5

Invariant measures for Markov semigroups

We are given a Hilbert space H (inner product $\langle \cdot, \cdot \rangle$, norm $|\cdot|$). We shall use the following notations.

- $B(x, r)$ is the open ball in H with centre x and radius $r > 0$.
- $C_b(H)$ (resp. $B_b(H)$) is the Banach space of all uniformly continuous and bounded mappings (resp. Borel bounded mappings) $\varphi \colon H \to \mathbb{R}$ endowed with the norm
$$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|.$$
- $L(C_b(H))$ (resp. $L(B_b(H))$) is the space of all linear bounded operators from $C_b(H)$ (resp. $B_b(H)$) into itself.
- $C_b^+(H)$ (resp. $B_b^+(H)$) represents the cone in $C_b(H)$ (resp. $C_b(H)$) consisting of all non-negative functions, and $\mathbf{1}$ the function on H identically equal to 1.
- $C_b(H)^*$ is the topological dual of $C_b(H)$.
- $\mathscr{P}(H)$ is the space of all probability measures on $(H, \mathscr{B}(H))$ where $\mathscr{B}(H)$ is the σ-algebra of all Borel subsets of H.
  There is a natural embedding of $\mathscr{P}(H)$ into $C_b(H)^*$. Namely, for any $\mu \in \mathscr{P}(H)$ we set
$$F_\mu(\varphi) = \int_H \varphi(x)\mu(dx), \quad \varphi \in C_b(H).$$
  In the following we shall often identify μ with F_μ.

5.1 Markov semigroups

Definition 5.1 *A* Markov semigroup P_t *on* $B_b(H)$ *is a mapping*
$$[0, +\infty) \to L(B_b(H)), \quad t \mapsto P_t,$$

such that

(i) $P_0 = 1$, $P_{t+s} = P_t P_s$ *for all* $t, s \geq 0$.
(ii) *For any* $t \geq 0$ *and* $x \in H$ *there exists a probability measure* $\pi_t(x, \cdot) \in \mathscr{P}(H)$ *such that*

$$P_t\varphi(x) = \int_H \varphi(y)\pi_t(x, dy) \quad \text{for all } \varphi \in B_b(H). \tag{5.1}$$

(iii) *For any* $\varphi \in C_b(H)$ *(resp.* $B_b(H)$*) and* $x \in H$*, the mapping* $t \mapsto P_t\varphi(x)$ *is continuous (resp. Borel).*

Obviously, by (5.1) it follows that for $t = 0$,

$$\pi_0(x, \cdot) = \delta_x, \quad x \in H,$$

where δ_x is the Dirac measure at x.

We notice that in the literature one requires usually only (i) and (ii) in the definition of Markov semigroup P_t. In this case condition (iii) means that P_t is *stochastically continuous*, see e.g. [10].

Definition 5.2 *Let* P_t *be a Markov semigroup.*

(i) P_t *is* Feller *if* $P_t\varphi \in C_b(H)$ *for any* $\varphi \in C_b(H)$ *and any* $t \geq 0$.
(ii) P_t *is* strong Feller *if* $P_t\varphi \in C_b(H)$ *for any* $\varphi \in B_b(H)$ *and any* $t > 0$.
(iii) P_t *is* irreducible *if* $P_t\mathbf{1}_{B(x_0,r)}(x) > 0$ *for all* $x, x_0 \in H$, $r > 0$ *and any* $t \geq 0$.

Let us give some general properties of a Markov semigroup P_t. First, notice that by (5.1) we have $P_t\mathbf{1} = \mathbf{1}$ for all $t \geq 0$ and that P_t *preserves positivity*, that is $P_t\varphi \in B_b^+(H)$ for all $\varphi \in B_b^+(H)$.

Moreover, since, for any $\varphi \in C_b(H)$,

$$-\|\varphi\|_0 \leq \varphi(x) \leq \|\varphi\|_0, \quad x \in H,$$

we have

$$|P_t\varphi(x)| \leq \|\varphi\|_0, \quad x \in H.$$

Consequently $\|P_t\|_{L(B_b(H))} \leq 1$, for any $t \geq 0$. That is P_t is a semigroup of contractions on $B_b(H)$.

Let us give now some properties of the family of measures $\pi_t(x, \cdot)$ (called a *probability kernel*).

