).$$

Such a solution is periodic if and only if $\lambda = -1$ or equivalently if $\nu = 0$. In this case the solution is $2T$-periodic. If $\nu \neq 0$, then the solution is unbounded. If $\nu < 0$, then the solution is asymptotic to zero as $t \to \infty$. On the other hand, if $\nu > 0$, then the solution is unbounded as $t \to \infty$.

If λ is complex, then we have $\mu = \alpha + i\beta$ and there is a solution given by

$$x(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(r(t) + is(t)).$$

Thus, there are real solutions

$$\begin{aligned} x_1(t) &= e^{\alpha t}(r(t)\cos\beta t - s(t)\sin\beta t), \\ x_2(t) &= e^{\alpha t}(r(t)\sin\beta t + s(t)\cos\beta t). \end{aligned}$$

If $\alpha \neq 0$, then both solutions are unbounded. But, if $\alpha < 0$, then these solutions are asymptotic to zero as $t \to \infty$. On the other hand, if $\alpha > 0$, then these solutions are unbounded as $t \to \infty$. If $\alpha = 0$ and there are relatively prime positive integers m and n such that $2\pi m/\beta = nT$, then the solution is nT-periodic. If no such integers exist, then the solution is called *quasi-periodic*.

We will prove in Section 2.4.4 that the stability of a periodic orbit is determined by the stability of the corresponding fixed point of a Poincaré map defined on a Poincaré section that meets the periodic orbit. Generically, the stability of the fixed point of the Poincaré map is determined by the eigenvalues of its derivative at the fixed point. For example, if the eigenvalues of the derivative of the Poincaré map at the fixed point corresponding to the periodic orbit are all inside the unit circle, then the periodic orbit is asymptotically stable. It turns out that the eigenvalues of the derivative of the Poincaré map are closely related to the characteristic multipliers of a time-periodic system, namely, the variational equation along the periodic orbit. We will have much more to say about the general case later. Here we will illustrate the idea for an example where the Poincaré map is easy to compute.

Suppose that

$$\dot{u} = f(u,t), \qquad u \in \mathbb{R}^n \tag{2.29}$$

is a smooth nonautonomous differential equation. If there is some $T > 0$ such that $f(u, t + T) = f(u, t)$ for all $u \in \mathbb{R}^n$ and all $t \in \mathbb{R}$, then the system (2.29) is called T-periodic.

The nonautonomous system (2.29) is made "artificially" autonomous by the addition of a new equation as follows:

$$\dot{u} = f(u,\psi), \qquad \dot{\psi} = 1 \tag{2.30}$$

where ψ may be viewed as an angular variable modulo T. In other words, we can consider $\psi + nT = \psi$ whenever n is an integer. The phase cylinder for system (2.30) is $\mathbb{R}^n \times \mathbb{T}$, where $\mathbb{T}$ (topologically the unit circle) is defined to be $\mathbb{R}$ modulo T. This autonomous system provides the correct geometry with which to define a Poincaré map.

For each $\xi \in \mathbb{R}^n$, let $t \mapsto u(t,\xi)$ denote the solution of the differential equation (2.29) such that $u(0,\xi) = \xi$, and note that $t \mapsto (u(t,\xi), t)$ is the corresponding solution of the system (2.30). The set $\Sigma := \{(\xi,\psi) : \psi = 0\}$ is a Poincaré section, and the corresponding Poincaré map is given by $\xi \mapsto u(T,\xi)$.

If there is a point $p \in \mathbb{R}^n$ such that $f(p,t) = 0$ for all $t \in \mathbb{R}$, then the function $t \mapsto (p,t)$, or equivalently $t \mapsto (u(t,p), t)$, is a periodic solution of the system (2.30) with period T. Moreover, let us note that $u(T,p) = p$. Thus, the periodic solution corresponds to a fixed point of the Poincaré map as it should.

The derivative of the Poincaré map at p is the linear transformation of $\mathbb{R}^n$ given by the partial derivative $u_\xi(T,p)$. Moreover, by differentiating both the differential equation (2.29) and the initial condition $u(0,\xi) = \xi$ with respect to ξ, it is easy to see that the matrix function $t \mapsto u_\xi(t,p)$ is the principal fundamental matrix solution at $t = 0$ of the (T-periodic linear) variational initial value problem

$$\dot{W} = f_u(u(t,p),t)W, \qquad W(0) = I. \tag{2.31}$$

If the solution of system (2.31) is represented in the Floquet normal form $u_\xi(t,p) = P(t)e^{tB}$, then the derivative of the Poincaré map is given by $u_\xi(T,p) = e^{TB}$. In particular, the characteristic multipliers of the variational equation (2.31) coincide with the eigenvalues of the derivative of the Poincaré map. Thus, whenever the principle of linearized stability is valid, the stability of the periodic orbit is determined by the characteristic multipliers of the periodic variational equation (2.31).

As an example, consider the pendulum with oscillating support

$$\ddot{\theta} + (1 + a\cos\omega t)\sin\theta = 0.$$

The zero solution, given by $\theta(t) \equiv 0$, corresponds to a $2\pi/\omega$-periodic solution of the associated autonomous system. A calculation shows that the variational equation along this periodic solution is equivalent to the second order differential equation

$$\ddot{x} + (1 + a\cos\omega t)x = 0,$$

called a *Mathieu equation.* The normal form for the Mathieu equation is

$$\ddot{x} + (a - 2q\cos 2t)x = 0,$$

where a and q are parameters.

Since, as we have just seen (see also Exercise 2.91), equations of Mathieu type arise frequently in applications, the stability analysis of such equations is important (see, for example, [12], [18], [99], [125], [147], and [234]). In Section 2.4.2 we will show how the stability of the zero solution of the Mathieu equation, and, in turn, the stability of the zero solution of the pendulum with oscillating support, is related in a delicate manner to the amplitude a and the frequency ω of the periodic displacement.

Exercise 2.96. This is a continuation of Exercise 2.56. Suppose that $v : \mathbb{R} \to \mathbb{R}^3$ is a periodic function. Consider the differential equation

$$\dot{x} = v(t) \times x$$

and discuss the stability of its periodic solutions.

Exercise 2.97. Determine the stability type of the periodic orbit discussed in Exercise 2.90.

Exercise 2.98. (a) Prove that the system

$$\begin{aligned}
\dot{x} &= x - y - x(x^2 + y^2),\\
\dot{y} &= x + y - y(x^2 + y^2),\\
\dot{z} &= z + xz - z^3
\end{aligned}$$

has periodic orbits. Hint: Change to cylindrical coordinates, show that the cylinder (with radius one whose axis of symmetry is the z-axis) is invariant, and recall the analysis of equation (1.43). (b) Prove that there is a stable periodic orbit. (c) The stable periodic orbit has three Floquet multipliers. Of course, one of them is unity. Find (exactly) a vector v such that $\Phi(T)v = v$, where T is the period of the periodic orbit and $\Phi(t)$ is the principal fundamental matrix solution at $t = 0$ of the variational equation along the stable periodic solution. (d) Approximate the remaining two multipliers. Note: It is possible to represent these multipliers with integrals, but they are easier to approximate using a numerical method.

2.4.1 Lyapunov Exponents

An important generalization of Floquet exponents, called Lyapunov exponents, are introduced in this section. This concept is used extensively in the theory of dynamical systems (see, for example, [101], [146], [173], and [230]).

Consider a (nonlinear) differential equation

$$\dot{u} = f(u), \qquad u \in \mathbb{R}^n \tag{2.32}$$

with flow φ_t. If $\epsilon \in \mathbb{R}$, ξ, $v \in \mathbb{R}^n$, and $\eta := \xi + \epsilon v$, then the two solutions

$$t \mapsto \varphi_t(\xi), \qquad t \mapsto \varphi_t(\xi + \epsilon v)$$

start at points that are $O(\epsilon)$ close; that is, the absolute value of the difference of the two points in $\mathbb{R}^n$ is bounded by the usual norm of v times ϵ. Moreover, by Taylor expansion at $\epsilon = 0$, we have that

$$\varphi_t(\xi + \epsilon v) - \varphi_t(\xi) = \epsilon D\varphi_t(\xi)v + O(\epsilon^2)$$

where $D\varphi_t(\xi)$ denotes the derivative of the function $u \mapsto \varphi_t(u)$ evaluated at $u = \xi$. Thus, the first order approximation of the difference of the solutions at time t is $\epsilon D\varphi_t(\xi)v$ where $t \mapsto D\varphi_t(\xi)$ is the principal fundamental matrix solution at $t = 0$ of the linearized equation

$$\dot{W} = Df(\varphi_t(\xi))W$$

along the solution of the original system (2.32) starting at ξ. To see this fact, just note that

$$\dot{\varphi}_t(u) = f(\varphi_t(u))$$

and differentiate both sides of this identity with respect to u at $u = \xi$.

