Chapter 2

Equations. Functions of one variable. Complex numbers

2.1 $ax^2 + bx + c = 0 \iff x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The roots of the general *quadratic* equation. They are real provided $b^2 \geq 4ac$ (assuming that a, b, and c are real).

2.2 If x_1 and x_2 are the roots of $x^2 + px + q = 0$, then

$$x_1 + x_2 = -p, \qquad x_1 x_2 = q$$

Viète's rule.

2.3 $ax^3 + bx^2 + cx + d = 0$

The general *cubic* equation.

2.4 $x^3 + px + q = 0$

(2.3) reduces to the form (2.4) if x in (2.3) is replaced by $x - b/3a$.

2.5 $x^3 + px + q = 0$ with $\Delta = 4p^3 + 27q^2$ has

- three different real roots if $\Delta < 0$;
- three real roots, at least two of which are equal, if $\Delta = 0$;
- one real and two complex roots if $\Delta > 0$.

Classification of the roots of (2.4) (assuming that p and q are real).

2.6 The solutions of $x^3 + px + q = 0$ are $x_1 = u + v$, $x_2 = \omega u + \omega^2 v$, and $x_3 = \omega^2 u + \omega v$, where $\omega = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$, and

$$u = \sqrt[3]{-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{4p^3 + 27q^2}{27}}}$$

$$v = \sqrt[3]{-\frac{q}{2} - \frac{1}{2}\sqrt{\frac{4p^3 + 27q^2}{27}}}$$

Cardano's formulas for the roots of a cubic equation. i is the imaginary unit (see (2.75)) and ω is a complex third root of 1 (see (2.88)). (If complex numbers become involved, the cube roots must be chosen so that $3uv = -p$. Don't try to use these formulas unless you have to!)

2.7 If x_1, x_2, and x_3 are the roots of the equation $x^3 + px^2 + qx + r = 0$, then

$$x_1 + x_2 + x_3 = -p$$
$$x_1x_2 + x_1x_3 + x_2x_3 = q$$
$$x_1x_2x_3 = -r$$

Useful relations.

2.8 $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$

A *polynomial* of degree n. ($a_n \neq 0$.)

2.9 For the polynomial $P(x)$ in (2.8) there exist constants $x_1, x_2, \ldots, x_n$ (real or complex) such that

$$P(x) = a_n(x - x_1)\cdots(x - x_n)$$

The *fundamental theorem of algebra.* $x_1, \ldots, x_n$ are called *zeros* of $P(x)$ and *roots* of $P(x) = 0$.

2.10
$$x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n}$$
$$x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n = \sum_{i<j} x_ix_j = \frac{a_{n-2}}{a_n}$$
$$x_1x_2\cdots x_n = (-1)^n\frac{a_0}{a_n}$$

Relations between the roots and the coefficients of $P(x) = 0$, where $P(x)$ is defined in (2.8). (Generalizes (2.2) and (2.7).)

2.11 If $a_{n-1}, \ldots, a_1, a_0$ are all integers, then any integer root of the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

must divide a_0.

Any integer solutions of $x^3 + 6x^2 - x - 6 = 0$ must divide -6. (In this case the roots are ± 1 and -6.)

2.12 Let k be the number of changes of sign in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ in (2.8). The number of positive real roots of $P(x) = 0$, counting the multiplicities of the roots, is k or k minus a positive even number. If $k = 1$, the equation has exactly one positive real root.

Descartes's rule of signs.

2.13 The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is

- an ellipse, a point or empty if $4AC > B^2$;
- a parabola, a line, two parallel lines, or empty if $4AC = B^2$;
- a hyperbola or two intersecting lines if $4AC < B^2$.

Classification of *conics*. A, B, C not all 0.

2.14 $x = x' \cos\theta - y' \sin\theta, \quad y = x' \sin\theta + y' \cos\theta$ with $\cot 2\theta = (A - C)/B$

Transforms the equation in (2.13) into a quadratic equation in x' and y', where the coefficient of $x'y'$ is 0.

2.15 $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The (Euclidean) *distance* between the points (x_1, y_1) and (x_2, y_2).

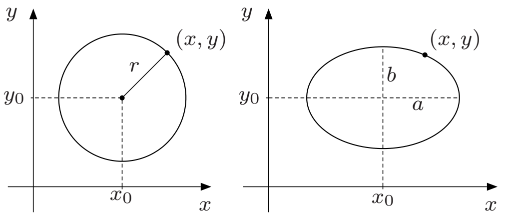
2.16 $(x - x_0)^2 + (y - y_0)^2 = r^2$

Circle with center at (x_0, y_0) and radius r.

