1

Complex Numbers

The most compact equation in all of mathematics is surely

$$e^{i\pi} + 1 = 0. \tag{1.1}$$

In this equation, the five fundamental constants coming from four major branches of classical mathematics – arithmetic $(0, 1)$, algebra (i), geometry (π), and analysis (e), – are connected by the three most important mathematic operations – addition, multiplication, and exponentiation – into two nonvanishing terms.

The reader is probably aware that (1.1) is but one of the consequences of the miraculous Euler formula (discovered around 1740 by Leonhard Euler)

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.2}$$

When $\theta = \pi,\ \cos\pi = -1,$ and $\sin\pi = 0$, it follows that $e^{i\pi} = -1$.

Much of the computations involving complex numbers are based on the Euler formula. To provide a proper setting for the discussion of this formula, we will first present a sketch of our number system and some historic background. This will also give us a framework to review some of the basic mathematical operations.

1.1 Our Number System

Any one who encounters for the first time these equations cannot help but be intrigued by the strange properties of the numbers such as e and i. But strange is relative, with sufficient familiarity, the strange object of yesterday becomes the common thing of today. For example, nowadays no one will be bothered by the negative numbers, but for a long time negative numbers were regarded as "strange" or "absurd." For 2000 years, mathematics thrived without negative. The Greeks did not recognize negative numbers and did not need them. Their main interest was geometry, for the description of which positive numbers are entirely sufficient. Even after Hindu mathematician Brahmagupta "invented" zero around 628, and negative numbers were interpreted as a loss instead of a gain in financial matters, medieval Europe mostly ignored them.

Indeed, so long as one regards subtraction as an act of "taken away," negative numbers are absurd. One cannot take away, say, three apples from two.

Only after the development of the axiomatic algebra, the full acceptance of negative numbers into our number system was made possible. It is also within the framework of axiomatic algebra, irrational numbers and complex numbers are seen to be natural parts of our number system.

By axiomatic method, we mean the step by step development of a subject from a small set of definitions and a chain of logical consequences derived from them. This method had long been followed in geometry, ever since the Greeks established it as a rigorous mathematical discipline.

1.1.1 Addition and Multiplication of Integers

We start with the assumption that we know what integers are, what zero is, and how to count. Although mathematicians could go even further back and describe the theory of sets in order to derive the properties of integers, we are not going in that direction.

We put the integers on a line with increasing order as in the following diagram:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\cdots \\ & & \downarrow & & & \uparrow & & \\ & & 2 & - & - & - & & \end{array}$$

If we start with certain integer a, and we count successively one unit b times to the right, the number we arrive at we call $a+b$, and that defines *addition* of integers. For example, starting at 2, and going up 3 units, we arrive at 5. So 5 is equal to $2+3$.

Once we have defined addition, then we can consider this: if we start with nothing and add a to it, b times in succession, we call the result *multiplication* of integers; we call it b *times* a.

Now as a consequence of these definitions it can be easily shown that these operations satisfy certain simple rules concerning the order in which the computations can proceed. They are the familiar commutative, associative, and distributive laws

$$\begin{array}{cc} a+b=b+a & \text{Commutative Law of Addition} \\ a+(b+c)=(a+b)+c & \text{Associative Law of Addition} \\ ab=ba & \text{Commutative Law of Multiplication} \\ (ab)c=a(bc) & \text{Associative Law of Multiplication} \\ a(b+c)=ab+ac & \text{Distributive Law.} \end{array} \tag{1.3}$$

These rules characterize the *elementary algebra.* We say elementary algebra because there is a branch of mathematics called modern algebra in which some of the rules such as $ab = ba$ are abandoned, but we shall not discuss that.

Among the integers, 1 and 0 have special properties:

$$a + 0 = a,$$
$$a \cdot 1 = a.$$

So 0 is the additive identity and 1 is the multiplicative identity. Furthermore

$$0 \cdot a = 0$$

and if $ab = 0$, either a or/and b is zero.

Now we can also have a succession of multiplications: if we start with 1 and multiply by a, b times in succession, we call that *raising to power*: a^b. It follows from this definition that

$$(ab)^c = a^c b^c,$$
$$a^b a^c = a^{(b+c)},$$
$$(a^b)^c = a^{(bc)}.$$

These results are well known and we shall not belabor them.

1.1.2 Inverse Operations

In addition to the direct operation of addition, multiplication, and raising to a power, we have also the inverse operations, which are defined as follows. Let us assume a and c are given, and that we wish to find what values of b satisfy such equations as $a + b = c,\ ab = c,\ b^a = c.$

If $a+b = c,\ b$ is defined as $c-a$, which is called *subtraction.* The operation called division is also clear: if $ab = c$, then $b = c/a$ defines division – a solution of the equation $ab = c$ “backwards.”

Now if we have a power $b^a = c$ and we ask ourselves, “What is b?,” it is called ath root of c : $b = \sqrt[a]{c}$. For instance, if we ask ourselves the following question, “What integer, raised to third power, equals 8?,” then the answer is cube root of 8; it is 2. The direct and inverse operations are summarized as follows:

Operation		Inverse Operation	
(a) addition :	$a + b = c$	(a') subtraction :	$b = c - a$
(b) multiplication :	$ab = c$	(b') division :	$b = c/a$
(c) power :	$b^a = c$	(c') root :	$b = \sqrt[a]{c}$

Insoluble Problems

When we try to solve simple algebraic equations using these definitions, we soon discover some insoluble problems, such as the following. Suppose we try to solve the equation $b = 3 - 5$. That means, according to our definition of subtraction, that we must find a number which, when added to 5, gives 3. And of course there is no such number, because we consider only positive integers; this is an insoluble problem.

1.1.3 Negative Numbers

In the grand design of algebra, the way to overcome this difficulty is to broaden the number system through *abstraction* and *generalization*. We abstract the original definitions of addition and multiplication from the rules and integers. We assume the rules to be true in general on a wider class of numbers, even though they are originally derived on a smaller class. Thus, rather using the integers to symbolically define the rules, we use the rules as the definition of the symbols, which then represent a more general kind of number. As an example, by working with the rules alone we can show that $3 - 5 = 0 - 2$. In fact we can show that one can make all subtractions, provided we define a whole set of new numbers: $0-1,\ 0-2,\ 0-3,\ 0-4$, and so on (abbreviated as $-1,\ -2,\ -3,\ -4, \ldots$), called the negative numbers.

So we have increased the range of objects over which the rules work, but the meaning of the symbols is different. One cannot say, for instance, that -2 times 5 really means to add 5 together successively -2 times. That means nothing. But we require the negative numbers to obey all the rules.

For example, we can use the rules to show that -3 times -5 is equal to 15. Let $x = -3(-5)$, this is equivalent to $x + 3(-5) = 0$, or $x + 3\,(0-5) = 0$. By the rules, we can write this equation as

$$x + 0 - 15 = (x + 0) - 15 = x - 15 = 0.$$

Thus, $x = 15$. Therefore negative a times negative b is equal to positive ab,

$$(-a)(-b) = ab.$$

An interesting problem comes up in taking powers. Suppose we wish to discover what $a^{(3-5)}$ means. We know that $3-5$ is a solution of the problem, $(3-5)+5 = 3$. Therefore

$$a^{(3-5)+5} = a^3.$$

Since

$$a^{(3-5)+5} = a^{(3-5)}a^5 = a^3$$

it follows that:

$$a^{(3-5)} = a^3/a^5.$$

Thus, in general

$$a^{n-m} = \frac{a^n}{a^m}.$$

If $n = m$, we have

$$a^0 = 1.$$

In addition, we found out what it means to raise a negative power. Since

$$3 - 5 = -2, \quad a^3/a^5 = \frac{1}{a^2}.$$

So

$$a^{-2} = \frac{1}{a^2}.$$

If our number system consists of only positive and negative integers, then $1/a^2$ is a meaningless symbol, because if a is a positive or negative integer, the square of it is greater than 1, and we do not know what we mean by 1 divided by a number greater than 1! So this is another insoluble problem.

1.1.4 Fractional Numbers

The great plan is to continue the process of generalization; whenever we find another problem that we cannot solve we extend our realm of numbers. Consider division: we cannot find a number which is an integer, even a negative integer, which is equal to the result of dividing 3 by 5. So we simply say that $3/5$ is another number, called fraction number. With the fraction number defined as a/b where a and b are integers and $b \neq 0$, we can talk about multiplying and adding fractions. For example, if $A = a/b$ and $B = c/b$, then by definition $bA = a$, $bB = c$, so $b(A + B) = a + c$.Thus, $A + B = (a + c)/b$. Therefore

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}.$$

Similarly, we can show

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}.$$

It can also be readily shown that fractional numbers satisfy the rules defined in (1.3). For example, to prove the commutative law of multiplication, we can start with

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{c}{d} \times \frac{a}{b} = \frac{ca}{db}.$$

Since a, b, c, d are integers, so $ac = ca$ and $bd = db$. Therefore $\frac{ac}{bd} = \frac{ca}{db}$. It follows that:

$$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}.$$

Take another example of powers: What is $a^{3/5}$? We know only that $(3/5)5 = 3$, since that was the definition of $3/5$. So we know also that

$$(a^{(3/5)})^5 = a^{(3/5)(5)} = a^3.$$

Then by the definition of roots we find that

$$a^{(3/5)} = \sqrt[5]{a^3}.$$

In this way we can define what we mean by putting fractions in the various symbols. It is a remarkable fact that all the rules still work for positive and negative integers, as well as for fractions!

Historically, the positive integers and their ratios (the fractions) were embraced by the ancients as natural numbers. These natural numbers together with their negative counter parts are known as *rational numbers* in our present day language.

The Greeks, under the influence of the teaching of Pythagoras, elevated fractional numbers to the central pillar of their mathematical and philosophical system. They believed that fractional numbers are prime cause behind everything in the world, from the laws of musical harmony to the motion of planets. So it was quite a shock when they found that there are numbers that cannot be expressed as a fraction.

1.1.5 Irrational Numbers

The first evidence of the existence of the irrational number (a number that is not a rational number) came from finding the length of the diagonal of a unit square. If the length of the diagonal is x, then by Pythagorean theorem $x^2 = 1^2 + 1^2 = 2$. Therefore $x = \sqrt{2}$. When people assumed this number is equal to some fraction, say m/n where m and n have no common factors, they found this assumption leads to a contradiction.

The argument goes as follows. If $\sqrt{2} = m/n$, then $2 = m^2/n^2$, or $2n^2 = m^2$. This means m^2 is an even integer. Furthermore, m itself must also be an even integer, since the square of an odd number is always odd. Thus $m = 2k$ for some integer k. It follows that $2n^2 = (2k)^2$, or $n^2 = 2k^2$. But this means n is also an even integer. Therefore, m and n have a common factor of 2, contrary to the assumption that they have no common factors. Thus $\sqrt{2}$ cannot be a fraction.

This was shocking to the Greeks, not only because of philosophical arguments, but also because mathematically, fractions form a dense set of numbers. By this we mean that between any two fractions, no matter how close, we can always squeeze in another. For example

$$\frac{1}{100}=\frac{2}{200}>\frac{2}{201}>\frac{2}{202}=\frac{1}{101}.$$

So we find $\frac{2}{201}$ between $\frac{1}{100}$ and $\frac{1}{101}$. Now between $\frac{1}{100}$ and $\frac{2}{201}$, we can squeeze in $\frac{4}{401}$, since

$$\frac{1}{100}=\frac{4}{400}>\frac{4}{401}>\frac{4}{402}=\frac{2}{201}.$$

This process can go on ad infinitum. So it seems only natural to conclude – as the Greeks did – that fractional numbers are continuously distributed on the number line. However, the discovery of irrational numbers showed that fractions, despite of their density, leave "holes" along the number line.

To bring the irrational numbers into our number system is in fact quite the most difficult step in the processes of generalization. A fully satisfactory theory of irrational numbers was not given until 1872 by Richard Dedekind (1831–1916), who made a careful analysis of continuity and ordering. To make the set of real numbers a continuum, we need the irrational numbers to fill the "holes" left by the rational numbers on the number line. A real number is any number that can be written as a decimal. There are three types of decimals: terminating, nonterminating but repeating, and nonterminating and nonrepeating. The first two types represent rational numbers, such as $\frac{1}{4}=0.25$; $\frac{2}{3}=0.666\ldots$. The third type represents irrational numbers, like $\sqrt{2}=1.4142135\ldots$.

From a practical point of view, we can always approximate an irrational number by truncating the unending decimal. If higher accuracy is needed, we simply take more decimal places. Since any decimal when stopped somewhere is rational, this means that an irrational number can be represented by a sequence of rational numbers with progressively increasing accuracy. This is good enough for us to perform mathematical operations with irrational numbers.

1.1.6 Imaginary Numbers

We go on in the process of generalization. Are there any other insoluble equations? Yes, there are. For example, it is impossible to solve this equation: $x^2=-1$. The square of no rational, of no irrational, of nothing that we have discovered so far, is equal to -1. So again we have to generalize our numbers to still a wider class.

This time we extend our number system to include the solution of this equation, and introduce the symbol i for $\sqrt{-1}$ (engineers call it j to avoid confusion with current). Of course some one could call it $-\mathrm{i}$ since it is just as good a solution. The only property of i is that $\mathrm{i}^2 = -1$. Certainly, $x = -\mathrm{i}$ also satisfies the equation $x^2 + 1 = 0$. Therefore it must be true that any equation we can write is equally valid if the sign of i is changed everywhere. This is called taking the *complex conjugate*.

We can make up numbers by adding successively i's, and multiplying i's by numbers, and adding other numbers and so on, according to all our rules. In this way we find that numbers can all be written as $a + \mathrm{i}b$, where a and b are real numbers, i.e., the numbers we have defined up until now. The number i is called the *unit imaginary number*. Any real multiple of i is called *pure imaginary*. The most general number is of course of the form $a + \mathrm{i}b$ and is called a *complex number*. Things do not get any worse if we add and multiply two such numbers. For example

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}. \tag{1.4}$$

In accordance with the distributive law, the multiplication of two complex number is defined as

$$\begin{aligned}(a + b\mathrm{i})\,(c + d\mathrm{i}) &= ac + a(d\mathrm{i}) + (b\mathrm{i})c + (b\mathrm{i})(d\mathrm{i}) \\ &= ac + (ad)\mathrm{i} + (bc)\mathrm{i} + (bd)\mathrm{ii} = (ac - bd) + (ad + bc)\mathrm{i},\end{aligned} \tag{1.5}$$

since $\mathrm{ii} = \mathrm{i}^2 = -1$. Therefore all the numbers have this mathematical form.

It is customary to use a single letter, z, to denote a complex number $z = a + b\mathrm{i}$. Its real and imaginary parts are written as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. With this notation, $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$. The equation $z_1 = z_2$ holds if and only if

$$\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2) \quad \text{and} \quad \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)\,.$$

Thus any equation involving complex numbers can be interpreted as a pair of real equations.

The complex conjugate of the number $z = a + b\mathrm{i}$ is usually denoted as either z^*, or $\overline{z}$, and is given by $z^* = a - b\mathrm{i}$. An important relation is that the product of a complex number and its complex conjugate is a real number

$$zz^* = (a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + b^2.$$

With this relation, the division of two complex numbers can also be written as the sum of a real part and an imaginary part

$$\frac{a + b\mathrm{i}}{c + d\mathrm{i}} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}}\frac{c - d\mathrm{i}}{c - d\mathrm{i}} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\mathrm{i}.$$

Example 1.1.1. Express the following in the form of $a + b\mathrm{i}$:

(a) $(6 + 2\mathrm{i}) - (1 + 3\mathrm{i})$, (b) $(2 - 3\mathrm{i})(1 + \mathrm{i})$,

(c) $\left(\dfrac{1}{2 - 3\mathrm{i}}\right)\left(\dfrac{1}{1 + \mathrm{i}}\right)$.