If we view v as a vector in the tangent space to $\mathbb{R}^n$ at ξ, denoted $T_\xi\mathbb{R}^n$, then $D\varphi_t(\xi)v$ is a vector in the tangent space $T_{\varphi_t(\xi)}\mathbb{R}^n$. For each such v, if $v \neq 0$, then it is natural to define a corresponding linear operator L, from the linear subspace of $T_\xi\mathbb{R}^n$ generated by v to the linear subspace of $T_{\varphi_t(\xi)}\mathbb{R}^n$ generated by $D\varphi_t(\xi)v$, defined by $L(av) = D\varphi_t(\xi)av$ where $a \in \mathbb{R}$. Let us note that the norm of this operator measures the relative "expansion" or "contraction" of the vector v; that is,

$$\|L\| = \sup_{a\neq 0} \frac{|D\phi_t(\xi)av|}{|av|} = \frac{|D\phi_t(\xi)v|}{|v|}.$$

Our two solutions can be expressed in integral form; that is,

$$\varphi_t(\xi) = \xi + \int_0^t f(\varphi_s(\xi))\,ds,$$
$$\varphi_t(\xi+\epsilon v) = \xi + \epsilon v + \int_0^t f(\varphi_s(\xi+\epsilon v))\,ds.$$

Hence, as long as we consider a finite time interval or a solution that is contained in a compact subset of $\mathbb{R}^n$, there is a Lipschitz constant $\operatorname{Lip}(f) > 0$ for the function f, and we have the inequality

$$|\varphi_t(\xi+\epsilon v) - \varphi_t(\xi)| \le \epsilon|v| + \operatorname{Lip}(f)\int_0^t |\varphi_s(\xi+\epsilon v) - \varphi_s(\xi)|\,ds.$$

By Gronwall's inequality, the separation distance between the solutions is bounded by an exponential function of time. In fact, we have the estimate

$$|\varphi_t(\xi+\epsilon v) - \varphi_t(\xi)| \le \epsilon|v|e^{t\operatorname{Lip}(f)}.$$

The above computation for the norm of L and the exponential bound for the separation rate between two solutions motivates the following definition (see [146]).

Definition 2.99. Suppose that $\xi \in \mathbb{R}^n$ and the solution $t \mapsto \varphi_t(\xi)$ of the differential equation (2.32) is defined for all $t \ge 0$. Also, let $v \in \mathbb{R}^n$ be a nonzero vector. The *Lyapunov exponent* at ξ in the direction v for the flow φ_t is defined to be

$$\chi(p,v) := \limsup_{t\to\infty} \frac{1}{t}\ln\Big(\frac{|D\phi_t(\xi)v|}{|v|}\Big).$$

As a simple example, let us consider the planar system

$$\dot{x} = -ax, \qquad \dot{y} = by$$

where a and b are positive parameters, and let us note that its flow is given by

$$\varphi_t(x,y) = (e^{-at}x, e^{bt}y).$$

By an easy computation using the definition of the Lyapunov exponents, it follows that if v is given by $v = (w,z)$ and $z \neq 0$, then $\chi(\xi, v) = b$. If

$z = 0$ and $w \neq 0$, then $\chi(\xi, v) = -a$. In particular, there are exactly two Lyapunov exponents for this system. Of course, the Lyapunov exponents in this case correspond to the eigenvalues of the system matrix.

Although our definition of Lyapunov exponents is for autonomous systems, it should be clear that since the definition only depends on the fundamental matrix solutions of the associated variational equations along orbits of the system, we can define the same notion for solutions of abstract time-dependent linear systems. Indeed, for a T-periodic linear system

$$\dot{u} = A(t)u, \qquad u \in \mathbb{R}^n \tag{2.33}$$

with principal fundamental matrix $\Phi(t)$ at $t = 0$, the Lyapunov exponent defined with respect to the nonzero vector $v \in \mathbb{R}^n$ is

$$\chi(v) := \limsup_{t\to\infty} \frac{1}{t} \ln \Big(\frac{|\Phi(t)v|}{|v|}\Big).$$

Proposition 2.100. *If μ is a Floquet exponent of the system (2.33), then the real part of μ is a Lyapunov exponent.*

Proof. Let us suppose that the principal fundamental matrix $\Phi(t)$ is given in Floquet normal form by

$$\Phi(t) = P(t)e^{tB}.$$

If $\mu = a + bi$ is a Floquet exponent, then there is a corresponding vector v such that $e^{TB}v = e^{\mu T}v$. Hence, using the Floquet normal form, we have that

$$\Phi(T)v = e^{\mu T}v.$$

If $t \geq 0$, then there is a nonnegative integer n and a number r such that $0 \leq r < T$ and

$$\begin{aligned}\frac{1}{t} \ln \Big(\frac{|\Phi(t)v|}{|v|}\Big) &= \frac{1}{T}\Big(\frac{nT}{nT+r}\Big)\Big(\frac{1}{n} \ln \Big(\frac{|P(nT+r)e^{rB}e^{n\mu T}v|}{|v|}\Big)\Big)\\ &= \frac{1}{T}\Big(\frac{nT}{nT+r}\Big)\Big(\frac{1}{n} \ln |e^{nTa}| + \frac{1}{n} \ln \Big(\frac{|P(r)e^{rB}v|}{|v|}\Big)\Big).\end{aligned}$$

Clearly, $n \to \infty$ as $t \to \infty$. Thus, it is easy to see that

$$\lim_{t\to\infty} \frac{1}{T}\Big(\frac{nT}{nT+r}\Big)\Big(\frac{1}{n} \ln |e^{nTa}| + \frac{1}{n} \ln \Big(\frac{|P(r)e^{rB}v|}{|v|}\Big)\Big) = a. \qquad \square$$

Let us suppose that a differential equation has a compact invariant set that contains an orbit whose closure is dense in the invariant set. Then, the existence of a positive Lyapunov exponent for this orbit ensures that nearby orbits tend to separate exponentially fast from the dense orbit. But, since these orbits are confined to a compact invariant set, they must also be bounded. This suggests that each small neighborhood in the invariant set undergoes both stretching and folding as it evolves with the flow. The subsequent kneading of the invariant set due to this stretching and folding would tend to mix the evolving neighborhoods so that they eventually intertwine in a complicated manner. For this reason, the existence of a positive Lyapunov exponent is often taken as a signature of "chaos." While this criterion is not always valid, the underlying idea that the stretching implied by a positive Lyapunov exponent is associated with complex motions is important in the modern theory of dynamical systems.

Exercise 2.101. Show that if two points are on the same orbit, then the corresponding Lyapunov exponents are the same.

Exercise 2.102. Prove the "converse" of Proposition 2.100; that is, every Lyapunov exponent for a time-periodic system is a Floquet exponent.

Exercise 2.103. If $\dot{x} = f(x)$, determine the Lyapunov exponent $\chi(\xi, f(\xi))$.

Exercise 2.104. How many Lyapunov exponents are associated with an orbit of a differential equation in an n-dimensional phase space.

Exercise 2.105. Suppose that x is in the omega limit set of an orbit. Are the Lyapunov exponents associated with x the same as those associated with the original orbit?

Exercise 2.106. In all the examples in this section, the lim sup can be replaced by lim. Are there examples where the superior limit is a finite number, but the limit does not exist? This is (probably) a challenging exercise! For an answer see [146] and [173].

