1

Random Fluctuations

1.1 Introduction

The sensitivity and accuracy of any detection system is limited by random fluctuations that always accompany the measurement. It also sets a limit to the minimum detectable signal. These random fluctuations or disturbing signals, called noise, can be divided into two categories depending on their nature. A part is not inherently connected to the detection principle but to the environment. Instrumental imperfections, atmospheric turbulence, vibrating mechanical constructions, 50 or 60 Hz and higher harmonics from the power line, radio and television stations, building vibrations, and temperature fluctuations all fall in this category. These environmental disturbances are in most cases occasional, peculiar to the surrounding, and not statistical. They can in principle be reduced to arbitrarily small values, but in practice, they can be very annoying and difficult to eliminate entirely. Reductions are often obtained by shielding or instrumental improvements.

The other category is fundamental from nature and inherently connected to the physical process that underlies the detection. For instance, through any conductor there is always a small fluctuating current due to the random thermal motion of the free electrons of this conductor. Other typical fluctuations arise from the fact that electrical currents are built up of irreducible elementary units, the charge of an electron. Similar effects occur for radiation as a flow of photons with discrete values. For this reason also thermal background radiation contains fluctuations and the temperature of a body is essentially not constant. Even in systems that filter out electronically the contributions of thermal background, a part of their fluctuations, are still present and mixed with the signal. The amount of this noise depends on fundamental physical quantities and sets the ultimate limit to the minimum detectable signal, which cannot be surpassed. Modern measuring instruments work close to their ultimate limits. Furthermore to exploit the sensitivity of a detection system, we must also ascertain the fundamental nature of the applied physical processes on which the detection is based.

For this reason we discuss in this chapter the fundamental aspects of noise. It will be done for thermal background radiation and for noise connected with the elementary units of radiation and charge carriers of the detection circuit. In detector circuits the electrons are not only driven by the incident radiation but also by random thermal motion. Further, the circuit currents from the signal of various detection systems have fluctuations due to the discreteness of the charge carriers and their random time distribution. Photoconductors produce additional noise by the random thermal process of generating carriers. Although various types of noise are fundamentally present in any detector system the interesting question is how can these contributions be minimized. For this purpose the physical nature of various noise sources will be treated in a qualitative and quantitative way. It will be done for background radiation, thermal electron motion present in any conductor, current fluctuations due to the discreteness of the electron charge and random photon emission in diodes, and for the generation and recombination processes in photoconductors. The derivation of the spectrum density and of the mean square fluctuations of the noise current turns out to be most relevant to detection systems. The total noise power still present in the final signal power of a detector system is then proportional to the frequency bandwidth of the system. The ratio of the output signal power to this noise power will be considered as the quality factor for the detection.

1.2 Thermal Noise of Resistance

Due to random motion of the electrons there are always fluctuations of the local charge density in any element of an electronic circuit. These charge densities cause voltage gradients which drive on their turn fluctuating currents. The average values of these fluctuations over large periods are, of course, zero, but this is not the case for a limited period. Intuitively one may say the smaller this time period or the larger the frequency bandwidth of the observation, the larger the fluctuations. The smallest period or upper frequency limit of this increasing thermal noise is set by the electron collision frequency which is roughly 10^{13} Hz. The thermal noise of conductors is the so-called Johnson noise. A quantitative treatment of this thermal noise can be carried out in different ways [1–3]. It is found that the thermal noise *power* of a resistive element with real impedance does not depend on resistivity, material, dimensions, or its surrounding but solely on its temperature and the frequency domain of the observation. The derivation is as follows.

Consider a closed loop containing a transmission line of length l connecting on both sides two identical resistors with resistivity R as illustrated in Fig. 1.1. The random thermal fluctuations of the electrons in the resistors can support traveling voltage waves in this closed loop. By choosing the characteristic impedance R_0 of the transmission line equal to R there are no reflections of waves at the ends. The natural frequencies of the loop correspond to

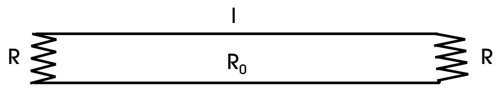

Fig. 1.1. Closed loop of transmission line

waves – called frequency modes – that are periodic in the round trip distance. Thus the wavelength is given by

$$\lambda_n = \frac{2l}{n}\,, \tag{1.1}$$

where n is an integer. The natural frequencies are then

$$\nu_n = \frac{nc}{2l}\,. \tag{1.2}$$

So the frequency spacing between these frequencies is $c/2l$. The number of traveling waves within a frequency bandwidth $\Delta\nu$ is then

$$N = \frac{2l\Delta\nu}{c}\,. \tag{1.3}$$

At thermal equilibrium the average energy of a single frequency mode is according to Planck's law

$$E_{h\nu} = \frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1}\,. \tag{1.4}$$

The total energy E_t in the bandwidth $\Delta\nu$ becomes $NE_{h\nu}$ or

$$E_t = \frac{2lh\nu\Delta\nu}{c(\mathrm{e}^{h\nu/kT} - 1)}\,. \tag{1.5}$$

This energy moves with the velocity c so that the round trip time is $2l/c$. The power P flowing in each direction of the transmission line is then

$$P = \frac{h\nu\Delta\nu}{\mathrm{e}^{h\nu/kT} - 1} \tag{1.6}$$

Usually the frequency of interest is much smaller than kT/h so that the spectral power density equal to $\mathrm{d}P/\mathrm{d}\nu$ can be considered constant and the noise is therefore often called "white noise" and is given by

$$P = kT\Delta\nu\,. \tag{1.7}$$

Since there are no reflections at the ends of the line the incoming power of a resistor is dissipated. Then an equal amount of power must also be generated by a resistor in order to have a balance of power. This power is apparently the thermal noise power of a resistor.

Let us now describe the resistor with its noise power by its resistance R in series with a noise generator having a mean square voltage amplitude $\overline{v_\mathrm{n}^2}$.