By (5.1) it follows that for any $E \in \mathscr{B}(H)$ we have

$$\pi_t(x, E) = P_t\mathbf{1}_E(x), \quad t \geq 0,\ x \in H. \tag{5.2}$$

Moreover, the following useful result holds.

Proposition 5.3 *For any* $t, s \geq 0,\ x \in H$ *and any* $E \in \mathscr{B}(H)$ *we have*

$$\pi_{t+s}(x, E) = \int_H \pi_s(y, E)\pi_t(x, dy). \tag{5.3}$$

Proof. We have in fact, taking into account the semigroup property of P_t, (5.2) and (5.1),

$$\pi_{t+s}(x, E) = P_{t+s}\mathbf{1}_E(x) = P_t\pi_s(\cdot, E)(x) = \int_H \pi_s(y, E)\pi_t(x, dy).$$

□

Example 5.4 Let us consider the differential equation

$$\begin{cases} X'(t) = b(X(t)), \\ X(0) = x, \end{cases} \tag{5.4}$$

on $H = \mathbb{R}^n$ where $b\colon H \to H$ is Lipschitz continuous. As is well known, there exists a unique solution $X(t, x)$ of problem (5.4). Set

$$\pi_t(x, \cdot) = \delta_{X(t,x)}, \quad x \in \mathbb{R}^n.$$

Then it is easy to see that the transition semigroup

$$P_t\varphi(x) = \varphi(X(t, x)), \quad \varphi \in B_b(\mathbb{R}^n) \tag{5.5}$$

is a Markov semigroup.

Exercise 5.5 (i) Prove that semigroup P_t, defined by (5.5), is Feller. Is P_t strong Feller?

(ii) Prove that P_t is strongly continuous in $C_b(H)$ if and only if b is bounded.

Example 5.6 Let us consider the stochastic differential equation

$$\begin{cases} dX = b(X)dt + \sqrt{C}\, dB(t), \\ \\ X(0) = x, \end{cases} \tag{5.6}$$

on $H = \mathbb{R}^n$ where B is a standard Brownian motion in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in H, $b\colon H \to H$ is locally Lipschitz continuous, $C \in L(H)$ and Hypothesis 4.23 is fulfilled.

Then by Proposition 4.3 there exists a unique continuous stochastic process $X(\cdot, x)$, the solution of problem (5.6). Set

$$\pi_t(x, E) = (X(t,x)_{\#}\mathbb{P})(E), \quad x \in \mathbb{R}^n,\ E \in \mathscr{B}(\mathbb{R}^n).$$

Then the transition semigroup

$$P_t\varphi(x) = \mathbb{E}\left[\varphi(X(t,x))\right] = \int_{\mathbb{R}} \varphi(y)\pi_t(x, dy), \quad \varphi \in B_b(H), \tag{5.7}$$

is a Markov semigroup as easily checked.

Exercise 5.7 Prove that the semigroup P_t, defined by (5.7), is Feller.

5.2 Invariant measures

In this section P_t represents a Markov semigroup on H. A probability measure $\mu \in \mathscr{P}(H)$ is said to be *invariant* for P_t if

$$\int_H P_t\varphi d\mu = \int_H \varphi d\mu \quad \text{ for all } \varphi \in B_b(H) \text{ and } t \geq 0. \tag{5.8}$$

If P_t is Feller this condition is clearly equivalent (identifying μ with F_μ) to

$$P_t^*\mu = \mu \quad \text{for all } t \geq 0, \tag{5.9}$$

where P_t^* is the transpose operator of P_t, defined as

$$\langle \varphi, P_t^* F\rangle = \langle P_t\varphi, F\rangle,$$

for all $\varphi \in C_b(H)$, $F \in C_b(H)^*$. (1)

If $\mu \in \mathscr{P}(H)$ is invariant for P_t we have

$$\mu(A) = P_t^*\mu(A) = \int_H P_t\mathbf{1}_A(x)\mu(dx), \quad A \in \mathscr{B}(H),$$

from which, recalling (5.8),

$$\mu(A) = \int_H \pi_t(x, A)\mu(dx), \quad A \in \mathscr{B}(H). \tag{5.10}$$

A first basic result is the following.

(1) $\langle \cdot, \cdot \rangle$ represent the duality between $C_b(H)$ and $C_b(H)^*$.

