1

Vectors

Vectors are used when both the magnitude and the direction of some physical quantity are required. Examples of such quantities are velocity, acceleration, force, electric and magnetic fields. A quantity that is completely characterized by its magnitude is known as a scalar. Mass and temperature are scalar quantities.

A vector is characterized by both magnitude and direction, but not all quantities that have magnitude and direction are vectors. For example, in the study of strength of materials, stress has both magnitude and direction. But stress is a second rank tensor, which we will study in a later chapter.

Vectors can be analyzed either with geometry or with algebra. The algebraic approach centers on the transformation properties of vectors. It is capable of generalization and leads to tensor analysis. Therefore it is fundamentally important in many problems of mathematical physics.

However, for pedagogical reasons we will begin with geometrical vectors, since they are easier to visualize. Besides, most readers probably already have some knowledge of the graphical approach of vector analysis.

A vector is usually indicated by a boldfaced letter, such as $\mathbf{V}$, or an arrow over a letter $\overrightarrow{V}$. While there are other ways to express a vector, whatever convention you choose, it is very important that vector and scalar quantities are represented by different types of symbols. A vector is graphically represented by a directed line segment. The length of the segment is proportional to the magnitude of the vector quantity with a suitable scale. The direction of the vector is indicated by an arrowhead at one end of the segment, which is known as the tip of the vector. The other end is called the tail. The magnitude of the vector is called the norm of the vector. In what follows, the letter V is used to mean the norm of $\mathbf{V}$. Sometimes, the norm of $\mathbf{V}$ is also represented by $|\mathbf{V}|$ or $\|\mathbf{V}\|$.

1.1 Bound and Free Vectors

There are two kinds of vectors; *bound vector* and *free vector*. Bound vectors are fixed in position. For example, in dealing with forces whose points of application or lines of action cannot be shifted, it is necessary to think of them as bound vectors. Consider the cases shown in Fig. 1.1. Two forces of the same magnitude and direction act at two different points along a beam. Clearly the torques produced at the supporting ends and the displacements at the free ends are totally different in these two cases. Therefore these forces are bound vectors. Usually in statics, structures, and strength of materials, forces are bound vectors; attention must be paid to their magnitude, direction, and the point of application.

A *free vector* is completely characterized by its magnitude and direction. These vectors are the ones discussed in mathematical analysis. In what follows, *vectors are understood to be free vectors* unless otherwise specified.

Two free vectors whose magnitudes, or lengths, are equal and whose directions are the same are said to be equal, regardless of the points in space from which they may be drawn. In other words, a vector quantity can be represented equally well by any of the infinite many line segments, all having the same length and the same direction. It is, therefore, customary to say that a vector can be moved parallel to itself without change.

1.2 Vector Operations

Mathematical operations defined for scalars, such as addition and multiplication, are not applicable to vectors, since vectors not only have magnitude but also direction. Therefore a set of vector operations must be introduced. These operations are the rules of combining a vector with another vector or a vector with a scalar. There are various ways of combining them. Some useful combinations are defined in this section.

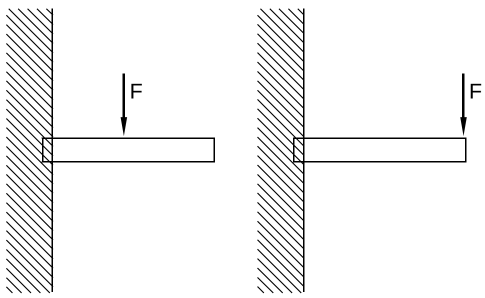

Fig. 1.1. Bound vectors representing the forces acting on the beam cannot be moved parallel to themselves

1.2.1 Multiplication by a Scalar

If c is a positive number, the equation

$$\mathbf{A} = c\mathbf{B} \tag{1.1}$$

means that the direction of the vector **A** is the same as that of **B**, and the magnitude of **A** is c times that of **B**. If c is negative, the equation means that the direction of **A** is opposite to that of **B** and the magnitude of **A** is c times that of **B**.

1.2.2 Unit Vector

A *unit vector* is a vector having a magnitude of one unit. If we divide a vector **V** by its magnitude V, we obtain a unit vector in the direction of **V**. Thus, the unit vector **n** in the direction of **V** is given by

$$\mathbf{n} = \frac{1}{V}\mathbf{V}. \tag{1.2}$$

Very often a hat is put on the vector symbol ($\widehat{\mathbf{n}}$) to indicate that it is a unit vector. Thus $\mathbf{A} = A\widehat{\mathbf{A}}$ and the statement "**n** is an unit vector in the direction of **A**" can be expressed as $\mathbf{n} = \widehat{\mathbf{A}}$.

1.2.3 Addition and Subtraction

Two vectors **A** and **B** are added by placing the tip of one at the tail of the other, as shown in Fig. 1.2. The sum $\mathbf{A} + \mathbf{B}$ is the vector obtained by connecting the tail of the first vector to the tip of the second vector. In Fig. 1.2a, **B** is moved parallel to itself, in Fig. 1.2b **A** is moved parallel to itself. Clearly

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{1.3}$$

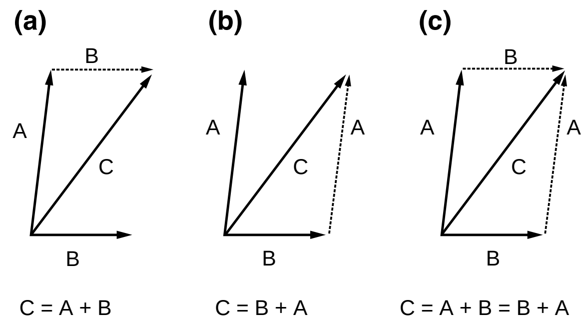

Fig. 1.2. Addition of two vectors: (**a**) connecting the tail of **B** to the tip of **A**; (**b**) connecting the tail of **A** to the tip of **B**; (**c**) parallelogram law which is valid for both free and bound vectors

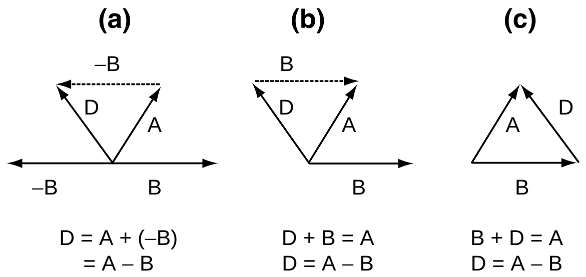

Fig. 1.3. Subtraction of two vectors: (**a**) as addition of a negative vector; (**b**) as an inverse of addition; (**c**) as the tip-to-tip vector which is the most useful interpretation of vector subtraction

If the two vectors to be added are considered to be the sides of a parallelogram, the sum is seen to be the diagonal as shown in Fig. 1.2c. This *parallelogram rule* is valid for both free vectors and bound vectors, and is often used to define the sum of two vectors. It is also the basis for decomposing a vector into its components.

Subtraction of vectors is illustrated in Fig. 1.3. In Fig. 1.3a subtraction is taken as a special case of addition

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})\,. \tag{1.4}$$

In Fig. 1.3b, subtraction is taken as an inverse operation of addition. Clearly, they are equivalent. The most often and the most useful definition of vector subtraction is illustrated in Fig. 1.3c, namely $\mathbf{A} - \mathbf{B}$ is the *tip-to-tip* vector $\mathbf{D}$, starting from the tip of $\mathbf{B}$ directed towards the tip of $\mathbf{A}$.

Graphically it can also be easily shown that vector addition is associative

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{B} + \mathbf{A}) + \mathbf{C}. \tag{1.5}$$

If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the three sides of a parallelepied, then $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is the vector along the longest diagonal.

1.2.4 Dot Product

The dot product (also known as the *scalar product*) of two vectors is defined to be

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta, \tag{1.6}$$

where θ is the angle that $\mathbf{A}$ and $\mathbf{B}$ form when placed tail-to-tail. Since it is a scalar, clearly the product is commutative

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \tag{1.7}$$

Geometrically, $\mathbf{A} \cdot \mathbf{B} = AB_A$ where B_A is the projection of $\mathbf{B}$ on $\mathbf{A}$, as shown in Fig. 1.4. It is also equal to BA_B, where A_B is the projection of $\mathbf{A}$ on $\mathbf{B}$.

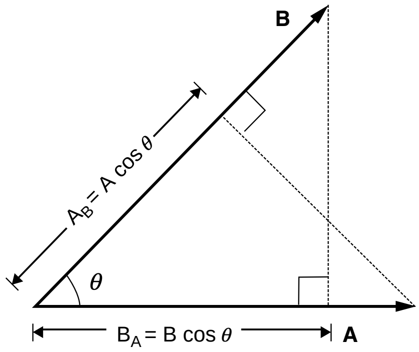

Fig. 1.4. Dot product of two vectors. $\mathbf{A}\cdot\mathbf{B} = AB_A = BA_B = AB\cos\theta$

If the two vectors are parallel, then $\theta = 0$ and $\mathbf{A}\cdot\mathbf{B} = AB$. In particular,

$$\mathbf{A}\cdot\mathbf{A} = A^2, \tag{1.8}$$

which says that the square of the magnitude of any vector is equal to its dot product with itself.