The noise power P of the resistor that is delivered to a transmission line with an arbitrary characteristic impedance R_0 is then given by

$$P = \frac{\overline{v_\mathrm{n}^2} R_0}{(R + R_0)^2} \,. \tag{1.8}$$

It is seen that the maximum value of P is obtained for $R = R_0$, so that using (1.7) we find for the mean square voltage amplitude of a resistor

$$\overline{v_\mathrm{n}^2} = 4kTR\Delta\nu \,. \tag{1.9}$$

An equivalent circuit of a resistor can be given by a noise current generator in parallel with the resistor. The mean square noise current is then

$$\overline{i_\mathrm{n}^2} = \frac{4kT\Delta\nu}{R} \,. \tag{1.10}$$

The two equivalent circuits are shown in Fig. 1.2. At room temperature the effective noise voltage $\sqrt{\overline{v_\mathrm{n}^2}}$ is about 0.13 nV $[\Omega^{-1/2}\,\mathrm{Hz}^{-1/2}]$ and the effective current $\sqrt{\overline{i_\mathrm{n}^2}}$ is about 0.13 nA $[\Omega^{1/2}\,\mathrm{Hz}^{-1/2}]$.

The fluctuating thermal noise voltage of a capacitor can be found by considering a closed circuit of a capacitor C in series with a resistor R as shown in Fig. 1.3. The mean square voltage amplitude over the capacity is given by

$$\overline{v_\mathrm{n}^2} = \int_0^\infty \frac{4kTR\,\mathrm{d}\nu}{1 + (2\pi\nu CR)^2} = \frac{kT}{C} \,. \tag{1.11}$$

Since R is not relevant to the result we find that the noise mean square voltage over a capacitor is given by (1.11). It should be noted that the same value for $\overline{v_\mathrm{n}^2}$ is found over a resistor connected to a capacitor. Alternatively, one can consider the RC circuit as a low-pass filter having a band width $\Delta\nu = 1/4RC$ for power transmission. Substituting this value of $\Delta\nu$ in (1.9) leads to the same result.

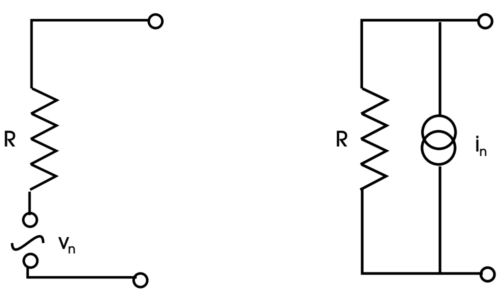

Fig. 1.2. Equivalent circuits

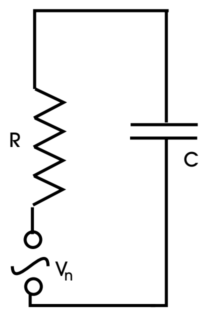

Fig. 1.3. Thermal noise of capacitor

1.3 Shot Noise

Emitted electrons from a thermal cathode or from a photo cathode traveling through a vacuum tube toward the anode produce a current in the external circuit only during their transit time. See Appendix A.1. We assume the total generated current low enough to neglect space charge effects of the electrons in the anode–cathode space so that there are no interactions between the various electrons. Each emitted electron gives a microcurrent pulse. The observed current in the external circuit is then simply the sum of all those randomly generated micropulses. This process also occurs when photons generate electron–hole pairs in a photoconductor or in a photodiode placed between electrodes. The current in the external circuit is only present during the traveling of free electrons to the positive electrode and the holes toward the negative electrode.

The external current due to a random generation of these charge carriers shows as a consequence of the individual pulses uncorrelated fluctuations which are called shot noise. In this section we consider photoemission and assume that each micropulse contains the charge of one electron and has constant duration time. (This is not the case for generation–recombination noise to be treated in Sect. 1.5.) Since the external current can be considered as a flow of electrons that passes a point, one expects by doing a large number of independent observations that the shorter the observation time for counting the number of passing electrons the larger the fluctuations of this number or the larger the shot noise and that by doing observations over large periods the fluctuations and thus the shot noise will approach zero. The analysis is as follows. Let we observe the mean square current fluctuations $\overline{i_\mathrm{n}^2}$ of an average current i_0 in the circuit during the time τ_ob. The average number of electrons is

$$\overline{n} = \frac{i_0 \tau_\mathrm{ob}}{e}\,. \tag{1.12}$$

The current fluctuation can be expressed as

$$\overline{i_\mathrm{n}^2} = \overline{(i - i_0)^2} = \frac{e^2}{\tau_\mathrm{ob}^2}\overline{(n - \overline{n})^2}\,. \tag{1.13}$$

With the assumption that the probability of creating a photoelectron depends on the incident radiation power it is derived in Appendix A.2.4 that for constant radiation power the number n obeys the Poisson statistics with the property

$$\overline{(n-\overline{n})^2} = \overline{n}\,. \tag{1.14}$$

Substituting (1.14) into (1.13) gives

$$\overline{i_{\mathrm{n}}^2} = \frac{i_0 e}{\tau_{\mathrm{ob}}}\,. \tag{1.15}$$

1.3.1 Spectral Distribution

In practice it is more useful to express the shot noise in terms of frequency instead of time. For this purpose the Fourier transform relations are used. The (real) function $f(t)$ and its Fourier transform are related by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t\,. \tag{1.16}$$

The inverse transform is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega\,. \tag{1.17}$$

Suppose that $f(t)$ is the current in the circuit then the average power over a period T dissipated through a resistor of $1\,\Omega$ is given by

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,\mathrm{d}t = \frac{1}{2\pi T}\int_{-T/2}^{T/2}\left\{f(t)\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega\right\}\mathrm{d}t\,. \tag{1.18}$$

If the current is only present or considered during the time T we find by using (1.16) and (1.17)

$$P = \frac{1}{2\pi T}\int_{-\infty}^{\infty} |F(\omega)|^2\,\mathrm{d}\omega = \frac{1}{\pi T}\int_{0}^{\infty} |F(\omega)|^2\,\mathrm{d}\omega\,. \tag{1.19}$$

As mentioned earlier the duration of the micropulses of the current flow in the external circuit are related to the time of flow of the generated charge carriers in the detection device. A micropulse current by an electron starting at t_n can be expressed as

$$i_{\mathrm{e}}(t) = ef(t-t_n)\,, \tag{1.20}$$

with the condition

$$\int_{-\infty}^{\infty} f(t-t_n)\,\mathrm{d}t = 1\,. \tag{1.21}$$

The total current is then

$$i(t) = e \sum_n f(t - t_n) \,, \tag{1.22}$$

where t_n is the random starting time of an electron.