2.17 $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} = 1$

Ellipse with center at (x_0, y_0) and axes parallel to the coordinate axes.

2.18



Graphs of (2.16) and (2.17).

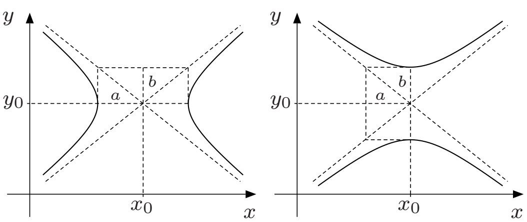
2.19 $\dfrac{(x - x_0)^2}{a^2} - \dfrac{(y - y_0)^2}{b^2} = \pm 1$

Hyperbola with center at (x_0, y_0) and axes parallel to the coordinate axes.

2.20 Asymptotes for (2.19): $y - y_0 = \pm \dfrac{b}{a}(x - x_0)$

Formulas for asymptotes of the hyperbolas in (2.19).

2.21



Hyperbolas with asymptotes, illustrating (2.19) and (2.20), corresponding to + and − in (2.19), respectively. The two hyperbolas have the same asymptotes.

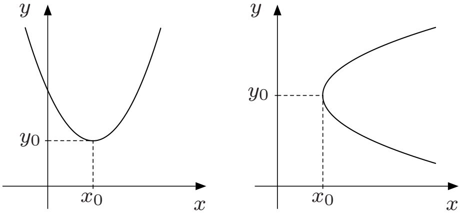
2.22 $y - y_0 = a(x - x_0)^2, \quad a \neq 0$

Parabola with vertex (x_0, y_0) and axis parallel to the y-axis.

2.23 $x - x_0 = a(y - y_0)^2, \quad a \neq 0$

Parabola with vertex (x_0, y_0) and axis parallel to the x-axis.

2.24



Parabolas illustrating (2.22) and (2.23) with $a > 0$.

2.25

A function f is

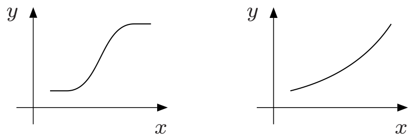
- *increasing* if
  $$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$$
- *strictly increasing* if
  $$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$
- *decreasing* if
  $$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$$
- *strictly decreasing* if
  $$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$
- *even* if $f(x) = f(-x)$ for all x
- *odd* if $f(x) = -f(-x)$ for all x
- *symmetric about the line* $x = a$ if
  $$f(a + x) = f(a - x) \text{ for all } x$$
- *symmetric about the point* $(a, 0)$ if
  $$f(a - x) = -f(a + x) \text{ for all } x$$
- *periodic* (with period k) if there exists a number $k > 0$ such that
  $$f(x + k) = f(x) \text{ for all } x$$

Properties of functions.

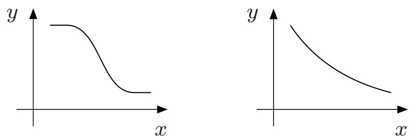
2.26

- If $y = f(x)$ is replaced by $y = f(x) + c$, the graph is moved upwards by c units if $c > 0$ (downwards if c is negative).
- If $y = f(x)$ is replaced by $y = f(x + c)$, the graph is moved c units to the left if $c > 0$ (to the right if c is negative).
- If $y = f(x)$ is replaced by $y = cf(x)$, the graph is stretched vertically if $c > 0$ (stretched vertically and reflected about the x-axis if c is negative).
- If $y = f(x)$ is replaced by $y = f(-x)$, the graph is reflected about the y-axis.

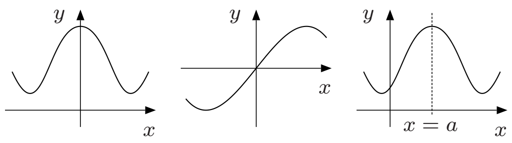
Shifting the graph of $y = f(x)$.

2.27 

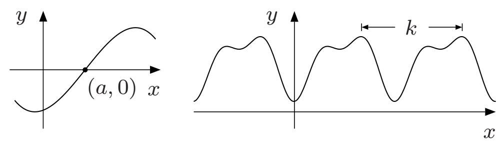
Graphs of increasing and strictly increasing functions.

2.28 

Graphs of decreasing and strictly decreasing functions.

2.29 

Graphs of even and odd functions, and of a function symmetric about $x = a$.

2.30 

Graphs of a function symmetric about the point $(a, 0)$ and of a function periodic with period k.