2.4.2 Hill's Equation

A famous example where Floquet theory applies to give good stability results is Hill's equation,

$$\ddot{u} + a(t)u = 0, \qquad a(t+T) = a(t).$$

It was introduced by George W. Hill in his study of the motions of the moon. Roughly speaking, the motion of the moon can be viewed as a harmonic oscillator in a periodic gravitational field. But this model equation arises in many areas of applied mathematics where the stability of periodic motions is an issue. A prime example, mentioned in the previous section, is the stability analysis of small oscillations of a pendulum whose length varies with time.

If we define

$$x := \begin{pmatrix} u \\ \dot{u} \end{pmatrix},$$

then Hill's equation is equivalent to the first order system $\dot{x} = A(t)x$ where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}.$$

We will apply linear systems theory, especially Floquet theory, to analyze the stability of the zero solution of this linear T-periodic system.

The first step in the stability analysis is an application of Liouville's formula (2.17). In this regard, you may recall from your study of scalar second order linear differential equations that if $\ddot{u} + p(t)\dot{u} + q(t)u = 0$ and the Wronskian of the two solutions u_1 and u_2 is defined by

$$W(t) := \det \begin{pmatrix} u_1(t) & u_2(t) \\ \dot{u}_1(t) & \dot{u}_2(t) \end{pmatrix},$$

then

$$W(t) = W(0)e^{-\int_0^t p(s)\,ds}. \tag{2.34}$$

Note that for the equivalent first order system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} x = B(t)x$$

with fundamental matrix $\Psi(t)$, formula (2.34) is a special case of Liouville's formula

$$\det \Psi(t) = \det \Psi(0) e^{\int_0^t \operatorname{tr} B(s) ds}.$$

At any rate, let us apply Liouville's formula to the principal fundamental matrix $\Phi(t)$ at $t = 0$ for Hill's system to obtain the identity $\det \Phi(t) \equiv 1$. Since the determinant of a matrix is the product of the eigenvalues of the matrix, we have an important fact: The product of the characteristic multipliers of the monodromy matrix, $\Phi(T)$, is 1.

Let the characteristic multipliers for Hill's equation be denoted by λ_1 and λ_2 and note that they are roots of the characteristic equation

$$\lambda^2 - (\operatorname{tr} \Phi(T))\lambda + \det \Phi(T) = 0.$$

For notational convenience let us set $2\phi = \operatorname{tr} \Phi(T)$ to obtain the equivalent characteristic equation

$$\lambda^2 - 2\phi\lambda + 1 = 0$$

whose solutions are given by

$$\lambda = \phi \pm \sqrt{\phi^2 - 1}.$$

There are several cases to consider depending on the value of ϕ.

Case 1: If $\phi > 1$, then λ_1 and λ_2 are distinct positive real numbers such that $\lambda_1\lambda_2 = 1$. Thus, we may assume that $0 < \lambda_1 < 1 < \lambda_2$ with $\lambda_1 = 1/\lambda_2$

and there is a real number $\mu > 0$ (a characteristic exponent) such that $e^{T\mu} = \lambda_2$ and $e^{-T\mu} = \lambda_1$. By Theorem 2.94 and Theorem 2.95, there is a fundamental set of solutions of the form

$$e^{-\mu t} p_1(t), \qquad e^{\mu t} p_2(t)$$

where the real functions p_1 and p_2 are T-periodic. In this case, the zero solution is unstable.

Case 2: If $\phi < -1$, then λ_1 and λ_2 are both real and both negative. Also, since $\lambda_1\lambda_2 = 1$, we may assume that $\lambda_1 < -1 < \lambda_2 < 0$ with $\lambda_1 = 1/\lambda_2$. Thus, there is a real number $\mu > 0$ (a characteristic exponent) such that $e^{2T\mu} = \lambda_1^2$ and $e^{-2T\mu} = \lambda_2^2$. As in Case 1, there is a fundamental set of solutions of the form

$$e^{\mu t} q_1(t), \qquad e^{-\mu t} q_2(t)$$

where the real functions q_1 and q_2 are $2T$-periodic. Again, the zero solution is unstable.

Case 3: If $-1 < \phi < 1$, then λ_1 and λ_2 are complex conjugates each with nonzero imaginary part. Since $\lambda_1\bar{\lambda}_1 = 1$, we have that $|\lambda_1| = 1$, and therefore both characteristic multipliers lie on the unit circle in the complex plane. Because both λ_1 and λ_2 have nonzero imaginary parts, one of these characteristic multipliers, say λ_1, lies in the upper half plane. Thus, there is a real number θ with $0 < \theta T < \pi$ and $e^{i\theta T} = \lambda_1$. In fact, there is a solution of the form $e^{i\theta t}(r(t) + is(t))$ with r and s both T-periodic functions. Hence, there is a fundamental set of solutions of the form

$$r(t)\cos\theta t - s(t)\sin\theta t, \quad r(t)\sin\theta t + s(t)\cos\theta t.$$

In particular, the zero solution is stable (see Exercise 2.112) but not asymptotically stable. Also, the solutions are periodic if and only if there are relatively prime positive integers m and n such that $2\pi m/\theta = nT$. If such integers exist, all solutions have period nT. If not, then these solutions are quasi-periodic.

We have just proved the following facts for Hill's equation: *Suppose that $\Phi(t)$ is the principal fundamental matrix solution of Hill's equation at $t = 0$. If $|\operatorname{tr}\Phi(T)| < 2$, then the zero solution is stable. If $|\operatorname{tr}\Phi(T)| > 2$, then the zero solution is unstable.*

Case 4: If $\phi = 1$, then $\lambda_1 = \lambda_2 = 1$. The nature of the solutions depends on the canonical form of $\Phi(T)$. If $\Phi(T)$ is the identity, then $e^0 = \Phi(T)$ and there is a Floquet normal form $\Phi(t) = P(t)$ where $P(t)$ is T-periodic and invertible. Thus, there is a fundamental set of periodic solutions and the zero solution is stable. If $\Phi(T)$ is not the identity, then there is a nonsingular matrix C such that

$$C\Phi(T)C^{-1} = I + N = e^N$$

where $N \neq 0$ is nilpotent. Thus, $\Phi(t)$ has a Floquet normal form $\Phi(t) = P(t)e^{tB}$ where $B := C^{-1}(\frac{1}{T}N)C$. Because

$$e^{tB} = C^{-1}(I + \frac{t}{T}N)C,$$

the matrix function $t \mapsto e^{tB}$ is unbounded, and therefore the zero solution is unstable.

Case 5: If $\phi = -1$, then the situation is similar to Case 4, except the fundamental matrix is represented by $Q(t)e^{tB}$ where $Q(t)$ is a $2T$-periodic matrix function.

By the results just presented, the stability of Hill's equation is reduced, in most cases, to a determination of the absolute value of the trace of its principal fundamental matrix evaluated after one period. While this is a useful fact, it leaves open an important question: Can the stability be determined without imposing a condition on the solutions of the equation? It turns out that in some special cases this is possible (see [147] and [234]). A theorem of Lyapunov [146] in this direction follows.

Theorem 2.107. *If $a : \mathbb{R} \to \mathbb{R}$ is a positive T-periodic function such that*

$$T \int_0^T a(t)\, dt \leq 4,$$

then all solutions of the Hill's equation $\ddot{x} + a(t)x = 0$ are bounded. In particular, the trivial solution is stable.

The proof of Theorem 2.107 is outlined in Exercises 2.112 and 2.115.

Exercise 2.108. Consider the second order system

$$\ddot{u} + \dot{u} + \cos(t)\, u = 0.$$

Prove: (a) If ρ_1 and ρ_2 are the characteristic multipliers of the corresponding first order system, then $\rho_1\rho_2 = \exp(-2\pi)$. (b) The Poincaré map for the system is dissipative; that is, it contracts area.

Exercise 2.109. Prove: The equation

$$\ddot{u} - (2\sin^2 t)\dot{u} + (1 + \sin 2t)u = 0.$$

does not have a fundamental set of periodic solutions. Does it have a nonzero periodic solution? Is the zero solution stable?

Exercise 2.110. Discuss the stability of the trivial solution of the scalar time-periodic system $\dot{x} = (\cos^2 t)x$.