Theorem 5.8 *Assume that μ is an invariant measure for P_t. Then for all $t \geq 0$, $p \geq 1$, P_t is uniquely extendible to a linear bounded operator on $L^p(H, \mu)$ that we still denote by P_t. Moreover*

$$\|P_t\|_{L(L^p(H,\mu))} \leq 1, \quad t \geq 0. \tag{5.11}$$

Finally, P_t is a strongly continuous semigroup in $L^p(H, \mu)$.

Proof. Let $\varphi \in C_b(H)$. By the Hölder inequality we have

$$|P_t\varphi(x)|^p \leq \int_H |\varphi(y)|^p \pi_t(x, dy) = P_t(|\varphi|^p)(x).$$

Integrating both sides of the above inequality with respect to μ over H yields

$$\int_H |P_t\varphi(x)|^p \mu(dx) \leq \int_H P_t(|\varphi|^p)(x)\mu(dx) = \int_H |\varphi(x)|^p \mu(dx)$$

in view of the invariance of μ. Since $C_b(H)$ is dense in $L^p(H, \mu)$, P_t is uniquely extendible to $L^p(H, \mu)$ and (5.11) follows.

Let us show finally that P_t is strongly continuous in $L^p(H, \mu)$. First let $\varphi \in C_b(H)$. Then, by property (iii) in Definition 5.1 of P_t we have that the function $t \to P_t\varphi(x)$ is continuous for any $x \in H$. Consequently, by the dominated convergence theorem

$$\lim_{t \to 0} P_t\varphi = \varphi \quad \text{in} \;\; L^p(H, \mu).$$

The same assertion follows easily when $\varphi \in L^p(H, \mu)$ by the density of $C_b(H)$ in $L^p(H, \mu)$. □

Let μ be an invariant measure for P_t. We are going to study the asymptotic behaviour of $P_t\varphi$, for $\varphi \in L^2(H, \mu)$. This is obvious when $P_t\varphi = \varphi$ for all $t > 0$. In this case we say that φ is *stationary*. In general, given $\varphi \in L^2(H, \mu)$, one can ask whether there exists the limit

$$\lim_{t \to +\infty} P_t\varphi(x), \tag{5.12}$$

or, if not, if there exists the limit of the means

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T P_s\varphi(x)ds. \tag{5.13}$$

We shall prove indeed that this limit always exists in $L^2(H, \mu)$ (*Von Neumann theorem*).

If in addition it happens that

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T P_t\varphi(x)dt = \int_H \varphi d\mu \quad \text{in } L^2(H,\mu), \tag{5.14}$$

for all $\varphi \in L^2(H,\mu)$, P_t is said to be *ergodic*. In this case the identity (5.14) is interpreted in physics by saying that the "temporal" average of $P_t\varphi$ coincides with the "spatial" average of φ.

It can also happen in particular that

$$\lim_{t\to+\infty} P_t\varphi(x) = \int_H \varphi d\mu \quad \text{in } L^2(H,\mu). \tag{5.15}$$

In this case P_t is said to be *strongly mixing*.

Existence and uniqueness of invariant measures will be proved in Chapter 7. We conclude this introduction by giving two examples of invariant measures.

Exercise 5.9 Consider the ordinary differential equation,

$$Z'(t) = Z(t) - Z^3(t), \quad Z(0) = x,$$

and the corresponding transition semigroup

$$P_t\varphi(x) = \varphi(Z(t,x)), \quad \varphi \in C_b(H).$$

Prove that P_t is a Markov semigroup and that $\pi_t(x,E) = \delta_{Z(t,x)}(E),\ E \in \mathscr{B}(\mathbb{R}),\ t \geq 0,\ x \in \mathbb{R}$.

Show moreover that measures δ_0, δ_1 and δ_{-1} are invariant, ergodic and strongly mixing.

Exercise 5.10 Consider the stochastic differential equation in $\mathbb{R}$,

$$dX(t) = -X(t)dt + dB(t), \quad X(0) = x,$$

whose solution $X(t,x)$ is given by the Ornstein–Uhlenbeck process (see Proposition 4.10),

$$X(t,x) = e^{-t}x + \int_0^t e^{-(t-s)}dB(s), \quad t \geq 0,\ x \in \mathbb{R}.$$

Prove that

$$\pi_t(x,\cdot) = N_{e^{-t}x, \frac{1}{2}(1-e^{-2t})}, \quad x \in \mathbb{R},\ t > 0.$$

Show moreover that the measure $\mu = N_{\frac{1}{2}}$ is invariant, ergodic and strongly mixing.