If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\theta = 90°$ and $\mathbf{A}\cdot\mathbf{B} = 0$. Conversely, if we can show $\mathbf{A}\cdot\mathbf{B} = 0$, then we have proved that $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

It is clear from Fig. 1.5 that

$$A(B+C)_A = AB_A + AC_A.$$

This shows that the distributive law holds for the dot product

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}. \tag{1.9}$$

With vector notations, many geometrical facts can be readily demonstrated.

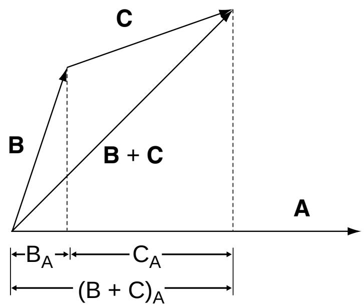

Fig. 1.5. Distributive law of dot product of two vectors. $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$

Example 1.2.1. **Law of cosines**. If A, B, C are the three sides of a triangle, and θ is the interior angle between **A** and **B**, show that

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

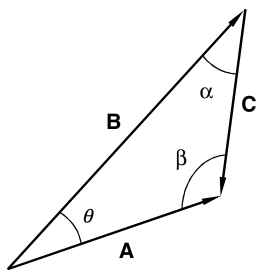

Fig. 1.6. The law of cosine can be readily shown with dot product of vectors, and the law of sine, with cross product

Solution 1.2.1. Let the triangle be formed by the three vectors **A**, **B**, and **C** as shown in Fig. 1.6. Since $\mathbf{C} = \mathbf{A} - \mathbf{B}$,

$$\mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B}.$$

It follows

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

Example 1.2.2. Prove that the diagonals of a parallelogram bisect each other.

Solution 1.2.2. Let the two adjacent sides of the parallelogram be represented by vectors **A** and **B** as shown in Fig. 1.7. The two diagonals are $\mathbf{A} - \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$. The vector from the bottom left corner to the mid-point of the diagonal $\mathbf{A} - \mathbf{B}$ is

$$\mathbf{B} + \frac{1}{2}(\mathbf{A}-\mathbf{B}) = \frac{1}{2}(\mathbf{A}+\mathbf{B}),$$

which is also the half of the other diagonal $\mathbf{A} + \mathbf{B}$. Therefore they bisect each other.

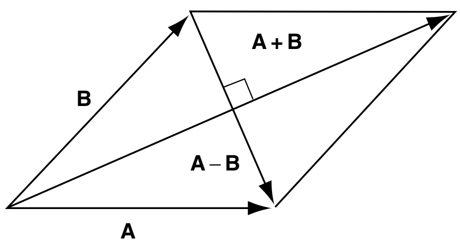

Fig. 1.7. Diagonals of a parallelogram bisect each other; diagonals of a rhombus ($A = B$) are perpendicular to each other

Example 1.2.3. Prove that the diagonals of a rhombus (a parallelogram with equal sides) are orthogonal (perpendicular to each other).

Solution 1.2.3. Again let the two adjacent sides be $\mathbf{A}$ and $\mathbf{B}$. The dot product of the two diagonals (Fig. 1.7) is

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{A}-\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{B} = A^2 - B^2.$$

For a rhombus, $A = B$. Therefore the dot product of the diagonals is equal to zero. Hence they are perpendicular to each other.

Example 1.2.4. Show that in a parallelogram, the two lines from one corner to the midpoints of the two opposite sides trisect the diagonal they cross.

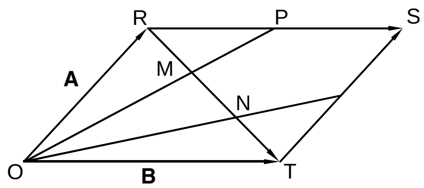

Fig. 1.8. Two lines from one corner of a parallelogram to the midpoints of the two opposite sides trisect the diagonal they cross

Solution 1.2.4. With the parallelogram shown in Fig. 1.8, it is clear that the line from O to the midpoint of RS is represented by the vector $\mathbf{A}+\frac{1}{2}\mathbf{B}$. A vector drawn from O to any point on this line can be written as

$$\mathbf{r}(\lambda) = \lambda\left(\mathbf{A}+\frac{1}{2}\mathbf{B}\right),$$

where λ is a real number which adjusts the length of OP. The diagonal RT is represented by the vector $\mathbf{B}-\mathbf{A}$. A vector drawn from O to any point on this diagonal is

$$\mathbf{r}(\mu) = \mathbf{A}+\mu\left(\mathbf{B}-\mathbf{A}\right),$$

where the parameter μ adjusts the length of the diagonal. The two lines meet when

$$\lambda\left(\mathbf{A}+\frac{1}{2}\mathbf{B}\right) = \mathbf{A}+\mu\left(\mathbf{B}-\mathbf{A}\right),$$

which can be written as

$$(\lambda-1+\mu)\,\mathbf{A}+\left(\frac{1}{2}\lambda-u\right)\mathbf{B} = \mathbf{0}.$$

This gives $\mu = \dfrac{1}{3}$ and $\lambda = \dfrac{2}{3}$, so the length of RM is one-third of RT. Similarly, we can show the length NT is one-third of RT.

Example 1.2.5. Show that an angle inscribed in a semicircle is a right angle.

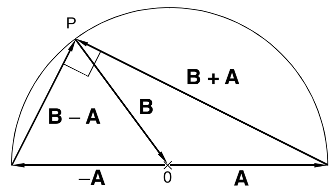

Fig. 1.9. The circum-angle of a semicircle is a right angle

Solution 1.2.5. With the semicircle shown in Fig. 1.9, it is clear the magnitude of $\mathbf{A}$ is the same as the magnitude of $\mathbf{B}$, since they both equal to the radius of the circle $A = B$. Thus $(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{B} + \mathbf{A}) = B^2 - A^2 = 0$. Therefore $(\mathbf{B} - \mathbf{A})$ is perpendicular to $(\mathbf{B} + \mathbf{A})$.

1.2.5 Vector Components

For algebraic description of vectors, we introduce a coordinate system for the reference frame, although it is important to keep in mind that the magnitude and direction of a vector is independent of the reference frame. We will first use the rectangular Cartesian coordinates to express vectors in terms of their components. Let $\mathbf{i}$ be a unit vector in the positive x direction, and $\mathbf{j}$ and $\mathbf{k}$ be unit vectors in the positive y and z directions. An arbitrary vector $\mathbf{A}$ can be expanded in terms of these *basis vectors* as shown in Fig. 1.10:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}, \tag{1.10}$$

where A_x, A_y, and A_z are the projections of $\mathbf{A}$ along the three coordinate axes, they are called components of $\mathbf{A}$.

Since $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular unit vectors, by the definition of dot product

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \tag{1.11}$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0. \tag{1.12}$$

Because the dot product is distributive, it follows that

$$\begin{aligned}\mathbf{A} \cdot \mathbf{i} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot \mathbf{i} \\ &= A_x\mathbf{i} \cdot \mathbf{i} + A_y\mathbf{j} \cdot \mathbf{i} + A_z\mathbf{k} \cdot \mathbf{i} = A_x,\end{aligned}$$

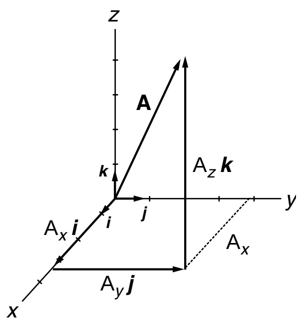

Fig. 1.10. Vector components. **i**, **j**, **k** are three unit vectors pointing in the direction of positive x-, y- and z-axis, respectively. A_x, A_y, A_z are the projections of **A** on these axes. They are components of **A** and $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$

$$\mathbf{A} \cdot \mathbf{j} = A_y, \qquad \mathbf{A} \cdot \mathbf{k} = A_z,$$

the dot product of **A** with any unit vector is the projection of A along the direction of that unit vector (or the component of **A** along that direction). Thus, (1.10) can be written as

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{i})\mathbf{i} + (\mathbf{A} \cdot \mathbf{j})\mathbf{j} + (\mathbf{A} \cdot \mathbf{k})\mathbf{k}. \tag{1.13}$$

Furthermore, using the distributive law of dot product and (1.11) and (1.12), we have

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x + A_yB_y + A_zB_z, \end{aligned} \tag{1.14}$$

and

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2 = A^2. \tag{1.15}$$

Since $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$, the angle between **A** and **B** is given by

$$\theta = \cos^{-1}\frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \cos^{-1}\left(\frac{A_xB_x + A_yB_y + A_zB_z}{AB}\right). \tag{1.16}$$

Example 1.2.6. Find the angle between $\mathbf{A} = 3\mathbf{i}+6\mathbf{j}+9\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

Solution 1.2.6.

$$A = (3^2 + 6^2 + 9^2)^{1/2} = 3\sqrt{14}; \quad B = \left((-2)^2 + 3^2 + 1^2\right)^{1/2} = \sqrt{14},$$

$$\mathbf{A} \cdot \mathbf{B} = 3 \times (-2) + 6 \times 3 + 9 \times 1 = 21$$

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{21}{3\sqrt{14}\sqrt{14}} = \frac{7}{14} = \frac{1}{2},$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ.$$

Example 1.2.7. Find the angle θ between the face diagonals **A** and **B** of a cube shown in Fig. 1.11.

