Invariant measures for Markov semigroups

We are given a Hilbert space $H$ (inner product $(\cdot, \cdot)$, norm $|\cdot|$). We shall use the following notations.

- $B(x, r)$ is the open ball in $H$ with centre $x$ and radius $r > 0$.
- $C_b(H)$ (resp. $B_b(H)$) is the Banach space of all uniformly continuous and bounded mappings (resp. Borel bounded mappings) $\varphi: H \to \mathbb{R}$ endowed with the norm
  $$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|.$$
- $L(C_b(H))$ (resp. $L(B_b(H))$) is the space of all linear bounded operators from $C_b(H)$ (resp. $B_b(H)$) into itself.
- $C_b^+(H)$ (resp. $B_b^+(H)$) represents the cone in $C_b(H)$ (resp. $C_b(H)$) consisting of all non-negative functions, and $1$ the function on $H$ identically equal to $1$.
- $C_b(H)^*$ is the topological dual of $C_b(H)$.
- $\mathcal{P}(H)$ is the space of all probability measures on $(H, \mathcal{B}(H))$ where $\mathcal{B}(H)$ is the $\sigma$-algebra of all Borel subsets of $H$.

There is a natural embedding of $\mathcal{P}(H)$ into $C_b(H)^*$. Namely, for any $\mu \in \mathcal{P}(H)$ we set
  $$F_\mu(\varphi) = \int_H \varphi(x) \mu(dx), \quad \varphi \in C_b(H).$$

In the following we shall often identify $\mu$ with $F_\mu$.

5.1 Markov semigroups

**Definition 5.1** A Markov semigroup $P_t$ on $B_b(H)$ is a mapping
  $$[0, +\infty) \to L(B_b(H)), \quad t \mapsto P_t,$$
such that

(i) $P_0 = 1$, $P_{t+s} = P_tP_s$ for all $t, s \geq 0$.

(ii) For any $t \geq 0$ and $x \in H$ there exists a probability measure $\pi_t(x, \cdot) \in \mathcal{P}(H)$ such that

$$P_t \varphi(x) = \int_H \varphi(y) \pi_t(x, dy) \quad \text{for all } \varphi \in B_b(H). \quad (5.1)$$

(iii) For any $\varphi \in C_b(H)$ (resp. $B_b(H)$) and $x \in H$, the mapping $t \mapsto P_t \varphi(x)$ is continuous (resp. Borel).

Obviously, by (5.1) it follows that for $t = 0$,

$$\pi_0(x, \cdot) = \delta_x, \quad x \in H,$$

where $\delta_x$ is the Dirac measure at $x$.

We notice that in the literature one requires usually only (i) and (ii) in the definition of Markov semigroup $P_t$. In this case condition (iii) means that $P_t$ is stochastically continuous, see e.g. [10].

**Definition 5.2** Let $P_t$ be a Markov semigroup.

(i) $P_t$ is Feller if $P_t \varphi \in C_b(H)$ for any $\varphi \in C_b(H)$ and any $t \geq 0$.

(ii) $P_t$ is strong Feller if $P_t \varphi \in C_b(H)$ for any $\varphi \in B_b(H)$ and any $t > 0$.

(iii) $P_t$ is irreducible if $P_t 1_{B(x_0, r)}(x) > 0$ for all $x, x_0 \in H$, $r > 0$ and any $t \geq 0$.

Let us give some general properties of a Markov semigroup $P_t$. First, notice that by (5.1) we have $P_t 1 = 1$ for all $t \geq 0$ and that $P_t$ preserves positivity, that is $P_t \varphi \in B^+_b(H)$ for all $\varphi \in B^+_b(H)$.

Moreover, since, for any $\varphi \in C_b(H)$,

$$-\|\varphi\|_0 \leq \varphi(x) \leq \|\varphi\|_0, \quad x \in H,$$

we have

$$|P_t \varphi(x)| \leq \|\varphi\|_0, \quad x \in H.$$ 

Consequently $\|P_t\|_{L(B_b(H))} \leq 1$, for any $t \geq 0$. That is $P_t$ is a semigroup of contractions on $B_b(H)$.