Hint. Check that (5.8) holds for $\varphi(x) = e^{ihx}$, where $h \in \mathbb{R}$.

In order to study the behaviour of $\lim_{T\to+\infty} \frac{1}{T} \int_0^T P_t\varphi dt$, we need some general result about the averages of the powers of a linear operator, proved in the next section.

5.3 Ergodic averages

We are given a linear bounded operator T on a Hilbert space E (norm $\|\cdot\|$, inner product $\langle\cdot,\cdot\rangle$).[2] We set

$$M_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k, \quad n \in \mathbb{N}.$$

Theorem 5.11 *Assume that* $\sup_{n\in\mathbb{N}} \|T^n\| < +\infty$. *Then there exists the limit*

$$\lim_{n\to\infty} M_n x := M_\infty x \quad \textit{for all } x \in E. \tag{5.16}$$

Moreover $M_\infty \in L(H)$, $M_\infty^2 = M_\infty$ *and* $M_\infty(E) = \mathrm{Ker}\,(1-T)$.

Proof. First notice that the limit of $(M_n x)$ certainly exists when either $x \in \mathrm{Ker}\,(1-T)$, or $x \in (1-T)(E)$. In fact in the first case we have obviously

$$\lim_{n\to\infty} M_n x = x \quad \text{for all } x \in \mathrm{Ker}\,(1-T),$$

and in the latter we have

$$\lim_{n\to\infty} M_n x = 0 \quad \text{for all } x \in (1-T)(E),$$

because

$$(1-T)M_n = M_n(1-T) = \frac{1}{n}\,(1-T^n), \quad n \in \mathbb{N}. \tag{5.17}$$

Consequently we also have

$$\lim_{n\to\infty} M_n x = 0 \quad \text{for all } x \in \overline{(1-T)(E)}, \tag{5.18}$$

where $\overline{(1-T)(E)}$ is the closure of $(1-T)(E)$.

Now let $x \in E$ be fixed. Since $\|M_n x\|_{n\in\mathbb{N}}$ is bounded by assumption, there exists a sub-sequence (n_k) of $\mathbb{N}$, and an element $y \in H$ such that $M_{n_k}x \to y$ weakly as $k \to \infty$. By (5.17) it follows also that $TM_{n_k}x \to Ty = y$, so that $y \in \mathrm{Ker}\,(1-T)$.

[2] Later we shall take $E = L^2(H,\mu)$.

Now we prove that $M_n x \to y$. First note that, since $y \in \mathrm{Ker}\,(1-T)$, we have $M_n y = y$, and so

$$M_n x = M_n y + M_n(x-y) = y + M_n(x-y). \tag{5.19}$$

We claim that $x - y \in \overline{(1-T)(E)}$, which will prove (5.17) by (5.16). We have in fact

$$x - y = \lim_{k\to\infty} (x - M_{n_k} x),$$

and $x - M_{n_k} x \in (1-T)(E)$ because

$$\begin{aligned} x - M_{n_k} x &= \frac{1}{n_k} \sum_{h=0}^{n_k-1} (1 - T^h)x \\ &= \frac{1}{n_k} \sum_{h=0}^{n_k-1} (1 + T + ... + T^{h-1})(1-T)x. \end{aligned}$$

Therefore (5.16) holds.

Finally, since $(1-T)M_n \to 0$, we have $M^\infty = TM^\infty$, so that $T^k M^\infty = M^\infty,\ k \in \mathbb{N}$, and $M^\infty = M_n M^\infty$, which yields as $n \to \infty$, $M^\infty = (M^\infty)^2$, as required. □

5.4 The Von Neumann theorem

In this section we assume that there is an invariant measure μ for the Markov semigroup P_t. This will allow us to extend the semigroup P_t to $L^2(H,\mu)$, as proved in Theorem 5.8.

We denote by Σ the set

$$\Sigma = \{f \in L^2(H,\mu) : P_t f = f,\ \mu\text{-a.e. for all } t \ge 0\} \tag{5.20}$$

of all *stationary* points of P_t. Clearly Σ is a closed subspace of $L^2(H,\mu)$ and $\mathbf{1} \in \Sigma$.