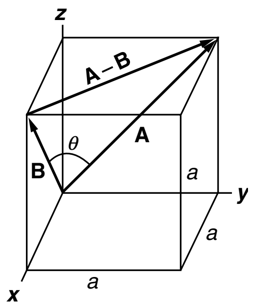

Fig. 1.11. The angle between the two face diagonals of a cube is 60°

Solution 1.2.7. The answer can be easily found from geometry. The triangle formed by **A**, **B** and **A**−**B** is clearly an equilateral triangle, therefore $\theta = 60°$. Now with dot product approach, we have

$$\mathbf{A} = a\mathbf{j} + a\mathbf{k}; \quad \mathbf{B} = a\mathbf{i} + a\mathbf{k}; \quad A = \sqrt{2}a = B,$$

$$\mathbf{A} \cdot \mathbf{B} = a \cdot 0 + 0 \cdot a + a \cdot a = a^2 = AB\cos\theta = 2a^2\cos\theta.$$

Therefore

$$\cos\theta = \frac{a^2}{2a^2} = \frac{1}{2}, \qquad \theta = 60°.$$

Example 1.2.8. If $\mathbf{A} = 3\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, find the projection of **A** on **B**.

Solution 1.2.8. The unit vector along **B** is

$$\mathbf{n} = \frac{\mathbf{B}}{B} = \frac{-2\mathbf{i}+3\mathbf{j}+\mathbf{k}}{\sqrt{14}}.$$

The projection of **A** on **B** is then

$$\mathbf{A} \cdot \mathbf{n} = \frac{1}{B}\mathbf{A} \cdot \mathbf{B} = \frac{1}{\sqrt{14}}(3\mathbf{i}+6\mathbf{j}+9\mathbf{k}) \cdot (-2\mathbf{i}+3\mathbf{j}+\mathbf{k}) = \frac{21}{\sqrt{14}}.$$

Example 1.2.9. The angles between the vector **A** and the three basis vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are, respectively, α, β, and γ. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Solution 1.2.9. The projections of $\mathbf{A}$ on $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are, respectively,

$$A_x = \mathbf{A} \cdot \mathbf{i} = A \cos \alpha; \quad A_y = \mathbf{A} \cdot \mathbf{j} = A \cos \beta; \quad A_z = \mathbf{A} \cdot \mathbf{k} = A \cos \gamma.$$

Thus

$$A_x^2 + A_y^2 + A_z^2 = A^2 \cos^2 \alpha + A^2 \cos^2 \beta + A^2 \cos^2 \gamma = A^2 \left(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma \right).$$

Since $A_x^2 + A_y^2 + A_z^2 = A^2$, therefore

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

The quantities $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are often denoted l, m, and n, respectively, and they are called the *direction cosine* of $\mathbf{A}$.

1.2.6 Cross Product

The vector cross product written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{1.17}$$

is another particular combination of the two vectors $\mathbf{A}$ and $\mathbf{B}$, which is also very useful. It is defined as a vector (therefore the alternative name: *vector product*) with a magnitude

$$C = AB \sin \theta, \tag{1.18}$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$, and a direction perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ in the sense of the advance of a right-hand screw as it is turned from $\mathbf{A}$ to $\mathbf{B}$. In other words, if the fingers of your right hand point in the direction of the first vector $\mathbf{A}$ and curl around toward the second vector $\mathbf{B}$, then your thumb will indicate the positive direction of $\mathbf{C}$ as shown in Fig. 1.12.

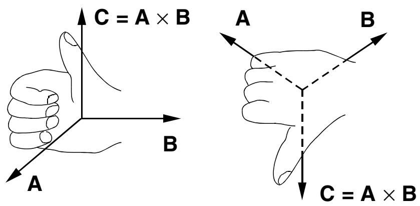

Fig. 1.12. Right-hand rule of cross product $\mathbf{A} \times \mathbf{B} = \mathbf{C}$. If the fingers of your right hand point in the direction of the first vector $\mathbf{A}$ and curl around toward the second vector $\mathbf{B}$, then your thumb will indicate the positive direction of $\mathbf{C}$

With this choice of direction, we see that cross product is anticommutative

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \tag{1.19}$$

It is also clear that if $\mathbf{A}$ and $\mathbf{B}$ are parallel, then $\mathbf{A} \times \mathbf{B} = 0$, since θ is equal to zero.

From this definition, the cross products of the basis vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) can be easily obtained

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \tag{1.20}$$

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j}. \end{aligned} \tag{1.21}$$

The following example illustrates the cross product of two nonorthogonal vectors. If $\mathbf{V}$ is a vector in the xz-plane and the angle between $\mathbf{V}$ and $\mathbf{k}$, the unit vector along the z-axis, is θ as shown in Fig. 1.13, then

$$\mathbf{k} \times \mathbf{V} = V \sin\theta \mathbf{j}.$$

Since $|\mathbf{k} \times \mathbf{V}| = |\mathbf{k}|\,|\mathbf{V}| \sin\theta = V \sin\theta$ is equal to the projection of $\mathbf{V}$ on the $\mathbf{xy}$-plane, the vector $\mathbf{k} \times \mathbf{V}$ is the result of rotating this projection 90° around the $\mathbf{z}$ axis.

With this understanding, we can readily demonstrate the distributive law of the cross product

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}. \tag{1.22}$$

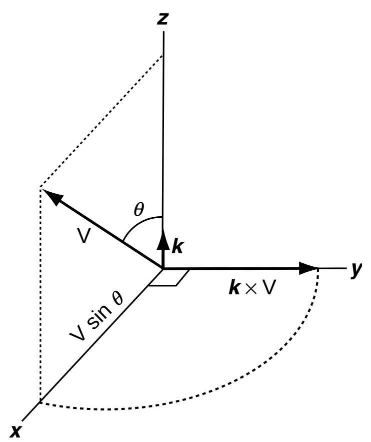

Fig. 1.13. The cross product of $\mathbf{k}$, the unit vector along the z-axis, and $\mathbf{V}$, a vector in the xz-plane

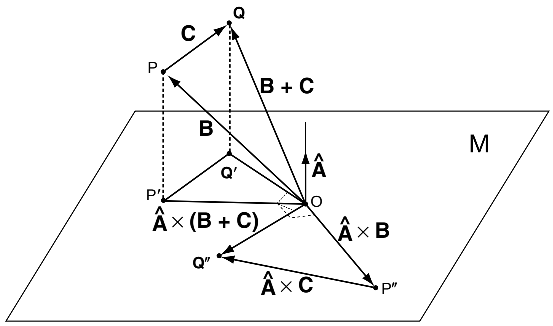

Fig. 1.14. Distributive law of cross product $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$

Let the triangle formed by the vectors $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{B} + \mathbf{C}$ be arbitrarily oriented with respect to the vector $\mathbf{A}$ as shown in Fig. 1.14. Its projection on the plane M perpendicular to $\mathbf{A}$ is the triangle OP′Q′. Turn this triangle 90° around $\mathbf{A}$, we obtain another triangle OP″Q″. The three sides of the triangle OP″Q″ are $\widehat{\mathbf{A}}\times\mathbf{B}$, $\widehat{\mathbf{A}}\times\mathbf{C}$, and $\widehat{\mathbf{A}}\times(\mathbf{B}+\mathbf{C})$, where $\widehat{\mathbf{A}}$ is the unit vector along the direction of $\mathbf{A}$. It follows from the rule of vector addition that

$$\widehat{\mathbf{A}}\times(\mathbf{B}+\mathbf{C}) = \widehat{\mathbf{A}}\times\mathbf{B} + \widehat{\mathbf{A}}\times\mathbf{C}.$$

Multiplying both sides by the magnitude A, we obtain (1.22).

With the distributive law and (1.20) and (1.21), we can easily express the cross product $\mathbf{A} \times \mathbf{B}$ in terms of the components of $\mathbf{A}$ and $\mathbf{B}$:

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= A_xB_x\mathbf{i}\times\mathbf{i} + A_xB_y\mathbf{i}\times\mathbf{j} + A_xB_z\mathbf{i}\times\mathbf{k} \\
&\quad + A_yB_x\mathbf{j}\times\mathbf{i} + A_yB_y\mathbf{j}\times\mathbf{j} + A_yB_z\mathbf{j}\times\mathbf{k} \\
&\quad + A_zB_x\mathbf{k}\times\mathbf{i} + A_zB_y\mathbf{k}\times\mathbf{j} + A_zB_z\mathbf{k}\times\mathbf{k} \\
&= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}. \qquad (1.23)
\end{aligned}$$

This cumbersome equation can be more neatly expressed as the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \qquad (1.24)$$

with the understanding that it is to be expanded about its first row. The determinant form is not only easier to remember but also more convenient to use.