Solution 1.1.1.

$$(a)\ \ (6+2\mathrm{i})-(1+3\mathrm{i})=(6-1)+\mathrm{i}(2-3)=5-\mathrm{i}.$$

$$\begin{aligned}(b)\ \ (2-3\mathrm{i})(1+\mathrm{i})&=2\,(1+\mathrm{i})-3\mathrm{i}(1+\mathrm{i})=2+2\mathrm{i}-3\mathrm{i}-3\mathrm{i}^2\\&=(2+3)+\mathrm{i}(2-3)=5-\mathrm{i}.\end{aligned}$$

$$\begin{aligned}(c)\ \ \left(\frac{1}{2-3\mathrm{i}}\right)\left(\frac{1}{1+\mathrm{i}}\right)&=\frac{1}{(2-3\mathrm{i})\,(1+\mathrm{i})}=\frac{1}{5-\mathrm{i}}\\&=\frac{5+\mathrm{i}}{(5-\mathrm{i})\,(5+\mathrm{i})}=\frac{5+\mathrm{i}}{5^2-\mathrm{i}^2}=\frac{5}{26}+\frac{1}{26}\mathrm{i}.\end{aligned}$$

Historically, Italian mathematician Girolamo Cardano was credited as the first to consider the square root of a negative number in 1545 in connection with solving quadratic equations. But after introducing the imaginary numbers, he immediately dismissed them as "useless." He had a good reason to think that way. At Cardano's time, mathematics was still synonymous with geometry. Thus the quadratic equation $x^2=mx+c$ was thought as a vehicle to find the intersection points of the parabola $y=x^2$ and the line $y=mx+c$. For an equation such as $x^2=-1$, the horizontal line $y=-1$ will obviously not intersect the parabola $y=x^2$ which is always positive. The absence of the intersection was thought as the reason of the occurrence of the imaginary numbers.

It was the cubic equation that forced complex numbers to be taken seriously. For a cubic curve $y=x^3$, the values of y go from $-\infty$ to $+\infty$. A line will always hit the curve at least once. In 1572, Rafael Bombeli considered the equation

$$x^3=15x+4,$$

which clearly has a solution of $x=4$. Yet at the time, it was known that this kind of equation could be solved by the following formal procedure. Let $x=a+b$, then

$$x^3=(a+b)^3=a^3+3ab(a+b)+b^3,$$

which can be written as

$$x^3=3abx+(a^3+b^3).$$

The problem will be solved, if we can find a set of values a and b satisfying the conditions

$$3ab=15 \quad \text{and} \quad a^3+b^3=4.$$

Since $a^3b^3 = 5^3$ and $b^3 = 4 - a^3$, we have

$$a^3(4 - a^3) = 5^3,$$

which is a quadratic equation in a^3

$$\left(a^3\right)^2 - 4a^3 + 125 = 0.$$

The solution of such an equation was known for thousands of years,

$$a^3 = \frac{1}{2}\left(4 \pm \sqrt{16 - 500}\right) = 2 \pm 11\mathrm{i}.$$

It follows that:

$$b^3 = 4 - a^3 = 2 \mp 11\mathrm{i}.$$

Therefore

$$x = a + b = (2 + 11\mathrm{i})^{1/3} + (2 - 11\mathrm{i})^{1/3}.$$

Clearly, the interpretation that the appearance of imaginary numbers signifies no solution of the geometric problem is not valid. In order to have the solution come out to equal 4, Bombeli assumed

$$(2 + 11\mathrm{i})^{1/3} = 2 + b\mathrm{i}; \qquad (2 - 11\mathrm{i})^{1/3} = 2 - b\mathrm{i}.$$

To justify this assumption, he had to use the rules of addition and multiplication of complex numbers. With the rules listed in (1.4) and (1.5), it can be readily shown that

$$\begin{aligned}(2 + b\mathrm{i})^3 &= 8 + 3\,(4)\,(b\mathrm{i}) + 3(2)\,(b\mathrm{i})^2 + (b\mathrm{i})^3 \\ &= \left(8 - 6b^2\right) + \left(12b - b^3\right)\mathrm{i}.\end{aligned}$$

With $b = \pm 1$, he obtained

$$(2 \pm \mathrm{i})^3 = 2 \pm 11\mathrm{i},$$

and

$$x = (2 + 11\mathrm{i})^{1/3} + (2 - 11\mathrm{i})^{1/3} = 2 + \mathrm{i} + 2 - \mathrm{i} = 4.$$

Thus he established that problems with real coefficients required complex arithmetic for solutions.

Despite Bombelli's work, complex numbers were greeted with suspicion, even hostility for almost 250 years. Not until the beginning of the 19th century, complex numbers were fully embraced as members of our number system. The acceptance of complex numbers was largely due to the work and reputation of Gauss.

Karl Friedrich Gauss (1777–1855) of Germany was given the title of "the prince of mathematics" by his contemporaries as a tribute to his great achievements in almost every branch of mathematics. At the age of 22, Gauss in his doctoral dissertation gave the first rigorous proof of what we now call the Fundamental Theorem of Algebra. It says that a polynomial of degree n always has exactly n complex roots. This shows that complex numbers are not only necessary to solve a general algebraic equation, they are also sufficient. In other words, with the invention of i, every algebraic equation can be solved. This is a fantastic fact. It is certainly not self-evident. In fact, the process by which our number system is developed would make us think that we will have to keep on inventing new numbers to solve yet unsolvable equations. It is a miracle that this is not the case. With the last invention of i, our number system is complete. Therefore a number, no matter how complicated it looks, can always be reduced to the form of $a + b\mathrm{i}$, where a and b are real numbers.

1.2 Logarithm

1.2.1 Napier's Idea of Logarithm

Rarely a new idea was embraced so quickly by the entire scientific community with such enthusiasm as the invention of logarithm. Although it was merely a device to simplify computation, its impact on scientific developments could not be overstated.

Before 17th century scientists had to spend much of their time doing numerical calculations. The Scottish baron, John Napier (1550–1617) thought to relieve this burden as he wrote: "Seeing there is nothing that is so troublesome to mathematical practice than multiplications, divisions, square and cubical extractions of great numbers,...... I began therefore in my mind by what certain and ready art I might remove those hinderance." His idea was this: if we could write any number as a power of some given, fixed number b (later to be called base), then multiplication of numbers would be equivalent to addition of their exponents. He called the power logarithm.

In modern notation, this works as follows. If

$$b^{x_1} = N_1; \qquad b^{x_2} = N_2$$

then by definition

$$x_1 = \log_b N_1; \quad x_2 = \log_b N_2.$$

Obviously

$$x_1 + x_2 = \log_b N_1 + \log_b N_2.$$

Since

$$b^{x_1+x_2} = b^{x_1} b^{x_2} = N_1 N_2$$

again by definition

$$x_1 + x_2 = \log_b N_1 N_2.$$

Therefore

$$\log_b N_1 N_2 = \log_b N_1 + \log_b N_2.$$

Suppose we have a table, in which N and $\log_b N$ (the power x) are listed side by side. To multiply two numbers N_1 and N_2, you first look up $\log_b N_1$ and $\log_b N_2$ in the table. You then add the two numbers. Next, find the number in the body of the table that matches the sum, and read backward to get the product $N_1 N_2$.

Similarly, we can show

$$\log_b \frac{N_1}{N_2} = \log_b N_1 - \log_b N_2,$$

$$\log_b N^n = n \log_b N, \qquad \log_b N^{1/n} = \frac{\log_b N}{n}.$$

Thus, division of numbers would be equivalent to subtraction of their exponents, raising a number to nth power would be equivalent to multiplying the exponent by n, and finding the nth root of a number would be equivalent to dividing the exponent by n. In this way the drudgery of computations is greatly reduced.

Now the question is, with what base b should we compute. Actually it makes no difference what base is used, as long as it is not exactly equal to 1. We can use the same principle all the time. Besides, if we are using logarithms to any particular base, we can find logarithms to any other base merely by multiplying a factor, equivalent to a change of scale. For example, if we know the logarithm of all numbers with base b, we can find the logarithm of N with base a. First if $a = b^x$, then by definition, $x = \log_b a$, therefore

$$a = b^{\log_b a}. \tag{1.6}$$

To find $\log_a N$, first let $y = \log_a N$. By definition $a^y = N$. With a given by (1.6), we have

$$\left(b^{\log_b a}\right)^y = b^{y \log_b a} = N.$$

Again by definition (or take logarithm of both sides of the equation)

$$y \log_b a = \log_b N.$$

Thus

$$y = \frac{1}{\log_b a} \log_b N.$$

Since $y = \log_a N$, it follows:

$$\log_a N = \frac{1}{\log_b a} \log_b N.$$

This is known as change of base. Having a table of logarithm with base b will enable us to calculate the logarithm to any other base.

In any case, the key is, of course, to have a table. Napier chose a number slightly less than one as the base and spent 20 years to calculate the table. He published his table in 1614. His invention was quickly adopted by scientists all across Europe and even in far away China. Among them was the astronomer Johannes Kepler, who used the table with great success in his calculations of the planetary orbits. These calculations became the foundation of Newton's classical dynamics and his law of gravitation.

1.2.2 Briggs' Common Logarithm

Henry Briggs (1561–1631), a professor of geometry in London, was so impressed by Napier's table, he went to Scotland to meet the great inventor in person. Briggs suggested that a table of base 10 would be more convenient. Napier readily agreed. Briggs undertook the task of additional computations. He published his table in 1624. For 350 years, the logarithmic table and the slide rule (constructed with the principle of logarithm) were indispensable tools of every scientist and engineer.

The logarithm in Briggs' table is now known as the common logarithm. In modern notation, if we write $x = \log N$ without specifying the base, it is understood that the base is 10, and $10^x = N$.

Today logarithmic tables are replaced by hand-held calculators, but logarithmic function remains central to mathematical sciences.

It is interesting to see how logarithms were first calculated. In addition to historic interests, it will help us to gain some insights into our number system.

Since a simple process for taking square roots was known, Briggs computed successive square roots of 10. A sample of the results is shown in Table 1.1. The powers (x) of 10 are given in the first column and the results, 10^x, are given in the second column. For example, the second row is the square root of 10, that is $10^{1/2} = \sqrt{10} = 3.16228$. The third row is the square root of the square root of 10, $\left(10^{1/2}\right)^{1/2} = 10^{1/4} = 1.77828$. So on and so forth, we get a series of successive square roots of 10. With a hand-held calculator, you can readily verify these results.

In the table we noticed that when 10 is raised to a very small power, we get 1 plus a small number. Furthermore, the small numbers that are added

Table 1.1. Successive square roots of ten

x ($\log N$)	10^x (N)	$(10^x-1)/x$
1	10.0	9.00
$\frac{1}{2}=0.5$	3.16228	4.32
$(\frac{1}{2})^2=0.25$	1.77828	3.113
$(\frac{1}{2})^3=0.125$	1.33352	2.668
$(\frac{1}{2})^4=0.0625$	1.15478	2.476
$(\frac{1}{2})^5=0.03125$	1.074607	2.3874
$(\frac{1}{2})^6=0.015625$	1.036633	2.3445
$(\frac{1}{2})^7=0.0078125$	1.018152	2.3234
$(\frac{1}{2})^8=0.00390625$	1.0090350	2.3130
$(\frac{1}{2})^9=0.001953125$	1.0045073	2.3077
$(\frac{1}{2})^{10}=0.00097656$	1.0022511	2.3051
$(\frac{1}{2})^{11}=0.00048828$	1.0011249	2.3038
$(\frac{1}{2})^{12}=0.00024414$	1.0005623	2.3032
$(\frac{1}{2})^{13}=0.00012207$	1.000281117	2.3029
$(\frac{1}{2})^{14}=0.000061035$	1.000140548	2.3027
$(\frac{1}{2})^{15}=0.0000305175$	1.000070272	2.3027
$(\frac{1}{2})^{16}=0.0000152587$	1.000035135	2.3026
$(\frac{1}{2})^{17}=0.0000076294$	1.0000175675	2.3026

to 1 begins to look as though we are merely dividing by 2 each time we take a square root. In other words, it looks that when x is very small, 10^x-1 is proportional to x. To find the proportionality constant, we list $(10^x-1)/x$ in column 3. At the top of the table, these ratios are not equal, but as they come down, they get closer and closer to a constant value. To the accuracy of five significant digits, the proportional constant is equal to 2.3026. So we find that when s is very small

$$10^s = 1 + 2.3026s. \tag{1.7}$$

Briggs computed successively 27 square roots of 10, and used (1.7) to obtain another 27 squares roots.

Since $10^x = N$ means $x = \log N$, the first column in Table 1.1 is also the logarithm of the corresponding number in the second column. For example, the second row is the square root of 10, that is $10^{1/2} = 3.16228$. Then by definition, we know

$$\log(3.16228) = 0.5.$$

If we want to know the logarithm of a particular number N, and N is not exactly the same as one of the entries in the second column, we have to break up N as a product of a series of numbers which are entries of the table. For example, suppose we want to know the logarithm of 1.2. Here is what we do. Let $N = 1.2$, and we are going to find a series of n_i in column 2 such that

$$N = n_1 n_2 n_3 \cdots .$$

Since all n_i are greater than one, so $n_i < N$. The number in column 2 closest to 1.2 satisfying this condition is 1.15478, So we choose $n_1 = 1.15478$, and we have

$$\frac{N}{n_1} = \frac{1.2}{1.15478} = 1.039159 = n_2 n_3 \cdots .$$

The number smaller than and closest to 1.039159 is 1.036633. So we choose $n_2 = 1.036633$, thus

$$\frac{N}{n_1 n_2} = \frac{1.039159}{1.036633} = 1.0024367.$$

With $n_3 = 1.0022511$, we have

$$\frac{N}{n_1 n_2 n_3} = \frac{1.0024367}{1.0022511} = 1.0001852.$$

The plan is to continue this way until the right-hand side is equal to one. But most likely, sooner or later, the right-hand side will fall beyond the table and is still not exactly equal to one. In our particular case, we can go down a couple of more steps. But for the purpose of illustration, let us stop here. So

$$N = n_1 n_2 n_3 (1 + \Delta n),$$

where $\Delta n = 0.0001852$. Now

$$\log N = \log n_1 + \log n_2 + \log n_3 + \log(1 + \Delta n).$$

The terms on the right-hand side, except the last one, can be read from the table. For the last term, we will make use of (1.7). By definition, if s is very small, (1.7) can be written as

$$s = \log(1 + 2.3026 s).$$

Let $\Delta n = 2.3026 s$, so $s = \frac{\Delta n}{2.3026} = \frac{0.0001852}{2.3026} = 8.04 \times 10^{-5}$. It follows:

$$\log(1 + \Delta n) = \log[1 + 2.3026 \left(8.04 \times 10^{-5}\right)] = 8.04 \times 10^{-5}.$$

With $\log n_1 = 0.0625$, $\log n_2 = 0.015625$, $\log n_3 = 0.0009765$ from the table, we arrived at

$$\log(1.2) = 0.0625 + 0.015625 + 0.0009765 + 0.0000804 = 0.0791819.$$

The value of $\log(1.2)$ should be 0.0791812. Clearly if we have a larger table we can have as many accurate digits as we want. In this way Briggs calculated the logarithms to 16 decimal places and reduced them to 14 when he published his table, so there were no rounding errors. With minor revisions, Briggs' table remained the basis for all subsequent logarithmic tables for the next 300 years.