Exercise 2.111. Prove: The zero solution is unstable for the system $\dot{x} = A(t)x$ where

$$A(t) := \begin{pmatrix} 1/2 - \cos t & 12 \\ 147 & 3/2 + \sin t \end{pmatrix}.$$

Exercise 2.112. Prove: If all solutions of the T-periodic system $\dot{x} = A(t)x$ are bounded, then the trivial solution is Lyapunov stable.

Exercise 2.113. For Hill's equation with period T, if the absolute value of the trace of $\Phi(T)$, where $\Phi(t)$ is the principal fundamental matrix at $t = 0$, is strictly less than two, show that there are no solutions of period T or $2T$. On the other hand, if the absolute value of the trace of $\Phi(T)$ is two, show that there is such a solution. Note that this property characterizes the boundary between the stable and unstable solutions.

Exercise 2.114. Prove: If $a(t)$ is an even T-periodic function, then Hill's equation has a fundamental set of solutions such that one solution is even and one is odd.

Exercise 2.115. Prove Theorem 2.107. Hint: If Hill's equation has an unbounded solution, then there is a real solution $t \mapsto x(t)$ and a real Floquet multiplier such that $x(t+T) = \lambda x(t)$. Define a new function $t \mapsto u(t)$ by

$$u(t) := \frac{\dot{x}(t)}{x(t)},$$

and show that u is a solution of the Riccati equation

$$\dot{u} = -a(t) - u^2.$$

Use the Riccati equation to prove that the solution x has at least one zero in the interval $[0, T]$. Also, show that x has two distinct zeros on some interval whose length does not exceed T. Finally, use the following proposition to finish the proof. If f is a smooth function on the finite interval $[\alpha, \beta]$ such that $f(\alpha) = 0$, $f(\beta) = 0$, and such that f is positive on the open interval (α, β), then

$$(\beta - \alpha) \int_\alpha^\beta \frac{|f''(t)|}{f(t)}\, dt > 4.$$

To prove this proposition, first suppose that f attains its maximum at γ and show that

$$\frac{4}{\beta - \alpha} \le \frac{1}{\gamma - \alpha} + \frac{1}{\beta - \gamma} = \frac{1}{f(\gamma)}\Big(\frac{f(\gamma) - f(\alpha)}{\gamma - \alpha} - \frac{f(\beta) - f(\gamma)}{\beta - \gamma}\Big).$$

Then, use the mean value theorem and the fundamental theorem of calculus to complete the proof.

Exercise 2.116. Prove: If $t \mapsto a(t)$ is negative, then the Hill's equation $\ddot{x} + a(t)x = 0$ has an unbounded solution. Hint: Multiply by x and integrate by parts.

2.4.3 Periodic Orbits of Linear Systems

In this section we will consider the existence and stability of periodic solutions of the time-periodic system

$$\dot{x} = A(t)x + b(t), \qquad x \in \mathbb{R}^n \tag{2.35}$$

where $t \mapsto A(t)$ is a T-periodic matrix function and $t \mapsto b(t)$ is a T-periodic vector function.

Theorem 2.117. *If the number one is not a characteristic multiplier of the T-periodic homogeneous system $\dot{x} = A(t)x$, then (2.35) has at least one T-periodic solution.*

Proof. Let us show first that if $t \mapsto x(t)$ is a solution of system (2.35) and $x(0) = x(T)$, then this solution is T-periodic. Define $y(t) := x(t+T)$. Note that $t \mapsto y(t)$ is a solution of (2.35) and $y(0) = x(0)$. Thus, by the uniqueness theorem $x(t+T) = x(t)$ for all $t \in \mathbb{R}$.

If $\Phi(t)$ is the principal fundamental matrix solution of the homogeneous system at $t = 0$, then, by the variation of parameters formula,

$$x(T) = \Phi(T)x(0) + \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds.$$

Therefore, $x(T) = x(0)$ if and only if

$$(I - \Phi(T))x(0) = \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds.$$

This equation for $x(0)$ has a solution whenever the number one is not an eigenvalue of $\Phi(T)$. (Note that the map $x(0) \mapsto x(T)$ is the Poincaré map. Thus, our periodic solution corresponds to a fixed point of the Poincaré map).

By Floquet's theorem, there is a matrix B such that the monodromy matrix is given by

$$\Phi(T) = e^{TB}.$$

In other words, by the hypothesis, the number one is not an eigenvalue of $\Phi(T)$. □

Corollary 2.118. *If $A(t) = A$, a constant matrix such that A is infinitesimally hyperbolic (no eigenvalues on the imaginary axis), then the differential equation (2.35) has at least one T-periodic solution.*

Proof. The monodromy matrix e^{TA} does not have 1 as an eigenvalue. □

Exercise 2.119. Discuss the uniqueness of the T-periodic solutions of the system (2.35). Also, using Theorem 2.88, discuss the stability of the T-periodic solutions.

In system (2.35) if $b = 0$, then the trivial solution is a T-periodic solution. The next theorem states a general sufficient condition for the existence of a T-periodic solution.

Theorem 2.120. *If the T-periodic system (2.35) has a bounded solution, then it has a T-periodic solution.*

Proof. Consider the principal fundamental matrix solution $\Phi(t)$ at $t = 0$ of the homogeneous system corresponding to the differential equation (2.35). By the variation of parameters formula, we have the equation

$$x(T) = \Phi(T)x(0) + \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds.$$

Also, by Theorem 2.81, there is a constant matrix B such that $\Phi(T) = e^{TB}$. Thus, the stroboscopic Poincaré map P is given by

$$\begin{aligned} P(\xi) &:= \Phi(T)\xi + \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds \\ &= e^{TB}\Big(\xi + \int_0^T \Phi^{-1}(s)b(s)\,ds\Big). \end{aligned}$$

If the solution with initial condition $x(0) = \xi_0$ is bounded, then the sequence $\{P^j(\xi_0)\}_{j=0}^{\infty}$ is bounded. Also, P is an affine map; that is, $P(\xi) = L\xi + y$ where $L = e^{TB} = \Phi(T)$ is a real invertible linear map and y is an element of $\mathbb{R}^n$.

Note that if there is a point $x \in \mathbb{R}^n$ such that $P(x) = x$, then the system (2.35) has a periodic orbit. Thus, if we assume that there are no periodic orbits, then the equation

$$(I - L)\xi = y$$

has no solution ξ. In other words, y is not in the range $\mathcal{R}$ of the operator $I - L$.

There is some vector $v \in \mathbb{R}^n$ such that v is orthogonal to $\mathcal{R}$ and the inner product $\langle v, y\rangle$ does not vanish. Moreover, because v is orthogonal to the range, we have

$$\langle (I - L)\xi, v\rangle = 0$$

for each $\xi \in \mathbb{R}^n$, and therefore

$$\langle \xi, v\rangle = \langle L\xi, v\rangle. \tag{2.36}$$

Using the representation $P(\xi) = L\xi + y$ and an induction argument, it is easy to prove that if j is a nonnegative integer, then $P^j(\xi_0) = L^j\xi_0 + \sum_{k=0}^{j-1} L^k y$. By taking the inner product with v and repeatedly applying the reduction formula (2.36), we have

$$\langle P^j(\xi_0), v\rangle = \langle \xi_0, v\rangle + (j - 1)\langle y, v\rangle.$$

Moreover, because $\langle v, y\rangle \neq 0$, it follows immediately that

$$\lim_{j\to\infty} \langle P^j(\xi_0), v\rangle = \infty,$$

and therefore the sequence $\{P^j(\xi_0)\}_{j=0}^{\infty}$ is unbounded, in contradiction. □

2.4.4 Stability of Periodic Orbits

Consider a (nonlinear) autonomous system of differential equations on $\mathbb{R}^n$ given by $\dot{u} = f(u)$ with a periodic orbit Γ. Also, for each $\xi \in \mathbb{R}^n$, define the vector function $t \mapsto u(t, \xi)$ to be the solution of this system with the initial condition $u(0, \xi) = \xi$.

If $p \in \Gamma$ and $\Sigma' \subset \mathbb{R}^n$ is a section transverse to $f(p)$ at p, then, as a corollary of the implicit function theorem, there is an open set $\Sigma \subseteq \Sigma'$ and a function $T : \Sigma \to \mathbb{R}$, the time of first return to Σ', such that for each $\sigma \in \Sigma$, we have $u(T(\sigma), \sigma) \in \Sigma'$. The map $\mathcal{P}$, given by $\sigma \mapsto u(T(\sigma), \sigma)$, is the Poincaré map corresponding to the Poincaré section Σ.