Let us consider the average

$$M(T)\varphi = \frac{1}{T}\int_0^T P_t\varphi dt, \quad \varphi \in L^2(H,\mu),\ T > 0.$$

Theorem 5.12 *There exists the limit*

$$\lim_{T\to\infty} M(T)\varphi =: M_\infty\varphi \quad \textit{in } L^2(H,\mu). \tag{5.21}$$

Moreover M_∞ is a projection operator on Σ, and

$$\int_H M_\infty \varphi d\mu = \int_H \varphi d\mu. \tag{5.22}$$

Proof. For all $T > 0$ write

$$T = n_T + r_T, \quad n_T \in \mathbb{N} \cup \{0\},\ r_T \in [0,1).$$

For $\varphi \in L^2(H,\mu)$ we have

$$\begin{aligned}
M(T)\varphi &= \frac{1}{T}\sum_{k=0}^{n_T-1}\int_k^{k+1} P_s\varphi ds + \frac{1}{T}\int_{n_T}^{T} P_s\varphi ds \\
&= \frac{1}{T}\sum_{k=0}^{n_T-1}\int_0^{1} P_{s+k}\varphi ds + \frac{1}{T}\int_0^{r_T} P_{s+n(T)}\varphi ds \\
&= \frac{n_T}{T}\,\frac{1}{n_T}\sum_{k=0}^{n_T-1}(P_1)^k M(1)\varphi + \frac{r_T}{T}(P_1)^{n_T} M(r_T)\varphi. \qquad (5.23)
\end{aligned}$$

Since

$$\lim_{T\to\infty}\frac{n_T}{T} = 1, \quad \lim_{T\to\infty}\frac{r_T}{T} = 0,$$

letting $n \to \infty$ in (5.23) and invoking Theorem 5.11, we get (5.21).

We prove now that for all $t \geq 0$

$$M_\infty P_t = P_t M_\infty = M_\infty. \tag{5.24}$$

In fact, given $t \geq 0$ we have

$$\begin{aligned}
M_\infty P_t\varphi &= \lim_{T\to\infty}\frac{1}{T}\int_0^T P_{t+s}\varphi ds = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} P_s\varphi ds \\
&= \lim_{T\to\infty}\frac{1}{T}\left\{\int_0^T P_s\varphi ds - \int_0^t P_s\varphi ds + \int_T^{T+t} P_s\varphi ds\right\} \\
&= M_\infty\varphi
\end{aligned}$$

and this yields (5.24).

By (5.24) it follows that $M_\infty f \in \Sigma$ for all $f \in L^2(H,\mu)$, and moreover that

$$M_\infty M(T) = M(T)P_\infty = M_\infty,$$

which yields, letting $T \to \infty$, $M_\infty^2 = M_\infty$. Finally, (5.22) follows, by integrating (5.21) with respect to μ. □

5.5 Ergodicity

Let μ be an invariant measure for P_t. We say that μ is *ergodic* if

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T P_t\varphi dt=\overline{\varphi}\quad \text{for all } \varphi\in L^2(H,\mu), \tag{5.25}$$

where

$$\overline{\varphi}=\int_H \varphi(x)\mu(dx).$$

Proposition 5.13 *Let μ be an invariant measure for P_t. Then μ is ergodic if and only if the dimension of the linear space Σ of all stationary elements of $L^2(H,\mu)$ defined by* (5.20) *is* 1.

Proof. If μ is ergodic it follows from (5.25) that any element in Σ is constant, so that dimension of Σ is 1. Conversely assume that dimension of Σ is 1. Then there is a linear bounded functional F on $L^2(H,\mu)$ such that

$$M_\infty\varphi=F(\varphi)\mathbf{1}.$$

By the Riesz representation theorem there exists an element $\varphi_0\in L^2(H,\mu)$ such that $F(\varphi)=\langle\varphi,\varphi_0\rangle$. Integrating this equality on H with respect to μ and taking into account the invariance of M_∞ (see (5.22)), yields

$$\int_H M_\infty\varphi d\mu=\int_H\varphi d\mu=\langle\varphi,\mathbf{1}\rangle=\langle\varphi,\varphi_0\rangle,\quad \varphi\in L^2(H,\mu).$$

Therefore $\varphi_0=\mathbf{1}$. □

Let μ be an invariant measure for P_t. A Borel set $\Gamma\in\mathscr{B}(H)$ is said to be *invariant* for P_t if its characteristic function $\mathbf{1}_\Gamma$ belongs to Σ. If $\mu(\Gamma)$ is equal to either 0 or 1, we say that Γ is *trivial*, otherwise it is *nontrivial*.