1.3 A Peculiar Number Called e

1.3.1 The Unique Property of e

Equation (1.7) expresses a very interesting property of our number system. If we let $t = 2.3026s$, then for a very small t, (1.7) becomes

$$10^{\frac{t}{2.3026}} = 1 + t. \tag{1.8}$$

To simplify the writing, let us denote

$$10^{\frac{1}{2.3026}} = \mathrm{e}. \tag{1.9}$$

Thus (1.8) says that e raised to a very small power is equal to one plus the small power

$$\mathrm{e}^t = 1 + t \quad \text{for } t \to 0. \tag{1.10}$$

Because of this, we find the derivative of e^x is equal to itself.

Recall the definition of the derivative of a function:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

So

$$\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\mathrm{e}^{x+\Delta x} - \mathrm{e}^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\mathrm{e}^x(\mathrm{e}^{\Delta x} - 1)}{\Delta x}.$$

Now Δx approaches zero as a limit, certainly it is very small, so we can write (1.10) as

$$\mathrm{e}^{\Delta x} = 1 + \Delta x.$$

Thus

$$\frac{\mathrm{e}^x(\mathrm{e}^{\Delta x} - 1)}{\Delta x} = \frac{\mathrm{e}^x(1 + \Delta x - 1)}{\Delta x} = \mathrm{e}^x.$$

Therefore

$$\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} = \mathrm{e}^x. \tag{1.11}$$

The function e^x (or written as $\exp(x)$) is generally called the natural exponential function, or simply the exponential function. Not only is the exponential function equal to its own derivative, it is the only function (apart from a multiplication constant) that has this property. Because of this, the exponential function plays a central role in mathematics and sciences.

1.3.2 The Natural Logarithm

If $\mathrm{e}^y = x$, then by definition

$$y = \log_{\mathrm{e}} x.$$

The logarithm to the base e is known as the natural logarithm. It appears with amazing frequency in mathematics and its applications. So we give it a special symbol. It is written as $\ln x$. That is

$$y = \log_{\mathrm{e}} x = \ln x.$$

Thus

$$\mathrm{e}^{\ln x} = x.$$

Furthermore,

$$\ln \mathrm{e}^x = x \ln \mathrm{e} = x.$$

In this sense, the exponential function and the natural logarithm are inverses of each other.

Example 1.3.1. Find the value of $\ln 10$.

Solution 1.3.1. Since

$$10^{\frac{1}{2.3026}} = \mathrm{e}, \quad \Rightarrow \quad 10 = \mathrm{e}^{2.3026}$$

it follows:

$$\ln 10 = \ln \mathrm{e}^{2.3026} = 2.3026.$$

The derivative of $\ln x$ is of special interests.

$$\frac{\mathrm{d}(\ln x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x},$$

$$\ln(x + \Delta x) - \ln x = \ln \frac{x + \Delta x}{x} = \ln \left(1 + \frac{\Delta x}{x}\right).$$

Now (1.10) can be written as

$$t = \ln(1 + t)$$

for a very small t. Since Δx approaches zero as a limit, for any fixed x, $\frac{\Delta x}{x}$ can certainly be made as small as we wish. Therefore, we can set $\frac{\Delta x}{x} = t$, and conclude

$$\frac{\Delta x}{x} = \ln\left(1 + \frac{\Delta x}{x}\right).$$

Thus,

$$\frac{\mathrm{d}(\ln x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\frac{\Delta x}{x} = \frac{1}{x}.$$

This in turn means

$$\mathrm{d}\,(\ln x) = \frac{\mathrm{d}x}{x}$$

or

$$\int \frac{\mathrm{d}x}{x} = \ln x + c, \tag{1.12}$$

where c is the constant of integration. It is well known that because of

$$\frac{\mathrm{d}x^{n+1}}{\mathrm{d}x} = (n+1)x^n,$$

we have

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{(n+1)} + c.$$

This formula holds for all values of n except for $n = -1$, since then the denominator $n + 1$ is zero. This had been a difficult problem, but now we see that (1.12) provides the "missing case."

In numerous phenomena, ranging from population growth to the decay of radioactive material, in which the rate of change of some quantity is proportional to the quantity itself. Such phenomenon is governed by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky,$$

where k is a constant that is positive if y is increasing and negative if y is decreasing. To solve this equation, we write it as

$$\frac{\mathrm{d}y}{y} = k\,\mathrm{d}t$$

and then integrate both sides to get

$$\ln y = kt + c, \quad \textit{or} \quad y = \mathrm{e}^{kt+c} = \mathrm{e}^{kt}\mathrm{e}^{c}.$$

If y_0 denotes the value of y when $t = 0$, then $y_0 = \mathrm{e}^c$ and

$$y = y_0\,\mathrm{e}^{kt}.$$

This equation is called the law of exponential change.

1.3.3 Approximate Value of e

The number e is found of such great importance, but what is the numerical value of e, which we have, so far, defined as $10^{(1/2.3025)}$? We can use our table of successive square root of 10 to calculate this number. The powers of 10 are given in the first column of Table 1.1. If we can find a series of numbers $n_1,\ n_2,\ n_3, \ldots$ in this column, such that

$$\frac{1}{2.3026} = n_1 + n_2 + n_3 + \cdots ,$$

then

$$10^{\frac{1}{2.3026}} = 10^{n_1+n_2+n_3+\cdots} = 10^{n_1}10^{n_2}10^{n_3}\cdots .$$

We can read from the second column of the table 10^{n_1}, and 10^{n_2}, and 10^{n_3} and so on, and multiply them together. Let us do just that.

$$\begin{aligned}\frac{1}{2.3026} = 0.43429 = {} & 0.25 + 0.125 + 0.03125 + 0.015625 \\ & + 0.0078125 + 0.00390625 + 0.00048828 + 0.00012207 \\ & + 0.000061035 + 0.000026535.\end{aligned}$$

From the table we find $10^{0.25} = 1.77828,\ 10^{0.125} = 1.33352, \ldots$ etc. except for the last term for which we use (1.7). Thus

$$\begin{aligned}\mathrm{e} = 10^{\frac{1}{2.3026}} & = 1.77828 \times 1.33352 \times 1.074607 \times 1.036633 \times 1.018152 \\ & \times 1.009035 \times 1.0011249 \times 1.000281117 \times 1.000140548 \\ & \times (1 + 2.3026 \times 0.000026535) = 2.71826.\end{aligned}$$

Since $\frac{1}{2.3026}$ is only accurate to 5 significant digits, we cannot expect our result to be accurate more than that. (The accurate result is $2.71828\cdots$) Thus what we get is only an approximation. Is there a more precise definition of e? The answer is yes. We will discuss this question in the next section.

1.4 The Exponential Function as an Infinite Series

1.4.1 Compound Interest

The origins of the number e are not very clear. The existence of this peculiar number could be extracted from the logarithmic table as we did. In fact there is an indirect reference to e in the second edition of Napier's table. But most probably the peculiar property of the number e was noticed even earlier in connection with compound interest.

A sum of money invested at x percent annual interest rate (x expressed as a decimal, for example $x = 0.06$ for 6%) means that at the end of the year the sum grows by a factor $(1+x)$. Some banks compute the accrued interest not once a year but several times a year. For example, if an annual interest rate of x percent is compounded semiannually, the bank will use one-half of the annual rate as the rate per period. Hence, if P is the original sum, at the end of the half-year, the sum grows to $P\left(1+\frac{x}{2}\right)$, and at the end of the year the sum becomes

$$\left[P\left(1+\frac{x}{2}\right)\right]\left(1+\frac{x}{2}\right)=P\left(1+\frac{x}{2}\right)^2.$$

In the banking industry one finds all kinds of compounding schemes – annually, semiannually, quarterly, monthly, weekly, and even daily. Suppose the compounding is done n times a year, at the end of the year, the principal P will yield the amount

$$S=P\left(1+\frac{x}{n}\right)^n.$$

It is interesting to compare the amounts of money a given principal will yield after one year for different conversion periods. Table 1.2 shows that the amounts of money one will get for \$100 invested for 1 year at 6% annual interest rate at different conversion periods. The result is quite surprising. As we see, a principal of \$100 compounded daily or weekly yield practically the same. But will this pattern go on? Is it possible that no matter how large n is, the values of $(1+\frac{x}{n})^n$ will settle on the same number? To answer this question, we must use methods other than merely computing individual values. Fortunately, such a method is available. With the binomial formula,

$$\begin{aligned}(a+b)^n &= a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2\\ &\quad+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+\cdots+b^n,\end{aligned}$$

we have

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n &= 1+n\left(\frac{x}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2\\ &\quad+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3+\cdots+\left(\frac{x}{n}\right)^n\\ &= 1+x+\frac{(1-1/n)}{2!}x^2+\frac{(1-1/n)(1-2/n)}{3!}x^3+\cdots\left(\frac{x}{n}\right)^n.\end{aligned}$$

Now as $n\to\infty$, $\frac{k}{n}\to 0$. Therefore

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots. \tag{1.13}$$

becomes an infinite series. Standard tests for convergence show that this is a convergent series for all real values of x. In other words, the value of $(1+\frac{x}{n})^n$ does settle on a specific limit as n increase without bound.

Table 1.2. The yields of \$100 invested for 1 year at 6% annual interest rate at different conversion periods

	n	x/n	$100(1+x/n)^n$
Annually	1	0.06	106.00
Semiannually	2	0.03	106.09
Quarterly	4	0.015	106.136
Monthly	12	0.005	106.168
Weekly	52	0.0011538	106.180
Daily	365	0.0001644	106.183

1.4.2 The Limiting Process Representing e

In early 18th century, Euler used the letter e to represent the series (1.13) for the case of $x = 1$,

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots . \tag{1.14}$$

This choice, like many other symbols of his, such as i, π , $f(x)$, became universally accepted.

It is important to note that when we say that the limit of $\frac{1}{n}$ as $n \to \infty$ is 0 it does not mean that $\frac{1}{n}$ itself will ever be equal to 0, in fact, it will not. Thus, if we let $t = \frac{1}{n}$, then as $n \to \infty,\ t \to 0$. So (1.14) can be written as

$$\mathrm{e} = \lim_{t\to 0}(1+t)^{1/t}.$$

In words, it says that if t is very small, then

$$\mathrm{e}^t = \left[(1+t)^{1/t}\right]^t = 1 + t, \qquad t \to 0.$$

This is exactly the same equation as shown in (1.10). Therefore, e is the same number previously written as $10^{1/2.3026}$. Now the formal definition of e is given by the limiting process

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

which can be written as an infinite series as shown in (1.14). The series converges rather fast. With seven terms, it gives us 2.71825. This approximation can be improved by adding more terms until the desired accuracy is reached. Since it is monotonely convergent, each additional term brings it closer to the limit: $2.71828\cdots$.

1.4.3 The Exponential Function e^x

Raising e to x power, we have

$$\mathrm{e}^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}.$$

Let $nx = m$, then $\frac{x}{m} = \frac{1}{n}$. As n goes to ∞, so does m. Thus the above equation becomes

$$\mathrm{e}^x = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}.$$

Now m may not be an integer, but the binomial formula is equally valid for noninteger power (one of the early discoveries of Isaac Newton). Therefore by the same reason as in (1.13), we can express the exponential function as an infinite series,

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{1.15}$$

It is from this series that the numerical values of e^x are most easily obtained, the first few terms usually suffice to obtain the desired accuracy.

We have shown in (1.11) that the derivative of e^x must be equal to itself. This is clearly the case as we take derivative of (1.15) term by term,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \mathrm{e}^x.$$

1.5 Unification of Algebra and Geometry

1.5.1 The Remarkable Euler Formula

Leonhard Euler (1707–1783) was born in Basel, a border town between Switzerland, France, and Germany. He is one of the great mathematicians and certainly the most prolific scientist of all times. His immense output fills at least 70 volumes. In 1771, after he became blind, he published three volumes of a profound treatise of optics. For almost 40 years after his death, the Academy at St. Petersburg continued to publish his manuscripts. Euler played with formulas like a child playing toys, making all kinds of substitutions until he got something interesting. Often the results were sensational.

He took the infinite series of e^x, and boldly replaced the real variable x in (1.15) with the imaginary expression $\mathrm{i}\theta$ and got

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \cdots.$$

Since $\mathrm{i}^2 = -1,\ \mathrm{i}^3 = -\mathrm{i},\ \mathrm{i}^4 = 1$, and so on, this equation became

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots .$$

He then changed the order of terms, collecting all the real terms separately from the imaginary terms, and arrived at the series

$$\mathrm{e}^{\mathrm{i}\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right).$$

Now it was already known in Euler's time that the two series appearing in the parentheses are the power series of the trigonometric functions $\cos\theta$ and $\sin\theta$, respectively. Thus Euler arrived at the remarkable formula (1.2)

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta,$$

which at once links the exponential function to ordinary trigonometry.

Strictly speaking, Euler played the infinite series rather loosely. Collecting all the real terms separately from the imaginary terms, he changed the order of terms. To do so with an infinite series can be dangerous. It may affect its sum, or even change a convergent series into a divergent series. But this result has withstood the test of rigor.

Euler derived hundreds of formulas, but this one is often called the most famous formula of all formulas. Feynman called it the amazing jewel.

1.5.2 The Complex Plane

The acceptance of complex number as a bona fide members of our number system was greatly helped by the realization that a complex number could be given a simple, concrete geometric interpretation. In a two-dimensional rectangular coordinate system, a point is specified by its x and y components. If we interpret the x and y axes as the real and imaginary axes, respectively, then the complex number $z = x + \mathrm{i}y$ is represented by the point (x, y). The horizontal position of the point is x, the vertical position of the point is y, as shown in Fig. 1.1. We can then add and subtract complex numbers by separately adding or subtracting the real and imaginary components. When thought in this way, the plane is called the complex plane, or the Argand plane.

This graphic representation was independently suggested around 1800 by Wessel of Norway, Argand of France, and Gauss. The publications by Wessel and by Argand went all but unnoticed, but the reputation of Gauss ensured wide dissemination and acceptance of the complex numbers as points in the complex plane.

At the time when this interpretation was suggested, the Euler formula (1.2) had already been known for at least 50 years. It might have played the

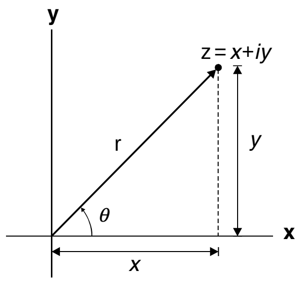

Fig. 1.1. Complex plane also known as Argand diagram. The real part of a complex number is along the x-axis, and the imaginary part, along the y-axis

role of guiding principle for this suggestion. The geometric interpretation of the complex number is certainly consistent with the Euler formula. We can derive the Euler formula by expressing $\mathrm{e}^{\mathrm{i}\theta}$ as a point in the complex plane.