Taking the Fourier transform of $i(t)$ we get

$$I(\omega) = e \sum_n \mathrm{e}^{-\mathrm{j}\omega t_n} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega t} f(t)\,\mathrm{d}t = eF(\omega) \sum_n \mathrm{e}^{-\mathrm{j}\omega t_n} \,. \tag{1.23}$$

The spectral power density $S_i(\omega) = \mathrm{d}P/\mathrm{d}\omega$ of the current $i(t)$ over a resistor of $1\,\Omega$ is according to (1.19)

$$S_i(\omega) = \frac{1}{\pi T} e^2 \left|F(\omega)\right|^2 \sum_{n,m} \mathrm{e}^{-\mathrm{j}\omega(t_n - t_m)} = i_0^2(\omega) \,. \tag{1.24}$$

Since the times t_n are random and we consider a very large number of electrons, the sum of the terms with $n \neq m$ and $\omega \neq 0$ will for constant production probability of the photoelectrons in average cancel. The summation term for $\omega \neq 0$ becomes equal to the total number of electrons or equal to $\frac{i_0 T}{e}$ where i_0 is the average current of the circuit. We now find for $i_0^2(\omega)$ its average value over T

$$\overline{i_0^2}(\omega) = \frac{1}{\pi} e i_0 \left|F(\omega)\right|^2 \,, \tag{1.25}$$

where the Fourier transform $F(\omega)$ of the micropulse contains the integration over its duration time τ.

The derivation of $\overline{i_0^2}(\omega)$ can be further extended with a description of the micropulse itself or if this is not known by the limitation of the considered bandwidth of the noise spectrum. Let us first consider any micropulse of duration τ for which the considered spectrum is restricted by $\omega\tau \ll 1$. In that case the Fourier transform approaches the unit impulse function i.e.,

$$F(\omega) = \int_0^\tau \mathrm{e}^{-\mathrm{j}\omega t} f(t)\,\mathrm{d}t \backsimeq 1 \,. \tag{1.26}$$

Thus in this case the spectral power density is practically flat, independent on frequency. This shot noise is therefore often considered as white noise. Substituting (1.26) into (1.25) we find the spectral power density of the current fluctuations as

$$\overline{i_0^2}(\omega) = \frac{e i_0}{\pi} \,. \tag{1.27}$$

By changing from radial frequency to Hertz frequency (ν) we have to multiply the last expression by 2π and we obtain

$$\overline{i_0^2}(\nu) = 2 e i_0 \,. \tag{1.28}$$

The current fluctuations or shot noise within a bandwidth B with the condition $2\pi B\tau \ll 1$ becomes

$$\overline{i_{\mathrm{n}}^2} = 2ei_0 B\,. \qquad (1.29)$$

Comparing (1.29) and (1.15) it is seen that the relation between the observation time and the bandwidth is given by $\frac{1}{\tau_{\mathrm{ob}}} = 2B$. In the following we specify the micropulse for two different situations.

The Charges Move with Constant Speed

Constant speed of created charge carriers by photoionization may occur for instance in a photoconductor or in the high-field region of the junction of a diode. The constant speed during τ gives $f(t) = \frac{1}{\tau}$, where τ is the transit time through the conductor or junction. The Fourier transform of the corresponding micropulse becomes

$$F(\omega) = \int_0^{\tau} \mathrm{e}^{-\mathrm{j}\omega t}\frac{1}{\tau}\,\mathrm{d}t = \frac{\sin(\omega\tau/2)}{\omega\tau/2}\mathrm{e}^{-\mathrm{j}\omega\tau/2}\,. \qquad (1.30)$$

Substituting (1.30) into (1.25) we obtain for the spectral power density of the shot noise

$$\overline{i_0^2}(\omega) = \frac{ei_0}{\pi}\frac{\sin^2(\omega\tau/2)}{(\omega\tau/2)^2}\,. \qquad (1.31)$$

This spectrum is shown in Fig. 1.4.

For practical purposes an effective bandwidth $\Delta\nu = \Delta\omega/2\pi$ is calculated for a rectangular spectrum of the same height at the center and of equal area as indicated in Fig. 1.4. The integral $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$ is equal to π. This gives an effective half width $\Delta\omega\tau/2 = \pi/2$. The effective maximum noise frequency $\Delta\nu_{\mathrm{m}}$ is then

$$\Delta\nu_{\mathrm{m}} = \frac{1}{2\tau}\,. \qquad (1.32)$$

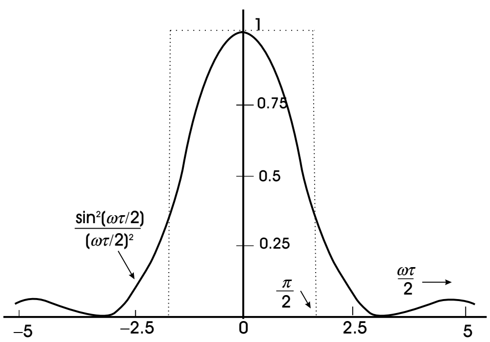

Fig. 1.4. Spectral power density of shot noise. The *dashed line* represents the equivalent rectangular spectrum with half width $\frac{\pi}{2}$

Thus for any noise bandwidth $B < \Delta\nu_{\rm m}$, the noise is given by (1.29). In practice the value for τ of a hole–electron pair in a diode is in the range of 1–0.01 ns so that $\Delta\nu_{\rm m}$ is roughly in the range of 1–100 GHz. The detection bandwidth limited by the electronic circuit is mostly much smaller than $\Delta\nu_{\rm m}$ so that (1.29) remains applicable.

The Charges Move with Constant Acceleration

Constant acceleration of electrons occurs in a vacuum photodiode where a linear potential field is applied between the electrodes so that the velocity of the electrons and thus the current increases linearly with the time of flight of the electrons. The current of the micropulse is then $2et/\tau^2$ where τ is the travel time of the electron from cathode to anode. So we now have $f(t) = 2t/\tau^2$ and the Fourier transform becomes

$$F(\omega) = \frac{2}{(\omega\tau)^2}\left[(1 + {\rm j}\omega\tau)\,{\rm e}^{-{\rm j}\omega\tau} - 1\right]. \tag{1.33}$$

Substituting (1.33) into (1.25) results in

$$\overline{i_0^2}(\omega) = \frac{4ei_0}{\pi(\omega\tau)^4}\left[4\sin^2\left(\frac{\omega\tau}{2}\right) + (\omega\tau)^2 - 2\omega\tau\sin\omega\tau\right]. \tag{1.34}$$

It is found again that $\overline{i_0^2}(0) = ei_0/\pi$.