The cross product has a useful geometrical interpretation. Figure 1.15 shows a parallelogram having $\mathbf{A}$ and $\mathbf{B}$ as co-terminal edges. The area of this parallelogram is equal to the base A times the height h. But $h = B\sin\theta$, so

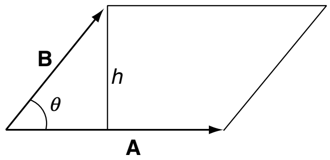

Fig. 1.15. The area of the parallelogram formed by **A** and **B** is equal to the magnitude of $\mathbf{A} \times \mathbf{B}$

$$\text{Parallelogram Area } = Ah = AB\sin\theta = |\mathbf{A} \times \mathbf{B}|. \tag{1.25}$$

Thus the magnitude of $\mathbf{A} \times \mathbf{B}$ is equal to the area of the parallelogram formed by **A** and **B**, its direction is normal to the plane of this parallelogram. This suggests that area may be treated as a vector quantity.

Since the area of the triangle formed by **A** and **B** as co-terminal edges is clearly half of the area of the parallelogram, so we also have

$$\text{Triangle Area} = \frac{1}{2}|\mathbf{A} \times \mathbf{B}|. \tag{1.26}$$

Example 1.2.10. **The law of sine.** With the triangle in Fig. 1.6, show that

$$\frac{\sin\theta}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}.$$

Solution 1.2.10. The area of the triangle is equal to $\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \frac{1}{2}AB\sin\theta$. The same area is also given by $\frac{1}{2}|\mathbf{A} \times \mathbf{C}| = \frac{1}{2}AC\sin\beta$. Therefore,

$$AB\sin\theta = AC\sin\beta.$$

It follows $\dfrac{\sin\theta}{C} = \dfrac{\sin\beta}{B}$. Similarly, $\dfrac{\sin\theta}{C} = \dfrac{\sin\alpha}{A}$. Hence

$$\frac{\sin\theta}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}.$$

Lagrange Identity

The magnitude of $|\mathbf{A} \times \mathbf{B}|$ can be expressed in terms of $\mathbf{A}, \mathbf{B}$, and $\mathbf{A} \cdot \mathbf{B}$ through the equation

$$|\mathbf{A} \times \mathbf{B}|^2 = A^2B^2 - (\mathbf{A} \cdot \mathbf{B})^2, \tag{1.27}$$

known as the *Lagrange identity.* This relation follows from the fact

$$|\mathbf{A}\times\mathbf{B}|^2 = (AB\sin\theta)^2 = A^2B^2(1-\cos^2\theta)$$
$$= A^2B^2 - A^2B^2\cos^2\theta = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2 .$$

This relation can also be shown by the components of the vectors. It follows from (1.24) that

$$|\mathbf{A}\times\mathbf{B}|^2 = (A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2 \quad (1.28)$$

and

$$A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2 = (A_x^2+A_y^2+A_z^2)(B_x^2+B_y^2+B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2. \quad (1.29)$$

Multiplying out the right-hand sides of these two equations, we see that they are identical term by term.

1.2.7 Triple Products

Scalar Triple Product

The combination $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ is known as the triple scalar product. $\mathbf{A}\times\mathbf{B}$ is a vector. The dot product of this vector with the vector $\mathbf{C}$ gives a scalar. The triple scalar product has a direct geometrical interpretation. The three vectors can be used to define a parallelopiped as shown in Fig. 1.16. The magnitude of $\mathbf{A}\times\mathbf{B}$ is the area of the parallelogram base and its direction is normal (perpendicular) to the base. The projection of $\mathbf{C}$ onto the unit normal of the base is the height h of the parallelopiped. Therefore, $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ is equal to the area of the base times the height which is the volume of the parallelopiped:

$$\text{Parallelopiped Volume} = \mathit{Area}\times h = |\mathbf{A}\times\mathbf{B}|\, h = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}.$$

The volume of a tetrahedron is equal to one-third of the height times the area of the triangular base. Thus the volume of the tetrahedron formed by the

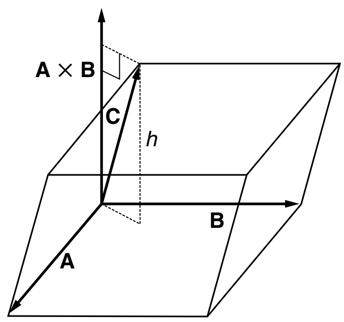

Fig. 1.16. The volume of the parallelopiped is equal to the triple scalar product of its edges as vectors

vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ as concurrent edges is equal to one-sixth of the scalar triple product of these three vectors:

$$\text{Tetrahedron Volume} = \frac{1}{3}h \times \frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \frac{1}{6}(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}.$$

In calculating the volume of the parallelopiped we can consider just as well $\mathbf{B} \times \mathbf{C}$ or $\mathbf{C} \times \mathbf{A}$ as the base. Since the volume is the same regardless of which side we choose as the base, we see that

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}. \tag{1.30}$$

The parentheses in this equation are often omitted, since the cross product must be performed first. If the dot product were performed first, the expression would become a scalar crossed into a vector, which is an undefined and meaningless operation. Without the parentheses, $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$, we see that in any scalar triple product, *the dot and the cross can be interchanged* without altering the value of the product. This is an easy way to remember this relation.

It is clear that if h is reduced to zero, the volume will become zero also. Therefore if $\mathbf{C}$ is in the same plane as $\mathbf{A}$ and $\mathbf{B}$, the scalar triple product $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ vanishes. In particular

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A} = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B} = 0. \tag{1.31}$$

A convenient expression in terms of components for the triple scalar product is provided by the determinant

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{1.32}$$

The rules for interchanging rows of a determinant provide another verification of (1.30)

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix}.$$

Vector Triple Product

The triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is a meaningful operation, because $\mathbf{B} \times \mathbf{C}$ is a vector, and can form cross product with $\mathbf{A}$ to give another vector (hence the name vector triple product). In this case, the parentheses are necessary, because $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ are two different vectors. For example,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \text{and} \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}.$$

The relation

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{1.33}$$

is a very important identity. Because of its frequent use in a variety of problems, this relation should be memorized. This relation (sometimes known as ACB–ABC rule) can be verified by the direct but tedious method of expanding both sides into their cartesian components. A vector equation is, of course, independent of any particular coordinate system. Therefore, it might be more instructive to prove (1.33) without coordinate components.

Let $(\mathbf{B} \times \mathbf{C}) = \mathbf{D}$, hence $\mathbf{D}$ is perpendicular to the plane of $\mathbf{B}$ and $\mathbf{C}$. Now the vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \times \mathbf{D}$ is perpendicular to $\mathbf{D}$, therefore it is in the plane of $\mathbf{B}$ and $\mathbf{C}$. Thus we can write

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha\mathbf{B} + \beta\mathbf{C}, \tag{1.34}$$

where α and β are scalar constants. Furthermore, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is also perpendicular to $\mathbf{A}$. So, the dot product of $\mathbf{A}$ with this vector must be zero:

$$\mathbf{A} \cdot [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] = \alpha\mathbf{A} \cdot \mathbf{B} + \beta\mathbf{A} \cdot \mathbf{C} = 0.$$

It follows that

$$\beta = -\alpha\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{C}}$$

and (1.34) becomes

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \frac{\alpha}{\mathbf{A} \cdot \mathbf{C}}[(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}]. \tag{1.35}$$

This equation is valid for any set of vectors. For the special case $\mathbf{B} = \mathbf{A}$, this equation reduces to

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = \frac{\alpha}{\mathbf{A} \cdot \mathbf{C}}[(\mathbf{A} \cdot \mathbf{C})\mathbf{A} - (\mathbf{A} \cdot \mathbf{A})\mathbf{C}]. \tag{1.36}$$

Take the dot product with $\mathbf{C}$, we have

$$\mathbf{C} \cdot [\mathbf{A} \times (\mathbf{A} \times \mathbf{C})] = \frac{\alpha}{\mathbf{A} \cdot \mathbf{C}}[(\mathbf{A} \cdot \mathbf{C})^2 - A^2C^2]. \tag{1.37}$$

Recall the property of the scalar triple product $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{D}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{D}$, with $\mathbf{D} = (\mathbf{A} \times \mathbf{C})$ the left-hand side of the last equation becomes

$$\mathbf{C} \cdot [\mathbf{A} \times (\mathbf{A} \times \mathbf{C})] = (\mathbf{C} \times \mathbf{A}) \cdot (\mathbf{A} \times \mathbf{C}) = -\left|\mathbf{A} \times \mathbf{C}\right|^2.$$

Using the Lagrange identity (1.27) to express $\left|\mathbf{A} \times \mathbf{C}\right|^2$, we have

$$\mathbf{C} \cdot [\mathbf{A} \times (\mathbf{A} \times \mathbf{C})] = -[(A^2C^2 - (\mathbf{A} \cdot \mathbf{C})^2]. \tag{1.38}$$

Comparing (1.37) and (1.38) we see that

$$\frac{\alpha}{\mathbf{A} \cdot \mathbf{C}} = 1,$$

and (1.35) reduces to the ACB–ABC rule of (1.33).