Since the most general number is a complex number in the form of a real part plus an imaginary part, so let us express $\mathrm{e}^{\mathrm{i}\theta}$ as

$$\mathrm{e}^{\mathrm{i}\theta} = a(\theta) + \mathrm{i}b(\theta)\,. \tag{1.16}$$

Note that both the real part a and the imaginary part b must be functions of θ. Here θ is any real number. Changing i to $-$i, in both sides of this equation, we get the complex conjugate

$$\mathrm{e}^{-\mathrm{i}\theta} = a(\theta) - \mathrm{i}b(\theta)\,.$$

Since

$$\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\theta-\mathrm{i}\theta} = \mathrm{e}^{0} = 1,$$

it follows that

$$\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = (a+\mathrm{i}b)(a-\mathrm{i}b) = a^2 + b^2 = 1.$$

Furthermore

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}(a+\mathrm{i}b) = \mathrm{i}a - b$$

but

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathrm{i}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}a + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}b = a' + \mathrm{i}b',$$

equating the real part to real part and imaginary part to imaginary part of the last two equations, we have

$$a' = \frac{\mathrm{d}}{\mathrm{d}\theta}a = -b, \quad b' = \frac{\mathrm{d}}{\mathrm{d}\theta}b = a.$$

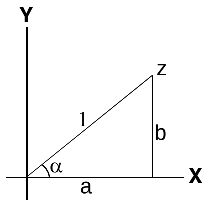

Fig. 1.2. The Argand diagram of the complex number $z = \mathrm{e}^{\mathrm{i}\theta} = a + \mathrm{i}b$. The distance between the origin and the point (a, b) must be 1

Thus

$$a'b = -b^2, \quad b'a = a^2$$

and

$$b'a - a'b = a^2 + b^2 = 1.$$

Now let $a(\theta)$ represent the abscissa (x-coordinate) and $b(\theta)$ represent the ordinate (y-coordinate) of a point in the complex plane as shown in Fig. 1.2. Let α be the angle between the x-axis and the vector from the origin to the point. Since the length of this vector is given by the Pythagorean theorem

$$r^2 = a^2 + b^2 = 1,$$

clearly

$$\cos\alpha = \frac{a(\theta)}{1} = a(\theta), \quad \sin\alpha = \frac{b(\theta)}{1} = b(\theta), \quad \tan\alpha = \frac{b(\theta)}{a(\theta)}. \tag{1.17}$$

Now

$$\frac{\mathrm{d}\tan\alpha}{\mathrm{d}\theta} = \frac{\mathrm{d}\tan\alpha}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = \frac{1}{\cos^2\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = \frac{1}{a^2}\frac{\mathrm{d}\alpha}{\mathrm{d}\theta},$$

but

$$\frac{\mathrm{d}\tan\alpha}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{b}{a}\right) = \frac{b'a - a'b}{a^2} = \frac{1}{a^2}.$$

It is clear from the last two equations that

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = 1.$$

In other words,

$$\alpha = \theta + c.$$

To determine the constant c, let us look at the case $\theta = 0$. Since $e^{i0} = 1 = a+ib$ means $a = 1$ and $b = 0$, in this case it is clear from the diagram that $\alpha = 0$. Therefore c must be equal to zero, so

$$\alpha = \theta.$$

It follows from (1.17) that:

$$a(\theta) = \cos\alpha = \cos\theta, \qquad b(\theta) = \sin\alpha = \sin\theta.$$

Putting them back to (1.16), we obtain again

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Note that we have derived the Euler formula without the series expansion. Previously we have derived this formula in a purely algebraic manner. Now we see that $\cos\theta$ and $\sin\theta$ are the cosine and sine functions naturally defined in geometry. This is the unification of algebra and geometry.

It took 250 years for mathematicians to get comfortable with complex numbers. Once fully accepted, the advance of theory of complex variables was rather rapid. In a short span of 40 years, Augustin Louis Cauchy (1789–1857) of France and Georg Friedrich Bernhard Riemann (1826–1866) of Germany developed a beautiful and powerful theory of complex functions, which we will describe in Chap. 2.

In this introductory chapter, we have presented some pieces of historic notes for showing that the logical structure of mathematics is as interesting as any other human endeavor. Now we must leave history behind because of our limited space. For more detailed information, we recommend the following references, from which much of our accounts are taken:

Richard Feynman, Robert B. Leighton, and Mathew Sands, *The Feynman Lectures on Phyics,* Vol. 1, Chapter 22, (1963) Addison Wesley

Eli Maor, *e: the Story of a Number,* (1994) Princeton University Press

Tristan Needham, *Visual Complex Analysis,* Chapter 1, (1997) Oxford University Press

1.6 Polar Form of Complex Numbers

In terms of polar coordinates (r, θ), the variable x and y are

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The complex variable z is then written as

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{1.18}$$

The quantity r, known as the modulus, is the absolute value of z and is given by

$$r = |z| = (zz^*)^{1/2} = \left(x^2 + y^2\right)^{1/2}.$$

The angle θ is known as the argument, or phase, of z. Measured in radians, it is given by

$$\theta = \tan^{-1} \frac{y}{x}.$$

If z is in the second or third quadrants, one has to use this equation with care. In the second quadrant, $\tan\theta$ is negative, but in a hand-held calculator, or a computer code, a negative arctangent is interpreted as an angle in the fourth quadrant. In the third quadrant, $\tan\theta$ is positive, but a calculator will interpret a positive arctangent as an angle in the first quadrant. Since an angle is fixed by its sine and cosine, θ is uniquely determined by the pair of equations

$$\cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}.$$

But in practice we usually compute $\tan^{-1}(y/x)$ and adjust for the quadrant problem by adding and subtracting π. Because of its identification as an angle, θ is determined only up to an integer multiple of 2π. We shall make the usual choice of limiting θ to the interval of $0 \leq \theta < 2\pi$ as its principal value. However, in computer codes the principal value is usually chosen in the open interval of $-\pi \leq \theta < \pi$.

Equation (1.18) is called the polar form of z. It is immediately clear that, the complex conjugate of z in the polar form is

$$z^*(r,\theta) = z(r,-\theta) = r\mathrm{e}^{-\mathrm{i}\theta}.$$

In the complex plane, z^* is the reflection of z across the x-axis.

It is helpful to always keep the complex plane in mind. As θ increases, $\mathrm{e}^{\mathrm{i}\theta}$ describes an unit circle in the complex plane as shown in Fig. 1.3. To reach a general complex number z, we must take the unit vector $\mathrm{e}^{\mathrm{i}\theta}$ that points at z and stretch it by the length $|z| = r$.

It is very convenient to multiply or divide two complex numbers in polar forms. Let

$$z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}, \quad z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2},$$

then

$$z_1 z_2 = r_1\mathrm{e}^{\mathrm{i}\theta_1} r_2\mathrm{e}^{\mathrm{i}\theta_2} = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} = r_1 r_2[\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)],$$

$$\frac{z_1}{z_2} = \frac{r_1\mathrm{e}^{\mathrm{i}\theta_1}}{r_2\mathrm{e}^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + \mathrm{i}\sin(\theta_1-\theta_2)].$$

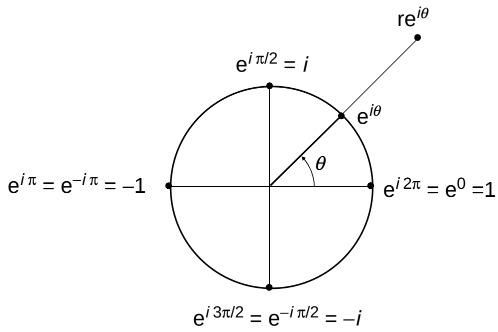

Fig. 1.3. Polar form of complex numbers. The unit circle in the complex plane is described by $e^{i\theta}$. A general complex number is given by $re^{i\theta}$

1.6.1 Powers and Roots of Complex Numbers

To obtain the nth power of a complex number, we take the nth power of the modulus and multiply the phase angle by n,

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta) .$$

This is a correct formula for both positive and negative integer n. But if n is a fraction number, we must use this formula with care. For example, we can interpret $z^{1/4}$ as the fourth root of z. In other words, we want to find a number whose 4th power is equal to z. It is instructive to work out the details for the case of $z = 1$. Clearly

$$\begin{aligned} 1^4 &= \left(e^{i0}\right)^4 = e^{i0} = 1, \\ i^4 &= \left(e^{i\pi/2}\right)^4 = e^{i2\pi} = 1, \\ (-1)^4 &= \left(e^{i\pi}\right)^4 = e^{i4\pi} = 1, \\ (-i)^4 &= \left(e^{i3\pi/2}\right)^4 = e^{i6\pi} = 1. \end{aligned}$$

Therefore there are four distinct answers

$$1^{1/4} = \begin{cases} 1 \ , \\ i \ , \\ -1 \ , \\ -i \ . \end{cases}$$

The multiplicity of roots is tied to the multiple ways of representing 1 in the polar form: e^{i0}, $e^{i2\pi}$, $e^{i4\pi}$, etc. Thus to compute all the nth roots of z, we must express z as

$$z = r\mathrm{e}^{\mathrm{i}\theta + \mathrm{i}k2\pi}, \quad (k = 0, 1, 2, \ldots, n-1)$$

and

$$z^{\frac{1}{n}} = \sqrt[n]{r}\mathrm{e}^{\mathrm{i}\theta/n + \mathrm{i}k2\pi/n}, \quad (k = 0, 1, 2, \ldots, n-1).$$

The reason that k stops at $n-1$ is because once k reaches n, $\mathrm{e}^{\mathrm{i}k2\pi/n} = \mathrm{e}^{\mathrm{i}2\pi} = 1$ and the root repeats itself. Therefore there are n distinct roots.

In general, if n and m are positive integers that have no common factor, then

$$z^{m/n} = \sqrt[n]{|z|^m}\mathrm{e}^{\mathrm{i}\frac{m}{n}(\theta + 2k\pi)} = \sqrt[n]{|z|^m}\left[\cos\frac{m}{n}(\theta + 2k\pi) + \mathrm{i}\sin\frac{m}{n}(\theta + 2k\pi)\right]$$

where $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$ and $k = 0, 1, 2, \ldots, n-1$.

Example 1.6.1. Express $(1+\mathrm{i})^8$ in the form of $a + b\mathrm{i}$.

Solution 1.6.1. Let $z = (1 + \mathrm{i}) = r\mathrm{e}^{\mathrm{i}\theta}$, where

$$r = (zz^*)^{1/2} = \sqrt{2}, \qquad \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}.$$

It follows that:

$$(1+\mathrm{i})^8 = z^8 = r^8\mathrm{e}^{\mathrm{i}8\theta} = 16\mathrm{e}^{\mathrm{i}2\pi} = 16.$$

Example 1.6.2. Express the following in the form of $a + b\mathrm{i}$:

$$\frac{\left(\frac{3}{2}\sqrt{3} + \frac{3}{2}\mathrm{i}\right)^6}{\left(\sqrt{\frac{5}{2}} + \mathrm{i}\sqrt{\frac{5}{2}}\right)^3}.$$

Solution 1.6.2. Let us denote

$$z_1 = \left(\frac{3}{2}\sqrt{3} + \frac{3}{2}\mathrm{i}\right) = r_1\mathrm{e}^{\mathrm{i}\theta_1},$$

$$z_2 = \left(\sqrt{\frac{5}{2}} + \mathrm{i}\sqrt{\frac{5}{2}}\right) = r_2\mathrm{e}^{\mathrm{i}\theta_2},$$

where

$$r_1 = (z_1 z_1^*)^{1/2} = 3, \qquad \theta_1 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6},$$

$$r_2 = (z_2 z_2^*)^{1/2} = \sqrt{5}, \quad \theta_2 = \tan^{-1}(1) = \frac{\pi}{4}.$$

Thus

$$\frac{\left(\frac{3}{2}\sqrt{3}+\frac{3}{2}\mathrm{i}\right)^6}{\left(\sqrt{\frac{5}{2}}+\mathrm{i}\sqrt{\frac{5}{2}}\right)^3}=\frac{z_1^6}{z_2^3}=\frac{\left(3\mathrm{e}^{\mathrm{i}\pi/6}\right)^6}{\left(\sqrt{5}\mathrm{e}^{\mathrm{i}\pi/4}\right)^3}=\frac{3^6\mathrm{e}^{\mathrm{i}\pi}}{\left(\sqrt{5}\right)^3\mathrm{e}^{\mathrm{i}3\pi/4}}$$

$$=\frac{729}{5\sqrt{5}}\mathrm{e}^{\mathrm{i}(\pi-3\pi/4)}=\frac{729}{5\sqrt{5}}\mathrm{e}^{\mathrm{i}\pi/4}$$

$$=\frac{729}{5\sqrt{5}}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)=\frac{729}{5\sqrt{10}}\left(1+\mathrm{i}\right).$$

Example 1.6.3. Find all the cube roots of 8.

Solution 1.6.3. Express 8 as a complex number z in the complex plane

$$z=8\mathrm{e}^{\mathrm{i}k2\pi},\quad k=0,1,2,\cdots.$$

Therefore

$$z^{1/3}=(8)^{1/3}\mathrm{e}^{\mathrm{i}k2\pi/3}=2\mathrm{e}^{\mathrm{i}k2\pi/3},\quad k=0,1,2.$$

$$z^{1/3}=\begin{cases} 2\mathrm{e}^{\mathrm{i}0}=2, & k=0\\ 2\mathrm{e}^{\mathrm{i}2\pi/3}=2\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)=-1+\mathrm{i}\sqrt{3}, & k=1\\ 2\mathrm{e}^{\mathrm{i}4\pi/3}=2\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right)=-1-\mathrm{i}\sqrt{3}, & k=2.\end{cases}$$

Note that the three roots are on a circle of radius 2 centered at the origin. They are 120° apart.

Example 1.6.4. Find all the cube roots of $\sqrt{2}+\mathrm{i}\sqrt{2}$.

Solution 1.6.4. The polar form of $\sqrt{2}+\mathrm{i}\sqrt{2}$ is

$$z=\sqrt{2}+\mathrm{i}\sqrt{2}=2\mathrm{e}^{\mathrm{i}\pi/4+\mathrm{i}k2\pi}.$$

The cube roots of $\sqrt{2}+\mathrm{i}\sqrt{2}$ are given by

$$z^{1/3}=\begin{cases} (2)^{1/3}\,\mathrm{e}^{\mathrm{i}\pi/12}=(2)^{1/3}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right), & k=0,\\ (2)^{1/3}\,\mathrm{e}^{\mathrm{i}(\pi/12+2\pi/3)}=(2)^{1/3}\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right), & k=1,\\ (2)^{1/3}\,\mathrm{e}^{\mathrm{i}(\pi/12+4\pi/3)}=(2)^{1/3}\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right), & k=2.\end{cases}$$

Again the three roots are on a circle 120° apart.

Example 1.6.5. Find all the values of z that satisfy the equation $z^4 = -64$.

Solution 1.6.5. Express -64 as a point in the complex plane

$$-64 = 64\mathrm{e}^{\mathrm{i}\pi + \mathrm{i}k2\pi}, \quad k = 0, 1, 2, \ldots .$$

It follows that:

$$z = (-64)^{1/4} = (64)^{1/4}\,\mathrm{e}^{\mathrm{i}(\pi+2k\pi)/4}, \quad k = 0, 1, 2, 3,$$

$$z = \begin{cases} 2\sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}) = 2 + 2\mathrm{i}, & k = 0 \\ 2\sqrt{2}(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}) = -2 + 2\mathrm{i}, & k = 1 \\ 2\sqrt{2}(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}) = -2 - 2\mathrm{i}, & k = 2 \\ 2\sqrt{2}(\cos\frac{7\pi}{4} + \mathrm{i}\sin\frac{7\pi}{4}) = 2 - 2\mathrm{i}, & k = 3. \end{cases}$$

Note that the four roots are on a circle of radius $\sqrt{8}$ centered at the origin. They are $90°$ apart.

Example 1.6.6. Find all the values of $(1-\mathrm{i})^{3/2}$.

