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Introduction

After the great scientific revolutions of Special Relativity and of Quantum Mechanics in the first half of the twentieth century, Relativistic Quantum Field Theory (QFT) was introduced to provide a synthesis of these two new paradigms (in Kuhn's words). As such, QFT is one of the major advances for theoretical physics in the second half of the last century. The main reason for emphasizing the importance of QFT lies in the impressive effort that it represents towards a unified understanding of the structure of matter at subatomic scales, as it emerges in the collision phenomena of particle physics in the range of energies presently explored in particle accelerators. But another reason is the extraordinary mathematical wealth of this theoretical framework, which makes it a fascinating domain of research for mathematical physics and may also stimulate the interest of mathematicians.

Foreseen by Dirac in 1933 and finaly established discovered by Feynman around 1949 for the treatment of quantum electrodynamics, the so-called "path-integral formalism" of quantum field theory is also now considered as a powerful tool for providing perturbative methods of computation in many problems of theoretical physics, namely in statistical mechanics and in string theory. However, the use of relativistic quantum fields as a basic concept of mathematical physics underlying all the phenomena of particle physics in a very large range of energies represents a much more ambitious program.

This program was indeed stimulated by the success of the quantum electrodynamics (QED) formalism for computing the electron-photon, electron-electron and electron-positron scattering amplitudes. Even today, it is by no means understood why the perturbative expansion of QED in powers of the coupling parameter, that is the electric charge of the electron, provides such a spectacular agreement with experimental data, although in practice it is reduced to the computation of the very first terms of the series (the only ones which are computable). This is even more surprising since we now know that the series cannot be convergent, so that, paraphrasing Wigner's words, one may wonder about the "unreasonable effectiveness" of perturbation theory in four-dimensional QED.

In the 1950s, however, the success of these computations was credited to the smallness of the coupling parameter of QED, and this situation stimulated research for other methods of investigation of QFT, which could apply to quantum field models with large coupling: it was in fact the very concept of a quantum field which appeared as the most powerful and promising one for explaining the phenomena of strong nuclear interaction of particle physics. During all that period the study of phenomena of weak nuclear interaction also benefitted, like QED, from the success of a perturbative QFT formalism reduced to very few terms.

The demand for a more general nonperturbative treatment of QFT motivated an important community of mathematical physicists to work out a model-independent *axiomatic approach to relativistic quantum field theory* [2,8,9,11,14]. Their first task was to provide a mathematically meaningful concept of a relativistic quantum field in terms of *"operator-valued distribution in the Hilbert space of quantum physical states"*, in such a way that the notion of ingoing and outgoing particle states could be introduced in terms of appropriate asymptotic forms of the field operators.

The precise mathematical formulation of fundamental physical principles of relativistic quantum theory in terms of field operators considered as basic entities of the system allowed the possibility to offer a general model-independent framework, in which one could propose a theoretical treatment of high-energy collision phenomena.

These phenomena involve not only the usual processes of "elastic scattering" preserving the number and nature of the particles, but also all possible processes of creation and annihilation of particles permitted by the kinematical laws of special relativity and the conservation laws of basic quantum numbers called "charges". In this mathematical framework, all the collision processes are encoded in a certain general unitary *scattering operator* S in the Hilbert space of states, according to the concept originally introduced by Heisenberg, and all the corresponding *scattering kernels* $S_{m,n}$ of $(m \to n)$-particle collision processes are related to general N-point structure functions of the quantum fields (with $N = m + n$) via the so-called LSZ *"reduction formulae"*. This formalism was fully justified at least for processes involving only massive particles; the inclusion of massless particles, such as the photons of QED, soon revealed the existence of hard mathematical problems of various types.

This general approach of QFT was followed by a conceptually important variant, in which the quantum fields are supposed to generate "algebras of local observables" attached to arbitrary regions of Minkowski spacetime. This is the *algebraic approach to quantum field theory* (Haag, Kastler 1964), whose most important developments between 1960 and 1990 have been presented in a book by Haag (1992) under the name of "local quantum physics" [6].