All higher vector products can be simplified by repeated application of scalar and vector triple products.

Example 1.2.11. Use the scalar triple product to prove the distributive law of cross product: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

Solution 1.2.11. First take a dot product $\mathbf{D} \cdot \mathbf{A} \times (\mathbf{B} + \mathbf{C})$ with an arbitrary vector $\mathbf{D}$, then regard $(\mathbf{B} + \mathbf{C})$ as one vector:

$$\begin{aligned} \mathbf{D} \cdot \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{D} \times \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{D} \times \mathbf{A} \cdot \mathbf{B} + \mathbf{D} \times \mathbf{A} \cdot \mathbf{C} \\ &= \mathbf{D} \cdot \mathbf{A} \times \mathbf{B} + \mathbf{D} \cdot \mathbf{A} \times \mathbf{C} = \mathbf{D} \cdot [\mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}]. \end{aligned}$$

(The first step is evident because dot and cross can be interchanged in the scalar triple product; in the second step we regard $\mathbf{D} \times \mathbf{A}$ as one vector and use the distributive law of the dot product; in the third step we interchange dot and cross again; in the last step we use again the distributive law of the dot product to factor out $\mathbf{D}$.) Since $\mathbf{D}$ can be any vector, it follows that $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

Example 1.2.12. Prove the general form of the Lagrange identity:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).$$

Solution 1.2.12. First regard $(\mathbf{C} \times \mathbf{D})$ as one vector and interchange the cross and the dot in the scalar triple product $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \times \mathbf{B} \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot \mathbf{B} \times (\mathbf{C} \times \mathbf{D}).$$

Then we expand the vector triple product $\mathbf{B} \times (\mathbf{C} \times \mathbf{D})$ in

$$\mathbf{A} \cdot \mathbf{B} \times (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot [(\mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{D}].$$

Since $(\mathbf{B} \cdot \mathbf{D})$ and $(\mathbf{B} \cdot \mathbf{C})$ are scalars, the distributive law of dot product gives

$$\mathbf{A} \cdot [(\mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{D}] = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).$$

Example 1.2.13. The dot and cross products of $\mathbf{u}$ with $\mathbf{A}$ are given by

$$\mathbf{A} \cdot \mathbf{u} = C; \qquad \mathbf{A} \times \mathbf{u} = \mathbf{B}.$$

Express $\mathbf{u}$ in terms of $\mathbf{A}, \mathbf{B}$, and C.

Solution 1.2.13.

$$\begin{aligned} \mathbf{A} \times (\mathbf{A} \times \mathbf{u}) &= (\mathbf{A} \cdot \mathbf{u})\,\mathbf{A} - (\mathbf{A} \cdot \mathbf{A})\,\mathbf{u} = C\mathbf{A} - A^2\mathbf{u}, \\ \mathbf{A} \times (\mathbf{A} \times \mathbf{u}) &= \mathbf{A} \times \mathbf{B}. \\ C\mathbf{A} - A^2\mathbf{u} &= A \times B \\ \mathbf{u} &= \frac{1}{A^2}[C\mathbf{A} - \mathbf{A} \times \mathbf{B}]. \end{aligned}$$

Example 1.2.14. The force $\mathbf{F}$ experienced by the charge q moving with velocity $\mathbf{V}$ in the magnetic field $\mathbf{B}$ is given by the Lorentz force equation

$$\mathbf{F} = q\,(\mathbf{V} \times \mathbf{B})\,.$$

In three separate experiments, it was found

$$\begin{aligned} \mathbf{V} &= \mathbf{i}, \qquad \mathbf{F}/q = 2\mathbf{k} - 4\mathbf{j}, \\ \mathbf{V} &= \mathbf{j}, \qquad \mathbf{F}/q = 4\mathbf{i} - \mathbf{k}, \\ \mathbf{V} &= \mathbf{k}, \qquad \mathbf{F}/q = \mathbf{j} - 2\mathbf{i}. \end{aligned}$$

From these results determine the magnetic field $\mathbf{B}$.

Solution 1.2.14. These results can be expressed as

$$\mathbf{i} \times \mathbf{B} = 2\mathbf{k} - 4\mathbf{j} \quad (1)\,; \quad \mathbf{j} \times \mathbf{B} = 4\mathbf{i} - \mathbf{k} \quad (2)\,; \quad \mathbf{k} \times \mathbf{B} = \mathbf{j} - 2\mathbf{i} \quad (3)\,.$$

From (1)

$$\begin{aligned} \mathbf{i} \times (\mathbf{i} \times \mathbf{B}) &= \mathbf{i} \times (2\mathbf{k} - 4\mathbf{j}) = -2\mathbf{j} - 4\mathbf{k}, \\ \mathbf{i} \times (\mathbf{i} \times \mathbf{B}) &= (\mathbf{i} \cdot \mathbf{B})\,\mathbf{i} - (\mathbf{i} \cdot \mathbf{i})\mathbf{B} = B_x\mathbf{i} - \mathbf{B}; \end{aligned}$$

therefore,

$$B_{\mathbf{x}}\mathbf{i} - \mathbf{B} = -\,2\mathbf{j} - 4\mathbf{k} \quad \textit{or} \quad \mathbf{B} = B_x\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

From (2)

$$\begin{aligned} \mathbf{k} \cdot (\mathbf{j} \times \mathbf{B}) &= \mathbf{k} \cdot (4\mathbf{i} - \mathbf{k}) = -1, \\ \mathbf{k} \cdot (\mathbf{j} \times \mathbf{B}) &= (\mathbf{k} \times \mathbf{j}) \cdot \mathbf{B} = -\mathbf{i} \cdot \mathbf{B}. \end{aligned}$$

Thus,

$$-\mathbf{i} \cdot \mathbf{B} = -1, \quad \text{or} \quad B_x = 1.$$

The final result is obtained just from these two conditions

$$\mathbf{B} = \mathbf{i} + \mathbf{2j} + \mathbf{4k}.$$

We can use the third condition as a consistency check

$$\mathbf{k} \times \mathbf{B} = \mathbf{k} \times (\mathbf{i} + \mathbf{2j} + \mathbf{4k}) = \mathbf{j} - \mathbf{2i},$$

which is in agreement with (3).

Example 1.2.15. **Reciprocal vectors.** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three noncoplanar vectors,

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \quad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \quad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}$$

are known as the reciprocal vectors. Show that any vector $\mathbf{r}$ can be expressed as

$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{a}')\,\mathbf{a} + \left(\mathbf{r}\cdot\mathbf{b}'\right)\mathbf{b} + (\mathbf{r}\cdot\mathbf{c}')\,\mathbf{c}.$$

Solution 1.2.15. Method I. Consider the vector product $(\mathbf{r}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{c})$. First, regard $(\mathbf{r}\times\mathbf{a})$ as one vector and expand

$$(\mathbf{r}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{c}) = [(\mathbf{r}\times\mathbf{a})\cdot\mathbf{c}]\,\mathbf{b} - [(\mathbf{r}\times\mathbf{a})\cdot\mathbf{b}]\,\mathbf{c}.$$

Then, regard $(\mathbf{b}\times\mathbf{c})$ as one vector and expand

$$(\mathbf{r}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{c}) = [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{r}]\,\mathbf{a} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{r}.$$

Therefore,

$$[(\mathbf{r}\times\mathbf{a})\cdot\mathbf{c}]\,\mathbf{b} - [(\mathbf{r}\times\mathbf{a})\cdot\mathbf{b}]\,\mathbf{c} = [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{r}]\,\mathbf{a} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{r}$$

or

$$[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{r} = [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{r}]\,\mathbf{a} - [(\mathbf{r}\times\mathbf{a})\cdot\mathbf{c}]\,\mathbf{b} + [(\mathbf{r}\times\mathbf{a})\cdot\mathbf{b}]\,\mathbf{c}.$$

