Numerical Analysis of Finite Element Methods for Eddy Current Problems. Applications to Electrode Simulation

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Summary. The objective of this work is to introduce and numerically solve a 3D-mathematical model for steady thermoelectrical behavior of electrodes in a metallurgical electric furnace. The mathematical model couples the time-harmonic eddy current model with the heat transfer equations in a bounded 3D-domain. An important part of the paper deals with the analysis and numerical solution of the eddy current model in a bounded domain.

1 Introduction

Silicon is produced industrially by reduction of silicon dioxide with carbon by a reaction which can be written in a simple way as follows:

$$\mathrm{Si\ O_2} + 2\mathrm{C} = \mathrm{Si} + 2\mathrm{CO}.$$

This reaction takes place in submerged arc furnaces which use three-phase alternating current. A simple sketch of the furnace can be seen in Figure 1. It consists of a cylindrical pot containing charge materials and three electrodes disposed conforming an equilateral triangle.

Electrodes are the main components of reduction furnaces and their purpose is to conduct the electric current which enters the electrode through the "contact clamps" (see Figure 1). The electric current goes down crossing the column length comprised between the contact clamps and the lower end of the column generating heat by Joule effect. At the tip of the electrode an electric arc is produced, reaching temperatures of about 2500 °C which are needed for the reduction chemical reactions to take place.

Classical electrodes extensively used in industry include *pure graphite*, *pre-baked* and *Søderberg* electrodes. The latter are the most used in ferro-silicon industry and they are composed by paste consisting of a carbon aggregate and a tar binding which are fed into a steel casing; the casing have steel

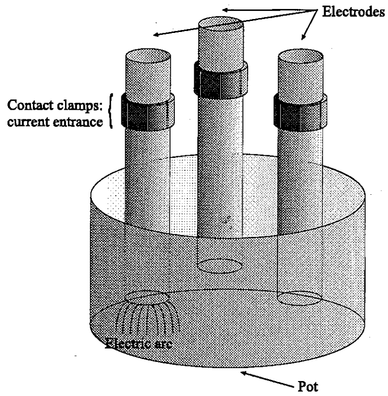

Fig. 1. A reduction furnace

fins attached to its inner part, which are placed radially in the cylinder. The great amount of heat generated by Joule effect is partially employed to bake the paste; this is a crucial process during which the initially soft/liquid non-conductive paste at the top of the electrode becomes a solid conductor. The advantages of Søderberg electrodes with respect to pure graphite or prebaked electrodes are that they are built in larger sizes and cost less. However, as the electrode is consumed, it has to be slipped and the steel casing moves with the carbon body so it melts and pollutes silicon. This is why they cannot be used to obtain *silicon metal* or silicon with metallurgical quality, which is used as alloying of other metals as aluminum. Thus prebaked electrodes have been for many years the only alternative for commercial silicon metal production.

In the early nineties, the Spanish company Ferroatlántica S.L. built a new compound electrode named ELSA ([14]) which serves for the production of silicon metal. It seems to be the solution for all silicon furnaces because its cost can be up to one third the price of a prebaked electrode.

ELSA electrode consists of a central column of baked carbonaceous material, graphite or similar, surrounded by a Søderberg-like paste (see Figure 2). There is a steel casing without fins that contains the paste until it is baked at the contact clamps zone. Two different slipping systems exist, one for the casing and another one for the central column; the combination of both systems is necessary so as to slip the casing as little as possible and also to carry out the correct extrusion of the carbon electrode. Then, unlike in the case of Søderberg electrodes, the casing is not consumed and it is possible to produce silicon with metallurgical quality. The result is that the furnace operation is similar to that of prebaked electrodes, but the compound electrode is less expensive. The disadvantage is that slipping velocity is not free as in prebaked electrodes, because the paste has to be baked before leaving the casing, so it is necessary a minimum period of time between slippages. Thus, baking of paste is a crucial point in the working of this type of electrodes.

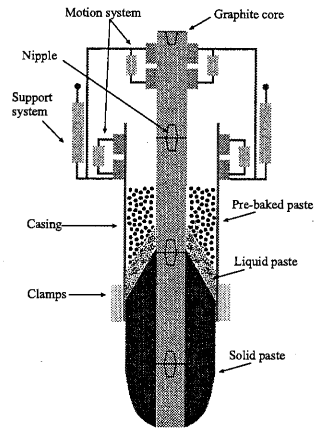

Fig. 2. Sketch of ELSA electrode

In general, the design and control parameters of electrodes are very complex and numerical simulation plays an important role at this point. Modeling the involved phenomena in a computer allows us to analyze the influence of changing a parameter without the need of expensive and difficult tests. Thus, during the last 20 years, an important number of mathematical models and computer programs have been developed in order to simulate the thermoelectrical behavior of classical electrodes (see for instance [15, 17, 18]). In particular, the mathematical models based on cylindrical symmetry have been the most extensively used. However, ELSA electrode works in a different manner from the classical electrodes. While classical electrode has only a constitutive material, compound electrode combines a good electric current conductor as graphite with a paste which becomes a good conductor only at high temperatures. Not only the core of graphite is important in the movement of the column but also in the distribution of current inside the electrode. Moreover, unlike Søderbeg electrodes, the non existence of fins gives a geometrical axisymmetry (see Figure 3).

This is why we first developed a finite element method based on cylindrical symmetry to compute the electric current and temperature distribution in a radial section of the electrode [5]. While the axisymmetric model has given valuable information on important electrode parameters, the assumption of cylindrical symmetry makes necessary to neglect the following facts:

- The electromagnetic effect caused on one electrode by the two others, that is the so called "proximity effect". This arises because the magnetic field generated by each electrode induces eddy currents in the two others.
- Thermal boundary conditions are not axisymmetric. Indeed, the temperature of the air around the electrode is greater on the surfaces oriented toward the furnace center.