From both mathematical and physical viewpoints, the visionary works of Von Neumann, which were contemporary to the birth of quantum mechanics, are at the origin of all these developments; however, the algebraic structures involved in relativistic QFT refer to the most advanced developments of the theory of Von Neumann algebras. Just as an example, it is worthwhile to recall that, in contrast with the case of nonrelativistic quantum mechanics, one is necessarily confronted by factors of type III in the standard classification of Von Neumann algebras. This occurs already in the simplest models of relativistic QFT, namely the free-fields. Moreover, important conceptual progress was made by the algebraic QFT viewpoint with respect to the primitive quantum-mechanical notion of the Hilbert-space of states of a given physical system. It consists in introducing algebras of local observables of the system as *abstract* C^*-*algebras* (and even more general $*$-algebras) and then representing all possible physical states of the system as linear functionals on such algebras. In this more general viewpoint, various types of physically relevant Hilbertian representations of the algebras may be introduced mathematically via GNS-type (Gelfand-Naimark-Segal) [6] constructions. As a typical illustration of this viewpoint, one can mention the Borchers-Uhlmann algebraic presentation of the original Wightman axiomatic QFT [3].

In these various general approaches to relativistic QFT aiming to provide a theoretical understanding of particle physics, a central role is played by a certain field-theoretical formulation of *Einstein causality* together with the relativistic principle of positivity of energy (also called *spectral condition*) and an appropriate expression of the covariance of the system under the Poincaré group.

The interplay of these postulates has generated rigorous proofs of important physical properties, thanks to the discovery of very rich *analyticity properties* of the N-point structure functions of the fields (elaborated in successive works by Ruelle, Steinmann, Araki, Bros, Epstein, Glaser, Stora). As a matter of fact, all these functions enjoy *two associated analytic structures*, which coexist in the complexified spacetime variables z and in the Fourier-conjugate complexified energy-momentum variables k; this double analytic structure is equipped with a peculiar type of algebraic relations between boundary values from tube domains and relevant (multiple) discontinuities, including various interconnections by Fourier-Laplace transformations in several variables. In both z-space and k-space, nontrivial problems of holomorphy envelopes of the so-called "edge-of-the-wedge" type which emerge from this double analytic structure have only been very partially solved. As typical illustrations of the latter, one can say that on the one hand the exploitation of the z-space analytic structure has provided proofs in general QFT of such important properties of particle physics as the PCT-symmetry and the spin-statistics connection . On the other hand, the exploitation of the k-space analytic structure has brought a general justification in QFT of important analytic continuation properties of the scattering kernels on the so-called *complex mass shell manifold* of various collision processes. As a typical example, let us mention the crossing property between the "particle-particle" and "particle-antiparticle" scattering processes, which can be geometrically expressed as the existence of a domain of analyticity on the complex mass shell manifold relating two different "physical regions" on the *real mass shell submanifold* (Bros, Epstein and Glaser). Such successful results, which were mostly obtained in the 1960s, correspond to the "golden age" of the so-called *dispersion relations* in particle physics. It is rather remarkable that such model-independent Cauchy-type integral relations were found to be in satisfactory agreement with the experimental data of two-particle collision processes and that in this analytic framework, important "high-energy bounds" could also be derived from general principles in basic works by Froissart and Martin [10].

Of course, as a natural complement to this conceptual progress, there was (and there is) the need to produce *nontrivial models* of relativistic quantum fields: for that matter, a basic criterion of nontriviality is the requirement that the associated scattering operator S be different from the identity operator. This accounts for the occurrence of another school of mathematical physicists, called *"Constructive quantum field theory"* and founded essentially by Glimm and Jaffe in the late 1960s [5], whose purpose was to produce rigorous existence proofs together with properties of the corresponding solutions for models of relativistic QFT associated with Lagrangians of suitable type. Ingenious methods of *"cluster expansions"* were invented which allowed one to treat successfully various classes of nontrivial models in two and three-dimensional Minkowski spacetime, thus displaying that (at least for these dimensions) the axiomatic framework was not a pure abstraction, but it contained a lot of mathematically well-defined Lagrangian theories. It was even shown that, for such models, the traditional perturbation expansion, defined as a formal power series in terms of a small coupling parameter, had Borel-summability properties which allowed one to reach the exact solution of the model [5, 7].