Plotting the curve of $\overline{i_0^2}(\omega)$ in Fig. 1.5 it is seen to have a broad maximum. The value of $\omega\tau$ for which it reaches its half maximum is $\approx\pi$ so that the maximum noise frequency $\Delta\nu_{\rm m}$ is again $\Delta\nu_{\rm m} = 1/2\tau$. Changing again from radial frequency to Hertz frequency we have to multiply (1.34) by 2π. For $\nu < 1/2\tau$ the spectral power density is then again given by (1.28). Consequently the shot noise for the bandwidth B is also given by (1.29).

In conclusion we mention that the spectral power density of the shot noise is determined by the *random* distribution of the micropulses, whereas its maximum frequency is determined by the *duration* of the micropulse which is of course also the maximum frequency response of the detector element.

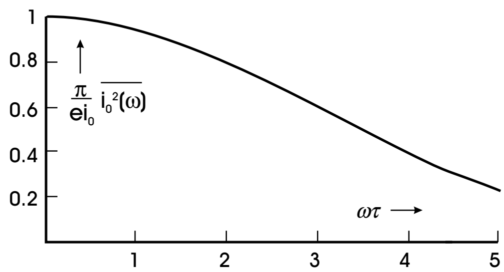

Fig. 1.5. Noise spectrum of rectangular micropulse

1.3.2 Photons

In the earlier analysis we have seen that the current fluctuations are due to the discreteness of the charges. A similar argument applies to a flux of photons. The spontaneously emitted photons of an incoherent radiation source also obey Poisson statistics. The photon fluctuations can then be calculated in a similar way and are obtained simply by replacing the electron charge by the photon energy and the current by the power. We then obtain for the power fluctuations of incoherent radiation having a narrow bandwidth B

$$\overline{\Delta P^2} = 2h\nu PB\,, \tag{1.35}$$

where $h\nu$ is the photon energy ($B \ll \nu$) and P the power of the beam.

For a coherent optical beam with its extremely narrow line width or very long temporal coherence the power fluctuations are negligible. However, the photon current generated by such a beam in any photon detector exhibits nevertheless shot noise as given by (1.29). The derivation of this noise is given in Appendix A.2.4.

1.4 Flicker Noise

Semiconductors and valves show relatively strong noise signals at low frequencies. This noise is usually called flicker noise, $1/f$-noise or excess noise. At low frequencies this noise can be considerably stronger than the shot noise. The observed strong noise signals at low frequencies cannot be fully explained by a description based on the motion of the generated charge carriers. There is more. The search for it has produced many theories based on lattice defects, diffusion of charge carriers, surface contact effects, and impurities. Its origin seems to be very complicated and a full understanding still remains unclear.

Semiempirical studies show a power spectrum more or less inversely proportional to the frequency and quadratic to the current. This frequency dependence remains up till very low frequencies and around 100 Hz it may be comparable with the shot noise. In practice most detection systems operate at frequencies high enough to neglect this type of noise. Therefore, in general it does not limit device performances.

1.5 Generation–Recombination Noise

The previously discussed shot noise is associated with the random generation of identical single charge micropulses. In case of semiconductors the created free carriers increase also the conductivity of the element during the life time of the carriers. As a result the charge of a micropulse initiated by the absorbed photon may be (much) more than that of a single electron. These generated multicharge micropulses are apart from their random distribution not identical because of the life time fluctuations of the carriers. Therefore additional noise is generated [4].

Photoconductors are divided in intrinsic and extrinsic types. In the case of an intrinsic photoconductor the absorbed photon creates a free electron in the conduction band and simultaneously a hole in the valence band. For the extrinsic semiconductor the conduction is produced by the photon absorption at the impurity levels. The photons create either free electrons in the conduction band, the so-called *n-type*, or holes in the valence band of the so-called *p-type*. In general the drift velocity of one type of carrier is much larger than the other one so that in fact the current is given by the dominating type of carrier. Usually the conductivity is mainly by the electrons with their much larger mobility, particularly for intrinsic and n-type extrinsic semiconductors.

Let us consider the drift of the carriers produced by the absorption of photons in a semiconductor crystal connected in series with a battery. See Fig. 1.6. For the optical beam of power P incident on the semiconductor the production rate is $\eta P/h\nu$ electron–hole pairs where η is the quantum efficiency. In steady state the production rate is equal to the recombination rate N/τ_l where N is the number of pairs and τ_l the recombination or life time. Thus we have

$$N = \frac{\eta P \tau_\mathrm{l}}{h\nu} \,. \qquad (1.36)$$

Due to the applied field the free carriers drift with constant velocity v between the contacts. Each drifting pair gives rise to an (external) current $i_e = ev/d$ where d is the distance between the contacts with the external leads. The total current using (1.36) becomes

$$i_0 = \frac{e\eta P}{h\nu}\left(\frac{\tau_\mathrm{l}}{\tau_\mathrm{d}}\right), \qquad (1.37)$$

where $\tau_d = d/v$ is the drift time between the contacts. The process can be seen as a carrier, for instance a free electron, that drifts toward the positive contact and leaves the semiconductor. At the same time, because of charge neutrality, a replacement electron enters the semiconductor at the negative contact. This goes on during the life time of the excited charge carrier. The

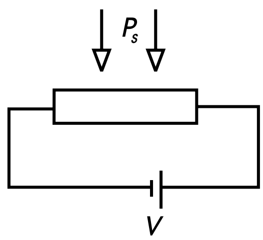

Fig. 1.6. Circuit with semiconductor

micropulse in the external circuit has the duration τ_l and an effective charge $(\tau_l/\tau_d)e$ per photoinduced charge carrier. The total external current is again the sum of all micropulses that originate from the individually photoinduced charge carriers.