Let us give now some properties of the family of measures $\pi_t(x, \cdot)$ (called a probability kernel).

By (5.1) it follows that for any $E \in \mathcal{B}(H)$ we have

$$\pi_t(x, E) = P_t 1_E(x), \quad t \geq 0, \quad x \in H. \quad (5.2)$$

Moreover, the following useful result holds.
Proposition 5.3 For any $t, s \geq 0$, $x \in H$ and any $E \in \mathcal{B}(H)$ we have

$$\pi_{t+s}(x, E) = \int_H \pi_s(y, E) \pi_t(x, dy).$$

(5.3)

Proof. We have in fact, taking into account the semigroup property of $P_t$, (5.2) and (5.1),

$$\pi_{t+s}(x, E) = P_{t+s}1_E(x) = P_t \pi_s(\cdot, E)(x) = \int_H \pi_s(y, E) \pi_t(x, dy).$$

□

Example 5.4 Let us consider the differential equation

$$\begin{cases}
X'(t) = b(X(t)), \\
X(0) = x,
\end{cases}$$

(5.4)
on $H = \mathbb{R}^n$ where $b: H \to H$ is Lipschitz continuous. As is well known, there exists a unique solution $X(t, x)$ of problem (5.4). Set

$$\pi_t(x, \cdot) = \delta_{X(t, x)}, \quad x \in \mathbb{R}^n.$$

Then it is easy to see that the transition semigroup

$$P_t \varphi(x) = \varphi(X(t, x)), \quad \varphi \in B_b(\mathbb{R}^n)$$

(5.5)
is a Markov semigroup.

Exercise 5.5 (i) Prove that semigroup $P_t$, defined by (5.5), is Feller. Is $P_t$ strong Feller?

(ii) Prove that $P_t$ is strongly continuous in $C_b(H)$ if and only if $b$ is bounded.

Example 5.6 Let us consider the stochastic differential equation

$$\begin{cases}
dX = b(X) dt + \sqrt{C} dB(t), \\
X(0) = x,
\end{cases}$$

(5.6)
on $H = \mathbb{R}^n$ where $B$ is a standard Brownian motion in a probability space $(\Omega, \mathcal{F}, P)$ with values in $H$, $b: H \to H$ is locally Lipschitz continuous, $C \in L(H)$ and Hypothesis 4.23 is fulfilled.
Then by Proposition 4.3 there exists a unique continuous stochastic process \( X(\cdot, x) \), the solution of problem (5.6). Set 
\[
\pi_t(x, E) = (X(t, x)_\# P)(E), \quad x \in \mathbb{R}^n, \ E \in \mathscr{B}(\mathbb{R}^n).
\]
Then the transition semigroup
\[
P_t \varphi(x) = \mathbb{E} [\varphi(X(t, x))] = \int_{\mathbb{R}} \varphi(y) \pi_t(x, dy), \quad \varphi \in B_b(H), \quad (5.7)
\]
is a Markov semigroup as easily checked.

**Exercise 5.7** Prove that the semigroup \( P_t \), defined by (5.7), is Feller.

### 5.2 Invariant measures

In this section \( P_t \) represents a Markov semigroup on \( H \). A probability measure \( \mu \in \mathscr{P}(H) \) is said to be **invariant** for \( P_t \) if
\[
\int_H P_t \varphi d\mu = \int_H \varphi d\mu \quad \text{for all } \varphi \in B_b(H) \text{ and } t \geq 0. \quad (5.8)
\]
If \( P_t \) is Feller this condition is clearly equivalent (identifying \( \mu \) with \( F_{\mu} \)) to
\[
P_t^* \mu = \mu \quad \text{for all } t \geq 0, \quad (5.9)
\]
where \( P_t^* \) is the transpose operator of \( P_t \), defined as
\[
\langle \varphi, P_t^* F \rangle = \langle P_t \varphi, F \rangle,
\]
for all \( \varphi \in C_b(H) \), \( F \in C_b(H)^* \).