Since

$$\begin{aligned} -(\mathbf{r}\times\mathbf{a})\cdot\mathbf{c} &= -\mathbf{r}\cdot(\mathbf{a}\times\mathbf{c}) = \mathbf{r}\cdot(\mathbf{c}\times\mathbf{a}), \\ (\mathbf{r}\times\mathbf{a})\cdot\mathbf{b} &= \mathbf{r}\cdot(\mathbf{a}\times\mathbf{b}), \end{aligned}$$

it follows that

$$\begin{aligned} \mathbf{r} &= \frac{\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c})}{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}}\mathbf{a} + \frac{\mathbf{r}\cdot(\mathbf{c}\times\mathbf{a})}{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}}\mathbf{b} + \frac{\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}}\mathbf{c} \\ &= (\mathbf{r}\cdot\mathbf{a}')\,\mathbf{a} + \left(\mathbf{r}\cdot\mathbf{b}'\right)\mathbf{b} + (\mathbf{r}\cdot\mathbf{c}')\,\mathbf{c}. \end{aligned}$$

Method II. Let

$$\mathbf{r} = q_1\mathbf{a} + q_2\mathbf{b} + q_3\mathbf{c}.$$

$$\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}) = q_1\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + q_2\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) + q_3\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}).$$

Since $(\mathbf{b}\times\mathbf{c})$ is perpendicular to $\mathbf{b}$ and perpendicular to $\mathbf{c}$, therefore

$$\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) = 0, \qquad c\cdot(\mathbf{b}\times\mathbf{c}) = 0.$$

Thus

$$q_1 = \frac{\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})} = \mathbf{r}\cdot\mathbf{a}'.$$

Similarly,

$$q_2 = \frac{\mathbf{r}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})} = \frac{\mathbf{r}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})} = \mathbf{r}\cdot\mathbf{b}',$$

$$q_3 = \frac{\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})} = \mathbf{r}\cdot\mathbf{c}'.$$

It follows that

$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{a}')\,\mathbf{a} + \left(\mathbf{r}\cdot\mathbf{b}'\right)\mathbf{b} + (\mathbf{r}\cdot\mathbf{c}')\,\mathbf{c}.$$

1.3 Lines and Planes

Much of analytic geometry can be simplified by the use of vectors. In analytic geometry, a point is a set of three coordinates (x, y, z). All points in space can be defined by the *position vector* $\mathbf{r}(x, y, z)$ (or just $\mathbf{r}$)

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \tag{1.39}$$

drawn from the origin to the point (x, y, z). To specify a particular point (x_0, y_0, z_0), we use the notation $\mathbf{r}_0(x_0, y_0, z_0)$

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}. \tag{1.40}$$

With these notations, we can define lines and planes in space.

1.3.1 Straight Lines

There are several ways to specify a straight line in space. Let us first consider a line through a given point (x_0, y_0, z_0) in the direction of a known vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. If $\mathbf{r}(x, y, z)$ is any other point on the line, the vector $\mathbf{r} - \mathbf{r}_0$ is parallel to $\mathbf{v}$. Thus, we can write the equation of a straight line as

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}, \tag{1.41}$$

where t is any real number. This equation is called the parametric form of a straight line. It is infinitely long and fixed in space, as shown in Fig. 1.17. It cannot be moved parallel to itself as a free vector. This equation in the form of its components

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = ta\mathbf{i} + tb\mathbf{j} + tc\mathbf{k} \tag{1.42}$$

represents three equations

$$(x - x_0) = ta, \qquad (y - y_0) = tb, \qquad (z - z_0) = tc. \tag{1.43}$$

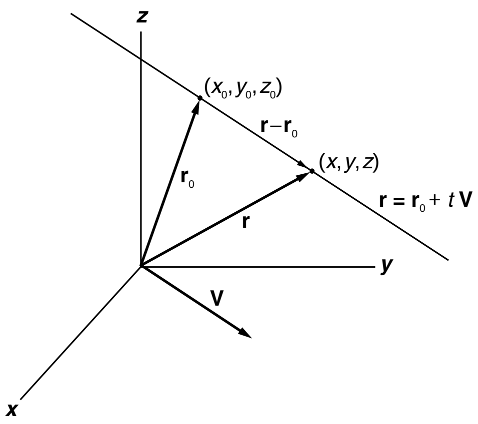

Fig. 1.17. A straight line in the parametric form

Now if a, b, c are not zero, we can solve for t in each of the three equations. The solutions must be equal to each other, since they are all equal to the same t.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{1.44}$$

This is called the symmetric form of the equation of a line. If $\mathbf{v}$ is a normalized unit vector, then a, b, c are the direction cosines of the line.

If c happens to be zero, then (1.43) should be written as

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}; \quad z = z_0. \tag{1.45}$$

The equation $z = z_0$ means that the line lies in the plane perpendicular to the z-axis, and the slope of the line is $\frac{b}{a}$. If both b and c are zero, then clearly the line is the intersection of the planes $y = y_0$ and $z = z_0$.

The parametric equation (1.41) has a useful interpretation when the parameter t means time. Consider a particle moving along this straight line. The equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ indicates when $t = 0$, the particle is at $\mathbf{r}_0$. As time goes on, the particle is moving with a constant velocity $\mathbf{v}$, or $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$.

Perpendicular Distance Between Two Skew Lines

Two lines which are not parallel and which do not meet are said to be skew lines. To find the perpendicular distance between them is a difficult problem in analytical geometry. With vectors, it is relatively easy.

Let the equations of two such lines be

$$\mathbf{r} = \mathbf{r}_1 + t\mathbf{v}_1, \tag{1.46}$$

$$\mathbf{r} = \mathbf{r}_2 + t'\mathbf{v}_2. \tag{1.47}$$

Let a on line 1 and b on line 2 be the end points of the common perpendicular on these two lines. We shall suppose that the position vector $\mathbf{r}_a$ from origin to a is given by (1.46) with $t = t_1$, and the position vector $\mathbf{r}_b$, by (1.47) with $t' = t_2$. Accordingly

$$\mathbf{r}_a = \mathbf{r}_1 + t_1\mathbf{v}_1, \tag{1.48}$$

$$\mathbf{r}_b = \mathbf{r}_2 + t_2\mathbf{v}_2. \tag{1.49}$$

Since $\mathbf{r}_b - \mathbf{r}_a$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$, it must be in the direction of $(\mathbf{v}_1 \times \mathbf{v}_2)$. If d is the length of $\mathbf{r}_b - \mathbf{r}_a$, then

$$\mathbf{r}_b - \mathbf{r}_a = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|}d.$$

Since $\mathbf{r}_b - \mathbf{r}_a = \mathbf{r}_2 - \mathbf{r}_1 + t_2\mathbf{v}_2 - t_1\mathbf{v}_1$,

$$\mathbf{r}_2 - \mathbf{r}_1 + t_2\mathbf{v}_2 - t_1\mathbf{v}_1 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|}d. \tag{1.50}$$

Then take the dot product with $\mathbf{v}_1 \times \mathbf{v}_2$ on both sides of this equation. Since $\mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{v}_2 = 0$, the equation becomes

$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = |\mathbf{v}_1 \times \mathbf{v}_2|\, d;$$

therefore,

$$d = \frac{(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)}{|\mathbf{v}_1 \times \mathbf{v}_2|}. \tag{1.51}$$

This must be the perpendicular distance between the two lines. Clearly, if $d = 0$, the two lines meet. Therefore the condition for the two lines to meet is

$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = 0. \tag{1.52}$$

To determine the coordinates of a and b, take the dot product of (1.50) first with $\mathbf{v}_1$, then with $\mathbf{v}_2$

$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{v}_1 + t_2\mathbf{v}_2 \cdot \mathbf{v}_1 - t_1\mathbf{v}_1 \cdot \mathbf{v}_1 = 0, \tag{1.53}$$

$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{v}_2 + t_2\mathbf{v}_2 \cdot \mathbf{v}_2 - t_1\mathbf{v}_1 \cdot \mathbf{v}_2 = 0. \tag{1.54}$$

These two equations can be solved for t_1 and t_2. With them, $\mathbf{r}_a$ and $\mathbf{r}_b$ can be found from (1.48) and (1.49).

Example 1.3.1. Find the coordinates of the end points a and b of the common perpendicular to the following two lines

$$\mathbf{r} = 9\mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + \mathbf{k}),$$

$$\mathbf{r} = -6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k} + t'(-3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}).$$

Solution 1.3.1. The first line passes through the point $\mathbf{r}_1(0, 9, 2)$ in the direction of $\mathbf{v}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$. The second line passes the point $\mathbf{r}_2(-6, -5, 10)$ in the direction of $\mathbf{v}_2 = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. From (1.53) and (1.54) ,

$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{v}_1 + t_2\mathbf{v}_2 \cdot \mathbf{v}_1 - t_1\mathbf{v}_1 \cdot \mathbf{v}_1 = 4 - 7t_2 - 11t_1 = 0,$$
$$(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{v}_2 + t_2\mathbf{v}_2 \cdot \mathbf{v}_2 - t_1\mathbf{v}_1 \cdot \mathbf{v}_2 = 22 + 29t_2 + 7t_1 = 0.$$

The solution of these two equations is

$$t_1 = 1, \quad t_2 = -1.$$

Therefore, by (1.48) and (1.49) ,

$$\mathbf{r}_a = \mathbf{r}_1 + t_1\mathbf{v}_1 = 3\mathbf{i} + 8\mathbf{j} + 3\mathbf{k},$$
$$\mathbf{r}_b = \mathbf{r}_2 + t_2\mathbf{v}_2 = -3\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}.$$

Example 1.3.2. Find the perpendicular distance between the two lines of the previous example, and an equation for the perpendicular line.