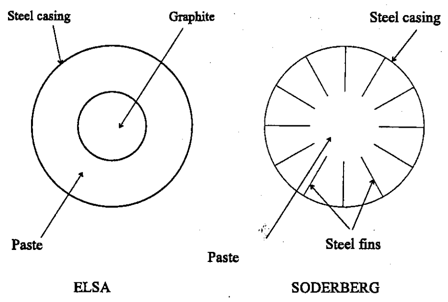

Fig. 3. Cross section of ELSA and Søderberg electrode

- The current entrance in the electrode through the contact clamps is not axisymmetric. The current is transferred to the contact clamps through copper bus tubes which in its turn are connected to three transformers with different phases. Then, in each electrode, half of the clamps receive current from one transformer while the other ones are connected to a second transformer.

These points can only be considered by using a pure three-dimensional model. Moreover, 3D-models are always needed to simulate Søderberg electrodes because the presence of fins breaks cylindrical symmetry (see Figure 3). Thus, we have developed a three dimensional thermoelectrical model which is enough general to model any kind of electrodes and even the complete furnace. In this paper, we describe two different mathematical models and analyze them from mathematical and numerical points of view.

The electromagnetic problem is obtained from the time-harmonic Maxwell equations assuming the frequency is low enough as to neglect the term involving the displacement current in Ampere's law. This is the so-called *eddy current* model. Because of many interesting applications in electrical engineering, numerical simulation of eddy current problems have led to a great number of publications in recent years (see for instance [1, 2, 3, 10, 11, 12, 13]). We notice that Maxwell equations concern the whole space, but we are interested in solving the problem in a bounded domain, so we have to define suitable boundary conditions and this need represents the main difficulty to study the problem in a bounded domain. Thus, we start introducing the eddy current problem in the whole furnace, including the electrodes and the air around, and defining natural and essential boundary conditions. In a second step we change this model, by introducing realistic boundary conditions, to compute the electromagnetic fields in only one electrode. Finally, we couple the electromagnetic model with a thermal one. Coupling between Maxwell and heat transfer equations is due to Joule effect which is the source term in the heat equation, and to the fact that thermoelectrical parameters depend on temperature.

The outline of the paper is as follows: In Section 2 we deal with the mathematical and numerical analysis of the electromagnetic problem in a bounded 3D domain which includes conductors and dielectrics. We introduce a weak formulation which involves the magnetic field in the conductor domain and a scalar magnetic potential in the dielectric one. This hybrid formulation is discretized by using Nédélec edge finite elements for the magnetic field and standard piecewise linear continuous elements for the magnetic potential. The resulting discrete problems are studied and error estimates are obtained under mild smoothness assumptions on the solution. Section 3 is devoted to propose and analyze a finite element method to solve the electromagnetic problem only in one electrode. We introduce a weak formulation of the problem in terms of the magnetic field and deal with boundary conditions directly related with the intensities which enter the domain. Lagrange multipliers are introduced to impose these "non standard" boundary conditions and the resulting mixed formulations are studied following classical techniques. In Section 4, we couple the electromagnetic problem with the thermal one and give a result concerning existence of solution. We end the paper by reporting, in Section 5, some numerical results obtained for ELSA and Søderberg electrodes.

2 The electromagnetic problem in the whole furnace

In order to consider all the facts which are neglected in the axisymmetric models, we start proposing a model to solve the eddy current problem in a bounded domain like the one presented in Figure 4, which includes not only *conductors* (the electrodes and wires supplying the electric current), but *dielectrics* as well (the air).

2.1 The eddy current problem

Eddy currents are usually modeled by the low-frequency harmonic Maxwell equations. Assuming alternating electric current of angular frequency ω, they are

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{1}$$
$$\mathrm{i}\omega\mu\mathbf{H} + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{2}$$
$$\operatorname{div}\mathbf{B} = 0, \tag{3}$$
$$\operatorname{div}\mathbf{D} = \rho, \tag{4}$$

with

$$\mathbf{B} = \mu\mathbf{H},\quad \mathbf{D} = \epsilon\mathbf{E},\qquad \mathbf{J} = \sigma\mathbf{E}, \tag{5}$$

where $\mathbf{H}$, $\mathbf{J}$, $\mathbf{B}$, $\mathbf{E}$, and $\mathbf{D}$ are the complex amplitudes associated with the magnetic field, the current density, the magnetic induction, the electric field and the electric displacement, respectively; ρ is the electric charge density, μ is the magnetic permeability, ϵ is the electric permittivity and σ is the electric conductivity.

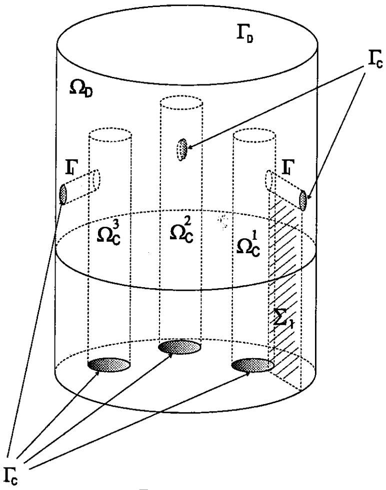

Fig. 4. Sketch of the furnace

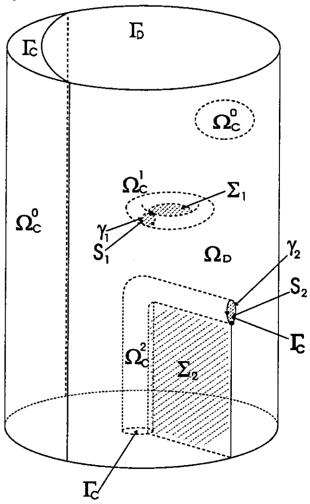

Fig. 5. Sketch of a general domain

We will solve these equations in a bounded domain Ω, which consists of two parts, Ω_C and Ω_D, occupied by conductors and dielectrics, respectively (see Figure 5). The boundary of the domain Ω also splits into two parts: $\Gamma_C := \partial\Omega_C \cap \partial\Omega$ and $\Gamma_D := \partial\Omega_D \cap \partial\Omega$. Finally, we denote by $\Gamma_I := \partial\Omega_C \cap \partial\Omega_D$, the interface between dielectric and conductors. The boundary conditions added to the eddy current model are

$$\mathbf{E} \times \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_C, \tag{6}$$

$$\mathbf{H} \times \mathbf{n} = \mathbf{f} \quad \text{on } \Gamma_D, \tag{7}$$

with $\mathbf{g}$ and $\mathbf{f}$ being given tangential vector fields (i.e., satisfying $\mathbf{g}\cdot\mathbf{n}=0$ on Γ_{C} and $\mathbf{f}\cdot\mathbf{n}=0$ on Γ_{D}) and $\mathbf{n}$ an outward unit normal vector to $\partial\Omega$.