Solution 1.6.6.

$$(1-\mathrm{i}) = \sqrt{2}\mathrm{e}^{\mathrm{i}\theta}, \qquad \theta = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

$$(1-\mathrm{i})^3 = 2\sqrt{2}\mathrm{e}^{\mathrm{i}3\theta+\mathrm{i}k2\pi}, \quad k = 0, 1, 2, \ldots .$$

$$(1-\mathrm{i})^{3/2} = \sqrt[4]{8}\mathrm{e}^{\mathrm{i}(3\theta/2+k\pi)}, \quad k = 0, 1.$$

$$(1-\mathrm{i})^{3/2} = \begin{cases} \sqrt[4]{8}\left[\cos\left(-\frac{3\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{8}\right)\right], & k = 0 \\ \sqrt[4]{8}\left[\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right], & k = 1. \end{cases}$$

1.6.2 Trigonometry and Complex Numbers

Many trigonometric identities can be most elegantly proved with complex numbers. For example, taking the complex conjugate of the Euler formula

$$(\mathrm{e}^{\mathrm{i}\theta})^* = (\cos\theta + \mathrm{i}\sin\theta)^*,$$

we have

$$\mathrm{e}^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta.$$

It is interesting to write this equation as

$$\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(-\theta)} = \cos(-\theta) + \mathrm{i}\sin(-\theta).$$

Comparing the last two equations, we find that

$$\begin{aligned}\cos(-\theta) &= \cos\theta \\ \sin(-\theta) &= -\sin\theta\end{aligned}$$

which is consistent with what we know about the cosine and sine functions of trigonometry.

Adding and subtracting $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$, we have

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} &= (\cos\theta + \mathrm{i}\sin\theta) + (\cos\theta - \mathrm{i}\sin\theta) = 2\cos\theta, \\ \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} &= (\cos\theta + \mathrm{i}\sin\theta) - (\cos\theta - \mathrm{i}\sin\theta) = 2\mathrm{i}\sin\theta.\end{aligned}$$

Using them one can easily express the powers of cosine and sine in terms of $\cos n\theta$ and $\sin n\theta$. For example, with $n = 2$

$$\begin{aligned}\cos^2\theta &= \left[\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^2 = \frac{1}{4}\left(\mathrm{e}^{\mathrm{i}2\theta} + 2\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) \\ &= \frac{1}{2}\left[\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + \mathrm{e}^{\mathrm{i}0}\right] = \frac{1}{2}\left(\cos 2\theta + 1\right),\end{aligned}$$

$$\begin{aligned}\sin^2\theta &= \left[\frac{1}{2i}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^2 = -\frac{1}{4}\left(\mathrm{e}^{\mathrm{i}2\theta} - 2\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) \\ &= \frac{1}{2}\left[-\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + \mathrm{e}^{\mathrm{i}0}\right] = \frac{1}{2}\left(-\cos 2\theta + 1\right).\end{aligned}$$

To find an identity for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$, we can view them as components of $\exp[\mathrm{i}(\theta_1 + \theta_2)]$. Since

$$\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2} = \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2),$$

and

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2} &= [\cos\theta_1 + \mathrm{i}\sin\theta_1][\cos\theta_2 + \mathrm{i}\sin\theta_2] \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2),\end{aligned}$$

equating the real and imaginary parts of these equivalent expressions, we get the familiar formulas

$$\begin{aligned}\cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \\ \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.\end{aligned}$$

From these two equations, it follows that:

$$\tan(\theta_1 + \theta_2) = \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} = \frac{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}.$$

Dividing top and bottom by $\cos\theta_1 \cos\theta_2$, we obtain

$$\tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}.$$

This formula can be derived directly with complex numbers. Let z_1 and z_2 be two points in the complex plane whose x components are both equal to 1.

$$z_1 = 1+\mathrm{i}y_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}, \qquad \tan\theta_1 = \frac{y_1}{1} = y_1,$$
$$z_2 = 1+\mathrm{i}y_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}, \qquad \tan\theta_2 = \frac{y_2}{1} = y_2.$$

The product of the two is given by

$$z_1z_2 = r_1r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}, \qquad \tan(\theta_1+\theta_2) = \frac{\mathrm{Im}(z_1z_2)}{\mathrm{Re}(z_1z_2)}.$$

But

$$z_1z_2 = (1+\mathrm{i}y_1)(1+\mathrm{i}y_2) = (1-y_1y_2)+\mathrm{i}(y_1+y_2),$$

therefore

$$\tan(\theta_1+\theta_2) = \frac{\mathrm{Im}(z_1z_2)}{\mathrm{Re}(z_1z_2)} = \frac{y_1+y_2}{1-y_1y_2} = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}.$$

These identities can, of course, be demonstrated geometrically. However, it is much easier to prove them algebraically with complex numbers.

Example 1.6.7. Prove De Moivre formula

$$(\cos\theta+\mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

Solution 1.6.7. Since $(\cos\theta+\mathrm{i}\sin\theta) = \mathrm{e}^{\mathrm{i}\theta}$, it follows: that

$$\begin{aligned}(\cos\theta+\mathrm{i}\sin\theta)^n &= \left(\mathrm{e}^{\mathrm{i}\theta}\right)^n = \mathrm{e}^{\mathrm{i}n\theta} \\ &= \cos n\theta + \mathrm{i}\sin n\theta.\end{aligned}$$

This theorem was published in 1707 by Abraham De Moivre, a French mathematician working in London.

Example 1.6.8. Use De Moivre's theorem and binomial expansion to express $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Solution 1.6.8.

$$\begin{aligned}\cos 4\theta + \mathrm{i}\sin 4\theta &= \mathrm{e}^{\mathrm{i}4\theta} = \left(\mathrm{e}^{\mathrm{i}\theta}\right)^4 = (\cos\theta + \mathrm{i}\sin\theta)^4 \\ &= \cos^4\theta + 4\cos^3\theta(\mathrm{i}\sin\theta) + 6\cos^2\theta(\mathrm{i}\sin\theta)^2 \\ &\quad + 4\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4 \\ &= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) \\ &\quad + \mathrm{i}\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)\end{aligned}$$

Equating the real and imaginary parts of these complex expressions, we obtain

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \\ \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.\end{aligned}$$

Example 1.6.9. Express $\cos^4\theta$ and $\sin^4\theta$ in terms of multiples of θ.

Solution 1.6.9.

$$\begin{aligned}\cos^4\theta &= \left[\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^4 \\ &= \frac{1}{16}\left[\left(\mathrm{e}^{\mathrm{i}4\theta} + \mathrm{e}^{-\mathrm{i}4\theta}\right) + 4\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + 6\right] \\ &= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.\end{aligned}$$

$$\begin{aligned}\sin^4\theta &= \left[\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}\right)\right]^4 \\ &= \frac{1}{16}\left[\left(\mathrm{e}^{\mathrm{i}4\theta} + \mathrm{e}^{-\mathrm{i}4\theta}\right) - 4\left(\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}\right) + 6\right] \\ &= \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}.\end{aligned}$$

Example 1.6.10. Show that

$$\begin{aligned}\cos(\theta_1+\theta_2+\theta_3) &= \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3 \\ &\quad - \sin\theta_1\sin\theta_3\cos\theta_2 - \sin\theta_2\sin\theta_3\cos\theta_1, \\ \sin(\theta_1+\theta_2+\theta_3) &= \sin\theta_1\cos\theta_2\cos\theta_3 + \sin\theta_2\cos\theta_1\cos\theta_3 \\ &\quad + \sin\theta_3\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\sin\theta_3.\end{aligned}$$

Solution 1.6.10.

$$\cos(\theta_1+\theta_2+\theta_3)+\mathrm{i}\sin(\theta_1+\theta_2+\theta_3)=\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2+\theta_3)}=\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}\mathrm{e}^{\mathrm{i}\theta_3},$$

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}\mathrm{e}^{\mathrm{i}\theta_3} &= (\cos\theta_1+\mathrm{i}\sin\theta_1)(\cos\theta_2+\mathrm{i}\sin\theta_2)(\cos\theta_3+\mathrm{i}\sin\theta_3)\\ &= \cos\theta_1(1+\mathrm{i}\tan\theta_1)\cos\theta_2(1+\mathrm{i}\tan\theta_2)\cos\theta_3(1+\mathrm{i}\tan\theta_3).\end{aligned}$$

Since

$$(1+a)(1+b)(1+c)=1+(a+b+b)+(ab+bc+ca)+abc,$$

$$\begin{aligned}&(1+\mathrm{i}\tan\theta_1)(1+\mathrm{i}\tan\theta_2)(1+\mathrm{i}\tan\theta_3)=1+\mathrm{i}(\tan\theta_1+\tan\theta_2+\tan\theta_3)\\ &+\mathrm{i}^2(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)+\mathrm{i}^3\tan\theta_1\tan\theta_2\tan\theta_3\\ &\qquad=[1-(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)]\\ &\qquad\quad+\mathrm{i}[(\tan\theta_1+\tan\theta_2+\tan\theta_3)-\tan\theta_1\tan\theta_2\tan\theta_3].\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{\mathrm{i}\theta_2}\mathrm{e}^{\mathrm{i}\theta_3} &= \cos\theta_1\cos\theta_2\cos\theta_3\\ &\times\left\{\begin{array}{l}[1-(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)]\\ +\mathrm{i}[(\tan\theta_1+\tan\theta_2+\tan\theta_3)-\tan\theta_1\tan\theta_2\tan\theta_3]\end{array}\right\}.\end{aligned}$$

Equating the real and imaginary parts

$$\begin{aligned}\cos(\theta_1+\theta_2+\theta_3) &= \cos\theta_1\cos\theta_2\cos\theta_3\\ &\quad\times[1-(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\tan\theta_3\tan\theta_1)]\\ &= \cos\theta_1\cos\theta_2\cos\theta_3-\sin\theta_1\sin\theta_2\cos\theta_3\\ &\quad-\sin\theta_1\sin\theta_3\cos\theta_2-\sin\theta_2\sin\theta_3\cos\theta_1,\end{aligned}$$

$$\begin{aligned}\sin(\theta_1+\theta_2+\theta_3) &= \cos\theta_1\cos\theta_2\cos\theta_3\\ &\quad\times[(\tan\theta_1+\tan\theta_2+\tan\theta_3)-\tan\theta_1\tan\theta_2\tan\theta_3]\\ &= \sin\theta_1\cos\theta_2\cos\theta_3+\sin\theta_2\cos\theta_1\cos\theta_3\\ &\quad+\sin\theta_3\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\sin\theta_3.\end{aligned}$$

Example 1.6.11. If $\theta_1,\theta_2,\theta_3$ are the three interior angles of a triangle, show that

$$\tan\theta_1+\tan\theta_2+\tan\theta_3=\tan\theta_1\tan\theta_2\tan\theta_3.$$

Solution 1.6.11. Since

$$\tan(\theta_1+\theta_2+\theta_3)=\frac{\sin(\theta_1+\theta_2+\theta_3)}{\cos(\theta_1+\theta_2+\theta_3)},$$

using the results of the previous problem and dividing the top and bottom by $\cos\theta_1\cos\theta_2\cos\theta_3$, we have

$$\tan(\theta_1+\theta_2+\theta_3)=\frac{\tan\theta_1+\tan\theta_2+\tan\theta_3-\tan\theta_1\tan\theta_2\tan\theta_3}{1-\tan\theta_1\tan\theta_2-\tan\theta_2\tan\theta_3-\tan\theta_3\tan\theta_1}.$$

Now $\theta_1,\theta_2,\theta_3$ are the three interior angles of a triangle, so $\theta_1+\theta_2+\theta_3=\pi$ and $\tan(\theta_1+\theta_2+\theta_3)=\tan\pi=0$. Therefore

$$\tan\theta_1+\tan\theta_2+\tan\theta_3=\tan\theta_1\tan\theta_2\tan\theta_3.$$

Example 1.6.12. Show that

$$\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2n-1)\theta=\frac{\sin n\theta\cos n\theta}{\sin\theta},$$
$$\sin\theta+\sin 3\theta+\sin 5\theta+\cdots+\sin(2n-1)\theta=\frac{\sin^2 n\theta}{\sin\theta}.$$

Solution 1.6.12. Let

$$C=\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2n-1)\theta,$$
$$S=\sin\theta+\sin 3\theta+\sin 5\theta+\cdots+\sin(2n-1)\theta.$$

$$\begin{aligned}Z=C+\mathrm{i}S&=(\cos\theta+\mathrm{i}\sin\theta)+(\cos 3\theta+\mathrm{i}\sin 3\theta)\\&\quad+(\cos 5\theta+\mathrm{i}\sin 5\theta)+\cdots+(\cos(n-1)\theta+\mathrm{i}\sin(2n-1)\theta)\\&=\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}3\theta}+\mathrm{e}^{\mathrm{i}5\theta}+\cdots+\mathrm{e}^{\mathrm{i}(2n-1)\theta}.\end{aligned}$$

$$\mathrm{e}^{\mathrm{i}2\theta}Z=\mathrm{e}^{\mathrm{i}3\theta}+\mathrm{e}^{\mathrm{i}5\theta}+\cdots+\mathrm{e}^{\mathrm{i}(2n-1)\theta}+\mathrm{e}^{\mathrm{i}(2n+1)\theta},$$

$$Z-\mathrm{e}^{\mathrm{i}2\theta}Z=\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}(2n+1)\theta},$$

$$\begin{aligned}Z&=\frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}(2n+1)\theta}}{1-\mathrm{e}^{\mathrm{i}2\theta}}=\frac{\mathrm{e}^{\mathrm{i}\theta}(1-\mathrm{e}^{\mathrm{i}2n\theta})}{\mathrm{e}^{\mathrm{i}\theta}\left(\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\right)}=\frac{\mathrm{e}^{in\theta}\left(\mathrm{e}^{-in\theta}-\mathrm{e}^{in\theta}\right)}{\left(\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}\right)}\\&=\frac{\mathrm{e}^{in\theta}\sin n\theta}{\sin\theta}=\frac{\cos n\theta\sin n\theta}{\sin\theta}+i\frac{\sin n\theta\sin n\theta}{\sin\theta}.\end{aligned}$$

Therefore

$$C=\frac{\cos n\theta\sin n\theta}{\sin\theta},\qquad S=\frac{\sin^2 n\theta}{\sin\theta}.$$

Example 1.6.13. For $r < 1$, show that

$$\left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2 = \frac{1}{1 - 2r^2 \cos\theta + r^4}.$$

Solution 1.6.13. Let

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} r^{2n} \cos n\theta + \mathrm{i} \sum_{n=0}^{\infty} r^{2n} \sin n\theta \\ &= \sum_{n=0}^{\infty} r^{2n} (\cos n\theta + \mathrm{i} \sin n\theta) = \sum_{n=0}^{\infty} r^{2n} \mathrm{e}^{\mathrm{i}n\theta} \\ &= 1 + r^2 \mathrm{e}^{\mathrm{i}\theta} + r^4 \mathrm{e}^{\mathrm{i}2\theta} + r^6 \mathrm{e}^{\mathrm{i}3\theta} + \cdots . \end{aligned}$$

Since $r < 1$, so this is a convergent series

$$r^2 \mathrm{e}^{\mathrm{i}\theta} Z = r^2 \mathrm{e}^{\mathrm{i}\theta} + r^4 \mathrm{e}^{\mathrm{i}2\theta} + r^6 \mathrm{e}^{\mathrm{i}3\theta} + \cdots .$$

$$Z - r^2 \mathrm{e}^{\mathrm{i}\theta} Z = 1,$$

$$Z = \frac{1}{1 - r^2 \mathrm{e}^{\mathrm{i}\theta}}.$$

$$\begin{aligned} |Z|^2 = ZZ^* &= \frac{1}{1 - r^2 \mathrm{e}^{\mathrm{i}\theta}} \times \frac{1}{1 - r^2 \mathrm{e}^{-\mathrm{i}\theta}} \\ &= \frac{1}{1 - r^2(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) + r^4} = \frac{1}{1 - 2r^2 \cos\theta + r^4}, \end{aligned}$$

But

$$\begin{aligned} |Z|^2 &= \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta + \mathrm{i} \sum_{n=0}^{\infty} r^{2n} \sin n\theta\right) \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta - \mathrm{i} \sum_{n=0}^{\infty} r^{2n} \sin n\theta\right) \\ &= \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2 . \end{aligned}$$

Therefore

$$\left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2 = \frac{1}{1 - 2r^2 \cos\theta + r^4}.$$

This is the intensity of the light, transmitted through a film after multiple reflections at the surfaces of the film and r is the fraction of light reflected each time.