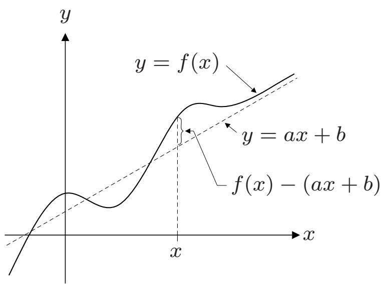
2.31 $y = ax + b$ is a *nonvertical asymptote* for the curve $y = f(x)$ if

$$\lim_{x\to\infty}\big(f(x) - (ax+b)\big) = 0$$

or

$$\lim_{x\to-\infty}\big(f(x) - (ax+b)\big) = 0$$

Definition of a nonvertical asymptote.

2.32 

$y = ax + b$ is an asymptote for the curve $y = f(x)$.

2.33 How to find a nonvertical asymptote for the curve $y = f(x)$ as $x \to \infty$:

- Examine $\lim_{x\to\infty} \big(f(x)/x\big)$. If the limit does not exist, there is no asymptote as $x \to \infty$.
- If $\lim_{x\to\infty} \big(f(x)/x\big) = a$, examine the limit $\lim_{x\to\infty} \big(f(x) - ax\big)$. If this limit does not exist, the curve has no asymptote as $x \to \infty$.
- If $\lim_{x\to\infty} \big(f(x) - ax\big) = b$, then $y = ax + b$ is an *asymptote* for the curve $y = f(x)$ as $x \to \infty$.

Method for finding nonvertical asymptotes for a curve $y = f(x)$ as $x \to \infty$. Replacing $x \to \infty$ by $x \to -\infty$ gives a method for finding nonvertical asymptotes as $x \to -\infty$.

To find an approximate root of $f(x) = 0$, define x_n for $n = 1, 2, \ldots$, by

2.34 $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If x_0 is close to an actual root x^*, the sequence $\{x_n\}$ will usually converge rapidly to that root.

Newton's approximation method. (A rule of thumb says that, to obtain an approximation that is correct to n decimal places, use Newton's method until it gives the same n decimal places twice in a row.)

2.35

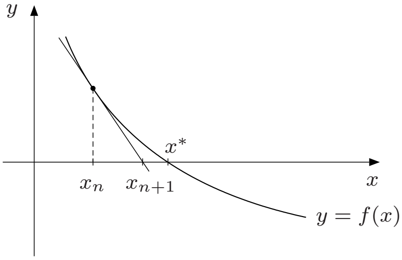


Illustration of Newton's approximation method. The tangent to the graph of f at $(x_n, f(x_n))$ intersects the x-axis at $x = x_{n+1}$.

2.36 Suppose in (2.34) that $f(x^*) = 0$, $f'(x^*) \neq 0$, and that $f''(x^*)$ exists and is continuous in a neighbourhood of x^*. Then there exists a $\delta > 0$ such that the sequence $\{x_n\}$ in (2.34) converges to x^* when $x_0 \in (x^* - \delta, x^* + \delta)$.

Sufficient conditions for convergence of Newton's method.

Suppose in (2.34) that f is twice differentiable with $f(x^*) = 0$ and $f'(x^*) \neq 0$. Suppose further that there exist a $K > 0$ and a $\delta > 0$ such that for all x in $(x^* - \delta, x^* + \delta)$,

2.37 $$\frac{|f(x)f''(x)|}{f'(x)^2} \leq K|x - x^*| < 1$$

Then if $x_0 \in (x^* - \delta, x^* + \delta)$, the sequence $\{x_n\}$ in (2.34) converges to x^* and

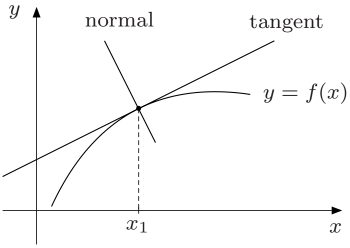
$$|x_n - x^*| \leq (\delta K)^{2^n}/K$$

A precise estimation of the accuracy of Newton's method.

2.38 $y - f(x_1) = f'(x_1)(x - x_1)$ — The equation for the *tangent* to $y = f(x)$ at $(x_1, f(x_1))$.

2.39 $y - f(x_1) = -\dfrac{1}{f'(x_1)}(x - x_1)$ — The equation for the *normal* to $y = f(x)$ at $(x_1, f(x_1))$.

2.40



The tangent and the normal to $y = f(x)$ at $(x_1, f(x_1))$.