A micropulse starting at t_n can be expressed as

$$i_e(t) = \frac{\tau_l}{\tau_d} e f(t - t_n) \,, \tag{1.38}$$

with the condition given by (1.21). By considering constant drift velocity in the crystal we have a rectangular pulse with $f(t) = \frac{1}{\tau_l}$ for $0 \leq t \leq \tau_l$ otherwise $f(t) = 0$. The Fourier transform of rectangular pulse is given by (1.30). Substituting this result into (1.25) and now using the effective charge of the pulse equal to $\frac{\tau_l}{\tau_d}e$ we obtain

$$\overline{i_0^2}(\omega) = \frac{e i_0}{\pi} \left(\frac{\tau_l}{\tau_d}\right) \frac{\sin^2(\omega\tau_l/2)}{(\omega\tau_l/2)^2} \,, \tag{1.39}$$

where i_0 is now given by (1.37).

So far the current fluctuations are identical to the shot noise derived in Sect. 1.3 except for the effective charge. The life time of the charge carriers is, however, the result of a spontaneous recombination process and thus it may fluctuate and give rise to additional noise. To include this part of the noise we make the usual assumption that the probability function $F(\tau)$ of the life time is given by

$$F(\tau) = \frac{1}{\tau_l} \mathrm{e}^{-\tau/\tau_l} \,, \tag{1.40}$$

so that the average life time is

$$\int_0^\infty \tau F(\tau)\, \mathrm{d}\tau = \tau_l \,. \tag{1.41}$$

Since a life time of a charge carrier is the same as the duration of the corresponding micropulse we can describe the micropulses by the same probability distribution. Then the shot noise produced by the micropulses with duration between τ and $\tau + \mathrm{d}\tau$ becomes by using (1.39) and (1.37)

$$\mathrm{d}\overline{i_0^2}(\omega) = \frac{e^2 \eta P}{\pi h \nu} \left(\frac{\tau}{\tau_d}\right)^2 \frac{\sin^2(\omega\tau/2)}{(\omega\tau/2)^2} \frac{1}{\tau_l} \mathrm{e}^{-\tau/\tau_l}\, \mathrm{d}\tau \,. \tag{1.42}$$

Integrating over τ we get the total spectral noise power density

$$\overline{i_0^2}(\omega) = \frac{2 e g i_0}{\pi} \left[\frac{1}{1 + (\omega\tau_l)^2}\right] \,, \tag{1.43}$$

where i_0 is given by (1.37) and $g = \tau_l/\tau_d$. Changing to Hertz frequency it becomes

$$\overline{i_0^2}(\nu) = 4 e g i_0 \left[\frac{1}{1 + (2\pi\nu\tau_l)^2}\right] \,. \tag{1.44}$$

The term within the brackets indicates the bandwidth limitation due to the carrier life time.

Comparing this g–r noise with the shot noise we see the similarity by noticing that for the g–r noise the "unit of spread" is the charge, ge, of the micropulse whereas for the shot noise it is e. The additional factor 2 in the g–r noise apparently comes from the life time spread of the carriers.

It is seen that the mechanism of producing carriers is not relevant in the derivation of the noise current.[1] The g–r noise is, therefore, also present in the so called "dark" current normally conducted by the thermally excited free carriers and driven by an applied field between the contacts with the external leads. This current consists again of a set of pulses with random arrival times and fluctuating pulse widths because of the statistical behavior of the recombination process. Since the probability of creating thermal free carriers depends on the temperature the dark current consists also of micropulses that obey Poisson statistics provided the temperature is constant. Its noise is therefore also given by (1.44) except that i_0 is now replaced by the dark current i_d induced by the applied field between the contacts and the life time τ_l by the life time of the thermally excited carriers. Calculating the noise the signal and dark currents are in practice often taken as total current in (1.44) assuming a single process for the thermal excitation and the same life time as for the signal carriers. If there are more thermal excitation processes the dark current noise is the sum of the individual contributions. A semiconductor with several g–r processes will make the treatment more complicated.

The dark current noise can also be seen as due to the fluctuations of the resistance, R, of the semiconductor because of the random generation and recombination process of the thermal free carriers. The semiconductor is as a resistor also subjected to the thermal motion of the free carriers colliding with the lattice. Usually the collision time is much shorter than the life time of the carrier so that thermal equilibrium exists with the temperature of the lattice. Thus the semiconductor behaves in addition to the g–r noise also as a resistor with Johnson or thermal noise with an amount given by (1.10).

1.6 Thermal Radiation and Its Fluctuations

According to Planck's radiation law the thermal radiation power at equilibrium temperature T incident on the area A within the small solid angle $\mathrm{d}\Omega$ and frequency interval $\mathrm{d}\nu$ is given by

$$\mathrm{d}P_{\nu,\Omega} = \cos\theta \overline{B} A \,\mathrm{d}\nu \,\mathrm{d}\Omega \,, \tag{1.45}$$

[1] Although the production mechanism is irrelevant the derived Poisson distribution of the micropulses is based on the assumption of constant radiation power. This implies strictly speaking a coherent radiation source. See Appendix A.2.4

where the average brightness $\overline{B}$ is

$$\overline{B} = \frac{2h\nu^3}{c^2(\mathrm{e}^{h\nu/kT} - 1)} \tag{1.46}$$

and θ the angle between the rays and the normal on A. Integrating (1.45) over Ω with $\mathrm{d}\Omega = 2\pi \sin\theta \, \mathrm{d}\theta$ we obtain for the incident thermal power within the frequency interval $\mathrm{d}\nu$ confined within the solid angle Ω_0 of a circular cone with half-angle θ_0

$$\mathrm{d}P_\nu = \frac{2\pi h A \sin^2\theta_0}{c^2} \frac{\nu^3 \, \mathrm{d}\nu}{\mathrm{e}^{h\nu/kT} - 1} \, . \tag{1.47}$$

In the ideal case the incident radiation is fully absorbed at the surface (black body). Since in equilibrium the wall temperature remain constant, the surface must emit the same amount. Thus in the ideal case the radiation power emitted in the frequency interval $\mathrm{d}\nu$ and area A at temperature T is also given by (1.47). In general the surface is not ideal and it emits less power. Then in order to maintain the constant temperature the incident power must be partly reflected. With a power reflectivity coefficient $\rho(\nu)$ averaged over θ we have for the emission coefficient $\epsilon(\nu)$ the relation

$$\epsilon(\nu) = 1 - \rho(\nu) \, . \tag{1.48}$$

Integrating (1.47) over the whole spectrum yields at thermal equilibrium for $\theta_0 = \pi/2$ the total incident power on the surface A or emitted average black body power, $\overline{P_\mathrm{t}}$, according to the Stefan–Boltzmann law

$$\overline{P_\mathrm{t}} = \sigma A T^4 \, , \tag{1.49}$$

where $\sigma = 2\pi^5 k^4/15c^2h^3 = 5.67 \times 10^{-8} \, [\mathrm{W\,m^{-2}\,K^{-4}}]$.