1.6.3 Geometry and Complex Numbers

There are three geometric representations of the complex number $z = x + \mathrm{i}y$:

(a) as the point $P\,(x, y)$ in the xy plane,

(b) as the vector OP from the origin to the point P,

(c) as any vector that is of the same length and same direction as OP.

For example, $z_A = 3 + \mathrm{i}$ can be represented by the point A in Fig. 1.4. It can also be represented by the vector z_A. Similarly $z_B = -2 + 3\mathrm{i}$ can be represented by the point B and the vector z_B. Now let us define z_C as $z_A + z_B$,

$$z_C = z_A + z_B = (3 + \mathrm{i}) + (-2 + 3\mathrm{i}) = 1 + 4\mathrm{i}.$$

So z_C is represented by the point C and the vector z_C. Clearly the two shaded triangles in Fig. 1.4 are identical. The vector AC (from A to C) is not only parallel to z_B, it is also of the same length as z_B. In this sense, we say that the vector AC can also represent z_B. Thus z_A, z_B, and $z_A + z_B$ are three sides of the triangle OAC. Since the sum of two sides of a triangle must be greater or equal to the third side, it follows that

$$|z_A| + |z_B| \geq |z_A + z_B|\,.$$

Since $z_B = z_C - z_A$, and z_B is the same as the AC, we can interpret $z_C - z_A$ as the vector from the tip of z_A to the tip of z_C. The distance between C and A is simply $|z_C - z_A|$.

If z is a variable and z_A is fixed, then a circle of radius r centered at z_A is described by the equation

$$|z - z_A| = r.$$

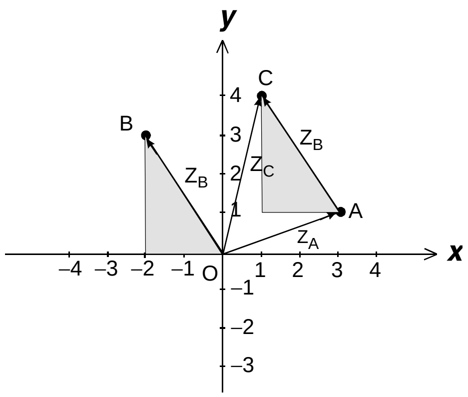

Fig. 1.4. Addition and subtraction of complex numbers in the complex plane. A complex number can be represented by a point in the complex plane, or by the vector from the origin to that point. The vector can be moved parallel to itself

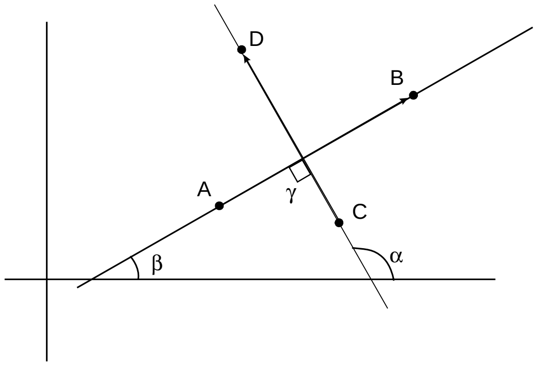

Fig. 1.5. Perpendicular segments. If AB and CD are perpendicular, then the ratio of $z_B - z_A$ and $z_D - z_C$ must be purely imaginary

If the two segments AB and CD are parallel, then

$$z_B - z_A = k\,(z_D - z_C)\,,$$

where k is a real number. If $k = 1$, then A, B, C, D must be the vertices of a parallelogram.

If the two segments AB and CD are perpendicular to each other, then the ratio $\frac{z_D - z_C}{z_B - z_A}$ must be a pure imaginary number. This can be seen as follows.

The segment AB in Fig. 1.5 can be expressed as

$$z_B - z_A = |z_B - z_A|\, \mathrm{e}^{\mathrm{i}\beta},$$

and segment CD as

$$z_D - z_C = |z_D - z_C|\, \mathrm{e}^{\mathrm{i}\alpha}.$$

So

$$\frac{z_D - z_C}{z_B - z_A} = \frac{|z_D - z_C|\, \mathrm{e}^{\mathrm{i}\alpha}}{|z_B - z_A|\, \mathrm{e}^{\mathrm{i}\beta}} = \frac{|z_D - z_C|}{|z_B - z_A|} \mathrm{e}^{\mathrm{i}(\alpha-\beta)}.$$

It is well known that the exterior angle is equal to the sum of the two interior angles, that is, in Fig. 1.5 $\alpha = \beta + \gamma$, or $\gamma = \alpha - \beta$. If AB is perpendicular to CD, then $\gamma = \frac{\pi}{2}$, and

$$\mathrm{e}^{\mathrm{i}(\alpha-\beta)} = \mathrm{e}^{\mathrm{i}\gamma} = \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}.$$

Thus

$$\frac{z_D - z_C}{z_B - z_A} = \frac{|z_D - z_C|}{|z_B - z_A|}\mathrm{i}.$$

Since $\frac{|z_D - z_C|}{|z_B - z_A|}$ is real, so $\frac{z_D - z_C}{z_B - z_A}$ must be imaginary.

The following examples will illustrate how to use these principles to solve problems in geometry.

Example 1.6.14. Determine the curve in the complex plane that is described by

$$\left|\frac{z+1}{z-1}\right| = 2.$$

Solution 1.6.14. $\left|\frac{z+1}{z-1}\right| = 2$ can be written as $|z+1| = 2\,|z-1|$. With $z = x + \mathrm{i}y$, this equation becomes

$$|(x+1) + \mathrm{i}y| = 2\,|(x-1) + \mathrm{i}y|\,,$$

$$\{[(x+1) + \mathrm{i}y]\,[(x+1) - \mathrm{i}y]\}^{1/2} = 2\,\{[(x-1) + \mathrm{i}y]\,[(x-1) - \mathrm{i}y]\}^{1/2}\,.$$

Square both sides

$$(x+1)^2 + y^2 = 4\,(x-1)^2 + 4y^2.$$

This gives

$$3x^2 - 10x + 3y^2 + 3 = 0,$$

which can be written as

$$\left(x - \frac{5}{3}\right)^2 + y^2 - \left(\frac{5}{3}\right)^2 + 1 = 0,$$

or

$$\left(x - \frac{5}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2.$$

This represents a circle of radius $\frac{4}{3}$ with a center at $\left(\frac{5}{3}, 0\right)$.

Example 1.6.15. In the parallelogram shown in Fig. 1.6, the base is fixed along the x-axis and is of length a. The length of the other side is b. As the angle θ between the two sides changes, determine the locus of the center of the parallelogram.

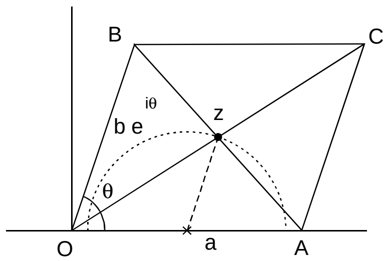

Fig. 1.6. The curve described by the center of a parallelogram. If the base is fixed, the locus of the center is a circle

Solution 1.6.15. Let the origin of the coordinates be at the left bottom corner of the parallelogram. So

$$z_A = a, \qquad z_B = b\mathrm{e}^{\mathrm{i}\theta}.$$

Let the center of the parallelogram be z which is at the midpoint of the diagonal OC. Thus

$$z = \frac{1}{2}(z_A + z_B) = \frac{1}{2}a + \frac{1}{2}b\mathrm{e}^{\mathrm{i}\theta}$$

or

$$z - \frac{1}{2}a = \frac{1}{2}b\mathrm{e}^{\mathrm{i}\theta}$$

It follows that:

$$\left| z - \frac{1}{2}a \right| = \left| \frac{1}{2}b\mathrm{e}^{\mathrm{i}\theta} \right| = \frac{1}{2}b.$$

Therefore the locus of the center is a circle of radius $\frac{1}{2}b$ centered at $\frac{1}{2}a$. Half of the circle is shown in Fig. 1.6.

Example 1.6.16. If E, F, G, H are midpoints of the quadrilateral $ABCD$. Prove that $EFGH$ is a parallelogram.

Solution 1.6.16. Let the vector from origin to any point P be z_P, then from Fig. 1.7 we see that

$$z_E = z_A + \frac{1}{2}(z_B - z_A),$$
$$z_F = z_B + \frac{1}{2}(z_C - z_B),$$

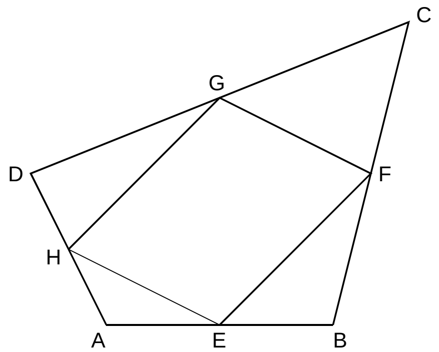

Fig. 1.7. Parallelogram formed by the midpoints of a quadrilateral

$$z_F - z_E = z_B + \frac{1}{2}(z_C - z_B) - z_A - \frac{1}{2}(z_B - z_A) = \frac{1}{2}(z_C - z_A).$$

$$z_G = z_D + \frac{1}{2}(z_C - z_D),$$
$$z_H = z_A + \frac{1}{2}(z_D - z_A),$$

$$z_G - z_H = z_D + \frac{1}{2}(z_C - z_D) - z_A - \frac{1}{2}(z_D - z_A) = \frac{1}{2}(z_C - z_A).$$

Thus

$$z_F - z_E = z_G - z_H.$$

Therefore $EFGH$ is a parallelogram.

Example 1.6.17. Use complex number to show that the diagonals of a rhombus (a parallelogram with equal sides) are perpendicular to each other, as shown in Fig. 1.8.

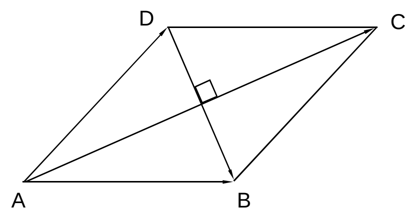

Fig. 1.8. The diagonals of a rhombus are perpendicular to each other

Solution 1.6.17. The diagonal AC is given by $z_C - z_A$, and the diagonal DB is given by $z_B - z_D$. Let the length of each side of the rhombus be a, and the origin of the coordinates coincide with A. Furthermore let the x-axis be along the line AB. Thus

$$z_A = 0, \quad z_B = a, \quad z_D = a\mathrm{e}^{\mathrm{i}\theta}.$$

Furthermore,

$$z_C = z_B + z_D = a + a\mathrm{e}^{\mathrm{i}\theta}.$$

Therefore

$$\begin{aligned} z_C - z_A &= a + a\mathrm{e}^{\mathrm{i}\theta} = a\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right), \\ z_B - z_D &= a - a\mathrm{e}^{\mathrm{i}\theta} = a\left(1 - \mathrm{e}^{\mathrm{i}\theta}\right). \end{aligned}$$

Thus

$$\begin{aligned} \frac{z_C - z_A}{z_B - z_D} &= \frac{a\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right)}{a\left(1 - \mathrm{e}^{\mathrm{i}\theta}\right)} = \frac{\left(1 + \mathrm{e}^{\mathrm{i}\theta}\right)\left(1 - \mathrm{e}^{-\mathrm{i}\theta}\right)}{\left(1 - \mathrm{e}^{\mathrm{i}\theta}\right)\left(1 - \mathrm{e}^{-\mathrm{i}\theta}\right)} \\ &= \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2 - \left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right)} = \mathrm{i}\frac{\sin\theta}{1 - \cos\theta}. \end{aligned}$$

Since $\frac{\sin\theta}{1-\cos\theta}$ is real, $\frac{z_C - z_A}{z_D - z_B}$ is purely imaginary. Hence AC is perpendicular to DB.

Example 1.6.18. In the triangle AOB, shown in Fig. 1.9, the angle between AO and OB is $90°$ and the length of AO is the same as the length of OB. The point D trisects the line AB such that $AD = 2DB$, and C is the midpoint of OB. Show that AC is perpendicular to OD.

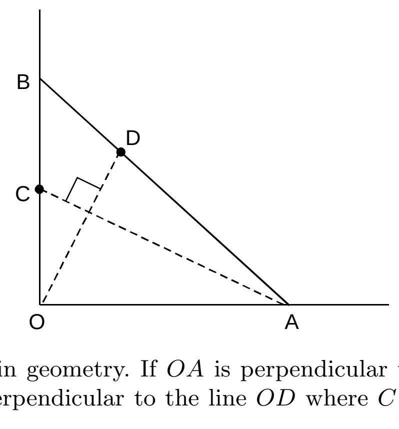

Fig. 1.9. A problem in geometry. If OA is perpendicular to OB and $OA = OB$, then the line CA is perpendicular to the line OD where C is the midpoint of OB and $DA = 2BD$

Solution 1.6.18. Let the real axis be along OA and the imaginary axis along OB. Let the length of OA and OB be a. Thus

$$z_O = 0, \quad z_A = a, \quad z_B = a\mathrm{i}, \quad z_C = \frac{1}{2}a\mathrm{i},$$
$$z_D = z_A + \frac{2}{3}(z_B - z_A) = a + \frac{2}{3}(a\mathrm{i} - a) = \frac{1}{3}a(1 + 2\mathrm{i}),$$
$$z_D - z_O = \frac{1}{3}a(1 + 2\mathrm{i}) - 0 = \frac{1}{3}a(1 + 2\mathrm{i}).$$

The vector AC is given by $z_C - z_A$,

$$z_C - z_A = \frac{1}{2}a\mathrm{i} - a = \mathrm{i}\frac{1}{2}a(1 + 2\mathrm{i}).$$

Thus

$$\frac{z_C - z_A}{z_D - z_O} = \mathrm{i}\frac{3}{2}.$$

Since this is purely imaginary, therefore AC is perpendicular to OD.

1.7 Elementary Functions of Complex Variable

1.7.1 Exponential and Trigonometric Functions of *z*

The exponential function e^z is of fundamental importance, not only for its own sake, but also as a basis for defining all the other elementary functions. The exponential function of real variable is well known. Now we wish to give meaning to e^z when $z = x + \mathrm{i}y$. In the spirit of Euler, we can work our way in a purely manipulative manner. Assuming that e^z obeys all the familiar rules of the exponential function of a real number, we have

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x\mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y). \tag{1.19}$$

Thus we can define e^z as $\mathrm{e}^x(\cos y + \mathrm{i}\sin y)$. It reduces to e^x when the imaginary part of z vanishes. It is also easy to show that

$$\mathrm{e}^{z_1}\mathrm{e}^{z_2} = \mathrm{e}^{z_1+z_2}.$$

Furthermore, in Chap. 2 we shall consider in detail the meaning of derivatives with respect to a complex z. Now it suffices to know that

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^z = \mathrm{e}^z.$$

Therefore the definition of (1.19) preserves all the familiar properties of the exponential function.