However, it became patent that all the models of interacting relativistic quantum fields in the physical four-dimensional spacetime are afflicted by very hard mathematical problems which have to do with unavoidable short-distance as well as long-distance divergent behaviours of the structure functions of the fields. These difficulties are closely related to the *renormalization* problems in the standard perturbative approach of QFT. In the 1980s, in spite of the fact that no completely rigorous proof of it could be produced, there appeared a wide consensus among theoretical physicists that the QFT model with quartic interaction, that is the "simplest" Lagrangian model of scalar fields to be considered, probably did not exist in four-dimensional spacetime, but only reduced to a pure formal series in the sense of the perturbative approach.

The same consensual opinion also applies to QED, which therefore makes it still more thrilling to have in mind the physical success of its perturbation series. The additional intriguing aspect of QED is the fact that it incorporates the fundamental physical notion of *electric charge*. As a matter of fact, the horizons of QFT were considerably renewed in the 1970s by the generalization of the notion of *charge* in particle physics and the associated concept of gauge symmetry groups for the quantum field description of charged states. From this viewpoint, in comparison with all the models studied in constructive QFT, the Lagrangian of QED appeared as the simplest one to incorporate a *local gauge symmetry group*, namely the gauge group based on the abelian group $U(1)$. A very important event was then the discovery of more complicated Lagrangians of relativistic fields, symmetric under *nonabelian gauge groups*, in particular the model called *quantum chromodynamics (QCD)*. The successful exploitation of the latter for the phenomenology of particle physics opened a way that is still used nowadays in describing elementary particle interactions. In fact, for theoreticians of particle physics, the early success of QED in perturbation theory was renewed in the 1970s and extended to all the phenomena of weak and strong nuclear interactions: this was mainly due to the success of the so-called "renormalization group" treatment of QCD for the understanding of strong nuclear interactions, and of the *standard model*, whose fundamental set of boson and fermion quantum fields are supposed to take into account all the basic interactions of matter but gravitation, thus providing a unified treatment of the electroweak and strong interactions together. In particular, it is the discovery of the basic notion of *asymptotic freedom* which allowed one to give a sense to expansions of the field-theoretical structure functions in terms of a large coupling parameter, and therefore to renew to some extent with QCD the old success of QED in the spirit of perturbation theory.

In that regard, however, the point of view of mathematical physicists remained somewhat different and cultivated a still larger ambition in the line of its tradition. As ingenious as they are, the computational techniques of QCD and of the standard model treat the quantum fields of those models as very formal objects. In these approaches, whose computational efficiency is the main criterion to be fulfilled, the problem of the mathematical existence of fields has not been really considered. A pragmatic attitude consists in being satisfied with a certain type of approximate computations, whose recipes pertain to a weakened conceptual framework of quantum field theory, which is sometimes called "effective quantum field theory" by the tenants of these approaches. However, the impressive adequacy of some of the results obtained with the experimental data is stimulating for the mathematical physicist whose hope is to obtain a fully coherent understanding of this fundamental branch of physics in terms of mathematically well-defined objects. There are basic mathematical problems of the QFT framework which remain unsolved and whose solutions might shed new light on deep conceptual aspects of high-energy particle physics.

For a large part of the community of theoreticians of particle physics, increasing evidence has been accumulated since the 1970s that nonabelian gauge quantum field theory might be the only class of theories relevant for elementary particle physics, and whose mathematical existence in four-dimensional Minkowskian spacetime might hopefully be established some day. As an abelian gauge theory, QED might have a less favourable mathematical status as an isolated theory, and (according to certain arguments) should be more credibly thought as "embedded" in a larger nonabelian theory (such as the "standard model"). During the last thirty years, these considerations have been taken into account by the community of QFT mathematical physicists.

On the one hand, the problem of treating charged states in the case of a *global gauge symmetry group* (namely an internal symmetry group independent of spacetime localization) gave rise to deep conceptual progress in the general approach of algebraic QFT. This was due to the analysis of charge sectors in the Hilbert space of states by Doplicher, Haag and Roberts in the 1970s, and to the axiomatic introduction of nonlocal charge-carrying quantum fields such as the "string-localized fields" of Buchholz-Fredenhagen (1982). This contributed to enlarging considerably the concept of a quantum field and its relationship with the concept of a particle and thereby to opening new ways of research for QFT in the spirit of mathematical physics.