If \( \mu \in \mathscr{P}(H) \) is invariant for \( P_t \) we have
\[
\mu(A) = P_t^* \mu(A) = \int_H P_t 1_{A}(x) \mu(dx), \quad A \in \mathscr{B}(H),
\]
from which, recalling (5.8),
\[
\mu(A) = \int_H \pi_t(x, A) \mu(dx), \quad A \in \mathscr{B}(H). \quad (5.10)
\]
A first basic result is the following.

---

\( \langle \cdot, \cdot \rangle \) represent the duality between \( C_b(H) \) and \( C_b(H)^* \).
Chapter 5

Theorem 5.8 Assume that \( \mu \) is an invariant measure for \( P_t \). Then for all \( t \geq 0, p \geq 1 \), \( P_t \) is uniquely extendible to a linear bounded operator on \( L^p(H, \mu) \) that we still denote by \( P_t \). Moreover

\[
\|P_t\|_{L^p(H, \mu)} \leq 1, \quad t \geq 0.
\] (5.11)

Finally, \( P_t \) is a strongly continuous semigroup in \( L^p(H, \mu) \).

Proof. Let \( \varphi \in C_b(H) \). By the Hölder inequality we have

\[
|P_t \varphi(x)|^p \leq \int_H |\varphi(y)|^p \pi_t(x, dy) = P_t(|\varphi|^p)(x).
\]

Integrating both sides of the above inequality with respect to \( \mu \) over \( H \) yields

\[
\int_H |P_t \varphi(x)|^p \mu(dx) \leq \int_H P_t(|\varphi|^p)(x) \mu(dx) = \int_H |\varphi(x)|^p \mu(dx)
\]

in view of the invariance of \( \mu \). Since \( C_b(H) \) is dense in \( L^p(H, \mu) \), \( P_t \) is uniquely extendible to \( L^p(H, \mu) \) and (5.11) follows.

Let us show finally that \( P_t \) is strongly continuous in \( L^p(H, \mu) \). First let \( \varphi \in C_b(H) \). Then, by property (iii) in Definition 5.1 of \( P_t \) we have that the function \( t \to P_t \varphi(x) \) is continuous for any \( x \in H \). Consequently, by the dominated convergence theorem

\[
\lim_{t \to 0} P_t \varphi = \varphi \quad \text{in} \quad L^p(H, \mu).
\]

The same assertion follows easily when \( \varphi \in L^p(H, \mu) \) by the density of \( C_b(H) \) in \( L^p(H, \mu) \). \( \Box \)

Let \( \mu \) be an invariant measure for \( P_t \). We are going to study the asymptotic behaviour of \( P_t \varphi \), for \( \varphi \in L^2(H, \mu) \). This is obvious when \( P_t \varphi = \varphi \) for all \( t > 0 \). In this case we say that \( \varphi \) is stationary. In general, given \( \varphi \in L^2(H, \mu) \), one can ask whether there exists the limit

\[
\lim_{t \to +\infty} P_t \varphi(x),
\] (5.12)

or, if not, if there exists the limit of the means

\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T P_s \varphi(x) ds.
\] (5.13)

We shall prove indeed that this limit always exists in \( L^2(H, \mu) \) (Von Neumann theorem).
If in addition it happens that
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t \varphi(x) dt = \int_H \varphi d\mu \quad \text{in } L^2(H, \mu),
\] (5.14)
for all \( \varphi \in L^2(H, \mu) \), \( P_t \) is said to be \textit{ergodic}. In this case the identity (5.14) is interpreted in physics by saying that the “temporal” average of \( P_t \varphi \) coincides with the “spatial” average of \( \varphi \).

It can also happen in particular that
\[
\lim_{t \to +\infty} P_t \varphi(x) = \int_H \varphi d\mu \quad \text{in } L^2(H, \mu).
\] (5.15)
In this case \( P_t \) is said to be \textit{strongly mixing}.