We have already seen that

$$\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}), \qquad \sin\theta = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}).$$

On the basis of these equations, we extend the definitions of the cosine and sine into the complex domain. Thus we define

$$\cos z = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}), \qquad \sin z = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}).$$

The rest of the trigonometric functions of z are defined in a usual way. For example,

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}, \\ \sec z &= \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}. \end{aligned}$$

With these definitions we can show that all the familiar formulas of trigonometry remain valid when real variable x is replaced by complex variable z:

$$\cos(-z) = \cos z, \qquad \sin(-z) = -\sin z,$$

$$\cos^2 z + \sin^2 z = 1,$$

$$\begin{aligned} \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, \\ \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \\ \frac{\mathrm{d}}{\mathrm{d}z}\cos z &= -\sin z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z. \end{aligned}$$

To prove them, we must start with their definitions. For example,

$$\begin{aligned} \cos^2 z + \sin^2 z &= \left[\frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})\right]^2 + \left[\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})\right]^2 \\ &= \frac{1}{4}(\mathrm{e}^{\mathrm{i}2z} + 2 + \mathrm{e}^{-\mathrm{i}2z}) - \frac{1}{4}(\mathrm{e}^{\mathrm{i}2z} - 2 + \mathrm{e}^{-\mathrm{i}2z}) = 1. \end{aligned}$$

Example 1.7.1. Express $\mathrm{e}^{1-\mathrm{i}}$ in the form of $a + b\mathrm{i}$, accurate to three decimal places.

Solution 1.7.1.

$$\mathrm{e}^{1-\mathrm{i}} = \mathrm{e}^{1}\mathrm{e}^{-\mathrm{i}} = e\,(\cos 1 - \mathrm{i}\sin 1)\,.$$

Using a hand-held calculator, we find

$$\begin{aligned} \mathrm{e}^{1-\mathrm{i}} &\simeq 2.718(0.5403 - 0.8415\mathrm{i}) \\ &= 1.469 - 2.287\mathrm{i}. \end{aligned}$$

Example 1.7.2. Show that

$$\sin 2z = 2\sin z \cos z.$$

Solution 1.7.2.

$$\begin{aligned} 2\sin z\cos z &= 2\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})\frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \\ &= \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}2z} - \mathrm{e}^{-\mathrm{i}2z}) = \sin 2z. \end{aligned}$$

Example 1.7.3. Compute $\sin(1-\mathrm{i})$.

Solution 1.7.3. By definition

$$\begin{aligned} \sin(1-\mathrm{i}) &= \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}(1-\mathrm{i})} - \mathrm{e}^{-\mathrm{i}(1-\mathrm{i})}\right) = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{1+\mathrm{i}} - \mathrm{e}^{-1-\mathrm{i}}\right) \\ &= \frac{1}{2\mathrm{i}}\left\{\mathrm{e}\left[\cos(1) + \mathrm{i}\sin(1)\right]\right\} - \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{-1}\left[\cos(1) - \mathrm{i}\sin(1)\right]\right\} \\ &= \frac{1}{2\mathrm{i}}\left(\mathrm{e} - \mathrm{e}^{-1}\right)\cos(1) + \frac{1}{2}\left(\mathrm{e} + \mathrm{e}^{-1}\right)\sin(1). \end{aligned}$$

We can get the same result by using the trigonometric addition formula.

1.7.2 Hyperbolic Functions of z

The following particular combinations of exponentials arise frequently,

$$\cosh z = \frac{1}{2}\left(\mathrm{e}^{z} + \mathrm{e}^{-z}\right), \qquad \sinh z = \frac{1}{2}\left(\mathrm{e}^{z} - \mathrm{e}^{-z}\right).$$

They are called hyperbolic cosine (*abbreviated* cosh) and hyperbolic sine (*abbreviated* sinh) . Clearly

$$\cosh(-z) = \cosh z, \qquad \sinh(-z) = -\sinh z.$$

The other hyperbolic functions are defined in a similar way to parallel the trigonometric functions:

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{1}{\tanh z}, \\ \sec hz &= \frac{1}{\cosh z}, \qquad \csc hz = \frac{1}{\sinh z}. \end{aligned}$$

With these definitions, all identities involving hyperbolic functions of real variable are preserved when the variable is complex. For example,

$$\cosh^2 z - \sinh^2 z = \frac{1}{4}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right)^2 - \frac{1}{4}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right)^2 = 1,$$
$$\sinh 2z = \frac{1}{2}\left(\mathrm{e}^{2z} - \mathrm{e}^{-2z}\right) = \frac{1}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right)\left(\mathrm{e}^z + \mathrm{e}^{-z}\right)$$
$$= 2\sinh z \cosh z.$$

There is a close relationship between the trigonometric and hyperbolic functions when the variable is complex. For example,

$$\sin \mathrm{i}z = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}(\mathrm{i}z)} - \mathrm{e}^{-\mathrm{i}(\mathrm{i}z)}\right) = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{-z} - \mathrm{e}^{z}\right)$$
$$= \frac{\mathrm{i}}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right) = \mathrm{i}\sinh z.$$

Similarly we can show

$$\cos \mathrm{i}z = \cosh z,$$
$$\sinh \mathrm{i}z = \mathrm{i}\sin z, \quad \cosh \mathrm{i}z = \cos z.$$

Furthermore,

$$\sin z = \sin(x + \mathrm{i}y) = \sin x \cos \mathrm{i}y + \cos x \sin \mathrm{i}y$$
$$= \sin x \cosh y + \mathrm{i}\cos x \sinh y,$$
$$\cos z = \cos x \cosh y - \mathrm{i}\sin x \sinh y.$$

Example 1.7.4. Show that

$$\frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \cosh z.$$

Solution 1.7.4.

$$\frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right) = \frac{1}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right) = \sinh z,$$
$$\frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\left(\mathrm{e}^z - \mathrm{e}^{-z}\right) = \frac{1}{2}\left(\mathrm{e}^z + \mathrm{e}^{-z}\right) = \cosh z.$$

Example 1.7.5. Evaluate $\cos(1 + 2\mathrm{i})$.

Solution 1.7.5.

$$\cos(1 + 2\mathrm{i}) = \cos 1 \cosh 2 - \mathrm{i}\sin 1 \sinh 2$$
$$= (0.5403)\,(3.7622) - \mathrm{i}\,(0.8415)\,(3.6269) = 2.033 - 3.052\mathrm{i}.$$

Example 1.7.6. Evaluate $\cos(\pi - \mathrm{i})$.

Solution 1.7.6. By definition,

$$\begin{aligned}\cos(\pi - \mathrm{i}) &= \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}(\pi-\mathrm{i})} + \mathrm{e}^{-\mathrm{i}(\pi-\mathrm{i})}\right) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\pi+1} + \mathrm{e}^{-\mathrm{i}\pi-1}\right)\\ &= \frac{1}{2}(-\mathrm{e} - \mathrm{e}^{-1}) = -\cosh(1) = -1.543.\end{aligned}$$

We get the same result by the expansion,

$$\begin{aligned}\cos(\pi - \mathrm{i}) &= \cos\pi\cosh(1) + \mathrm{i}\sin\pi\sinh(1)\\ &= -\cosh(1) = -1.543.\end{aligned}$$

1.7.3 Logarithm and General Power of z

The natural logarithm of $z = x + \mathrm{i}y$ is denoted $\ln z$ and is defined in a similar way as in the real variable, namely as the inverse of the exponential function. However, there is an important difference. A real valued exponential $y = \mathrm{e}^x$ is a one to one function, since two different x always produce two different values of y. Strictly speaking, only one to one function has an inverse, because only then will each value of y can be the image of exactly one x value. But the complex exponential e^z is a multivalued function, since

$$\mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y).$$

When y is increased by an integer multiple of 2π, the exponential returns to its original value. Therefore to define a complex logarithm we have to relax the one to one restriction. Thus,

$$w = \ln z$$

is defined for $z \neq 0$ by the relation

$$\mathrm{e}^w = z.$$

If we set

$$w = u + \mathrm{i}v, \qquad z = r\mathrm{e}^{\mathrm{i}\theta},$$

this becomes

$$\mathrm{e}^w = \mathrm{e}^{u+\mathrm{i}v} = \mathrm{e}^u\mathrm{e}^{\mathrm{i}v} = r\mathrm{e}^{\mathrm{i}\theta}.$$

Since

$$|\mathrm{e}^w| = \left[(\mathrm{e}^w)(\mathrm{e}^w)^*\right]^{1/2} = \left(\mathrm{e}^{u+\mathrm{i}v}\mathrm{e}^{u-\mathrm{i}v}\right)^{1/2} = \mathrm{e}^u,$$

$$|\mathrm{e}^w| = \left[\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^*\right]^{1/2} = \left[\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\left(r\mathrm{e}^{-\mathrm{i}\theta}\right)\right]^{1/2} = r,$$

Therefore

$$\mathrm{e}^u = r.$$

By definition,

$$u = \ln r.$$

Since $\mathrm{e}^w = z$,

$$\mathrm{e}^u \mathrm{e}^{\mathrm{i}v} = r\mathrm{e}^{\mathrm{i}v} = r\mathrm{e}^{\mathrm{i}\theta}$$

it follows that:

$$v = \theta.$$

Thus

$$w = u + \mathrm{i}v = \ln r + \mathrm{i}\theta.$$

Therefore the rule of logarithm is preserved,

$$\ln z = \ln r\, \mathrm{e}^{\mathrm{i}\theta} = \ln r + \mathrm{i}\theta. \tag{1.20}$$

Since θ is the polar angle, after it is increased by 2π in the z complex plane, it comes back to the same point and z will have the same value. However, the logarithm of z will not return to its original value. Its imaginary part will increase by $2\pi i$. If the argument of z in a particular interval of 2π is denoted as θ_0, then (1.20) can be written as

$$\ln z = \ln r + \mathrm{i}(\theta_0 + 2\pi n), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

By specifying such an interval, we say that we have selected a particular branch of θ as the principal branch. The value corresponding to $n = 0$ is known as the principal value and is commonly denoted as $Ln\ z$, that is

$$Ln\ z = \ln r + \mathrm{i}\theta_0.$$

The choice of the principal branch is somewhat arbitrary.

Figure 1.10 illustrates two possible branch selections. Figure 1.10a depicts the branch that selects the value of the argument of z from the interval $-\pi < \theta \leq \pi$. The values in this branch are most commonly used in complex algebra computer codes. The argument θ is inherently discontinuous, jumping by 2π as z crosses the negative x-axis. This line of discontinuities is known as the branch cut. The cut ends at the origin, which is known as the branch point.

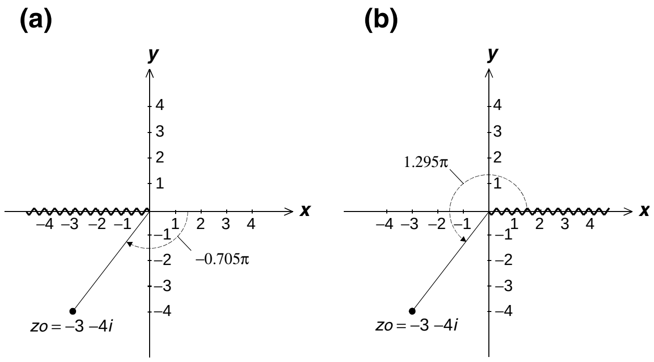

Fig. 1.10. Two possible branch selections. (**a**) Branch cut on the negative x-axis. The point $-3-4\mathrm{i}$ has argument -0.705π. (**b**) Branch cut on the positive x-axis. The point $-3-4\mathrm{i}$ has argument 1.295π

With the branch cut along the negative real axis, the principal value of the logarithm of $z_0 = -3-4\mathrm{i}$ is given by $\ln(|z_0|\,\mathrm{e}^{\mathrm{i}\theta})$ where $\theta = \tan^{-1}\frac{4}{3} - \pi$, thus the principal value is

$$\ln(-3-4\mathrm{i}) = \ln 5\mathrm{e}^{\mathrm{i}\theta} = \ln 5 \ + \mathrm{i}(\tan^{-1}\frac{4}{3} - \pi) = 1.609 - 0.705\pi\mathrm{i}.$$

However, if we select the interval $0 \le \theta < 2\pi$ as the principal branch, then the branch cut is along the positive x-axis, as shown in Fig. 1.10b. In this case the principal value of the logarithm of z_0 is

$$\ln(-3-4\mathrm{i}) = \ln 5 \ + \mathrm{i}(\tan^{1}\frac{4}{3} + \pi) = 1.609 + 1.295\pi\mathrm{i}.$$

Unless otherwise specified, we shall use the interval $0 \le \theta < 2\pi$ as the principal branch.

It can be easily checked that the familiar laws of logarithm which hold for real variables can be established for complex variables as well. For example,

$$\begin{aligned}
\ln z_1 z_2 &= \ln r_1\,\mathrm{e}^{\mathrm{i}\theta_1} r_2\,\mathrm{e}^{\mathrm{i}\theta_2} = \ln r_1 r_2 + \mathrm{i}\,(\theta_1 + \theta_2) \\
&= \ln r_1 + \ln r_2 + \mathrm{i}\theta_1 + \mathrm{i}\theta_2 \\
&= (\ln r_1 + \mathrm{i}\theta_1) + (\ln r_2 + \mathrm{i}\theta_2) = \ln z_1 + \ln z_2.
\end{aligned}$$

This relation is always true as long as infinitely many values of logarithms are taken into consideration. However, if only the principal values are taken, then the sum of the two principal values $\ln z_1 + \ln z_2$ may fall outside of the principal branch of $\ln(z_1 z_2)$.

Example 1.7.7. Find all values of $\ln 2$.

Solution 1.7.7. The real number 2 is also the complex number $2 + \mathrm{i}0$, and

$$2 + \mathrm{i}0 = 2\mathrm{e}^{\mathrm{i}n2\pi}, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Thus

$$\begin{aligned} \ln 2 &= Ln\ 2 + n2\pi\mathrm{i} \\ &= 0.693 + n2\pi\mathrm{i}, \qquad n = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

Even positive real numbers now have infinitely many logarithms. Only one of them is real, corresponding to $n = 0$ principal value.

Example 1.7.8. Find all values of $\ln(-1)$.

Solution 1.7.8.

$$\ln(-1) = \ln \mathrm{e}^{\mathrm{i}(\pi \pm 2\pi n)} = \mathrm{i}(\pi + 2\pi n), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

The principal value is $\mathrm{i}\pi$ for $n = 0$.

Since $\ln a = x$ means $\mathrm{e}^x = a$, so long as the variable x is real, a is always positive. Thus, in the domain of real numbers, the logarithm of a negative number does not exist. Therefore the answer must come from the complex domain. The situation was still sufficiently confused in the 18th century that it was possible for so great a mathematician as D'Alembert (1717–1783) to think $\ln(-x) = \ln(x)$, so $\ln(-1) = \ln(1) = 0$. His reason was the following. Since $(-x)(-x) = x^2$, therefore $\ln[(-x)(-x)] = \ln x^2 = 2\ln x$. But $\ln[(-x)(-x)] = \ln(-x) + \ln(-x) = 2\ln(-x)$, so we get $\ln(-x) = \ln x$. This is incorrect, because it applies the rule of ordinary algebra to the domain of complex numbers. It was Euler who pointed out that $\ln(-1)$ must be equal to the complex number $\mathrm{i}\pi$, which is in accordance with his equation $\mathrm{e}^{\mathrm{i}\pi} = -1$.

Example 1.7.9. Find the principal value of $\ln(1 + \mathrm{i})$.

Solution 1.7.9. Since

$$1 + \mathrm{i} = \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4},$$

$$\ln(1 + \mathrm{i}) = \ln\sqrt{2} + \frac{\pi}{4}\mathrm{i} = 0.3466 + 0.7854\mathrm{i}.$$

We are now in a position to consider the general power of a complex number. First let us see how to find i^{i}. Since

$$\mathrm{i} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi n\right)},$$

$$\mathrm{i}^{\mathrm{i}} = \left[\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi n\right)}\right]^{\mathrm{i}} = \mathrm{e}^{-\left(\frac{\pi}{2}+2\pi n\right)}, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

We get infinitely many values – all of them real. In a literal sense, Euler showed that imaginary power of an imaginary number can be real.