On the other hand, the mathematical study of QFTs involving a *local* gauge group (like QED and QCD) remains a much harder task. Indeed the very existence of a local gauge symmetry renders the axioms incompatible and one has to choose whether to work in local and covariant gauges and abandon Hilbertian positivity, and therefore a direct quantum mechanical interpretation, or to work in positive gauges where all the standard wisdom of QFT fails (in particular the expression of Einstein causality). A general approach to these problems was introduced and developed in the papers by Strocchi and Wightman in the 1970s. Still harder problems concern the construction of charged states for nonabelian gauge QFTs, the "confinement problem" being a particular instance of it [12] (see also further remarks below). The importance of this stream of research may be symbolized by the inscription of the study of Yang-Mills Quantum Field Theory (namely the "gluonic part" of QCD) in the seven Millennium Prize Problems proposed by the Clay Mathematical Institute. We refer to the presentation of this subject by A. Jaffe and E. Witten for a description of the type of hard mathematical problems to be solved in that context.

In their spirit, all the articles presented in this book consider quantum fields as genuine mathematical objects, whose various properties and relevant physical interpretations have to be studied in a well-defined mathematical framework. They therefore pertain to that tradition of "Rigorous Quantum Field Theory", which traces back to the basic axiomatic settings, supplemented by relevant constructive approaches, that we have previously mentioned. In this spirit, the most recent investigations of QFT may be characterized by the conceptual needs of extending the general axiomatic framework at least in two directions: i) taking into account the inclusion of internal (abelian and nonabelian) gauge groups as basic structures suggested by particle physics; ii) going beyond the Minkowskian spacetime of special relativity by incorporating more general notions concerning space and time. The latter investigations include, on the one hand, the consideration of QFT on *curved spacetime* manifolds, stimulated by various currents of research in general relativity, astrophysics and cosmology; in this context, unexpected relationships with basic concepts of "thermal QFT", namely field theory in a medium with a large number of particles, have also been exhibited. They also include, on the other hand, the recent development of QFT on *noncommutative spacetimes*, stimulated by new insights on the very-short-range structure of space and time. From these various viewpoints, it seems that one has now entered a new phase, namely the genesis of a deep renewal of the fundamental structures of QFT, in which the most advanced progress in pure mathematics may have its role to play.

We shall end this introduction with a very brief overview of the different topics which are represented in the various texts of this book. These topics can be grossly regrouped under seven titles, whose order is by no means significant and which do not exclude overlap:

1. *Analytic Structures of QFT.*
   The analytic structures of QFT in complex energy-momentum space offer microlocal aspects and global aspects. While the microlocal aspects concern the analytic wave-fronts of N-point functions and of general multiparticle collision amplitudes, the global aspects involve such general concepts as holomorphy envelopes in $\mathbf{C}^n$, Fredholm-type equations with floating cycles in complex manifolds (i.e., the so-called Bethe-Salpeter-type equations), polar singularities in the space $\mathbf{C}^2$ of the complex energy and angular momentum variables, interpreted as Regge particles in QFT.
2. *Renormalization group methods.*
   A version of rigorous renormalization theory based on the flow equations of the Wilson renormalization group is presented and illustrated in particular by considering the case of the scalar field with quartic coupling. More general results of the rigorous approach of renormalization group methods are also described. Moreover a new look at the renormalization group from the point of view of Algebraic Quantum Field Theory is proposed which results in a consistent definition of local algebras of observables and of interacting fields in renormalized perturbative QFT, involving an appropriate use of the classical action of the models.
3. *New investigations and results related to Gauge QFT.*
   Some basic unsolved problems of Quantum Electrodynamics which concern the formulations of that theory in different gauges are investigated from the viewpoint of the Wightman functions of the fields.

   Yang-Mills field equations are the starting point of a far-reaching mathematical study on "Yang-Mills algebra", "quadratic self-duality algebras" and "super" versions of them as applications of the theory of homogeneous algebras.