Existence and uniqueness of invariant measures will be proved in Chapter 7. We conclude this introduction by giving two examples of invariant measures.

\textbf{Exercise 5.9} Consider the ordinary differential equation,
\[
Z'(t) = Z(t) - Z^3(t), \quad Z(0) = x,
\]
and the corresponding transition semigroup
\[
P_t \varphi(x) = \varphi(Z(t, x)), \quad \varphi \in C_b(H).
\]
Prove that \( P_t \) is a Markov semigroup and that \( \pi_t(x, E) = \delta_{Z(t,x)}(E), \ E \in \mathcal{B}(\mathbb{R}), \ t \geq 0, \ x \in \mathbb{R} \).

Show moreover that measures \( \delta_0, \delta_1 \) and \( \delta_{-1} \) are invariant, ergodic and strongly mixing.

\textbf{Exercise 5.10} Consider the stochastic differential equation in \( \mathbb{R} \),
\[
dX(t) = -X(t) dt + dB(t), \quad X(0) = x,
\]
whose solution \( X(t, x) \) is given by the Ornstein–Uhlenbeck process (see Proposition 4.10),
\[
X(t, x) = e^{-t} x + \int_0^t e^{-(t-s)} dB(s), \quad t \geq 0, \ x \in \mathbb{R}.
\]
Prove that
\[
\pi_t(x, \cdot) = N e^{-t x} \frac{1}{2} (1 - e^{-2t}), \quad x \in \mathbb{R}, \ t > 0.
\]
Show moreover that the measure \( \mu = N \frac{1}{2} \) is invariant, ergodic and strongly mixing.

\textbf{Hint.} Check that (5.8) holds for \( \varphi(x) = e^{ihx} \), where \( h \in \mathbb{R} \).
In order to study the behaviour of \(\lim_{T \to +\infty} \frac{1}{T} \int_0^T P_t \varphi dt\), we need some general result about the averages of the powers of a linear operator, proved in the next section.

5.3 Ergodic averages

We are given a linear bounded operator \(T\) on a Hilbert space \(E\) (norm \(\| \cdot \|\), inner product \(\langle \cdot, \cdot \rangle\)).\(^{(2)}\) We set

\[
M_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k, \quad n \in \mathbb{N}.
\]

**Theorem 5.11** Assume that \(\sup_{n\in\mathbb{N}} \|T^n\| < +\infty\). Then there exists the limit

\[
\lim_{n \to \infty} M_n x := M_\infty x \quad \text{for all } x \in E.
\]

Moreover \(M_\infty \in L(H), \ M_\infty^2 = M_\infty\) and \(M_\infty(E) = \text{Ker } (1 - T)\).

**Proof.** First notice that the limit of \((M_n x)\) certainly exists when either \(x \in \text{Ker } (1 - T)\), or \(x \in (1 - T)(E)\). In fact in the first case we have obviously

\[
\lim_{n \to \infty} M_n x = x \quad \text{for all } x \in \text{Ker } (1 - T),
\]

and in the latter we have

\[
\lim_{n \to \infty} M_n x = 0 \quad \text{for all } x \in (1 - T)(E),
\]

because

\[
(1 - T)M_n = M_n (1 - T) = \frac{1}{n} (1 - T^n), \quad n \in \mathbb{N}.
\]

Consequently we also have

\[
\lim_{n \to \infty} M_n x = 0 \quad \text{for all } x \in (1 - T)(E),
\]

where \((1 - T)(E)\) is the closure of \((1 - T)(E)\).

Now let \(x \in E\) be fixed. Since \(\|M_n x\|_{n\in\mathbb{N}}\) is bounded by assumption, there exists a sub-sequence \((n_k)\) of \(\mathbb{N}\), and an element \(y \in H\) such that \(M_{n_k} x \to y\) weakly as \(k \to \infty\). By (5.17) it follows also that \(TM_{n_k} x \to Ty = y\), so that \(y \in \text{Ker } (1 - T)\).