The Poincaré map is defined only on Σ, a manifold contained in $\mathbb{R}^n$. It is convenient to avoid choosing local coordinates on Σ. Thus, we will view the elements in Σ also as points in the ambient space $\mathbb{R}^n$. In particular, if $v \in \mathbb{R}^n$ is tangent to Σ at p, then the derivative of $\mathcal{P}$ in the direction v is given by

$$D\mathcal{P}(p)v = (dT(p)v)f(p) + u_\xi(T(p), p)v. \tag{2.37}$$

The next proposition relates the spectrum of $D\mathcal{P}(p)$ to the Floquet multipliers of the first variational equation

$$\dot{W} = Df(u(t, p))W.$$

Proposition 2.121. *If Γ is a periodic orbit and $p \in \Gamma$, then the union of the set of eigenvalues of the derivative of a Poincaré map at $p \in \Gamma$ and the singleton set $\{1\}$ is the same as the set of characteristic multipliers of the first variational equation along Γ. In particular, zero is not an eigenvalue.*

Proof. Recall that $t \mapsto u_\xi(t, \xi)$ is the principal fundamental matrix solution at $t = 0$ of the first variational equation and, since

$$\frac{d}{dt} f(u(t, \xi)) = Df(u(t, \xi)u_t(t, \xi) = Df(u(t, \xi)f(u(t, \xi)),$$

the vector function $t \mapsto f(u(t, \xi))$ is the solution of the variational equation with the initial condition $W(0) = f(\xi)$. In particular,

$$u_\xi(T(p), p)f(p) = f(u(T(p), p)) = f(p),$$

and therefore $f(p)$ is an eigenvector of the linear transformation $u_\xi(T(p), p)$ with eigenvalue the number one.

Since Σ is transverse to $f(p)$, there is a basis of $\mathbb{R}^n$ of the form

$$f(p), s_1, \ldots, s_{n-1}$$

with s_i tangent to Σ at p for each $i = 1, \ldots, n-1$. It follows that the matrix $u_\xi(T(p), p)$ has block form, relative to this basis, given by

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

where a is $1 \times (n-1)$ and b is $(n-1) \times (n-1)$. Moreover, each $v \in \mathbb{R}^n$ that is tangent to Σ at p has block form (the transpose of) $(0, v_\Sigma)$. As a result, we have the equality

$$u_\xi(T(p), p)v = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ v_\Sigma \end{pmatrix}.$$

The range of $D\mathcal{P}(p)$ is tangent to Σ at p. Thus, using equation (2.37) and the block form of $u_\xi(T(p), p)$, it follows that

$$D\mathcal{P}(p)v = \begin{pmatrix} dT(p)v + av_\Sigma \\ bv_\Sigma \end{pmatrix} = \begin{pmatrix} 0 \\ bv_\Sigma \end{pmatrix}.$$

In other words, the derivative of the Poincaré map may be identified with b and the differential of the return time map with $-a$. In particular, the eigenvalues of the derivative of the Poincaré map coincide with the eigenvalues of b. □

Exercise 2.122. Prove that the characteristic multipliers of the first variational equation along a periodic orbit do not depend on the choice of $p \in \Gamma$.

Most of the rest of this section is devoted to a proof of the following fundamental theorem.

Theorem 2.123. *Suppose that Γ is a periodic orbit for the autonomous differential equation $\dot{u} = f(u)$ and $\mathcal{P}$ is a corresponding Poincaré map defined on a Poincaré section Σ such that $p \in \Gamma \cap \Sigma$. If the eigenvalues of the derivative $D\mathcal{P}(p)$ are inside the unit circle in the complex plane, then Γ is asymptotically stable.*

There are several possible proofs of this theorem. The approach used here is adapted from [121].

To give a complete proof of Theorem 2.123, we will require several preliminary results. Our first objective is to show that the point p is an asymptotically stable fixed point of the dynamical system defined by the Poincaré map on Σ.

Let us begin with a useful simple replacement of the Jordan normal form theorem that is adequate for our purposes here (see [127]).

Proposition 2.124. *An $n \times n$ (possibly complex) matrix A is similar to an upper triangular matrix whose diagonal elements are the eigenvalues of A.*

Proof. Let v be a nonzero eigenvector of A corresponding to the eigenvalue λ. The vector v can be completed to a basis of $\mathbb{C}^n$ that defines a matrix

Q partitioned by the corresponding column vectors $Q := [v, y_1, \dots, y_{n-1}]$. Moreover, Q is invertible and

$$[Q^{-1}v, Q^{-1}y_1, \dots, Q^{-1}y_{n-1}] = [\mathbf{e}_1, \dots, \mathbf{e}_n]$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the usual basis elements.

Note that

$$\begin{aligned} Q^{-1}AQ &= Q^{-1}[\lambda v, Ay_1, \dots, Ay_{n-1}] \\ &= [\lambda \mathbf{e}_1, Q^{-1}Ay_1, \dots, Q^{-1}Ay_{n-1}]. \end{aligned}$$

In other words, the matrix $Q^{-1}AQ$ is given in block form by

$$Q^{-1}AQ = \begin{pmatrix} \lambda & * \\ 0 & \tilde{A} \end{pmatrix}$$

where $\tilde{A}$ is an $(n-1)\times(n-1)$ matrix. In particular, this proves the theorem for all 2×2 matrices.

By induction, there is an $(n-1) \times (n-1)$ matrix $\tilde{R}$ such that $\tilde{R}^{-1}\tilde{A}\tilde{R}$ is upper triangular. The matrix $(QR)^{-1}AQR$ where

$$R = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{R} \end{pmatrix}$$

is an upper triangular matrix with the eigenvalues of A as its diagonal elements, as required. □

Let $\rho(A)$ denote the *spectral radius* of A, that is, the maximum modulus of the eigenvalues of A.

Proposition 2.125. *Suppose that A is an $n \times n$ matrix. If $\epsilon > 0$, then there is a norm on $\mathbb{C}^n$ such that $\|A\|_\epsilon < \rho(A) + \epsilon$. If A is a real matrix, then the restriction of the "ϵ-norm" to $\mathbb{R}^n$ is a norm on $\mathbb{R}^n$ with the same property.*

Proof. The following proof is adapted from [127]. By Proposition 2.124, there is a matrix Q such that

$$QAQ^{-1} = D + N$$

where D is diagonal with the eigenvalues of A as its diagonal elements, and N is upper triangular with each of its diagonal elements equal to zero.

Let $\mu > 0$, and define a new diagonal matrix S with diagonal elements

$$1, \mu^{-1}, \mu^{-2}, \dots, \mu^{1-n}.$$

A computation shows that

$$S(D + N)S^{-1} = D + SNS^{-1}.$$

Also, it is easy to show—by writing out the formulas for the components—that every element of the matrix SNS^{-1} is $O(\mu)$.

Define a norm on $\mathbb{C}^n$, by the formula

$$|v|_\mu := |SQv| = \langle SQv, SQv\rangle$$

where the angle brackets on the right hand side denote the usual Euclidean inner product on $\mathbb{C}^n$. It is easy to verify that this procedure indeed defines a norm on $\mathbb{C}^n$ that depends on the parameter μ.