We remark that (6) is the natural condition for the conducting part of the boundary, while (7) is imposed on the dielectric part and allows taking into account all of the electromagnetic effects outside the domain.

We will introduce and analyze a finite element method to solve this problem in domains of general topology. To attain this goal, we will consider a formulation introduced by Bossavit and Vérité [13], which involves the magnetic field in the conductor domain and a scalar magnetic potential in the dielectric one. Then, as a first step, we start analyzing a weak formulation of the problem in terms of the magnetic field.

2.2 A magnetic field formulation of the eddy current problem

Let us assume that Ω is simply connected, with a Lipschitz-continuous connected boundary. The subdomains Ω_{C} and Ω_{D} are also assumed to have Lipschitz-continuous boundaries, although not necessarily connected. Finally, the boundaries of Γ_{C}, Γ_{D}, and Γ_{I} are also assumed to have Lipschitz-continuous boundaries.

Let us consider the following closed subspaces of $\mathrm{H}(\mathbf{curl},\Omega)$,

$$\mathcal{V}=\{\mathbf{G}\in\mathrm{H}(\mathbf{curl},\Omega):\ \mathbf{curl}\,\mathbf{G}=\mathbf{0}\text{ in }\Omega_{\mathrm{D}}\},$$
$$\mathcal{V}^0=\left\{\mathbf{G}\in\mathcal{V}:\ \mathbf{G}\times\mathbf{n}=\mathbf{0}\text{ in }\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})^3\right\},$$

where $\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})^3$ denotes the dual space of $\mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{D}})^3$ which, in its turn, is the space of functions defined on Γ_{D} that extended by $\mathbf{0}$ on $\partial\Omega\setminus\Gamma_{\mathrm{D}}$ belong to $\mathrm{H}^{1/2}(\partial\Omega)^3$. We assume that $\mu,\epsilon,\sigma\in\mathrm{L}^\infty(\Omega)$, and that there exist constants, $\underline{\mu}$, $\underline{\epsilon}$, and $\underline{\sigma}$, such that

$$\mu(\mathbf{x})\geq\underline{\mu}>0,\qquad \epsilon(\mathbf{x})\geq\underline{\epsilon}>0,\quad \text{a.e. in }\Omega,$$
$$\sigma(\mathbf{x})\geq\underline{\sigma}>0,\quad \text{a.e. in }\Omega_{\mathrm{C}},\qquad \sigma(\mathbf{x})=0\quad\text{in }\Omega_{\mathrm{D}}.$$

We suppose that the boundary data $\mathbf{g}$ satisfies $\mathbf{g}\times\mathbf{n}\in\mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{C}})^3$. On the other hand, concerning the boundary data $\mathbf{f}$, we suppose there exists a field $\mathbf{H_f}\in\mathcal{V}$ such that

$$\mathbf{H_f}\times\mathbf{n}=\mathbf{f}\text{ in }\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})^3.$$

Then, multiplying the equation (2) by a test function of the space $\mathcal{V}^0$, integrating in Ω, and using Green's formula, (1), (6), and (7), we obtain the following weak formulation in terms of the magnetic field $\mathbf{H}$.

Problem MP.- *To find* $\mathbf{H}\in\mathcal{V}$ *such that*

$$\mathbf{H}\times\mathbf{n}=\mathbf{f}\ \textit{in}\ \mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})^3,\tag{8}$$
$$\mathrm{i}\omega\int_\Omega\mu\mathbf{H}\cdot\bar{\mathbf{G}}+\int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H}\cdot\mathbf{curl}\,\bar{\mathbf{G}}=\langle\mathbf{g}\times\mathbf{n},\mathbf{G}\times\mathbf{n}\rangle_{\Gamma_{\mathrm{C}}}\quad\forall\mathbf{G}\in\mathcal{V}^0.\tag{9}$$

Theorem 1. *If there exists* $\mathbf{H_f} \in \mathcal{V}$ *such that* $\mathbf{H_f} \times \mathbf{n} = \mathbf{f}$ *in* $\mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})^3$*, then problem* **MP** *has a unique solution.*

Once the magnetic field **H** is known, the current density **J** and the electric field **E** can be computed in conductors, namely, $\mathbf{J} = \mathbf{curl}\,\mathbf{H}$ and $\mathbf{E} = (\frac{1}{\sigma}\mathbf{J})|_{\Omega_{\mathrm{C}}}$. These are the magnitudes actually needed in most applications and satisfy the Maxwell equations (1)–(5) and boundary conditions (6)–(7) (see Theorem 3.2 in [7]).

2.3 Introducing a magnetic potential

In this section we show how problem **MP** can be transformed by replacing the magnetic field in the dielectric domain Ω_{D} by a (scalar) magnetic potential.

Let $\Omega_{\mathrm{C}} = \bigcup_{j=0}^{j=J} \Omega_{\mathrm{C}}^j$, with Ω_{C}^0 being the union of all the connected components of Ω_{C} such that $\Omega \setminus \Omega_{\mathrm{C}}^0$ is simply connected, and Ω_{C}^j, $j = 1, \ldots, J$, the remaining connected components of Ω_{C} (see Figure 5).

We assume that for each Ω_{C}^j, $j = 1, \ldots, J$, there exists an open "cut" surface $\Sigma_j \subset \Omega_{\mathrm{D}}$ such that $\partial\Sigma_j \subset \partial\Omega_{\mathrm{D}}$ and $\widetilde{\Omega}_{\mathrm{D}} := \Omega_{\mathrm{D}} \setminus \bigcup_{j=0}^{j=J} \overline{\Sigma}_j$ is pseudo-Lipschitz and simply connected (see Figure 5). We also assume that each of these surfaces Σ_j is connected, and $\overline{\Sigma}_j \cap \overline{\Sigma}_k = \emptyset$ for $j \neq k$ (see, for instance, [4]).