\(^{(2)}\) Later we shall take \(E = L^2(H, \mu)\).
Now we prove that \( M_n x \to y \). First note that, since \( y \in \text{Ker } (1 - T) \), we have \( M_n y = y \), and so

\[
M_n x = M_n y + M_n(x - y) = y + M_n(x - y). \tag{5.19}
\]

We claim that \( x - y \in (1 - T)(E) \), which will prove (5.17) by (5.16). We have in fact

\[
x - y = \lim_{k \to \infty} (x - M_{n_k} x),
\]

and \( x - M_{n_k} x \in (1 - T)(E) \) because

\[
x - M_{n_k} x = \frac{1}{n_k} \sum_{h=0}^{n_k-1} (1 - T^h)x.
\]

Therefore (5.16) holds.

Finally, since \( (1 - T)M_n \to 0 \), we have \( M_\infty = TM_\infty \), so that \( T^k M_\infty = M_\infty \), \( k \in \mathbb{N} \), and \( M_\infty = M_n M_\infty \), which yields as \( n \to \infty \), \( M_\infty = (M_\infty)^2 \), as required. \( \square \)

### 5.4 The Von Neumann theorem

In this section we assume that there is an invariant measure \( \mu \) for the Markov semigroup \( P_t \). This will allow us to extend the semigroup \( P_t \) to \( L^2(H, \mu) \), as proved in Theorem 5.8.

We denote by \( \Sigma \) the set

\[
\Sigma = \{ f \in L^2(H, \mu) : P_t f = f, \mu\text{-a.e. for all } t \geq 0 \} \tag{5.20}
\]

of all stationary points of \( P_t \). Clearly \( \Sigma \) is a closed subspace of \( L^2(H, \mu) \) and \( 1 \in \Sigma \).

Let us consider the average

\[
M(T) \varphi = \frac{1}{T} \int_0^T P_t \varphi dt, \quad \varphi \in L^2(H, \mu), \quad T > 0.
\]

**Theorem 5.12** There exists the limit

\[
\lim_{T \to \infty} M(T) \varphi =: M_\infty \varphi \quad \text{in } L^2(H, \mu). \tag{5.21}
\]
Moreover $M_\infty$ is a projection operator on $\Sigma$, and
\[ \int_H M_\infty \varphi d\mu = \int_H \varphi d\mu. \] (5.22)

Proof. For all $T > 0$ write
\[ T = n_T + r_T, \quad n_T \in \mathbb{N} \cup \{0\}, \quad r_T \in [0,1). \]

For $\varphi \in L^2(H,\mu)$ we have
\[ M(T)\varphi = \frac{1}{n_T} \sum_{k=0}^{n_T - 1} \int_k^{k+1} P_s \varphi ds + \frac{1}{n_T} \int_{n_T}^T P_s \varphi ds \]
\[ = \frac{1}{n_T} \sum_{k=0}^{n_T - 1} \int_0^1 P_{s+k} \varphi ds + \frac{1}{n_T} \int_0^{r_T} P_{s+n(T)} \varphi ds \]
\[ = \frac{n_T}{T} \frac{1}{n_T} \sum_{k=0}^{n_T - 1} (P_1)^k M(1) \varphi + \frac{r_T}{T} (P_1)^{n_T} M(r_T) \varphi. \] (5.23)

Since
\[ \lim_{T \to \infty} \frac{n_T}{T} = 1, \quad \lim_{T \to \infty} \frac{r_T}{T} = 0, \]
letting $n \to \infty$ in (5.23) and invoking Theorem 5.11, we get (5.21).

We prove now that for all $t \geq 0$
\[ M_\infty P_t = P_t M_\infty = M_\infty. \] (5.24)

In fact, given $t \geq 0$ we have
\[ M_\infty P_t \varphi = \lim_{T \to \infty} \frac{1}{T} \int_0^T P_{t+s} \varphi ds = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} P_s \varphi ds \]
\[ = \lim_{T \to \infty} \frac{1}{T} \left\{ \int_0^T P_s \varphi ds - \int_0^t P_s \varphi ds + \int_t^{T+t} P_s \varphi ds \right\} \]
\[ = M_\infty \varphi \]
and this yields (5.24).