Post multiplication by SQ of both sides of the equation

$$SQAQ^{-1}S^{-1} = D + SNS^{-1}$$

yields the formula

$$SQA = (D + SNS^{-1})SQ.$$

Using this last identity we have that

$$|Av|_\mu^2 = |SQAv|^2 = |(D + SNS^{-1})SQv|^2.$$

Let us define $w := SQv$ and then expand the last norm into inner products to obtain

$$\begin{aligned}|Av|_\mu^2 = \langle Dw, Dw\rangle &+ \langle SNS^{-1}w, Dw\rangle \\ &+\langle Dw, SNS^{-1}w\rangle + \langle SNS^{-1}w, SNS^{-1}w\rangle.\end{aligned}$$

A direct estimate of the first inner product together with an application of the Schwarz inequality to each of the other inner products yields the following estimate:

$$|Av|_\mu^2 \le (\rho^2(A) + O(\mu))|w|^2.$$

Moreover, we have that $|v|_\mu = |w|$. In particular, if $|v|_\mu = 1$ then $|w| = 1$, and it follows that

$$\|A\|_\mu^2 \le \rho^2(A) + O(\mu).$$

Thus, if $\mu > 0$ is sufficiently small, then $\|A\|_\mu < \rho(A) + \epsilon$, as required. □

Corollary 2.126. *If all the eigenvalues of the $n \times n$ matrix A are inside the unit circle in the complex plane, then there is an "adapted norm" and a number λ, with $0 < \lambda < 1$, such that $|Av|_a < \lambda|v|_a$ for all vectors v, real or complex. In particular A is a contraction with respect to the adapted norm. Moreover, for each norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, there is a positive number C such that $|A^n v| \le C\lambda^n|v|$ for all nonnegative integers n.*

Proof. Under the hypothesis, we have $\rho(A) < 1$; thus, there is a number λ such that $\rho(A) < \lambda < 1$. Using Proposition 2.125, there is an adapted norm so that $\|A\|_a < \lambda$. This proves the first part of the corollary. To prove the second part, recall that all norms on a finite dimensional space are equivalent. In particular, there are positive numbers C_1 and C_2 such that

$$C_1|v| \leq |v|_a \leq C_2|v|$$

for all vectors v. Thus, we have

$$C_1|A^n v| \leq |A^n v|_a \leq |A|_a^n |v|_a \leq C_2 \lambda^n |v|.$$

After dividing both sides of the last inequality by $C_1 > 0$, we obtain the desired estimate. □

We are now ready to return to the dynamics of the Poincaré map $\mathcal{P}$ defined above. Recall that Γ is a periodic orbit for the differential equation $\dot{u} = f(u)$ and $\mathcal{P} : \Sigma \to \Sigma'$ is defined by $\mathcal{P}(\sigma) = u(T(\sigma), \sigma)$ where T is the return time function. Also, we have that $p \in \Gamma \cap \Sigma$.

Lemma 2.127. *Suppose that $V \subseteq \mathbb{R}^n$ is an open set with compact closure $\bar{V}$ such that $\Gamma \subset V$ and $\bar{V}$ is contained in the domain of the function f. If $t_* \geq 0$, then there is an open set $W \subseteq V$ that contains Γ and is such that, for each point $\xi \in W$, the solution $t \mapsto u(t, \xi)$ is defined and stays in V on the interval $0 \leq t \leq t_*$. Moreover, if ξ and ν are both in W and $0 \leq t \leq t_*$, then there is a number $L > 0$ such that*

$$|u(t, \xi) - u(t, \nu)| < |\xi - \nu| e^{L t_*}.$$

Proof. Note that $\bar{V}$ is a compact subset of the domain of the function f. By Lemma 2.74, f is globally Lipschitz on V with a Lipschitz constant $L > 0$. Also, there is a minimum *positive* distance m from the boundary of V to Γ.

An easy application of Gronwall's inequality can be used to show that if $\xi, \nu \in V$, then

$$|u(t, \xi) - u(t, \nu)| \leq |\xi - \nu| e^{Lt} \tag{2.38}$$

for all t such that both solutions are defined on the interval $[0, t]$.

Define the set

$$W_q := \{\xi \in \mathbb{R}^n : |\xi - q| e^{L t_*} < m\}$$

and note that W_q is open. If $\xi \in W_q$, then

$$|\xi - q| < m e^{-L t_*} < m.$$

Thus, it follows that $W_q \subseteq V$.

Using the extension theorem (Theorem 1.263), it follows that if $\xi \in W_q$, then the interval of existence of the solution $t \mapsto u(t, \xi)$ can be extended as long as the orbit stays in the compact set $\bar{V}$. The point q is on the periodic orbit Γ. Thus, the solution $t \to u(t, q)$ is defined for all $t \geq 0$. Using the definition of W_q and an application of the inequality (2.38) to the solutions starting at ξ and q, it follows that the solution $t \mapsto u(t, \xi)$ is defined and stays in V on the interval $0 \leq t \leq t_*$.

The union $W := \bigcup_{q \in \Gamma} W_q$ is an open set in V containing Γ with the property that all solutions starting in W remain in V at least on the time interval $0 \leq t \leq t_*$. □

Define the distance of a point $q \in \mathbb{R}^n$ to a set $S \subseteq \mathbb{R}^n$ by

$$\operatorname{dist}(q, S) = \inf_{x \in S} |q - x|$$

where the norm on the right hand side is the usual Euclidean norm. Similarly, the (minimum) distance between two sets is defined as

$$\operatorname{dist}(A, B) = \inf\{|a - b| : a \in A, b \in B\}.$$

(Warning: dist is not a metric.)

Proposition 2.128. *If $\sigma \in \Sigma$ and if $\lim_{n \to \infty} \mathcal{P}^n(\sigma) = p$, then*

$$\lim_{t \to \infty} \operatorname{dist}(u(t, \sigma), \Gamma) = 0.$$

Proof. Let $\epsilon > 0$ be given and let Σ_0 be an open subset of Σ such that $p \in \Sigma_0$ and such that $\bar{\Sigma}_0$, the closure of Σ_0, is a compact subset of Σ. The return time map T is continuous; hence, it is uniformly bounded on the set $\bar{\Sigma}_0$, that is,

$$\sup\{T(\eta) : \eta \in \bar{\Sigma}_0\} = T^* < \infty.$$

Let V be an open subset of $\mathbb{R}^n$ with compact closure $\bar{V}$ such that $\Gamma \subset V$ and $\bar{V}$ is contained in the domain of f. By Lemma 2.127, there is an open set $W \subseteq V$ such that $\Gamma \subset W$ and such that, for each $\xi \in W$, the solution starting at ξ remains in V on the interval $0 \leq s \leq T^*$.

Choose $\delta > 0$ so small that the set

$$\Sigma_\delta := \{\eta \in \Sigma : |\eta - p| < \delta\}$$

is contained in $W \cap \Sigma_0$, and such that

$$|\eta - p| e^{LT^*} < \min\{m, \epsilon\}$$

for all $\eta \in \Sigma_\delta$. By Lemma 2.127, if $\eta \in \Sigma_\delta$, then, for $0 \leq s \leq T^*$, we have that

$$|u(s, \eta) - u(s, p)| < \epsilon.$$

By the hypothesis, there is some integer $N > 0$ such that $\mathcal{P}^n(\sigma) \in \Sigma_\delta$ whenever $n \geq N$.

Using the group property of the flow, let us note that

$$\mathcal{P}^n(\sigma) = u(\sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)), \sigma).$$

Moreover, if $t \geq \sum_{j=0}^{N-1} T(\mathcal{P}^j(\sigma))$, then there is some integer $n \geq N$ and some number s such that $0 \leq s \leq T^*$ and

$$t = \sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)) + s.$$

For this t, we have $\mathcal{P}^n(\sigma) \in \Sigma_\delta$ and

$$\begin{aligned}
\operatorname{dist}(u(t,\sigma),\Gamma) &= \min_{q\in\Gamma} |u(t,\sigma) - q| \\
&\leq |u(t,\sigma) - u(s,p)| \\
&= |u(s, u(\sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)), \sigma)) - u(s,p)| \\
&= |u(s, P^n(\sigma)) - u(s,p)|.
\end{aligned}$$

It follows that $\operatorname{dist}(u(t,\sigma),\Gamma) < \epsilon$ whenever $t \geq \sum_{j=0}^{N-1} T(\mathcal{P}^j(\sigma))$. In other words,

$$\lim_{t\to\infty} \operatorname{dist}(u(t,\sigma),\Gamma) = 0,$$

as required. □

We are now ready for the proof of Theorem 2.123.

Proof. Suppose that V is a neighborhood of Γ. We must prove that there is a neighborhood U of Γ such that $U \subseteq V$ with the additional property that every solution of $\dot{u} = f(u)$ that starts in U stays in V and is asymptotic to Γ.