Let us now consider the average thermal power $\overline{P}$ incident normal to a detector surface A within the small solid angle $\Delta\Omega$ and bandwidth $\Delta\nu$. Taking $\cos\theta \approx 1$ and assuming $\Delta\nu$ is small compared to kT/h so that the considered thermal energy per unit frequency is constant we obtain from (1.45)

$$\overline{P} = \frac{\Delta\Omega A}{\lambda^2} \frac{2h\nu}{\mathrm{e}^{h\nu/kT} - 1} \Delta\nu \, . \tag{1.50}$$

The quantized average energy of a single frequency mode is equal to

$$\overline{E}_{h\nu} = \frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1} \, . \tag{1.51}$$

The factor 2 in (1.50) refers to the two independent polarizations of the field. We should keep in mind that $\Delta\nu$ is the selected optical bandwidth which is always much larger than the electronic bandwidth B of the detection system.

The incident thermal radiation can be derived by any set of orthogonal field functions that completely fills the space bounded by the plane containing the area A. If the radiation originates from a distance large enough to receive all wavefronts of the radiation at A parallel, the field components on A are coherent. Then, considering (1.50), the minimum space of photons, i.e., a single spatial mode, is bounded by the condition

$$\Delta\Omega A \approx \lambda^2 \, . \tag{1.52}$$

In case the incident radiation falls within an area–angle product larger than λ^2 its field is build up of several spatial modes. The number is given by

$$N_m \approx \frac{\Delta\Omega A}{\lambda^2} \, . \tag{1.53}$$

Each spatial mode contains many frequency modes. The power within a single spatial mode for one polarization is derived in Appendix A.4 and is given by

$$\overline{P}_m = \overline{E}_{h\nu}\Delta\nu \, . \tag{1.54}$$

By substituting (1.53) and (1.51) into (1.50) we get

$$\overline{P} = 2N_m\overline{E}_{h\nu}\Delta\nu \, . \tag{1.55}$$

The noise associated with this thermal radiation consists of two parts. One part is due to the quantization of the radiation. The radiation may be regarded as a stream of fluctuating photons. The thermally emitted photons at constant temperature with their finite lifetimes obey Poisson statistics. The noise power for a detection bandwidth B corresponds to shot noise, similar to what has been described for the random flow of charge particles in Sect. 1.3. Looking at (1.29) we have to replace for the similarity the current i_0 by the radiation power $\overline{P}$ and the charge e by the photon energy $h\nu$. Applying (1.35) we find for this part of the power fluctuations

$$\overline{\Delta P_{\mathrm{sh}}^2} = 2h\nu\overline{P}B \, . \tag{1.56}$$

In practice the band width B of the detection system is always much smaller than the optical bandwidth $\Delta\nu$ of the selected thermal radiation.

The other part results from the fluctuations of amplitude and phase. Within a single spatial mode the instantaneous radiation field results from the concerted action of a very large number of independent emitters. Therefore, from a statistical point of view the central limit theorem is appropriate and the resulting radiation field amplitude and its phase within a spatial mode are Gaussian processes. Their mutually independent fluctuations can be evaluated by considering the field as composed of two components in an arbitrary rectangular coordinate system and then apply the Gaussian process to each component. The Gaussian distribution of the field component v_x in the x-direction with the probability $F(v_x)\,\mathrm{d}v_x$ for having its value between v_x and $v_x + \mathrm{d}v_x$ at any time is given by

$$F(v_x)\,\mathrm{d}v_x = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-v_x^2/2\sigma^2}\,\mathrm{d}v_x \tag{1.57}$$

Similarly for $F(v_y)$ we have

$$F(v_y)\,\mathrm{d}v_y = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-v_y^2/2\sigma^2}\,\mathrm{d}v_y\,. \tag{1.58}$$

The probability $F(v_x + v_y)\,\mathrm{d}v_x\,\mathrm{d}v_y$ of finding the x-component between v_x and $v_x + \mathrm{d}v_x$ and the y-component between v_y and $v_y + \mathrm{d}v_y$ is then

$$F(v_x + v_y)\,\mathrm{d}v_x\,\mathrm{d}v_y = \frac{1}{2\pi\sigma^2}\,\mathrm{e}^{-(v_x^2+v_y^2)/2\sigma^2}\,\mathrm{d}v_x\,\mathrm{d}v_y\,. \tag{1.59}$$

Changing to the circular components v and φ with $v^2 = v_x^2+v_y^2$ and $\mathrm{d}v_x\,\mathrm{d}v_y = v\mathrm{d}\,v\mathrm{d}\varphi$ and integrating over φ we find the field probability of v

$$F(v)\,\mathrm{d}v = \frac{1}{\sigma^2}\,\mathrm{e}^{-v^2/2\sigma^2} v\,\mathrm{d}v\,. \tag{1.60}$$

The power P_m within a spatial mode is proportional to v^2. The average power $\overline{P_m}$ is then obtained by multiplying the last equation by v^2 and substituting $P_m = \alpha v^2$. We obtain

$$\overline{P_m} = \int_0^\infty \frac{P_m}{2\alpha\sigma^2}\,\mathrm{e}^{-P_m/2\alpha\sigma^2}\,\mathrm{d}P_m = 2\alpha\sigma^2\,. \tag{1.61}$$

Thus the radiation power probability distribution $F(P_m)$ of a single spatial mode, which may contain a set of frequency modes, is given by

$$F(P_m) = \frac{1}{\overline{P_m}}\,\mathrm{e}^{-P_m/\overline{P_m}} \tag{1.62}$$

which is called the Rayleigh distribution.