By (5.24) it follows that $M_\infty f \in \Sigma$ for all $f \in L^2(H,\mu)$, and moreover that
\[ M_\infty M(T) = M(T) P_\infty = M_\infty, \]
which yields, letting $T \to \infty$, $M_\infty^2 = M_\infty$. Finally, (5.22) follows, by integrating (5.21) with respect to $\mu$. □
5.5 Ergodicity

Let $\mu$ be an invariant measure for $P_t$. We say that $\mu$ is ergodic if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T P_t \varphi dt = \varphi$$

for all $\varphi \in L^2(H, \mu)$, (5.25)

where

$$\varphi = \int_H \varphi(x) \mu(dx).$$

**Proposition 5.13** Let $\mu$ be an invariant measure for $P_t$. Then $\mu$ is ergodic if and only if the dimension of the linear space $\Sigma$ of all stationary elements of $L^2(H, \mu)$ defined by (5.20) is 1.

**Proof.** If $\mu$ is ergodic it follows from (5.25) that any element in $\Sigma$ is constant, so that dimension of $\Sigma$ is 1. Conversely assume that dimension of $\Sigma$ is 1. Then there is a linear bounded functional $F$ on $L^2(H, \mu)$ such that

$$M_\infty \varphi = F(\varphi) 1.$$

By the Riesz representation theorem there exists an element $\varphi_0 \in L^2(H, \mu)$ such that $F(\varphi) = \langle \varphi, \varphi_0 \rangle$. Integrating this equality on $H$ with respect to $\mu$ and taking into account the invariance of $M_\infty$ (see (5.22)), yields

$$\int_H M_\infty \varphi d\mu = \int_H \varphi d\mu = \langle \varphi, 1 \rangle = \langle \varphi, \varphi_0 \rangle, \quad \varphi \in L^2(H, \mu).$$

Therefore $\varphi_0 = 1$. □

Let $\mu$ be an invariant measure for $P_t$. A Borel set $\Gamma \in \mathcal{B}(H)$ is said to be invariant for $P_t$ if its characteristic function $1_{\Gamma}$ belongs to $\Sigma$. If $\mu(\Gamma)$ is equal to either 0 or 1, we say that $\Gamma$ is trivial, otherwise it is nontrivial.

We now want to show that $\mu$ is ergodic if and only if all invariant sets are trivial. For this it is important to notice that $\Sigma$ is a lattice, as proved in the next proposition.

**Proposition 5.14** Assume that $\varphi$ and $\psi$ belong to $\Sigma$. Then the following statements hold.

(i) $|\varphi| \in \Sigma.$
(ii) $\varphi^+, \varphi^- \in \Sigma.$ (3)
(3) $\varphi^+ = \max\{\varphi, 0\}, \varphi^- = \max\{-\varphi, 0\}.$


\(\text{(iii) } \varphi \lor \psi, \varphi \land \psi \in \Sigma.\) 
\(\text{(iv) For any } a \in \mathbb{R} \text{ we have } 1_{\{x \in H : \varphi(x) > a\}} \in \Sigma.\)

**Proof.** Let us prove (i). Let \(t > 0\) and assume that \(\varphi \in \Sigma\), so that 
\[|\varphi(x)| = |P_t \varphi(x)| \leq P_t(|\varphi|)(x), \quad x \in H. \quad (5.26)\]

We claim that 
\[|\varphi(x)| = P_t(|\varphi|)(x), \quad \mu\text{-a.s.}\]

Assume by contradiction that there is a Borel subset \(I \subset H\) such that 
\[\mu(I) > 0 \quad \text{and} \quad |\varphi(x)| < P_t(|\varphi|)(x), \quad x \in I.\]

Then we have 
\[\int_H |\varphi(x)| \mu(dx) < \int_H P_t(|\varphi|)(x) \mu(dx).\]

Since, by the invariance of \(\mu\), 
\[\int_H P_t(|\varphi|)(x) \mu(dx) = \int_H |\varphi|(x) \mu(dx),\]

we find a contradiction.