For any function $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$, we denote by $[\![\widetilde{\Psi}]\!]_{\Sigma_j}$ the jump of $\widetilde{\Psi}$ through Σ_j. The gradient of $\widetilde{\Psi}$ in $\mathcal{D}'(\widetilde{\Omega}_{\mathrm{D}})$ can be extended to $\mathrm{L}^2(\Omega_{\mathrm{D}})^3$ and will be denoted by $\widetilde{\mathbf{grad}}\,\widetilde{\Psi}$.

Let Θ be the linear space of $\mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ defined by

$$\Theta = \left\{\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \ [\![\widetilde{\Psi}]\!]_{\Sigma_j} = \text{constant}, \ j = 1, \ldots, J\right\}.$$

Then, for $\widetilde{\Psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$, we have that $\widetilde{\mathbf{grad}}\,\widetilde{\Psi} \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{D}})$ if and only if $\widetilde{\Psi} \in \Theta$, in which case $\mathbf{curl}\,(\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) = \mathbf{0}$ (see Lemma 3.11 in [4]). Then, for all $\mathbf{G} \in \mathcal{V}$ there exist $\widetilde{\Psi} \in \Theta$ such that $\mathbf{G}|_{\Omega_{\mathrm{D}}} = \widetilde{\mathbf{grad}}\,\widetilde{\Psi}$.

We introduce the following notation: for $\mathbf{G}_{\mathrm{C}} \in \mathrm{L}^2(\Omega_{\mathrm{C}})^3$ and $\mathbf{G}_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})^3$, we denote by $(\mathbf{G}_{\mathrm{C}}|\mathbf{G}_{\mathrm{D}})$ the field $\mathbf{G} \in \mathrm{L}^2(\Omega)^3$ defined a.e. by

$$\mathbf{G}(\mathbf{x}) := \begin{cases} \mathbf{G}_{\mathrm{C}}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_{\mathrm{C}}, \\ \mathbf{G}_{\mathrm{D}}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_{\mathrm{D}}. \end{cases}$$

Let us denote by $\mathcal{W}$ the linear space given by

$$\mathcal{W} := \left\{(\mathbf{G}, \widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{C}}) \times (\Theta/\mathbb{C}) : \ (\mathbf{G}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Psi}) \in \mathrm{H}(\mathbf{curl}, \Omega)\right\}.$$

Similarly, we define the closed subspace of $\mathcal{W}$

$$\mathcal{W}^0 := \left\{(\mathbf{G}, \widetilde{\Psi}) \in \mathcal{W} : \ \widetilde{\mathbf{grad}}\,\widetilde{\Psi} \times \mathbf{n} = \mathbf{0} \text{ in } \mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})^3\right\}.$$

By using this notation we can define the following problem:
Problem HP.- *To find* $(\mathbf{H},\widetilde{\Phi}) \in \mathcal{W}$ *such that*

$$\widetilde{\mathbf{grad}}\,\widetilde{\Phi} \times \mathbf{n} = \mathbf{f} \ \textit{in}\ \mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})^3,$$

$$\mathrm{i}\omega \int_{\Omega_{\mathrm{C}}} \mu \mathbf{H} \cdot \bar{\mathbf{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\,\mathbf{H} \cdot \mathbf{curl}\,\bar{\mathbf{G}} + \mathrm{i}\omega \int_{\Omega_{\mathrm{D}}} \mu\, \widetilde{\mathbf{grad}}\,\widetilde{\Phi} \cdot \widetilde{\mathbf{grad}}\,\bar{\widetilde{\Psi}} =$$
$$= \langle \mathbf{g} \times \mathbf{n}, \mathbf{G} \times \mathbf{n} \rangle_{\Gamma_{\mathrm{C}}} \qquad \forall (\mathbf{G},\widetilde{\Psi}) \in \mathcal{W}^0.$$

This is the well known magnetic field/magnetic potential hybrid formulation of the eddy current problem introduced by Bossavit and Vérité [13]. The main advantage with respect to formulation (8)–(9) lies in the fact that a vector field is replaced by a scalar one in the dielectric domain.

Theorem 2. *Under the assumptions of Theorem 1, problem* **HP** *has a unique solution* $(\mathbf{H},\widetilde{\Phi})$, *with* $(\mathbf{H}|\,\widetilde{\mathbf{grad}}\,\widetilde{\Phi})$ *being the unique solution of problem* **MP**.

2.4 Numerical solution

In this section we first introduce a discretization of problem **MP** and then we obtain a discrete version of problem **HP** equivalent to the previous one.

Let us assume Ω, Ω_{C}, and Ω_{D} are Lipschitz polyhedra, and consider a family of regular tetrahedral meshes $\{\mathcal{T}_h\}$ of Ω such that, for every mesh $\mathcal{T}_h$, each element $K \in \mathcal{T}_h$ is contained either in $\overline{\Omega}_{\mathrm{C}}$ or in $\overline{\Omega}_{\mathrm{D}}$ (h stands as usual for the corresponding mesh-size).

The magnetic field is discretized by using Nédélec edge finite elements (see [19]). In particular, $\mathbf{H}$ is approximated in each tetrahedron K by a polynomial vector field in the space

$$\mathcal{N}(K) := \left\{ \mathbf{G}_h \in \mathcal{P}_1(K)^3 : \ \mathbf{G}_h(\mathbf{x}) = \mathbf{a} \times \mathbf{x} + \mathbf{b},\ \mathbf{a},\mathbf{b} \in \mathbb{C}^3,\ \mathbf{x} \in K \right\}.$$

Then, fields in $\mathrm{H}(\mathbf{curl},\Omega)$ will be approximated in the following finite dimensional space:

$$\mathcal{N}_h(\Omega) := \{ \mathbf{G}_h \in \mathrm{H}(\mathbf{curl},\Omega) : \ \mathbf{G}_h|_K \in \mathcal{N}(K)\ \forall K \in \mathcal{T}_h \}.$$

In order to use these elements to discretize problem **MP**, we have to use an approximation $\mathbf{f}_{\mathrm{I}}$ of the boundary data $\mathbf{f}$ such that a discrete version of equation (8) can hold true. To attain this goal, we will use the two-dimensional Nédélec interpolant of $\mathbf{n} \times \mathbf{f}$ on the triangular mesh induced by $\mathcal{T}_h$ on the polyhedral surface Γ_{D}. This interpolant and several of its properties are described in detail in [7].