We now want to show that μ is ergodic if and only if all invariant sets are trivial. For this it is important to notice that Σ is a lattice, as proved in the next proposition.

Proposition 5.14 *Assume that φ and ψ belong to Σ. Then the following statements hold.*

(i) $|\varphi|\in\Sigma$.
(ii) $\varphi^+,\varphi^-\in\Sigma$.(3)

(3) $\varphi^+=\max\{\varphi,0\}$, $\varphi^-=\max\{-\varphi,0\}$.

(iii) $\varphi \vee \psi,\ \varphi \wedge \psi \in \Sigma$.[4]
(iv) For any $a \in \mathbb{R}$ *we have* $\mathbf{1}_{\{x\in H:\, \varphi(x)>a\}} \in \Sigma$.

Proof. Let us prove (i). Let $t > 0$ and assume that $\varphi \in \Sigma$, so that $\varphi(x) = P_t\varphi(x)$. Then we have

$$|\varphi(x)| = |P_t\varphi(x)| \le P_t(|\varphi|)(x), \quad x \in H. \tag{5.26}$$

We claim that

$$|\varphi(x)| = P_t(|\varphi|)(x), \quad \mu\text{-a.s.}$$

Assume by contradiction that there is a Borel subset $I \subset H$ such that $\mu(I) > 0$ and

$$|\varphi(x)| < P_t(|\varphi|)(x), \quad x \in I.$$

Then we have

$$\int_H |\varphi(x)|\mu(dx) < \int_H P_t(|\varphi|)(x)\mu(dx).$$

Since, by the invariance of μ,

$$\int_H P_t(|\varphi|)(x)\mu(dx) = \int_H |\varphi|(x)\mu(dx),$$

we find a contradiction.

Statements (ii) and (iii) follow from the obvious identities

$$\varphi^+ = \frac{1}{2}(\varphi + |\varphi|),\ \ \varphi^- = \frac{1}{2}(\varphi - |\varphi|),$$

$$\varphi \vee \psi = (\varphi - \psi)^+ + \psi,\ \ \varphi \wedge \psi = -(\varphi - \psi)^+ + \varphi.$$

Finally let us prove (iv). It is enough to show that the set $\{\varphi > 0\}$ is invariant, or, equivalently, that $\mathbf{1}_{\{\varphi>0\}}$ belongs to Σ. We have in fact, as it is easily checked,

$$\mathbf{1}_{\{\varphi>0\}} = \lim_{n\to\infty} \varphi_n(x),\ \ x \in H,$$

where $\varphi_n = (n\varphi^+) \wedge \mathbf{1},\ n \in \mathbb{N}$, belongs to Σ by (ii) and (iii). Therefore $\{\varphi > 0\}$ is invariant. □

We are now ready to prove the following result.

Theorem 5.15 *Let* μ *be an invariant measure for* P_t. *Then* μ *is ergodic if and only if any invariant set is trivial.*

[4] $\varphi \vee \psi = \max\{\varphi, \psi\}$, $\varphi \wedge \psi = \min\{\varphi, \psi\}$.

Proof. Let Γ be invariant for μ. Then if μ is ergodic $\mathbf{1}_\Gamma$ must be constant (otherwise dim $\Sigma \geq 2$) and so Γ is trivial. Assume conversely that the only invariant sets for μ are trivial and, by contradiction, that μ is not ergodic. Then there exists a non-constant function $\varphi_0 \in \Sigma$. Therefore by Proposition 5.14 for some $\lambda \in \mathbb{R}$ the invariant set $\{\varphi_0 > \lambda\}$ is not trivial. □

5.6 Structure of the set of all invariant measures

We still assume that P_t is a Markov semigroup on H. We denote by Λ the set of all its invariant measures and we assume that Λ is non-empty. Clearly Λ is a convex subset of $C_b(H)^*$.

Theorem 5.16 *Assume that there is a unique invariant measure μ for P_t. Then μ is ergodic.*

Proof. Assume by contradiction that μ is not ergodic. Then there is a nontrivial invariant set Γ. Let us prove that the measure μ_Γ defined as

$$\mu_\Gamma(A) = \frac{1}{\mu(\Gamma)}\, \mu(A \cap \Gamma), \quad A \in \mathscr{B}(H),$$

belongs to Λ. This will give rise to a contradiction.