2.41

(i) $a^r \cdot a^s = a^{r+s}$ (ii) $(a^r)^s = a^{rs}$

(iii) $(ab)^r = a^r b^r$ (iv) $a^r / a^s = a^{r-s}$

(v) $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$ (vi) $a^{-r} = \dfrac{1}{a^r}$

Rules for powers. (r and s are arbitrary real numbers, a and b are positive real numbers.)

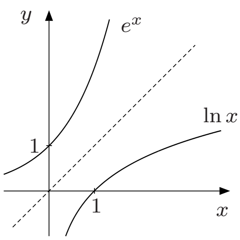
2.42

- $e = \lim_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n = 2.718281828459\ldots$
- $e^x = \lim_{n\to\infty}\left(1 + \dfrac{x}{n}\right)^n$
- $\lim_{n\to\infty} a_n = a \;\Rightarrow\; \lim_{n\to\infty}\left(1 + \dfrac{a_n}{n}\right)^n = e^a$

Important definitions and results. See (8.23) for another formula for e^x.

2.43 $e^{\ln x} = x$ — Definition of the natural logarithm.

2.44



The graphs of $y = e^x$ and $y = \ln x$ are symmetric about the line $y = x$.

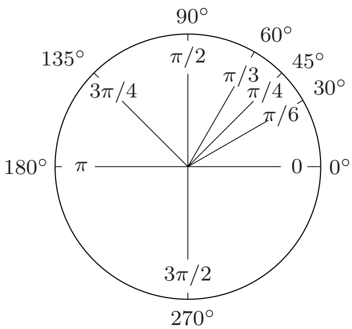
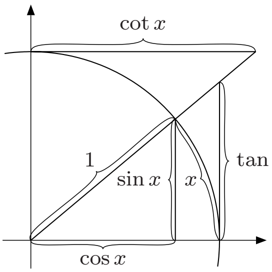
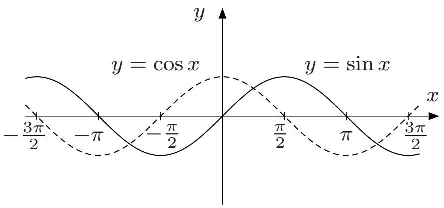
2.45

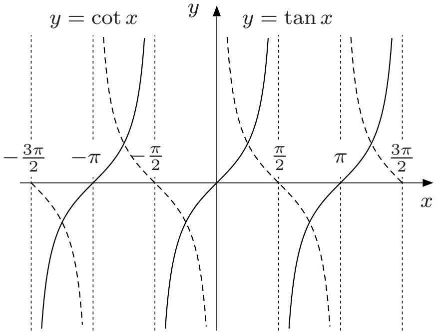
$\ln(xy) = \ln x + \ln y; \quad \ln\dfrac{x}{y} = \ln x - \ln y$

$\ln x^p = p\ln x; \quad \ln\dfrac{1}{x} = -\ln x$

Rules for the natural logarithm function. (x and y are positive.)

2.46 $a^{\log_a x} = x \;\; (a > 0,\; a \neq 1)$ — Definition of the *logarithm* to the base a.

2.47	$\log_a x = \dfrac{\ln x}{\ln a}$; $\log_a b \cdot \log_b a = 1$ $\log_e x = \ln x$; $\log_{10} x = \log_{10} e \cdot \ln x$	Logarithms with different bases.
2.48	$\log_a(xy) = \log_a x + \log_a y$ $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ $\log_a x^p = p \log_a x$, $\log_a \dfrac{1}{x} = -\log_a x$	Rules for logarithms. (x and y are positive.)
2.49	$1° = \dfrac{\pi}{180}$ rad, $1 \text{ rad} = \left(\dfrac{180}{\pi}\right)^{\circ}$	Relationship between degrees and radians (rad).
2.50		Relations between degrees and radians.
2.51		Definitions of the basic *trigonometric* functions. x is the length of the arc, and also the radian measure of the angle.
2.52		The graphs of $y = \sin x$ (—) and $y = \cos x$ (- - -). The functions sin and cos are periodic with period 2π: $\sin(x + 2\pi) = \sin x$, $\cos(x + 2\pi) = \cos x$.
2.53	$\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$	Definition of the *tangent* and *cotangent* functions.

2.54 

The graphs of $y = \tan x$ (—) and $y = \cot x$ (- - -). The functions tan and cot are periodic with period π: $\tan(x + \pi) = \tan x$, $\cot(x + \pi) = \cot x$.

2.55

x	0	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
$\cos x$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	$*$
$\cot x$	$*$	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0

* *not defined*

Special values of the trigonometric functions.