4. *New methods and results in Constructive QFT.*
   a) The invention of new procedures for constructing models of local relativistic quantum fields starting with a given scattering operator gives successful results for the case of factorizing S-matrices in two-dimensional spacetime. An intermediate step in the construction procedure makes use of nonlocal fields, called "Polarization-Free Generators" whose affiliated algebras are localized in wedge-shaped regions of spacetime rather than in bounded regions. The hope of obtaining similar results when starting from more general collision operators in four-dimensional spacetime by also incorporating the concept of "lightfront holography" is also discussed. In the same spirit, a general procedure for constructing stringly-localized fields from Wigner representations of the Poincaré group in d-dimensional spacetime is also presented.

   b) Thermal Quantum Field Theory, whose axiomatic status has been implemented in recent years on the basis of the KMS condition (starting from the general analysis of Haag, Hugenholtz and Winnink), presents a characteristic analytic structure in complex spacetime. This structure, which involves a periodicity in imaginary times given by the inverse of the temperature, is an alternative to the one prescribed by the spectral condition for the usual theories with a ground state (or zero-energy "vacuum"), thus appearing as the case of zero temperature. The construction of thermal field models in two-dimensional spacetime, associated with polynomial Lagrangians (of the type considered by the shool of constructive QFT in the 1970s) has been performed very recently, and it is proven here that they satisfy a relativistic form of the KMS-condition introduced earlier at the axiomatic level by Bros and Buchholz.
5. *Stability properties in QFT and extensions of the Axiomatic Framework of QFT.*
   Apart from the standard postulates of QFT expressed respectively by the spectral condition for theories with a ground state, and by the KMS-condition for thermal QFT, have both been proved to result from a general stability criterion called "passivity" (according to a basic work of Pusz and Woronowicz). Another type of stability condition is presented here, which is formulated in terms of "quantum energy inequalities" to be satisfied by the energy-momentum tensor of the theory. The links between this type of condition with the passivity condition are investigated, as also those with the important concept of "nuclearity" introduced in QFT by Buchholz.

   Another direction of research concerns the concepts of Anosov flows and of Kolmogorov systems, the important point being that they can be translated from classical to quantum systems. With some modifications necessary to keep the same clustering behavior as the typical one for classical Anosov systems, Anosov structure then appears rather naturally in a typeIII_1 algebra. Here Anosov structure and Kolmogorov structure with respect to modular evolution are even equivalent. The Rindler wedge of quantum field theory offers a typical example.

6. *QFT on Models of Curved Spacetimes.*
   Quantum field theory in curved spacetime is one important issue of modern theoretical physics, in that, while waiting for new high energy physics experiments in terrestrial laboratories, most of the new data come from observational cosmology. In particular, the evidence for a nonzero cosmological constant indicates that the de Sitter geometry plays and will play in future an important role. The study of soluble models on the two-dimensional de Sitter spacetime should be based on the two-dimensional massless scalar field. An original study of that model is presented here, that puts in evidence the anomaly of the equations of motion and fully characterizes the charge structure of the model.

   Anti-de Sitter Quantum Field theory is nowadays very popular because of its appearence in the context of string theory with the so-called "AdS-CFT correspondence". Euclidean AdS QFT can also provide a covariant regularization of flat QFT. The difficulty in studying AdS-QFT lies mainly in the absence of global hyperbolicity, caused by the presence of a boundary at spacelike infinity. A general rigorous approach to AdS QFT that bypasses these problems is presented here together with a collection of structural results that are implied by the chosen axioms.
7. *QFT on Noncommutative Minkowski Spacetime.*
   One of the fundamental questions in field theory on noncommutative spacetimes is how to find suitable generalizations of the local interaction terms we know from ordinary field theory. By construction, the notion of a point loses its meaning on such a spacetime and, not surprisingly, the principle of locality has to be modified. Depending on how this is done, various interaction terms are obtained. One can replace the idea of coinciding points in a way compatible with the uncertainty relations. The resulting interaction term resembles a point-split regularized product of fields and leads to an ultraviolet finite perturbation theory.

   The Wick reduction of *non-locally* time-ordered products of Wick monomials can be performed in a quantum field theory with a certain nonlocal self–interaction as introduced by Doplicher, Fredenhagen and Roberts, and simple Dyson diagrams are discussed.

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