Recalling (5.10), we have to show that

$$\mu_\Gamma(A) = \int_H \pi_t(x, A)\mu_\Gamma(dx), \quad A \in \mathscr{B}(H),$$

or, equivalently, that

$$\mu(A \cap \Gamma) = \int_\Gamma \pi_t(x, A)\mu(dx), \quad A \in \mathscr{B}(H). \tag{5.27}$$

In fact, since Γ is invariant, we have

$$P_t\mathbf{1}_\Gamma = \mathbf{1}_\Gamma, \quad P_t\mathbf{1}_{\Gamma^c} = \mathbf{1}_{\Gamma^c}, \quad t \geq 0,$$

and so

$$\pi_t(x, \Gamma) = \mathbf{1}_\Gamma(x), \quad \pi_t(x, \Gamma^c) = \mathbf{1}_{\Gamma^c}(x), \quad t \geq 0.$$

Consequently,

$$\pi_t(x, A \cap \Gamma^c) = 0, \quad \mu\text{-a.e. in } \Gamma \text{ and } \pi_t(x, A \cap \Gamma) = 0,\ \mu\text{-a.e. in } \Gamma^c,$$

and so

$$\int_\Gamma \pi_t(x,A)\mu(dx) = \int_\Gamma \pi_t(x,A\cap\Gamma)\mu(dx) + \int_\Gamma \pi_t(x,A\cap\Gamma^c)\mu(dx)$$

$$= \int_\Gamma \pi_t(x,A\cap\Gamma)\mu(dx) = \int_H \pi_t(x,A\cap\Gamma)\mu(dx) = \mu(A\cap\Gamma),$$

and (5.10) holds. □

We want now to show that the set of all extremal points of Λ is precisely the set of all ergodic measures of P_t. For this we need a lemma.

Lemma 5.17 *Let $\mu,\nu \in \Lambda$ with μ ergodic and ν absolutely continuous with respect to μ. Then $\mu = \nu$.*

Proof. Let $\Gamma \in \mathscr{B}(H)$. By the Von Neumann theorem there exists $T_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} P_t \mathbf{1}_\Gamma dt = \mu(\Gamma), \quad \mu\text{-a.e.} \tag{5.28}$$

Since $\nu \ll \mu$, identity (5.28) holds also ν-a.e. Now integrating (5.28) with respect to ν yields

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \left(\int_H P_t \mathbf{1}_\Gamma d\nu \right) dt = \nu(\Gamma), \;\; \mu\text{-a.e.}$$

Consequently $\nu(\Gamma) = \mu(\Gamma)$ as required. □

We can now prove the announced property of Λ.

Theorem 5.18 *The set of all invariant ergodic measures of P_t coincides with the set of all extremal points of Λ.*

Proof. We first prove that if μ is ergodic then it is an extremal point of Λ. Assume by contradiction that μ is ergodic and it is not an extremal point of Λ. Then there exist $\mu_1, \mu_2 \in \Lambda$ with $\mu_1 \neq \mu_2$, and $\alpha \in (0,1)$ such that

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2.$$

Then clearly $\mu_1 \ll \mu$ and $\mu_2 \ll \mu$. By Lemma 5.17 we get a contradiction.

We finally prove that if μ is an extremal point of Λ, then it is ergodic. Assume by contradiction that μ is not ergodic. Then there exists a nontrivial invariant set Γ. Consequently, arguing as in the proof of Theorem 5.16, we have $\mu_\Gamma, \mu_{\Gamma^c} \in \Lambda$. Since

$$\mu = \mu(\Gamma)\mu_\Gamma + (1 - \mu(\Gamma))\mu_{\Gamma^c},$$

we find that μ is not extremal, a contradiction. □

Theorem 5.19 *Assume that μ and ν are ergodic invariant measures with $\mu \neq \nu$. Then μ and ν are singular.*

Proof. Let $\Gamma \in \mathscr{B}(H)$ be such that $\mu(\Gamma) \neq \nu(\Gamma)$. From the Von Neumann theorem it follows that there exists $T_n \uparrow +\infty$ and two Borel sets M and N such that $\mu(M) = 1, \nu(N) = 1$, and

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} (P_t \mathbf{1}_\Gamma)(x)dt = \mu(\Gamma),\ \forall\, x \in M,$$

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} (P_t \mathbf{1}_\Gamma)(x)dt = \nu(\Gamma),\ \forall\, x \in N.$$

Since $\mu(\Gamma) \neq \nu(\Gamma)$ this implies that $M \cap N = \varnothing$, and so μ and ν are singular. □