Statements (ii) and (iii) follow from the obvious identities 
\[\varphi^+ = \frac{1}{2}(\varphi + |\varphi|), \quad \varphi^- = \frac{1}{2}(\varphi - |\varphi|),\]

\[\varphi \lor \psi = (\varphi - \psi)^+ + \psi, \quad \varphi \land \psi = -(\varphi - \psi)^+ + \varphi.\]

Finally let us prove (iv). It is enough to show that the set \(\{\varphi > 0\}\) is invariant, or, equivalently, that \(1_{\{\varphi > 0\}}\) belongs to \(\Sigma\). We have in fact, as it is easily checked, 
\[1_{\{\varphi > 0\}} = \lim_{n \to \infty} \varphi_n(x), \quad x \in H,\]

where \(\varphi_n = (n\varphi^+) \land 1, \quad n \in \mathbb{N}\), belongs to \(\Sigma\) by (ii) and (iii). Therefore 
\(\{\varphi > 0\}\) is invariant. \(\square\)

We are now ready to prove the following result.

**Theorem 5.15** Let \(\mu\) be an invariant measure for \(P_t\). Then \(\mu\) is ergodic if and only if any invariant set is trivial.

\(\text{(4)}\) \(\varphi \lor \psi = \max\{\varphi, \psi\}, \quad \varphi \land \psi = \min\{\varphi, \psi\}.\)
Proof. Let $\Gamma$ be invariant for $\mu$. Then if $\mu$ is ergodic $1_\Gamma$ must be constant (otherwise $\dim \Sigma \geq 2$) and so $\Gamma$ is trivial. Assume conversely that the only invariant sets for $\mu$ are trivial and, by contradiction, that $\mu$ is not ergodic. Then there exists a non-constant function $\varphi_0 \in \Sigma$. Therefore by Proposition 5.14 for some $\lambda \in \mathbb{R}$ the invariant set $\{\varphi_0 > \lambda\}$ is not trivial. □

5.6 Structure of the set of all invariant measures

We still assume that $P_t$ is a Markov semigroup on $H$. We denote by $\Lambda$ the set of all its invariant measures and we assume that $\Lambda$ is non-empty. Clearly $\Lambda$ is a convex subset of $C_b(H)^*$. 

Theorem 5.16 Assume that there is a unique invariant measure $\mu$ for $P_t$. Then $\mu$ is ergodic.

Proof. Assume by contradiction that $\mu$ is not ergodic. Then there is a nontrivial invariant set $\Gamma$. Let us prove that the measure $\mu_\Gamma$ defined as

$$\mu_\Gamma(A) = \frac{1}{\mu(\Gamma)} \mu(A \cap \Gamma), \quad A \in \mathcal{B}(H),$$

belongs to $\Lambda$. This will give rise to a contradiction. Recalling (5.10), we have to show that

$$\mu_\Gamma(A) = \int_H \pi_t(x, A) \mu(dx), \quad A \in \mathcal{B}(H),$$

or, equivalently, that

$$\mu(A \cap \Gamma) = \int_{\Gamma} \pi_t(x, A) \mu(dx), \quad A \in \mathcal{B}(H). \quad (5.27)$$