Solution 1.3.2. The perpendicular distance d is simply

$$d = |\mathbf{r}_a - \mathbf{r}_b| = |6\mathbf{i} + 15\mathbf{j} - 3\mathbf{k}| = 3\sqrt{30}.$$

It can be readily verified that this is the same as given by (1.51). The perpendicular line can be represented by the equation

$$\mathbf{r} = \mathbf{r}_a + t(\mathbf{r}_a - \mathbf{r}_b) = (3 + 6t)\,\mathbf{i} + (8 + 15t)\,\mathbf{j} + (3 - 3t)\,\mathbf{k},$$

or equivalently

$$\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}.$$

This line can also be represented by

$$\mathbf{r} = \mathbf{r}_b + s\,(\mathbf{r}_a - \mathbf{r}_b) = (-3 + 6s)\,\mathbf{i} + (-7 + 15s)\,\mathbf{j} + (6 - 3s)\,\mathbf{k},$$

or

$$\frac{x+3}{6} = \frac{y+7}{15} = \frac{z-6}{-3}.$$

Example 1.3.3. Find (a) the perpendicular distance of the point $(5, 4, 2)$ from the line

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1},$$

and (b) also the coordinates of the point where the perpendicular meets the line, and (c) an equation for the line of the perpendicular.

Solution 1.3.3. The line passes the point $\mathbf{r}_0 = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and is in the direction of $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. The parametric form of the line is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = -\,\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t\,(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,.$$

Let the position vector to $(5, 4, 2)$ be

$$\mathbf{r}_1 = 5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}.$$

The distance d from the point $\mathbf{r}_1(5, 4, 2)$ to the line is the cross product of $(\mathbf{r}_1 - \mathbf{r}_0)$ with the unit vector in the $\mathbf{v}$ direction

$$d = \left|(\mathbf{r}_1 - \mathbf{r}_0) \times \frac{\mathbf{v}}{v}\right| = \left|(6\mathbf{i} + \mathbf{j} + \mathbf{k}) \times \frac{2\mathbf{i} + 3\mathbf{j} - \mathbf{k}}{\sqrt{4 + 9 + 1}}\right| = 2\sqrt{6}.$$

Let p be the point where the perpendicular meets the line. Since p is on the given line, the position vector to p must satisfy the equation of the given line with a specific t. Let that specific t be t_1,

$$\mathbf{r}_p = -\,\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t_1\,(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,.$$

Since $(\mathbf{r}_p - \mathbf{r}_0)$ is perpendicular to $\mathbf{v}$, their dot product must be zero

$$(\mathbf{r}_1 - \mathbf{r}_p) \cdot \mathbf{v} = [6\mathbf{i} + \mathbf{j} + \mathbf{k} - t_1\,(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})] \cdot (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 0.$$

This gives $t_1 = 1$. It follows that

$$\mathbf{r}_p = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} + 1\,(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = \mathbf{i} + 6\mathbf{j}.$$

In other words, the coordinates of the foot of the perpendicular is $(1, 6, 0)$. The equation of the perpendicular can be obtained from the fact that it passes $\mathbf{r}_1$ (or $\mathbf{r}_p$) and is in the direction of the vector from $\mathbf{r}_p$ to $\mathbf{r}_1$. So the equation can be written as

$$\mathbf{r} = \mathbf{r}_1 + t\,(\mathbf{r}_1 - \mathbf{r}_p) = 5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + t(4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

or

$$\frac{x-5}{4} = \frac{y-4}{-2} = \frac{z-2}{2}.$$

1.3.2 Planes in Space

A set of parallel planes in space can be determined by a vector normal (perpendicular) to these planes. A particular plane can be specified by an additional condition, such as the perpendicular distance between the origin and the plane, or a given point that lies on the plane.

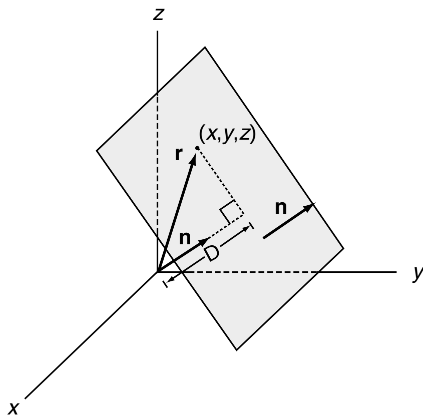

Fig. 1.18. A plane in space. The position vector $\mathbf{r}$ from the origin to any point on the plane must satisfy the equation $\mathbf{r}\cdot\mathbf{n}=D$ where $\mathbf{n}$ is the unit normal to the plane and D is the perpendicular distance between the origin and the plane

Suppose the unit normal to the plane is known to be

$$\mathbf{n}=A\mathbf{i}+B\mathbf{j}+C\mathbf{k} \tag{1.55}$$

and the distance between this plane and the origin is D as shown in Fig. 1.18. If (x,y,z) is a point (any point) on the plane, then it is clear from the figure that the projection of $\mathbf{r}(x,y,z)$ on $\mathbf{n}$ must satisfy the equation

$$\mathbf{r}\cdot\mathbf{n}=D. \tag{1.56}$$

Multiplying out its components, we have the familiar equation of a plane

$$Ax+By+Cz=D. \tag{1.57}$$

This equation will not be changed if both sides are multiplied by the same constant. The result represents, of course, the same plane. However, it is to be emphasized that if the right-hand side D is interpreted as the distance between the plane and the origin, then the coefficients A,B,C on the left-hand side must satisfy the condition $A^2+B^2+C^2=1$, since they should be the direction cosine of the unit normal.

The plane can also be uniquely specified if in addition to the unit normal $\mathbf{n}$, a point (x_0,y_0,z_0) that lies on the plane is known. In this case, the vector $\mathbf{r}-\mathbf{r}_0$ must be perpendicular to $\mathbf{n}$. That means the dot product with $\mathbf{n}$ must vanish,

$$(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n}=0. \tag{1.58}$$

Multiplying out the components, we can write this equation as

$$Ax+By+Cz=Ax_0+By_0+Cz_0. \tag{1.59}$$

This is in the same form of (1.57). Clearly the distance between this plane and the origin is

$$D = Ax_0 + By_0 + Cz_0. \tag{1.60}$$

In general, to find distances between points and lines or planes, it is far simpler to use vectors as compared with calculations in analytic geometry without vectors.

Example 1.3.4. Find the perpendicular distance from the point $(1, 2, 3)$ to the plane described by the equation $3x - 2y + 5z = 10$.

Solution 1.3.4. The unit normal to the plane is

$$\mathbf{n} = \frac{3}{\sqrt{9+4+25}}\mathbf{i} - \frac{2}{\sqrt{9+4+25}}\mathbf{j} + \frac{5}{\sqrt{9+4+25}}\mathbf{k}.$$

The distance from the origin to the plane is

$$D = \frac{10}{\sqrt{9+4+25}} = \frac{10}{\sqrt{38}}.$$

The length of the projection of $\mathbf{r}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ on $\mathbf{n}$ is

$$\ell = \mathbf{r}_1 \cdot \mathbf{n} = \frac{3}{\sqrt{38}} - \frac{4}{\sqrt{38}} + \frac{15}{\sqrt{38}} = \frac{14}{\sqrt{38}}.$$

The distance from $(1, 2, 3)$ to the plane is therefore

$$d = |\ell - D| = \frac{14}{\sqrt{38}} - \frac{10}{\sqrt{38}} = \frac{4}{\sqrt{38}}.$$

Another way to find the solution is to note that the required distance is equal to the projection on $\mathbf{n}$ of any vector joining the given point with a point on the plane. Note that

$$\mathbf{r}_0 = D\mathbf{n} = \frac{30}{38}\mathbf{i} - \frac{20}{38}\mathbf{j} + \frac{50}{38}\mathbf{k}$$

is the position vector of the foot of the perpendicular from the origin to the plane. Therefore

$$d = |(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}| = \frac{4}{\sqrt{38}}.$$

Example 1.3.5. Find the coordinates of the foot of the perpendicular from the point $(1, 2, 3)$ to the plane of the last example.