Then, in order to discretize problem **MP**, we introduce the following finite-dimensional spaces,

$$\mathcal{V}_h := \{\mathbf{G}_h \in \mathcal{N}_h(\Omega) : \ \mathbf{curl}\,\mathbf{G}_h = \mathbf{0} \ \text{ in } \Omega_{\mathrm{D}}\},$$
$$\mathcal{V}_h^0 := \{\mathbf{G}_h \in \mathcal{V}_h : \ \mathbf{G}_h \times \mathbf{n} = \mathbf{0} \ \text{ on } \Gamma_{\mathrm{D}}\},$$

and obtain the following discrete magnetic problem:

Problem DMP.- *Find* $\mathbf{H}_h \in \mathcal{V}_h$ *such that*

$$\mathbf{H}_h \times \mathbf{n} = \mathbf{f}_{\mathrm{I}} \quad \textit{on } \Gamma_{\mathrm{D}},$$
$$\mathrm{i}\omega \int_\Omega \mu \mathbf{H}_h \cdot \bar{\mathbf{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\,\mathbf{H}_h \cdot \mathbf{curl}\,\bar{\mathbf{G}}_h = \int_{\Gamma_{\mathrm{C}}} \mathbf{g} \times \mathbf{n} \cdot \bar{\mathbf{G}}_h \times \mathbf{n} \qquad \forall \mathbf{G}_h \in \mathcal{V}_h^0.$$

Theorem 3. *Let us assume that the solution* $\mathbf{H}$ *of problem* **MP** *satisfies* $\mathbf{H}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^r(\mathbf{curl}, \Omega_{\mathrm{C}})$ *and* $\mathbf{H}|_{\Omega_{\mathrm{D}}} \in \mathrm{H}^r(\Omega_{\mathrm{D}})^3$, *with* $r \in (\frac{1}{2}, 1]$. *Then,* $\mathbf{f}_{\mathrm{I}}$ *is well defined by the 2D Nédélec interpolant of* $\mathbf{n} \times \mathbf{f}$, *problem* **DMP** *has a unique solution* $\mathbf{H}_h$ *and the following error estimate holds:*

$$\|\mathbf{H} - \mathbf{H}_h\|_{\mathrm{H}(\mathbf{curl},\Omega)} \le C h^r \left[\|\mathbf{H}\|_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})} + \|\mathbf{H}\|_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right].$$

However, problem **DMP** is actually just a "theoretical" method in that its solution requires to impose somehow the curl-free condition in the definition of $\mathcal{V}_h$ to trial and test functions. Then, we will handle this curl-free condition by introducing a discrete multiple-valued magnetic potential in the dielectric domain.

We assume that the cut surfaces Σ_j are polyhedral and that the meshes are compatible with them, in the sense that each $\overline{\Sigma}_j$ is union of faces of tetrahedra $K \in \mathcal{T}_h$, for each mesh $\mathcal{T}_h$. Therefore, $\mathcal{T}_h^{\Omega_{\mathrm{D}}} := \{K \in \mathcal{T}_h : \ K \subset \overline{\Omega}_{\mathrm{D}}\}$ can also be seen as a mesh of $\widetilde{\Omega}_{\mathrm{D}}$.

In order to introduce an approximation of the space Θ, let us denote

$$\mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) := \left\{ \widetilde{\Psi}_h \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \ \widetilde{\Psi}_h|_K \in \mathcal{P}_1(K) \ \forall K \in \mathcal{T}_h^{\Omega_{\mathrm{D}}} \right\}.$$

Then, we consider the family of finite dimensional subspaces of Θ given by

$$\Theta_h := \{ \widetilde{\Psi}_h \in \mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) : \ [\![\widetilde{\Psi}_h]\!]_{\Sigma_j} = \text{constant}, \ j = 1, \dots, J \}.$$

The following lemma shows that each curl-free vector field in $\mathcal{N}_h(\Omega_{\mathrm{D}})$ admits a multiple-valued potential in Θ_h (see [7]).

Lemma 1. *Let* $\mathbf{G}_h \in \mathrm{L}^2(\Omega_{\mathrm{D}})^3$. *Then* $\mathbf{G}_h \in \mathcal{N}_h(\Omega_{\mathrm{D}})$ *with* $\mathbf{curl}\,\mathbf{G}_h = \mathbf{0}$ *in* Ω_{D} *if and only if there exists* $\widetilde{\Psi}_h \in \Theta_h$ *such that* $\mathbf{G}_h = \widetilde{\mathbf{grad}}\,\widetilde{\Psi}_h$ *in* Ω_{D}. *Such* $\widetilde{\Psi}_h$ *is unique up to an additive constant.*

Let us introduce the following families of finite-dimensional approximations of $\mathcal{W}$ and $\mathcal{W}^0$, respectively:

$$\mathcal{W}_h := \left\{ (\mathbf{G}_h, \widetilde{\Psi}_h) \in \mathcal{N}_h(\Omega_C) \times (\Theta_h/\mathbb{C}) : \ (\mathbf{G}_h | \, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h) \in \mathrm{H}(\mathbf{curl}, \Omega) \right\},$$

$$\mathcal{W}_h^0 := \left\{ (\mathbf{G}_h, \widetilde{\Psi}_h) \in \mathcal{W}_h : \ \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

Thus, we define the following discrete problem which is equivalent to problem **DMP**:

Problem DHP.- *To find* $(\mathbf{H}_h, \widetilde{\Phi}_h) \in \mathcal{W}_h$ *such that*

$$\widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h \times \mathbf{n} = \mathbf{f}_I \quad \text{on } \Gamma_D,$$

$$\mathrm{i}\omega \int_{\Omega_C} \mu \mathbf{H}_h \cdot \bar{\mathbf{G}}_h + \int_{\Omega_C} \frac{1}{\sigma} \mathbf{curl}\, \mathbf{H}_h \cdot \mathbf{curl}\, \bar{\mathbf{G}}_h + \mathrm{i}\omega \int_{\Omega_D} \mu\, \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h \cdot \widetilde{\mathbf{grad}}\, \bar{\widetilde{\Psi}}_h$$
$$= \int_{\Gamma_C} \mathbf{g} \times \mathbf{n} \cdot \bar{\mathbf{G}}_h \times \mathbf{n} \qquad \forall (\mathbf{G}_h, \widetilde{\Psi}_h) \in \mathcal{W}_h^0.$$

Theorem 4. *Let us assume that the solution* $(\mathbf{H}, \widetilde{\Phi})$ *of problem* **HP** *satisfies* $\mathbf{H} \in \mathrm{H}^r(\mathbf{curl}, \Omega_C)$ *and* $\widetilde{\mathbf{grad}}\, \widetilde{\Phi} \in \mathrm{H}^r(\Omega_D)^3$*, with* $r \in (\frac{1}{2}, 1]$*. Then, problem* **DHP** *is well posed, it has a unique solution* $(\mathbf{H}_h, \widetilde{\Phi}_h)$*, and*

$$\|\mathbf{H} - \mathbf{H}_h\|_{\mathrm{H}(\mathbf{curl}, \Omega_C)} + \| \widetilde{\mathbf{grad}}\, \widetilde{\Phi} - \widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h \|_{\mathrm{L}^2(\Omega_D)^3}$$
$$\leq C h^r \left[\|\mathbf{H}\|_{\mathrm{H}^r(\mathbf{curl}, \Omega_C)} + \| \widetilde{\mathbf{grad}}\, \widetilde{\Phi} \|_{\mathrm{H}^r(\Omega_D)^3} \right].$$

Effective procedures to solve numerically the problem **DMP** are described in [7]. In particular, numerical techniques to impose the following constraints are studied:

1. $(\mathbf{G}_h | \, \widetilde{\mathbf{grad}}\, \widetilde{\Psi}_h) \in \mathrm{H}(\mathbf{curl}, \Omega)$, which arise in the definition of $\mathcal{W}_h$.
2. $[\![\widetilde{\Psi}_h]\!]_{\Sigma_j} =$ constant, which arise in the definition of Θ_h.
3. The boundary condition $\widetilde{\mathbf{grad}}\, \widetilde{\Phi}_h \times \mathbf{n} = \mathbf{f}_I$ on Γ_D.

The first constraint is imposed by eliminating the degrees of freedom of $\mathbf{G}_h$ associated with the edges $\ell \in \Gamma_I$ in terms of those of $\widetilde{\Phi}_h$ corresponding to the vertices of the mesh on this interface.

The second constraint is handled by distinguishing the degrees of freedom of $\widetilde{\Psi}_h$ on one side of the surface Σ_j from those on the other side, and by eliminating ones of them in terms of the others and the current intensities through each conductor Ω_C^j.

The third constraint is imposed by means of a Lagrange multiplier, increasing in this way the number of unknowns but with the advantage that the computer implementation is quite straightforward.

We have developed a MATLAB code which implements the method described above. To validate the computer code and to test the performance and convergence properties of the method, we have solved a problem with known analytical solution (see [7] for further details).

3 The electromagnetic problem in one electrode

3.1 Statement of the problem

The model described in the previous section presents some drawbacks. First, it is highly complex and its numerical solution takes a lot of time. On the other hand, it is difficult to obtain the boundary data $\mathbf{f}$ from realistic data such as intensities or potentials, which usually are the only data we know. Then, we are going to propose an alternative approach which consists in solving the eddy current problem in one electrode which is a particular bounded conducting domain. We are going to analyze a weak formulation of this problem in terms of the magnetic field, considering realistic boundary conditions from the point of view of applications. In particular, following Bossavit [12], we will consider boundary conditions directly related with the input current intensities which enter the electrode. We will impose these boundary conditions by means of Lagrange multipliers and study the resulting mixed formulations.

Since we only consider the conducting domain, we will get an important saving in computer time when compared with the model of the whole furnace, and we will still be able to consider some important effects which are not taken into account by the axisymmetric models, although not the proximity effect.

We consider a bounded conducting domain Ω having a Lipschitz-continuous and connected boundary. However, it is not necessary that Ω be simply connected. Let $\partial\Omega$ be the boundary of the domain Ω which splits into two parts: $\partial\Omega = \bar{\Gamma}_{\mathrm{E}} \cup \bar{\Gamma}_{\mathrm{J}}$. The surface Γ_{E} corresponds to the tip of the electrode where the electric arc arises. In its turn, the rest of the electrode boundary splits as follows:

$$\bar{\Gamma}_{\mathrm{J}} = \bar{\Gamma}_{\mathrm{J}}^0 \cup \bar{\Gamma}_{\mathrm{J}}^1 \cup \cdots \cup \bar{\Gamma}_{\mathrm{J}}^N,$$

where Γ_{J}^n, $n = 1, \ldots, N$, are the parts of the boundary connected to the wires supplying electric current to the electrode, and $\Gamma_{\mathrm{J}}^0 = \Gamma_{\mathrm{J}} \setminus (\bar{\Gamma}_{\mathrm{J}}^1 \cup \cdots \cup \bar{\Gamma}_{\mathrm{J}}^N)$ is the remaining part (see Figure 6). We also assume $\bar{\Gamma}_{\mathrm{J}}^n \cap \bar{\Gamma}_{\mathrm{E}} = \emptyset$ and $\bar{\Gamma}_{\mathrm{J}}^n \cap \bar{\Gamma}_{\mathrm{J}}^m = \emptyset$, $m, n = 1, \ldots, N$, $m \neq n$.

Our goal is to solve the eddy current equations (1)–(5) subject to the following boundary conditions:

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{E}}, \tag{10}$$

$$\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\mathbf{H} \cdot \mathbf{n} = I_n, \quad n = 1, \ldots, N, \tag{11}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{J}}^n, \quad n = 1, \ldots, N, \tag{12}$$

$$\mathbf{curl}\,\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\mathrm{J}}^0, \tag{13}$$

$$\mu\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{14}$$

where the only data I_n, $n = 1, \ldots, N$, are the current intensities through each wire.