In fact, since $\Gamma$ is invariant, we have

$$P_t 1_\Gamma = 1_\Gamma, \quad P_t 1_{\Gamma^c} = 1_{\Gamma^c}, \quad t \geq 0,$$

and so

$$\pi_t(x, \Gamma) = 1_\Gamma(x), \quad \pi_t(x, \Gamma^c) = 1_{\Gamma^c}(x), \quad t \geq 0.$$ Consequently,

$$\pi_t(x, A \cap \Gamma^c) = 0, \quad \mu\text{-a.e. in } \Gamma \text{ and } \pi_t(x, A \cap \Gamma) = 0, \quad \mu\text{-a.e. in } \Gamma^c,$$
and so
\[
\int_{\Gamma} \pi_t(x, A) \mu(dx) = \int_{\Gamma} \pi_t(x, A \cap \Gamma) \mu(dx) + \int_{\Gamma} \pi_t(x, A \cap \Gamma^c) \mu(dx)
\]
\[= \int_{\Gamma} \pi_t(x, A \cap \Gamma) \mu(dx) = \int_{H} \pi_t(x, A \cap \Gamma) \mu(dx) = \mu(A \cap \Gamma),
\]
and (5.10) holds. □

We want now to show that the set of all extremal points of \( \Lambda \) is precisely the set of all ergodic measures of \( P_t \). For this we need a lemma.

**Lemma 5.17** Let \( \mu, \nu \in \Lambda \) with \( \mu \) ergodic and \( \nu \) absolutely continuous with respect to \( \mu \). Then \( \mu = \nu \).

**Proof.** Let \( \Gamma \in \mathcal{B}(H) \). By the Von Neumann theorem there exists \( T_n \uparrow \infty \) such that
\[
\lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} P_t 1_{\Gamma} dt = \mu(\Gamma), \quad \mu\text{-a.e.} \tag{5.28}
\]
Since \( \nu \ll \mu \), identity (5.28) holds also \( \nu \text{-a.e.} \) Now integrating (5.28) with respect to \( \nu \) yields
\[
\lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \left( \int_{H} P_t 1_{\Gamma} d\nu \right) dt = \nu(\Gamma), \quad \mu\text{-a.e.}
\]
Consequently \( \nu(\Gamma) = \mu(\Gamma) \) as required. □

We can now prove the announced property of \( \Lambda \).

**Theorem 5.18** The set of all invariant ergodic measures of \( P_t \) coincides with the set of all extremal points of \( \Lambda \).

**Proof.** We first prove that if \( \mu \) is ergodic then it is an extremal point of \( \Lambda \). Assume by contradiction that \( \mu \) is ergodic and it is not an extremal point of \( \Lambda \). Then there exist \( \mu_1, \mu_2 \in \Lambda \) with \( \mu_1 \neq \mu_2 \), and \( \alpha \in (0, 1) \) such that
\[
\mu = \alpha \mu_1 + (1 - \alpha) \mu_2.
\]
Then clearly \( \mu_1 \ll \mu \) and \( \mu_2 \ll \mu \). By Lemma 5.17 we get a contradiction.

We finally prove that if \( \mu \) is an extremal point of \( \Lambda \), then it is ergodic. Assume by contradiction that \( \mu \) is not ergodic. Then there exists a nontrivial invariant set \( \Gamma \). Consequently, arguing as in the proof
of Theorem 5.16, we have $\mu_\Gamma, \mu_{\Gamma^c} \in \Lambda$. Since
\[ \mu = \mu(\Gamma)\mu_\Gamma + (1 - \mu(\Gamma))\mu_{\Gamma^c}, \]
we find that $\mu$ is not extremal, a contradiction. \qed

**Theorem 5.19** Assume that $\mu$ and $\nu$ are ergodic invariant measures with $\mu \neq \nu$. Then $\mu$ and $\nu$ are singular.

**Proof.** Let $\Gamma \in B(H)$ be such that $\mu(\Gamma) \neq \nu(\Gamma)$. From the Von Neumann theorem it follows that there exists $T_n \uparrow +\infty$ and two Borel sets $M$ and $N$ such that $\mu(M) = 1, \nu(N) = 1,$ and
\[ \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} (P_t1_\Gamma)(x)dt = \mu(\Gamma), \forall x \in M, \]
\[ \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} (P_t1_\Gamma)(x)dt = \nu(\Gamma), \forall x \in N. \]
Since $\mu(\Gamma) \neq \nu(\Gamma)$ this implies that $M \cap N = \emptyset$, and so $\mu$ and $\nu$ are singular. \qed