We may as well assume that V has compact closure $\bar{V}$ and $\bar{V}$ is contained in the domain of f. Then, by Lemma 2.127, there is an open set W that contains Γ and is contained in the closure of V with the additional property that every solution starting in W exists and stay in V on the time interval $0 \leq t \leq 2\tau$ where $\tau := T(p)$ is the period of Γ.

Also, let us assume without loss of generality that our Poincaré section Σ is a subset of a hyperplane Σ' and that the coordinates on Σ' are chosen so that p lies at the origin. By our hypothesis, the linear transformation $D\mathcal{P}(0) : \Sigma' \to \Sigma'$ has its spectrum inside the unit circle in the complex plane. Thus, by Corollary 2.126, there is an adapted norm on Σ' and a number λ with $0 < \lambda < 1$ such that $\|D\mathcal{P}(0)\| < \lambda$.

Using the continuity of the map $\sigma \to D\mathcal{P}(\sigma)$, the return time map, and the adapted norm, there is an open ball $\Sigma_0 \subseteq \Sigma$ centered at the origin such that $\Sigma_0 \subset W$, the return time map T restricted to Σ_0 is bounded by 2τ, and $\|D\mathcal{P}(\sigma)\| < \lambda$ whenever $\sigma \in \Sigma_0$. Moreover, using the mean value theorem, it follows that

$$|\mathcal{P}(\sigma)| = |\mathcal{P}(\sigma) - \mathcal{P}(0)| < \lambda|\sigma|,$$

whenever $\sigma \in \Sigma_0$. In particular, if $\sigma \in \Sigma_0$, then $\mathcal{P}(\sigma) \in \Sigma_0$.

Let us show that all solutions starting in Σ_0 are defined for all positive time. To see this, consider $\sigma \in \Sigma_0$ and note that, by our construction, the solution $t \mapsto u(t, \sigma)$ is defined for $0 \le t \le T(\sigma)$ because $T(\sigma) < 2\tau$. We also have that $u(T(\sigma), \sigma) = \mathcal{P}(\sigma) \in \Sigma_0$. Thus, the solution $t \mapsto u(t, \sigma)$ can be extended beyond the time $T(\sigma)$ by applying the same reasoning to the solution $t \to u(t, \mathcal{P}(\sigma)) = u(t + u(T\sigma), \sigma))$. This procedure can be extended indefinitely, and thus the solution $t \to u(t, \sigma)$ can be extended for all positive time.

Define $U := \{u(t, \sigma) : \sigma \in \Sigma_0 \text{ and } t > 0\}$. Clearly, $\Gamma \subset U$ and also every solution that starts in U stays in U for all $t \ge 0$. We will show that U is open. To prove this fact, let $\xi := u(t, \sigma) \in U$ with $\sigma \in \Sigma_0$. If we consider the restriction of the flow given by $u : (0, \infty) \times \Sigma_0 \to U$, then, using the same idea as in the proof of the rectification lemma (Lemma 1.120), it is easy to see that the derivative $Du(t, \sigma)$ is invertible. Thus, by the inverse function theorem (Theorem 1.121), there is an open set in U at ξ diffeomorphic to a product neighborhood of (t, σ) in $(0, \infty) \times \Sigma_0$. Thus, U is open.

To show that $U \subseteq V$, let $\xi := u(t, \sigma) \in U$ with $\sigma \in \Sigma_0$. There is some integer $n \ge 0$ and some number s such that

$$t = \sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)) + s$$

where $0 \le s < T(\mathcal{P}^n(\sigma)) < 2\tau$. In particular, we have that $\xi = u(s, \mathcal{P}^n(\sigma))$. But since $0 \le s < 2\tau$ and $\mathcal{P}^n(\sigma) \in W$ it follows that $\xi \in V$.

Finally, for this same $\xi \in U$, we have as an immediate consequence of Proposition 2.128 that $\lim_{t\to\infty} \operatorname{dist}(u(t, \mathcal{P}^n(\xi)), \Gamma) = 0$. Moreover, for each $t \ge 0$, we also have that

$$\operatorname{dist}(u(t, \xi), \Gamma) = \operatorname{dist}(u(t, u(s, \mathcal{P}^n(\xi))), \Gamma) = \operatorname{dist}(u(s + t, \mathcal{P}^n(\xi)), \Gamma).$$

It follows that $\lim_{t\to\infty} \operatorname{dist}(u(t, \xi), \Gamma) = 0$, as required. □

A useful application of our results can be made for a periodic orbit Γ of a differential equation defined on the plane. In fact, there are exactly two characteristic multipliers of the first variational equation along Γ. Since one of these characteristic multipliers must be the number one, the product of the characteristic multipliers is the eigenvalue of the derivative of every Poincaré map defined on a section transverse to Γ. Because the determinant of a matrix is the product of its eigenvalues, an application of Liouville's formula proves the following proposition.

Proposition 2.129. *If Γ is a periodic orbit of period ν of the autonomous differential equation $\dot{u} = f(u)$ on the plane, and if $\mathcal{P}$ is a Poincaré map defined at $p \in \Gamma$, then, using the notation of this section, the eigenvalue λ_Γ of the derivative of $\mathcal{P}$ at p is given by*

$$\lambda_\Gamma = \det u_\xi(T(p), p) = e^{\int_0^\nu \operatorname{div} f(u(t,p))\, dt}.$$

In particular, if $\lambda_\Gamma < 1$ then Γ is asymptotically stable, whereas if $\lambda_\Gamma > 1$ then Γ is unstable.

The flow near an attracting limit cycle is very well understood. A next proposition states that the orbits of points in the basin of attraction of the limit cycle are "asymptotically periodic."

Proposition 2.130. *Suppose that Γ is an asymptotically stable periodic orbit with period T. There is a neighborhood V of Γ such that if $\xi \in V$, then $\lim_{t\to\infty} |u(t+T,\xi) - u(t,\xi)| = 0$ where $|\ |$ is an arbitrary norm on $\mathbb{R}^n$. (In this case, the point ξ is said to have asymptotic period T.)*

Proof. By Lemma 2.127, there is an open set W such that $\Gamma \subset W$ and the function $\xi \mapsto u(T,\xi)$ is defined for each $\xi \in W$. Using the continuity of this function, there is a number $\delta > 0$ such that $\delta < \epsilon/2$ and

$$|u(T,\xi) - u(T,\eta)| < \frac{\epsilon}{2}$$

whenever $\xi, \eta \in W$ and $|\xi - \eta| < \delta$.

By the hypothesis, there is a number T^* so large that $\operatorname{dist}(u(t,\xi),\Gamma) < \delta$ whenever $t \geq T^*$. In particular, for each $t \geq T^*$, there is some $q \in \Gamma$ such that $|u(t,\xi) - q| < \delta$. Using this fact and the group property of the flow, we have that

$$\begin{aligned} |u(t+T,\xi) - u(t,\xi)| &\leq |u(T,u(t,\xi)) - u(T,q)| + |q - u(t,\xi)| \\ &\leq \frac{\epsilon}{2} + \delta < \epsilon \end{aligned}$$

whenever $t \geq T^*$. Thus, $\lim_{t\to\infty} |u(t+T,\xi) - u(t,\xi)| = 0$, as required. □

A periodic orbit can be asymptotically stable without being hyperbolic. In fact, it is easy to construct a limit cycle in the plane that is asymptotically stable whose Floquet multiplier is the number one. By the last proposition, points in the basin of attraction of such an attracting limit cycle have asymptotic periods equal to the period of the limit cycle. But, if the periodic orbit is hyperbolic, then a stronger result is true: Not only does each point in the basin of attraction have an asymptotic period, each such point has an asymptotic phase. This is the content of the next result.

Theorem 2.131. *If* Γ *is an attracting hyperbolic periodic orbit, then there is a neighborhood* V *of* Γ *such that for each* $\xi \in V$ *there is some* $q \in \Gamma$ *such that* $\lim_{t\to\infty} |u(t,\xi) - u(t,q)| = 0$. *(In this case,* ξ *is said to have asymptotic phase* q*.)*

Proof. Let Σ be a Poincaré section at $p \in \Gamma$ such that Σ has compact closure. Moreover, let us suppose, without loss of generality, that Σ has the following additional properties: If $\sigma \in \Sigma$, then (1) $\lim_{n\to\infty} \mathcal{P}^n(\sigma) = p$; (2) $T(\sigma) < 2T(p)$; and (3) $\|DT(\sigma)\| < 2\|DT(p)\|$ where T is the return time function on Σ.