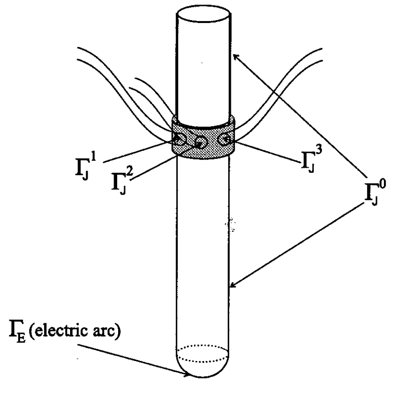

Fig. 6. Example of domain

Condition (10) is the natural one to model the free current exit on the electrode tip. Conditions (11) and (13) take into account the input intensities and the fact that there is no current flow through Γ_J^0, respectively. Conditions (12) and (14) have been proposed by Bossavit [12] in a more general setting. They will appear as natural boundary conditions of the weak formulation of our problem. The former implies the assumption that the electric current is normal to the surface on the current entrance, whereas the latter means that the magnetic field is tangential to the conductor surface. Of course, condition (14) is not always fulfilled, but it is a good approximation in our model problem.

Next, we analyze a weak formulation of this problem in terms of the magnetic field and propose a finite element method for its numerical solution.

3.2 Analysis of the weak formulation of the problem

To obtain a weak formulation of the eddy current problem (1)–(5) with boundary conditions (10)–(14) in terms of the magnetic field, we notice that the boundary condition (14) implies that the tangential component of $\mathbf{E}$ on the boundary of Ω is a gradient. In particular, we obtain that $\mathbf{E} \times \mathbf{n} = -\nabla\phi \times \mathbf{n}$ on $\partial\Omega$ for some scalar function ϕ with $\phi = 0$ on Γ_E, because of (10). Moreover, because of (12), $\phi|_{\Gamma_J^n}$ must be constant. Then, multiplying the equation (2) by a test function $\mathbf{G}$ such that $\mathbf{curl}\,\mathbf{G} \cdot \mathbf{n} = 0$ on Γ_J^0 and $\int_{\Gamma_J^n} \mathbf{curl}\,\mathbf{G} \cdot \mathbf{n} = 0$, $n = 1, \dots, N$, using Green's formula, and taking into account that $\mathbf{E} = \frac{1}{\sigma}\mathbf{curl}\,\mathbf{H}$, we obtain

$$\mathrm{i}\omega \int_\Omega \mu \mathbf{H} \cdot \bar{\mathbf{G}} + \int_\Omega \frac{1}{\sigma} \mathbf{curl}\,\mathbf{H} \cdot \mathbf{curl}\,\bar{\mathbf{G}} = 0.$$

Let $\boldsymbol{\mathcal{X}} := \mathrm{H}(\mathbf{curl}, \Omega)$ and $a\colon \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}} \longrightarrow \mathbb{C}$ be the sesquilinear continuous and elliptic form defined by

$$a(\mathbf{H},\mathbf{G}) := \mathrm{i}\omega \int_{\Omega} \mu \mathbf{H} \cdot \bar{\mathbf{G}} + \int_{\Omega} \frac{1}{\sigma} \mathbf{curl}\, \mathbf{H} \cdot \mathbf{curl}\, \bar{\mathbf{G}}.$$

Let $\mathcal{L}$ be the following closed subspace of $\mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{J}})$:

$$\mathcal{L} := \left\{ \nu \in \mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{J}}) : \ \nu|_{\Gamma_{\mathrm{J}}^{n}} = \text{constant}, \ n = 1, \dots, N \right\}.$$

Given $\mathbf{I} = (I_1, \dots, I_N) \in \mathbb{C}^N$, let us consider the closed linear manifold of $\boldsymbol{\mathcal{X}}$,

$$\boldsymbol{\mathcal{W}}(\mathbf{I}) := \left\{ \mathbf{G} \in \boldsymbol{\mathcal{X}} : \ \langle \mathbf{curl}\, \mathbf{G} \cdot \mathbf{n}, \nu \rangle_{\Gamma_{\mathrm{J}}} = \sum_{n=1}^{N} \int_{\Gamma_{\mathrm{J}}^{n}} I_n \, \bar{\nu} \quad \forall \nu \in \mathcal{L} \right\},$$

and its associated subspace

$$\boldsymbol{\mathcal{W}}(\mathbf{0}) = \left\{ \mathbf{G} \in \boldsymbol{\mathcal{X}} : \ \langle \mathbf{curl}\, \mathbf{G} \cdot \mathbf{n}, \nu \rangle_{\Gamma_{\mathrm{J}}} = 0 \quad \forall \nu \in \mathcal{L} \right\}.$$

We introduce the following problem:
Problem PI.- *For any* $\mathbf{I} \in \mathbb{C}^N$, *find* $\mathbf{H} \in \boldsymbol{\mathcal{W}}(\mathbf{I})$ *such that*

$$a(\mathbf{H},\mathbf{G}) = 0 \qquad \forall \mathbf{G} \in \boldsymbol{\mathcal{W}}(\mathbf{0}).$$

Theorem 5. *Given* $\mathbf{I} \in \mathbb{C}^N$, *problem* **PI** *has a unique solution* $\mathbf{H}$.

To avoid dealing with functions that satisfy the constraints involved in $\boldsymbol{\mathcal{W}}(\mathbf{I})$ and $\boldsymbol{\mathcal{W}}(\mathbf{0})$, we consider a mixed formulation of the problem. It consists in handling the boundary conditions (11) and (13) in a weak sense by introducing a Lagrange multiplier defined on Γ_{J}.