The power spread of a single spatial mode is given by

$$\overline{\Delta P_m^2} = \overline{\left(P_m - \overline{P_m}\right)^2} = \overline{P_m^2} - \overline{P_m}^2\,. \tag{1.63}$$

Using (1.62) we find $\overline{P_m^2}$ equal to $2\overline{P_m}^2$ so that

$$\overline{\Delta P_m^2} = \overline{P_m}^2\,. \tag{1.64}$$

Taking the sum of the energy spreads of all spatial modes we just multiply the last equation by $2N_m$ because the spatial mode are independent from each other. We obtain

$$\overline{\Delta P_{ray}^2} = 2N_m\overline{P_m}^2\,. \tag{1.65}$$

With the aid of (1.54) and (1.55) we get

$$\overline{\Delta P^2_{ray}} = \overline{PE}_{h\nu}\Delta\nu\,. \tag{1.66}$$

The last expression contains the noise power spread over its full optical spectrum $\Delta\nu$. We are now interested to know its frequency distribution because the detector with its much smaller bandwidth B receives only a small part of it. We assume $\Delta\nu$ to be small compared to kT/h so that the considered thermal power per unit frequency is constant. Following a classical description we note that the power fluctuations within a spatial mode correspond to beat frequencies between the field components of the frequency modes. Since the bandwidth of the selected radiation is $\Delta\nu$ the radiation power has components with beat frequencies ν_b ranging from zero to $\Delta\nu$. The number of beat components is highest for $\nu_{\mathrm{b}} = 0$ and the power of these beat components decreases linearly with $(1 - (\nu_b/\Delta\nu))$ to reach zero for $\nu_{\mathrm{b}} = \Delta\nu$. This is indicated in Fig. 1.7. It is seen from this figure that the noise content for a spectrum with $B \ll \Delta\nu$ is the fraction $2B/\Delta\nu$ of the total noise. Using (1.66) the Rayleigh noise within the bandwidth B becomes

$$\overline{\Delta P^2_{ray}} = 2B\overline{PE}_{h\nu}\,. \tag{1.67}$$

The total noise within the bandwidth B is the sum of parts given by, respectively, (1.56) and (1.67) or

$$\overline{\Delta P^2} = 2h\nu\overline{P}\left(1 + \frac{1}{\mathrm{e}^{h\nu/kT} - 1}\right)B\,, \tag{1.68}$$

where we used (1.51).

It is interesting that the result given by (1.68) can also be derived straightforward from statistical thermodynamics. This is done by starting from the partition function, $Z_{h\nu}$, of a radiation mode with photon energy $h\nu$ given by

$$Z_{h\nu} = \sum_{n=0}^{\infty} \mathrm{e}^{-nh\nu/kT} = \frac{1}{1 - \mathrm{e}^{-h\nu/kT}}\,. \tag{1.69}$$

The average energy, $\overline{E}_{h\nu} = \frac{1}{Z_{h\nu}}\sum_{n=0}^{\infty} nh\nu\,\mathrm{e}^{-nh\nu/kT}$, is given by

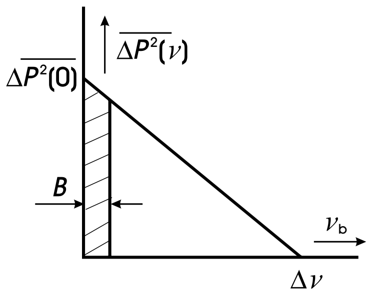

Fig. 1.7. Spectral distribution of thermal noise

$$\overline{E}_{h\nu} = \left[1 - \mathrm{e}^{-h\nu/kT}\right] kT^2 \frac{\mathrm{d}}{\mathrm{d}T} Z_{h\nu} = \frac{h\nu}{\mathrm{e}^{h\nu/kT} - 1} \,. \tag{1.70}$$

The average square of the energy, $\overline{E^2_{h\nu}} = \frac{1}{Z_{h\nu}} \sum_{n=0}^{\infty} (nh\nu)^2 \, \mathrm{e}^{-nh\nu/kT}$, is calculated similarly by

$$\overline{E^2_{h\nu}} = \left[1 - \mathrm{e}^{-h\nu/kT}\right] kT^2 \frac{\mathrm{d}}{\mathrm{d}T} \left[\frac{\overline{E}_{h\nu}}{1 - \mathrm{e}^{-h\nu/kT}}\right] = (h\nu)^2 \frac{\mathrm{e}^{h\nu/kT} + 1}{\left(\mathrm{e}^{h\nu/kT} - 1\right)^2} \,. \tag{1.71}$$

Further we have $\overline{\Delta E^2_{h\nu}} = \overline{\left(E_{h\nu} - \overline{E_{h\nu}}\right)^2} = \overline{E^2_{h\nu}} - \overline{E}^2_{h\nu}$. Substituting (1.70) and (1.71) we get

$$\overline{\Delta E^2_{h\nu}} = h\nu \overline{E}_{h\nu} \left(1 + \frac{1}{\mathrm{e}^{h\nu/kT} - 1}\right) \,. \tag{1.72}$$

The relation between the energy of a single frequency mode and the power of a spatial mode is derived in Appendix A.4. The fluctuations of the total power of the considered beam is the sum of the fluctuations of each spatial mode or by using (1.55)

$$\overline{(\Delta P^2)}_{total} = \overline{(P - \overline{P})^2} = 2N_m \, (\Delta\nu)^2 \, \overline{\left(E_{h\nu} - \overline{E_{h\nu}}\right)^2} = 2 \, (\Delta\nu)^2 \, N_m \overline{\Delta E^2_{h\nu}} \,, \tag{1.73}$$

where the number of independent spatial modes N_m is multiplied by 2 because of the two independent polarizations. Substituting (1.72) and (1.55) into (1.73) we get

$$\overline{(\Delta P^2)}_{\mathrm{total}} = h\nu \overline{P} \left(1 + \frac{1}{\mathrm{e}^{h\nu/kT} - 1}\right) \Delta\nu \,, \tag{1.74}$$

which contains the noise power spread over $\Delta\nu$. Since we are interested in the contributions within the band $B \ll \Delta\nu$ we follow the previous discussion to derive (1.67) and replace $\Delta\nu$ by $2B$. We obtain in agreement with (1.68)

$$\overline{\Delta P^2} = 2h\nu \overline{P} \left(1 + \frac{1}{\mathrm{e}^{h\nu/kT} - 1}\right) B \,. \tag{1.75}$$

In the optical region where $h\nu/k$ is much larger than T the second term within the brackets is negligible, but this is not the case for thermal radiation.