Using the implicit function theorem, it is easy to construct a neighborhood V of Γ such that for each $\xi \in V$, there is a number $t_\xi \geq 0$ with $\sigma_\xi := u(t_\xi, \xi) \in \Sigma$. Moreover, using Lemma 2.127, we can choose V such that every solution with initial point in V is defined on the time interval $0 \leq t \leq 2T(p)$ where $T(p)$ is the period of Γ.

We will show that if $\sigma_\xi \in \Sigma$, then there is a point $q_\xi \in \Gamma$ such that

$$\lim_{t\to\infty} |u(t,\sigma_\xi) - u(t,q_\xi)| = 0.$$

Using this fact, it follows that if $r := u(-t_\xi, q_\xi)$, then

$$\begin{aligned}\lim_{t\to\infty} |u(t,\xi) - u(t,r)| &= \lim_{t\to\infty} |u(t - t_\xi, u(t_\xi,\xi)) - u(t - t_\xi, q_\xi)| \\ &= \lim_{t\to\infty} |u(t - t_\xi, \sigma_\xi) - u(t - t_\xi, q_\xi)| = 0.\end{aligned}$$

Thus, it suffices to prove the theorem for a point $\sigma \in \Sigma$.

For a point $\sigma \in \Sigma$, consider the sequence $\{u(nT(p),\sigma)\}_{n=0}^\infty$ and note that if $n \geq 0$, then there is some number s_n such that

$$nT(p) = \sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)) + s_n$$

with $0 \leq s_n \leq T(\mathcal{P}^n(\sigma)) \leq 2T(p)$, and therefore

$$u(nT(p),\sigma) = u(s_n, \mathcal{P}^n(\sigma)).$$

Moreover, we have that

$$(n+1)T(p) - nT(p) = T(\mathcal{P}^n(\sigma)) + s_{n+1} - s_n,$$

and, as a result,

$$|s_{n+1} - s_n| = |T(p) - T(\mathcal{P}^n(\sigma))| \leq 2\|DT(p)\||p - \mathcal{P}^n(p)|.$$

By the hyperbolicity hypothesis, the spectrum of $D\mathcal{P}(p)$ is inside the unit circle, and therefore there is a number λ and a positive constant C such that $0 < \lambda < 1$ and

$$|p - \mathcal{P}^n(\sigma)| < C\lambda^n |p - \sigma|.$$

(Here we could use an adapted norm to make the computations more elegant, but perhaps less instructive.) Hence, there is a positive constant C_1 such that

$$|s_{n+1} - s_n| < C_1 \lambda^n$$

whenever $n \geq 0$.

Note that because $s_n = s_1 + \sum_{j=1}^{n-1}(s_{j+1} - s_j)$ and

$$\sum_{j=1}^{n-1} |s_{j+1} - s_j| < C_1 \sum_{j=1}^{n-1} \lambda^j < C_1 \frac{1}{1-\lambda},$$

the series $\sum_{j=1}^{\infty}(s_{j+1} - s_j)$ is absolutely convergent—its absolute partial sums form an increasing sequence that is bounded above. Thus, in fact, there is a number s such that $\lim_{n\to\infty} s_n = s$. Also, $0 \leq s \leq 2T(p)$.

Let $\epsilon > 0$ be given. By the compactness of its domain, the function

$$u : [0, 2T(p)] \times \bar{\Sigma} \to \mathbb{R}^n$$

is uniformly continuous. In particular, there is a number $\delta > 0$ such that if (t_1, σ_1) and (t_2, σ_2) are both in the domain and if $|t_1 - t_2| + |\sigma_1 - \sigma_2| < \delta$, then

$$|u(t_1, \sigma_1) - u(t_2, \sigma_2)| < \epsilon.$$

In view of the equality

$$|u(nT(p), \sigma) - u(s, p)| = |u(s_n, \mathcal{P}^n(\sigma)) - u(s, p)|$$

and the implication that if n is sufficiently large, then

$$|s_n - s| + |\mathcal{P}^n(\sigma) - p| < \epsilon,$$

it follows that

$$\lim_{n\to\infty} |u(nT(p), \sigma) - u(s, p)| = 0.$$

Also, for each $t \geq 0$, there is an integer $n \geq 0$ and a number $s(t)$ such that $0 \leq s(t) < T(p)$ and $t = nT(p) + s(t)$. Using this fact, we have the equation

$$|u(t, \sigma) - u(t, u(s, p))| = |u(s(t), u(nT(p), \sigma)) - u(s(t), u(nT(p), u(s, p))|.$$

Also, because $q := u(s, p) \in \Gamma$ and Lemma 2.127, there is a constant $L > 0$ such that

$$\begin{aligned}|u(t,\sigma) - u(t,q)| &= |u(s(t), u(nT(p),\sigma)) - u(s(t),q))| \\ &\le |u(nT(p),\sigma) - q|e^{LT(p)}.\end{aligned}$$

By passing to the limit as $n \to \infty$, we obtain the desired result. □

Necessary and sufficient conditions for the existence of asymptotic phase are known (see [47, 76]). An alternate proof of Theorem 2.131 is given in [47].

Exercise 2.132. Find a periodic solution of the system

$$\begin{aligned}\dot{x} &= x - y - x(x^2 + y^2), \\ \dot{y} &= x + y - y(x^2 + y^2), \\ \dot{z} &= -z,\end{aligned}$$

and determine its stability type. In particular, compute the Floquet multipliers for the monodromy matrix associated with the periodic orbit [126, p. 120].

Exercise 2.133. (a) Find an example of a planar system with a limit cycle such that some nearby solutions do not have an asymptotic phase. (b) Contrast and compare the asymptotic phase concept for the following planar systems that are defined in the punctured plane in polar coordinates:

1. $\dot{r} = r(1 - r), \quad \dot{\theta} = r,$
2. $\dot{r} = r(1 - r)^2, \quad \dot{\theta} = r,$
3. $\dot{r} = r(1 - r)^n, \quad \dot{\theta} = r.$

Exercise 2.134. Suppose that $v \neq 0$ is an eigenvector for the monodromy operator with associated eigenvalue λ_Γ as in Proposition 2.129. If $\lambda_\Gamma \neq 1$, then v and $f(p)$ are independent vectors that form a basis for $\mathbb{R}^2$. The monodromy operator expressed in this basis is diagonal. (a) Express the operators a and b defined in the proof of Proposition 2.121 in this basis. (b) What can you say about the derivative of the transit time map along a section that is tangent to v at p?

Exercise 2.135. This exercise is adapted from [232]. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and $A := \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ is a regular level set of f. (a) Prove that each bounded component of A is an attracting hyperbolic limit cycle for the differential equation

$$\dot{x} = -f_y - f f_x, \qquad \dot{y} = f_x - f f_y.$$

(b) Prove that the bounded components of A are the only periodic orbits of the system. (c) Draw and explain the phase portrait of the system for the case where

$$f(x, y) = ((x - \epsilon)^2 + y^2 - 1)(x^2 + y^2 - 9).$$

Exercise 2.136. Consider an attracting hyperbolic periodic orbit Γ for an autonomous system $\dot{u} = f(u)$ with flow φ_t, and for each point $p \in \Gamma$, let Γ_p denote the set of all points in the phase space with asymptotic phase p. (a) Construct Γ_p for each p on the limit cycle in the planar system

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

(b) Repeat the construction for the planar systems of Exercise 2.133. (c) Prove that $\mathcal{F} := \bigcup_{p\in\Gamma} \Gamma_p$ is an invariant foliation of the phase space in a neighborhood U of Γ. Let us take this to mean that every point in U is in one of the sets in the union $\mathcal{F}$ and the following invariance property is satisfied: If $\xi \in \Gamma_p$ and $s \in \mathbb{R}$, then $\varphi_s(\xi) \in \Gamma_{\varphi_s(p)}$. The second condition states that the flow moves fibers of the foliation (Γ_p is the fiber over p) to fibers of the foliation. (d) Are the fibers of the foliation smooth manifolds?