Let b be the sesquilinear form defined in $\boldsymbol{\mathcal{X}} \times \mathcal{L}$ by

$$b(\mathbf{G},\nu) := \langle \mathbf{curl}\, \mathbf{G} \cdot \mathbf{n}, \nu \rangle_{\Gamma_{\mathrm{J}}}.$$

The mixed problem associated with problem **PI** is the following:
Problem MPI.- *Given* $\mathbf{I} \in \mathbb{C}^N$, *find* $\mathbf{H} \in \boldsymbol{\mathcal{X}}$ *and* $\lambda \in \mathcal{L}$ *such that*

$$a(\mathbf{H},\mathbf{G}) + b(\bar{\mathbf{G}},\lambda) = 0 \qquad \forall \mathbf{G} \in \boldsymbol{\mathcal{X}},$$

$$b(\mathbf{H},\nu) = \sum_{n=1}^{N} \int_{\Gamma_{\mathrm{J}}^{n}} I_n \, \bar{\nu} \qquad \forall \nu \in \mathcal{L}.$$

Theorem 6. *Given* $I \in \mathbb{C}^N$, *let* $\mathbf{H} \in \boldsymbol{\mathcal{X}}$ *be the solution of problem* **PI**. *Then, there exists a unique* $\lambda \in \mathcal{L}$ *such that* $(\mathbf{H},\lambda)$ *is the only solution of problem* **MPI**. *Furthermore, the following estimate holds:*

$$\|\mathbf{H}\|_{\boldsymbol{\mathcal{X}}} + \|\lambda\|_{\mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{J}})} \leq C \, |\mathbf{I}| \, .$$

The proof is based on the classical Babuška-Brezzi theory. In particular we prove the inf-sup condition for the bilinear form b by using results concerning vector potentials in $\mathbb{R}^3$ (see [9]).

Theorem 3.5 in [9] shows that the solution of problem **MPI**, together with $\mathbf{E} = \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H}$ and $\mathbf{J} = \mathbf{curl}\,\mathbf{H}$, satisfy the Maxwell equations (1)–(5) and the boundary conditions (10)–(14) in a suitable weak sense. Moreover, from that theorem, we also have that the Lagrange multiplier is an *electric surface potential* on Γ_{J}, namely,

$$\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = -\mathbf{n} \times (\nabla \bar{\lambda}^* \times \mathbf{n}) =: -\,\mathbf{grad}_{\Gamma} \bar{\lambda}^* \quad \text{on } \Gamma_{\mathrm{J}},$$

λ^* being a lifting of λ to Ω such that $\lambda^* \in \mathrm{H}^1(\Omega)$ and $\lambda^*|_{\Gamma_{\mathrm{E}}} = 0$.

3.3 Finite element discretization

In this section we introduce a discretization of the mixed problem **MPI** and study its convergence properties. To this goal, we assume that Ω is a Lipschitz polyhedron and that Γ_{J}^n are polyhedral surfaces for all $n = 0, \dots, N$. Consequently, Γ_{E} is also a polyhedral surface. We also assume that σ is piecewise smooth (e.g., $\mathcal{C}^2$) on a polyhedral partition of Ω.

We consider a family of shape-regular tetrahedral meshes $\{\mathcal{T}_h\}$ of Ω. We assume that the meshes are compatible with the splitting of the boundary of the domain in the sense that, $\forall K \in \mathcal{T}_h$ with a face T lying on $\partial\Omega$,

- either $T \subset \bar{\Gamma}_{\mathrm{E}}$ or $T \subset \bar{\Gamma}_{\mathrm{J}}^n$ for some $n = 0, \dots, N$;
- $\sigma|_T$ is smooth.

The magnetic field, which is a function of $\boldsymbol{\mathcal{X}} = \mathrm{H}(\mathbf{curl}, \Omega)$, is discretized by the lowest-order Nédélec edge finite elements described in Section 2, i.e. we define $\boldsymbol{\mathcal{X}}_h = \boldsymbol{\mathcal{N}}_h(\Omega)$ as an approximation of $\boldsymbol{\mathcal{X}}$.

Let $\mathcal{T}_h^{\Gamma_{\mathrm{J}}}$ be the triangular mesh induced by $\mathcal{T}_h$ on the polyhedral surface Γ_{J} and consider the following finite-dimensional space:

$$\mathcal{Q}_h^1(\Gamma_{\mathrm{J}}) := \left\{ q_h \in \mathrm{H}_0^1(\Gamma_{\mathrm{J}}) : \ q_h|_T \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}_h^{\Gamma_{\mathrm{J}}} \right\}.$$

The Lagrange multiplier will be approximated in the finite dimensional space

$$\mathcal{L}_h := \left\{ \nu_h \in \mathcal{Q}_h^1(\Gamma_{\mathrm{J}}) : \ \nu_h|_{\Gamma_{\mathrm{J}}^n} = \text{constant}, \ n = 1, \dots, N \right\}.$$

We define the following discrete problem

Problem DMPI.- *Given* $\mathbf{I} \in \mathbb{C}^N$, *find* $\mathbf{H}_h \in \boldsymbol{\mathcal{X}}_h$ *and* $\lambda_h \in \mathcal{L}_h$ *such that*

$$a(\mathbf{H}_h, \mathbf{G}_h) + b(\bar{\mathbf{G}}_h, \lambda_h) = 0 \qquad \forall \mathbf{G}_h \in \boldsymbol{\mathcal{X}}_h,$$

$$b(\mathbf{H}_h, \nu_h) = \sum_{n=1}^{N} \int_{\Gamma_{\mathrm{J}}^n} I_n \, \bar{\nu}_h \qquad \forall \nu_h \in \mathcal{L}_h.$$

Theorem 7. *Given* $\mathbf{I} \in \mathbb{C}^N$, *problem* **DMPI** *attains a unique solution* $(\mathbf{H}_h, \lambda_h)$. *Furthermore, if the solution* $(\mathbf{H}, \lambda)$ *of problem* **MPI** *satisfies* $\mathbf{H} \in \mathrm{H}^r(\mathbf{curl}, \Omega)$ *with* $1/2 < r \leq 1$, *then the following error estimate holds true:*

$$\|\mathbf{H} - \mathbf{H}_h\|_{\boldsymbol{\mathcal{X}}} \leq Ch^r \|\mathbf{H}\|_{\mathrm{H}^r(\mathbf{curl},\Omega)}.$$

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