Solution 1.3.5. Let the position vector from the origin to the foot of the perpendicular be $\mathbf{r}_p$. The vector $\mathbf{r}_1 - \mathbf{r}_p$ is perpendicular to the plane, therefore it is parallel to the unit normal vector $\mathbf{n}$ of the plane,

$$\mathbf{r}_1 - \mathbf{r}_p = k\mathbf{n}.$$

It follows that $|\mathbf{r}_1 - \mathbf{r}_p| = k$. Since $|\mathbf{r}_1 - \mathbf{r}_p| = d$, so $k = d$. Thus,

$$\mathbf{r}_p = \mathbf{r}_1 - d\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} - \frac{4}{\sqrt{38}}\left(\frac{3}{\sqrt{38}}\mathbf{i} - \frac{2}{\sqrt{38}}\mathbf{j} + \frac{5}{\sqrt{9+4+25}}\mathbf{k}\right).$$

Hence, the coordinates of the foot of the perpendicular are $\left(\frac{26}{38}, \frac{84}{38}, \frac{94}{38}\right)$.

Example 1.3.6. A plane intersects the x, y, and z axes, respectively, at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ (Fig. 1.19). Find (a) a unit normal to this plane, (b) the perpendicular distance between the origin and this plane, (c) the equation for this plane.

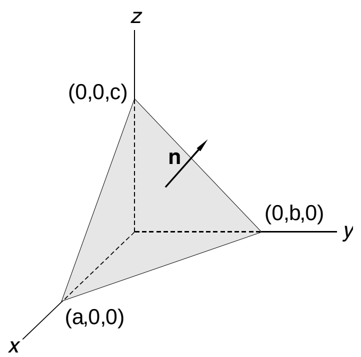

Fig. 1.19. The plane $bcx + acy + abz = abc$ cuts the three axes at $(a, 0, 0)$, $(0, a, 0)$, $(0, 0, a)$, respectively

Solution 1.3.6. Let $\mathbf{r}_1 = a\mathbf{i}$, $\mathbf{r}_2 = b\mathbf{j}$, $\mathbf{r}_3 = c\mathbf{k}$. The vector from $(a, 0, 0)$ to $(0, b, 0)$ is $\mathbf{r}_2 - \mathbf{r}_1 = b\mathbf{j} - a\mathbf{i}$, and the vector from $(a, 0, 0)$ to $(0, 0, c)$ is $\mathbf{r}_3 - \mathbf{r}_1 = c\mathbf{k} - a\mathbf{i}$. The unit normal to this plane must be in the same direction as the cross product of these two vectors:

$$\mathbf{n} = \frac{(b\mathbf{j} - a\mathbf{i}) \times (c\mathbf{k} - a\mathbf{i})}{|(b\mathbf{j} - a\mathbf{i}) \times (c\mathbf{k} - a\mathbf{i})|},$$

$$(b\mathbf{j} - a\mathbf{i}) \times (c\mathbf{k} - a\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k},$$

$$\mathbf{n} = \frac{bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}}{\left((bc)^2 + (ac)^2 + (ab)^2\right)^{1/2}}.$$

If the perpendicular distance from the origin to the plane is D, then

$$D = \mathbf{r}_1 \cdot \mathbf{n} = \mathbf{r}_2 \cdot \mathbf{n} = \mathbf{r}_3 \cdot \mathbf{n},$$

$$D = \frac{abc}{\left((bc)^2 + (ac)^2 + (ab)^2\right)^{1/2}}.$$

In general, the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to any point (x, y, z) on the plane must satisfy the equation

$$\mathbf{r} \cdot \mathbf{n} = D,$$

$$\frac{xbc + yac + zab}{\left((bc)^2 + (ac)^2 + (ab)^2\right)^{1/2}} = \frac{abc}{\left((bc)^2 + (ac)^2 + (ab)^2\right)^{1/2}}.$$

Therefore the equation of this plane can be written as

$$bcx + acy + abz = abc$$

or as

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Another way to find an equation for the plane is to note that the scalar triple product of three coplanar vectors is equal to zero. If the position vector from the origin to any point (x, y, z) on the plane is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then the three vectors $\mathbf{r} - a\mathbf{i}$, $b\mathbf{j} - a\mathbf{i}$ and $c\mathbf{k} - a\mathbf{i}$ are in the same plane. Therefore,

$$(\mathbf{r} - a\mathbf{i}) \cdot (b\mathbf{j} - a\mathbf{i}) \times (c\mathbf{k} - a\mathbf{i}) = \begin{vmatrix} x - a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$
$$= bc\,(x - a) + acy + abz = 0$$

or

$$bcx + acy + abz = abc.$$

Exercises

1. If the vectors $\mathbf{A} = 2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - \mathbf{k}$, find $|\mathbf{A}|$, $|\mathbf{B}|$, $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, and $\mathbf{A} \cdot \mathbf{B}$. What is the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$?
Ans. $\sqrt{13}, \sqrt{2}, 3\mathbf{i} + 2\mathbf{k},\ \mathbf{i} + 4\mathbf{k}, -1,\ 101°$.

2. For what value of c are the vectors $c\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $-\mathbf{i}+2\mathbf{k}$ perpendicular?
Ans. 2.

3. If $\mathbf{A}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\mathbf{B}=-6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, find the projection of $\mathbf{A}$ on $\mathbf{B}$, and the projection of $\mathbf{B}$ on $\mathbf{A}$.
Ans. 4/7, 4/3.

4. Show that the vectors $\mathbf{A}=3\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{B}=\mathbf{i}-3\mathbf{j}+5\mathbf{k}$, and $\mathbf{C}=2\mathbf{i}+\mathbf{j}-4\mathbf{k}$ form a right triangle.

5. Use vectors to prove that the line joining the midpoints of two sides of any triangle is parallel to the third side and half its length.

6. Use vectors to show that for any triangle, the medians (the three lines drawn from each vertex to the midpoint of the opposite side) all pass the same point. The point is at two-thirds of the way of the median from the vertex.

7. If $\mathbf{A}=2\mathbf{i}-3\mathbf{j}-\mathbf{k}$ and $\mathbf{B}=\mathbf{i}+4\mathbf{j}-2\mathbf{k}$, find $\mathbf{A}\times\mathbf{B}$ and $\mathbf{B}\times\mathbf{A}$.
Ans. $10\mathbf{i}+3\mathbf{j}+11\mathbf{k}$, $-10\mathbf{i}-3\mathbf{j}-11\mathbf{k}$.

8. Find the area of a parallelogram having diagonals $\mathbf{A}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{B}=\mathbf{i}-3\mathbf{j}+4\mathbf{k}$.
Ans. $5\sqrt{3}$.

9. Evaluate $(2\mathbf{i}-3\mathbf{j})\cdot[(\mathbf{i}+\mathbf{j}-\mathbf{k})\times(3\mathbf{j}-\mathbf{k})]$.
Ans. 4.

10. Find the volume of the parallelepied whose edges are represented by $\mathbf{A}=2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $\mathbf{B}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$, and $\mathbf{C}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}$.
Ans. 7.

11. Find the constant a such that the vectors $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+a\mathbf{j}+5\mathbf{k}$ are coplanar.
Ans. $a=-4$.

12. Show that (a) $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=\mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})$;
(b) $(\mathbf{a}\times\ \mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=(\mathbf{a}\ \cdot\mathbf{b}\times\mathbf{c})^2$.
Hint: to prove (a) first regard $(\mathbf{b}\times\mathbf{c})$ as one vector, then note $\mathbf{b}\times\mathbf{c}\cdot\mathbf{c}=0$

13. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar (so $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq \mathbf{0}$), and

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \quad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}}, \quad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}},$$

show that
(a) $\mathbf{a}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{b} = \mathbf{c}' \cdot \mathbf{c} = 1,$
(b) $\mathbf{a}' \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{c} = 0, \quad \mathbf{b}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{c} = 0, \quad \mathbf{c}' \cdot \mathbf{a} = \mathbf{c}' \cdot \mathbf{b} = 0,$
(c) if $\mathbf{a} \cdot \ \mathbf{b} \times \mathbf{c} = V$ then $\mathbf{a}' \cdot \ \mathbf{b}' \times \mathbf{c}' = 1/V.$

14. Find the perpendicular distance from the point $(-1, 0, 1)$ to the line $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t\,(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.
Ans. $\sqrt{10}$.

15. Find the coordinates of the foot of the perpendicular from the point $(1, 2, 1)$ to the line joining the origin to the point $(2, 2, 5)$.
Ans. $(2/3, 2/3, 5/3)$.

16. Find the length and equation of the line which is the common perpendicular to the two lines

$$\frac{x-4}{2} = \frac{y+2}{1} = \frac{z-3}{-1}, \qquad \frac{x+7}{3} = \frac{y+2}{2} = \frac{z-1}{1}.$$

Ans. $\sqrt{35}$, $\quad \dfrac{x-2}{3} = \dfrac{y+3}{-5} = \dfrac{z-4}{1}.$

17. Find the distance from $(-2, 4, 5)$ to the plane $2x + 6y - 3z = 10$.
Ans. 5/7

18. Find the equation of the plane that is perpendicular to the vector $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and passes through the point $(1, 2, 1)$.
Ans. $x + y - z = 2$.

19. Find an equation for the plane determined by the points $(2, -1, 1)$, $(3, 2, -1)$, and $(-1, 3, 2)$.
Ans. $11x + 5y + 13z = 30$.