17 A Mirror Conjecture

We can now make a mirror symmetry conjecture, which, while not totally precise, is the closest we will come to a precise statement. This will be enough for working out the best-known example of the quintic.

Conjecture 17.1 Let $f : \mathcal{X} \to (\Delta^*)^s$ be a family of Calabi–Yau 3-folds with $0 \in \Delta^s$ a large complex structure limit point. Then there exists a Calabi–Yau 3-fold $\check{X}$, called the *mirror* of the family f, a choice of framing $\Sigma \subseteq \mathcal{K}_{\check{X}}$ generated by a basis $e_1, \ldots, e_s \in \overline{\mathcal{K}}_{\check{X}}$, and a choice of canonical coordinates $q_1, \ldots, q_s$ on $(\Delta^*)^s$ with the following property: The basis $e_1, \ldots, e_s$ determines coordinates $\check{q}_1, \ldots, \check{q}_s$ on $\mathcal{M}_{Kah,\Sigma}(\check{X})$ and hence a map $m : (\Delta^*)^s \to \mathcal{M}_{Kah,\Sigma}(\check{X})$ taking a point with coordinates $(q_1, \ldots, q_s) \in (\Delta^*)^s$ to the point in $\mathcal{M}_{Kah,\Sigma}(\check{X})$ with coordinates $(\check{q}_1, \ldots, \check{q}_s) = (q_1, \ldots, q_s)$. Then

$$\left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right\rangle_p = \left\langle \frac{\partial}{\partial \check{q}_i}, \frac{\partial}{\partial \check{q}_j}, \frac{\partial}{\partial \check{q}_k} \right\rangle_{m(p)}.$$

Here the left hand side refers to the $(1,2)$-Yukawa coupling for X at $p \in (\Delta^*)^s$, while the right hand side refers to the $(1,1)$-Yukawa coupling on Kähler moduli space of $\check{X}$ at $m(p)$. The map m is called the *mirror map*.

Keep in mind that the choice of canonical coordinates means a choice of basis for W_2/W_0, which amounts to choosing an integral isomorphism between W_2/W_0 and $H^2(\check{X}, \mathbb{Q})$. However, the conjecture as stated doesn't specify how one chooses this isomorphism. In the toric situation there is a canonical isomorphism, but when we deal with the quintic, it won't be much of a problem since these spaces are one-dimensional.

The reader should not regard this version of mirror symmetry as being the final word defining mirror symmetry; it is in fact an early definition, first phrased by Morrison in [1], and can be refined. Ultimately, however, this conjecture is only really reflecting one symptom of mirror symmetry, and an eventual mathematical definition of mirror symmetry may prove to be quite different from the above conjecture, with the equality of Yukawa couplings being deduced in some deep way from the hypothetical eventual definition.