In general, since $a = \mathrm{e}^{\ln a}$, so

$$a^b = \left(\mathrm{e}^{\ln a}\right)^b = \mathrm{e}^{b \ln a}.$$

In this formula, both a and b can be complex numbers. For example, to find $(1+\mathrm{i})^{1-\mathrm{i}}$, first we write

$$(1+i)^{1-i} = \left[\mathrm{e}^{\ln(1+i)}\right]^{1-i} = \mathrm{e}^{(1-i)\ln(1+i)}.$$

Since

$$\ln(1+\mathrm{i}) = \ln \sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+2\pi n\right)} = \ln \sqrt{2} + \mathrm{i}\left(\frac{\pi}{4} + 2\pi n\right), \qquad n = 0, \pm 1, \pm 2, \ldots,$$

now

$$\begin{aligned}(1+\mathrm{i})^{1-\mathrm{i}} &= \mathrm{e}^{\left(\ln\sqrt{2}+\mathrm{i}\left(\frac{\pi}{4}+2\pi n\right)-\mathrm{i}\ln\sqrt{2}+\left(\frac{\pi}{4}+2\pi n\right)\right)} \\ &= \mathrm{e}^{\ln\sqrt{2}+\frac{\pi}{4}+2\pi n}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+2\pi n-\ln\sqrt{2}\right)}.\end{aligned}$$

Using

$$\mathrm{e}^{\mathrm{i}2\pi n} = 1, \qquad \mathrm{e}^{\ln\sqrt{2}} = \sqrt{2},$$

we have

$$(1+\mathrm{i})^{1-\mathrm{i}} = \sqrt{2}\mathrm{e}^{\frac{\pi}{4}+2\pi n}\left[\cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)\right].$$

Using a calculator, this expression is found to be

$$(1+\mathrm{i})^{1-\mathrm{i}} = \mathrm{e}^{2\pi n}\left(2.808 + 1.318\mathrm{i}\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

Example 1.7.10. Find all values of $\mathrm{i}^{1/2}$.

Solution 1.7.10.

$$\mathrm{i}^{1/2} = \left[\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi n\right)}\right]^{1/2} = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}n\pi}, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

Since $\mathrm{e}^{\mathrm{i}n\pi} = 1$ for n even, and $\mathrm{e}^{\mathrm{i}n\pi} = -1$ for n odd, thus

$$\mathrm{i}^{1/2} = \pm\mathrm{e}^{\mathrm{i}\pi/4} = \pm\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = \pm\frac{\sqrt{2}}{2}\left(1+\mathrm{i}\right).$$

Notice that although n can be any of the infinitely many integers, we find only two values for $\mathrm{i}^{1/2}$ as we should, for it is the square root of i.

Example 1.7.11. Find the principal value of 2^{i}.

Solution 1.7.11.

$$\begin{aligned}2^{\mathrm{i}} &= \left[\mathrm{e}^{\ln 2}\right]^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\ln 2} = \cos(\ln 2) + \mathrm{i}\sin(\ln 2) \\ &= 0.769 + 0.639\mathrm{i}.\end{aligned}$$

Example 1.7.12. Find the principal value of $(1+\mathrm{i})^{2-\mathrm{i}}$.

Solution 1.7.12.

$$(1+\mathrm{i})^{2-\mathrm{i}} = \exp\left[(2-\mathrm{i})\ln(1+\mathrm{i})\right].$$

The principal value of $\ln(1+\mathrm{i})$ is

$$\ln(1+\mathrm{i}) = \ln\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4} = \ln\sqrt{2} + \mathrm{i}\frac{\pi}{4}.$$

Therefore

$$\begin{aligned}(1+\mathrm{i})^{2-\mathrm{i}} &= \exp\left[(2-\mathrm{i})\left(\ln\sqrt{2} + \mathrm{i}\frac{\pi}{4}\right)\right] \\ &= \exp\left(2\ln\sqrt{2} + \frac{\pi}{4}\right)\exp\left[\mathrm{i}\left(\frac{\pi}{2} - \ln\sqrt{2}\right)\right] \\ &= 2\mathrm{e}^{\pi/4}\left[\cos\left(\frac{\pi}{2} - \ln\sqrt{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2} - \ln\sqrt{2}\right)\right] \\ &= 4.3866\,(\sin 0.3466 + \mathrm{i}\cos 0.3466) = 1.490 + 4.126\mathrm{i}.\end{aligned}$$

1.7.4 Inverse Trigonometric and Hyperbolic Functions

Starting from their definitions, we can work out sensible expressions for the inverse of trigonometric and inverse hyperbolic functions. For example, to find

$$w = \sin^{-1} z,$$

we write this as

$$z = \sin w = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}\right).$$

Multiplying $\mathrm{e}^{\mathrm{i}w}$, we have

$$z\mathrm{e}^{\mathrm{i}w} = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}2w} - 1\right).$$

Rearranging, we get a quadratic equation in e^{iw},

$$\left(e^{iw}\right)^2 - 2ize^{iw} - 1 = 0.$$

The solution of this equation is

$$e^{iw} = \frac{1}{2}\left(2iz \pm \sqrt{-4z^2+4}\right) = iz \pm \left(1-z^2\right)^{1/2}.$$

Taking logarithm of both sides

$$iw = \ln\left[iz \pm \left(1-z^2\right)^{1/2}\right].$$

Therefore

$$w = \sin^{-1} z = -i\ln\left[iz \pm \left(1-z^2\right)^{1/2}\right].$$

Because of logarithm, this expression is multivalued. Even in the principal branch, $\sin^{-1} z$ has two values for $z \neq 1$ because of the square roots.

Similarly, we can show

$$\cos^{-1} z = -i\ln\left[z \pm \left(z^2-1\right)^{1/2}\right],$$
$$\tan^{-1} z = \frac{i}{2}\ln\frac{i+z}{i-z},$$

$$\sinh^{-1} z = \ln\left[z \pm \left(1+z^2\right)^{1/2}\right],$$
$$\cosh^{-1} z = \ln\left[z \pm \left(z^2-1\right)^{1/2}\right],$$
$$\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}.$$

Example 1.7.13. Evaluate $\cos^{-1} 2$.

Solution 1.7.13. Let $w = \cos^{-1} 2$, so $\cos w = 2$. It follows:

$$\frac{1}{2}\left(e^{iw} + e^{-iw}\right) = 2$$

Multiplying e^{iw}, we have a quadratic equation in e^{iw}

$$(e^{iw})^2 + 1 = 4e^{iw}.$$

Solving for e^{iw}

$$e^{iw} = \frac{1}{2}(4 \pm \sqrt{16-4}) = 2 \pm \sqrt{3}.$$

Thus

$$\mathrm{i}w = \ln\left(2 \pm \sqrt{3}\right).$$

Therefore

$$\cos^{-1} 2 = w = -\mathrm{i}\ln\left(2 \pm \sqrt{3}\right).$$

Now

$$\ln(2+\sqrt{3}) = 1.317, \qquad \ln(2-\sqrt{3}) = -1.317.$$

Note only in this particular case, $-\ln(2+\sqrt{3}) = \ln(2-\sqrt{3})$, since

$$-\ln(2+\sqrt{3}) = \ln(2+\sqrt{3})^{-1} = \ln\frac{1}{2+\sqrt{3}} = \ln\frac{2-\sqrt{3}}{2^2-(\sqrt{3})^2} = \ln(2-\sqrt{3}).$$

Thus the principal values of $\ln\left(2 \pm \sqrt{3}\right) = \pm 1.317$. Therefore

$$\cos^{-1} 2 = \mp 1.317\mathrm{i} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots.$$

In real variable domain, the maximum value of cosine is one. Therefore we expect $\cos^{-1} 2$ to be complex numbers. Also note that $\pm$ solutions may be expected since $\cos(-z) = \cos(z)$.

Example 1.7.14. Show that

$$\tan^{-1} z = \frac{\mathrm{i}}{2}[\ln(\mathrm{i}+z) - \ln(\mathrm{i}-z)].$$

Solution 1.7.14. Let $w = \tan^{-1} z$, so

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}}{\mathrm{i}\left(\mathrm{e}^{\mathrm{i}w} + \mathrm{e}^{-\mathrm{i}w}\right)},$$

$$\mathrm{i}z\left(\mathrm{e}^{\mathrm{i}w} + \mathrm{e}^{-\mathrm{i}w}\right) = \mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w},$$

$$(\mathrm{i}z - 1)\,\mathrm{e}^{\mathrm{i}w} + (\mathrm{i}z + 1)\,\mathrm{e}^{-\mathrm{i}w} = 0.$$

Multiplying $\mathrm{e}^{\mathrm{i}w}$ and rearranging, we have

$$\mathrm{e}^{\mathrm{i}2w} = \frac{1+\mathrm{i}z}{1-\mathrm{i}z}.$$

Taking logarithm on both sides

$$\mathrm{i}\,2w = \ln\frac{1+\mathrm{i}z}{1-\mathrm{i}z} = \ln\frac{\mathrm{i}-z}{\mathrm{i}+z}.$$

Thus

$$w = \frac{1}{2\mathrm{i}}\ln\frac{\mathrm{i}-z}{\mathrm{i}+z} = -\frac{\mathrm{i}}{2}\ln\frac{\mathrm{i}-z}{\mathrm{i}+z} = \frac{\mathrm{i}}{2}\ln\frac{\mathrm{i}+z}{\mathrm{i}-z},$$

$$\tan^{-1} z = w = \frac{\mathrm{i}}{2}[\ln(\mathrm{i}+z) - \ln(\mathrm{i}-z)].$$

Exercises

1. Approximate $\sqrt{2}$ as 1.414 and use the table of successive square root of 10 to compute $10^{\sqrt{2}}$.

   Ans. 25.94

2. Use the table of successive square root of 10 to compute $\log 2$.

   Ans. 0.3010

3. How long will it take for a sum of money to double if invested at 20% interest rate compounded annually? (This question was posted in a clay tablet dated 1700 BC now at Louvre.)
   Hint: Solve $(1.2)^x = 2$.

   Ans. 3.8 years, or 3 years 8 months and 18 days.

4. Suppose the annual interest rate is fixed at 5%. Banks are competing by offering compound interests with increasing number of conversions, monthly, daily, hourly, and so on. With a principal of $100, what is the maximum amount of money one can get after 1 year?

   Ans. $100\,\mathrm{e}^{0.05} = 105.13$

5. Simplify (express it in the form of $a + \mathrm{i}b$)

   $$\frac{\cos 2\alpha + \mathrm{i}\sin 2\alpha}{\cos\alpha + \mathrm{i}\sin\alpha}.$$

   Ans. $\cos\alpha + \mathrm{i}\sin\alpha$.

6. Simplify (express it in the form of $a + \mathrm{i}b$)

   $$\frac{(\cos\theta - \mathrm{i}\sin\theta)^2}{(\cos\theta + \mathrm{i}\sin\theta)^3}.$$

   Ans. $\cos 5\theta - \mathrm{i}\sin 5\theta$.

7. Find the roots of

$$x^4 + 1 = 0.$$

Ans. $\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}.$

8. Find all the distinct fourth roots of $8 - \mathrm{i}8\sqrt{3}$.
Ans. $2\left(\cos\frac{5\pi}{12} + \mathrm{i}\sin\frac{5\pi}{12}\right), 2\left(\cos\frac{11\pi}{12} + \mathrm{i}\sin\frac{11\pi}{12}\right),$
$2\left(\cos\frac{17\pi}{12} + \mathrm{i}\sin\frac{17\pi}{12}\right), 2\left(\cos\frac{23\pi}{12} + \mathrm{i}\sin\frac{23\pi}{12}\right).$

9. Find all the values of the following in the form of $a + \mathrm{i}b$.

$$\text{(a) } \mathrm{i}^{2/3}, \quad \text{(b) } (-1)^{1/3}, \quad \text{(c) } (3 + 4\mathrm{i})^4.$$

Ans. (a) $-1,\ (1 \pm \mathrm{i}\sqrt{3})/2$, (b) $-1,\ (1 \pm \mathrm{i}\sqrt{3})/2$, (c) $-527 - 336\mathrm{i}$.

10. Use complex numbers to show

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

11. Use complex numbers to show

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1),$$
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

12. Show that

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\frac{1}{2}\theta},$$

$$\sum_{k=0}^{n} \sin k\theta = \frac{1}{2}\cot\frac{1}{2}\theta - \frac{\cos\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\frac{1}{2}\theta}.$$

13. Find the location of the center and the radius of the following circle:

$$\left|\frac{z-1}{z+1}\right| = 3.$$

Ans. $(-\frac{5}{4},\ 0)$ $r = \frac{3}{4}$.

14. Use complex numbers to show that the diagonals of a parallelogram bisect each other.

15. Use complex numbers to show that the line segment joining the two midpoints of two sides of any triangle is parallel to the third side and half its length.

16. Use complex numbers to prove that medians of a triangle intersect at a point two-thirds of the way from any vertex to the midpoint of the opposite side.

17. Let ABC be an isosceles triangle such that $AB = AC$. Use complex numbers to show that the line from A to the midpoint of BC is perpendicular to BC.

18. Express the principal value of the following in the form of $a + \mathrm{i}b$:

$$(a)\ \exp\left(\frac{\mathrm{i}\pi}{4} + \frac{\ln 2}{2}\right), \quad (b)\ \cos(\pi - 2\mathrm{i}\ln 3), \quad (c)\ \ln(-\mathrm{i}).$$

Ans. (a) $1 + \mathrm{i}$, (b) $-\frac{41}{9}$, (c) $-\mathrm{i}\frac{\pi}{2}$ or $\mathrm{i}\frac{3\pi}{2}$.

19. Express the principal value of the following in the form of $a + \mathrm{i}b$:

$$(\mathrm{a})\ \mathrm{i}^{3+\mathrm{i}},\ (\mathrm{b})\ (2\mathrm{i})^{1+\mathrm{i}}, (\mathrm{c})\ \left(\frac{1+\mathrm{i}\sqrt{3}}{2}\right)^{\mathrm{i}}.$$

Ans. (a) $-0.20788\mathrm{i}$, (b) $-0.2657 + 0.3189\mathrm{i}$, (c) 0.35092.

20. Find all the values of the following expressions:

$$(\mathrm{a})\ \sin\left(\mathrm{i}\ \ln\frac{1-\mathrm{i}}{1+\mathrm{i}}\right), (b)\ \tan^{-1}(2\mathrm{i}), (c)\ \cosh^{-1}\left(\frac{1}{2}\right)$$

Ans. (a) 1, (b) $\frac{1+2n}{2}\pi + \mathrm{i}\frac{1}{2}\ln 3$, (c) $\mathrm{i}\left(\pm\frac{\pi}{3} + 2n\pi\right)$.

21. With $z = x + \mathrm{i}y$, verify the following

$$\begin{aligned}
\sin z &= \sin x \cosh y + \mathrm{i}\cos x \sinh y,\\
\cos z &= \cos x \cosh y - \mathrm{i}\sin x \sinh y,\\
\sinh z &= \sinh x \cos y + \mathrm{i}\cosh x \sin y,\\
\cosh z &= \cosh x \cos y + \mathrm{i}\sinh x \sin y.
\end{aligned}$$

22. Show that

$$\begin{aligned}
\sin 2z &= 2\sin z \cos z,\\
\cos 2z &= \cos^2 z - \sin^2 z,\\
\cosh^2 z - \sinh^2 z &= 1.
\end{aligned}$$

23. Show that

$$\begin{aligned}
\cos^{-1} z &= -\mathrm{i}\ \ln\left[z + \left(z^2 - 1\right)^{1/2}\right],\\
\sinh^{-1} z &= \ln\left[z + \left(1 + z^2\right)^{1/2}\right].
\end{aligned}$$