1.7 Temperature Fluctuations of Small Bodies

The energy of a body is always in interaction with its surroundings and its transfer occurs by statistical processes of radiation, convection, and conduction. Even at equilibrium its value will fluctuate randomly about a mean value. The question is how large are those fluctuations and how do they depend on physical quantities. An elegant way to answer these questions is to apply statistical thermodynamics and derive the formula for the probability that a system has an energy E.

Let us have a large number of identical systems that can assume energy values $E_1, E_2, E_3, \ldots$ and which are all in thermal heat exchange with a large temperature bath kept at constant temperature T. According to Boltzmann the probability that a system has an energy E_i is $A\mathrm{e}^{-E_i/kT}$. The sum of all probabilities must be one, so that $\sum_i A\mathrm{e}^{-E_i/kT} = 1$. The average energy, $\overline{E}$, is obtained by

$$\overline{E} = \sum_i AE_i\mathrm{e}^{-E_i/kT} = \frac{\sum_i E_i\mathrm{e}^{-E_i/kT}}{\sum_i \mathrm{e}^{-E_i/kT}} \,. \tag{1.76}$$

Taking the temperature derivative of $\overline{E}$

$$\frac{\mathrm{d}\overline{E}}{\mathrm{d}T} = \frac{1}{kT^2}\left[\frac{\sum_i E_i^2\mathrm{e}^{-E_i/kT}}{\sum_i \mathrm{e}^{-E_i/kT}} - \left(\frac{\sum_i E_i\mathrm{e}^{-E_i/kT}}{\sum_i \mathrm{e}^{-E_i/kT}}\right)^2\right] = \frac{1}{kT^2}\left[\overline{E^2} - \overline{E}^2\right] \,, \tag{1.77}$$

which is the heat capacity of the system.

We now apply this result to a small body and describe thereby the heat content by its temperature so that energy fluctuations will be interpreted as temperature fluctuations according to $E - \overline{E} = C_{\mathrm{th}}(T - \overline{T})$ where C_{th} is the heat capacity. Then

$$\overline{E^2} - \overline{E}^2 = \overline{\left(E - \overline{E}\right)^2} = C_{\mathrm{th}}^2\overline{\left(T - \overline{T}\right)^2} = C_{\mathrm{th}}^2\overline{\Delta T^2} \,. \tag{1.78}$$

Substituting (1.78) into (1.77) we get

$$\overline{\Delta T^2} = \frac{kT^2}{C_{\mathrm{th}}} \,. \tag{1.79}$$

Next we want to describe the spectral density of the temperature fluctuations. For that purpose we realize that frequency fluctuations are damped by the thermal slowness of the system which has a time constant $\tau_{\mathrm{th}} = C_{\mathrm{th}}/\lambda$ where λ is the thermal conductance. It behaves analogously to a RC circuit in electronics. This is obvious if we relate ΔT to voltage, C_{th} to electrical capacity and λ to electrical conductivity. Since we describe mean-square fluctuations we obtain from the analogy the following frequency (f) dependence

$$\overline{\Delta T^2}(f) = \frac{\overline{\Delta T^2}(0)}{1 + (2\pi f\tau_{\mathrm{th}})^2} \tag{1.80}$$

Integrating the last equation over all frequencies we get again the total value given by (1.79). From this we find $\overline{\Delta T^2}(0) = 4kT^2/\lambda$ and write (1.80) as

$$\overline{\Delta T^2}(f) = \frac{4kT^2}{\lambda}\frac{1}{1 + (2\pi f\tau_{\mathrm{th}})^2} \,. \tag{1.81}$$

Integrating (1.81) over a bandwidth B much smaller than the reciprocal thermal time we get

$$\overline{\Delta T^2} = \frac{4kT^2B}{\lambda}\frac{1}{1 + (2\pi f\tau_{\mathrm{th}})^2} \,. \tag{1.82}$$

These temperature fluctuations limit the minimum detectable power of thermal detectors.

1.7.1 Absorption and Emission Fluctuations

Let us consider the situation that a black body is in equilibrium with the thermal radiation of the surrounding and that all energy transfer is by radiation only. The thermal fluctuations of the body are then related to both emission and absorption fluctuations. A small change of radiation transfer ΔP, either by absorption or emission, results in a small temperature change ΔT of the body. The relation between ΔP and ΔT derived from (1.49) yields

$$\lambda = \frac{\mathrm{d}P}{\mathrm{d}T} = 4\sigma A T^3 \,. \tag{1.83}$$

The mean square power fluctuations near $f = 0$ within the bandwidth B are obtained by substituting (1.83) into (1.82). We find

$$\overline{\Delta P^2} = 16 A B \sigma k T^5 \,. \tag{1.84}$$

The fluctuations of the incident absorbing radiation in the case of no reflection are obtained by integrating (1.68) over the full spectrum. This is done by substituting for $\overline{P}$ the expression $\mathrm{d}P_\nu$ from (1.47) and integrating over ν. We obtain

$$\overline{\Delta P^2_{\mathrm{abs}}} = \frac{4\pi A B h^2 \sin^2\theta_0}{c^2} \int_0^\infty \frac{\nu^4 \mathrm{e}^{h\nu/kT}}{\left(\mathrm{e}^{h\nu/kT} - 1\right)^2} \, \mathrm{d}\nu \tag{1.85}$$

or

$$\overline{\Delta P^2_{\mathrm{abs}}} = \frac{16\pi^5 A B k^5 T^5 \sin^2\theta_0}{15 c^2 h^3} = 8 \sin^2\theta_0 A B \sigma k T^5 \,, \tag{1.86}$$

where we have substituted $\sigma = 2\pi^5 k^4 / 15 c^2 h^3$. It is seen that for the total incident radiation with $\theta_0 = \pi/2$ we find just one-half of what is obtained for the total fluctuations given by (1.84). This can be understood by the fact that we have considered so far only the incident radiation. If the area is in thermal equilibrium with the radiation field, there will be an equal amount of power fluctuations emitted by the area A so that the total fluctuations are the same as derived by (1.84).