2.56

x	$\frac{3\pi}{4} = 135°$	$\pi = 180°$	$\frac{3\pi}{2} = 270°$	$2\pi = 360°$
$\sin x$	$\frac{1}{2}\sqrt{2}$	0	-1	0
$\cos x$	$-\frac{1}{2}\sqrt{2}$	-1	0	1
$\tan x$	-1	0	$*$	0
$\cot x$	-1	$*$	0	$*$

* *not defined*

2.57 $$\lim_{x \to 0} \frac{\sin ax}{x} = a$$

An important limit.

2.58 $$\sin^2 x + \cos^2 x = 1$$

Trigonometric formulas. (For series expansions of trigonometric functions, see Chapter 8.)

2.59 $$\tan^2 x = \frac{1}{\cos^2 x} - 1, \qquad \cot^2 x = \frac{1}{\sin^2 x} - 1$$

2.60
$$\begin{aligned}
\cos(x + y) &= \cos x \cos y - \sin x \sin y \\
\cos(x - y) &= \cos x \cos y + \sin x \sin y \\
\sin(x + y) &= \sin x \cos y + \cos x \sin y \\
\sin(x - y) &= \sin x \cos y - \cos x \sin y
\end{aligned}$$

2.61
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$
Trigonometric formulas.

2.62
$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin 2x = 2 \sin x \cos x$$

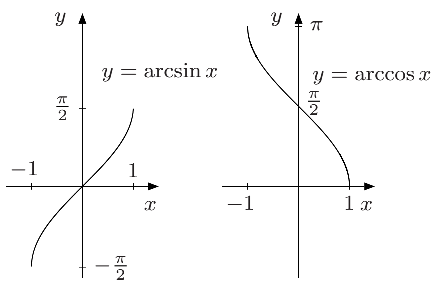
2.63
$$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}, \qquad \cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$$

2.64
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

2.65
$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

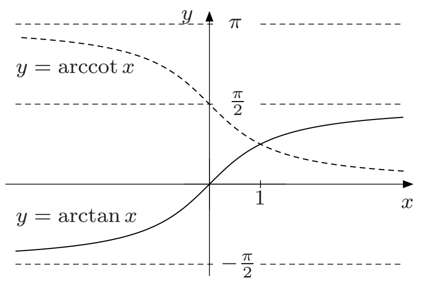
2.66
$$y = \arcsin x \Leftrightarrow x = \sin y,\ x \in [-1,1],\ y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$
$$y = \arccos x \Leftrightarrow x = \cos y,\ x \in [-1,1],\ y \in [0,\pi]$$
$$y = \arctan x \Leftrightarrow x = \tan y,\ x \in \mathbb{R},\ y \in (-\frac{\pi}{2}, \frac{\pi}{2})$$
$$y = \operatorname{arccot} x \Leftrightarrow x = \cot y,\ x \in \mathbb{R},\ y \in (0,\pi)$$
Definitions of the inverse trigonometric functions.

2.67

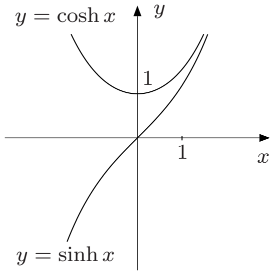


Graphs of the inverse trigonometric functions $y = \arcsin x$ and $y = \arccos x$.

2.68



Graphs of the inverse trigonometric functions $y = \arctan x$ and $y = \operatorname{arccot} x$.

2.69	$\arcsin x = \sin^{-1} x, \quad \arccos x = \cos^{-1} x$ $\arctan x = \tan^{-1} x, \quad \operatorname{arccot} x = \cot^{-1} x$	Alternative notation for the inverse trigonometric functions.
2.70	$\arcsin(-x) = -\arcsin x$ $\arccos(-x) = \pi - \arccos x$ $\arctan(-x) = \arctan x$ $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$ $\arcsin x + \arccos x = \frac{\pi}{2}$ $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$ $\arctan \frac{1}{x} = \frac{\pi}{2} - \arctan x, \quad x > 0$ $\arctan \frac{1}{x} = -\frac{\pi}{2} - \arctan x, \quad x < 0$	Properties of the inverse trigonometric functions.
2.71	$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$	*Hyperbolic* sine and cosine.
2.72		Graphs of the hyperbolic functions $y = \sinh x$ and $y = \cosh x$.
2.73	$\cosh^2 x - \sinh^2 x = 1$ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$ $\cosh 2x = \cosh^2 x + \sinh^2 x$ $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ $\sinh 2x = 2 \sinh x \cosh x$	Properties of hyperbolic functions.
2.74	$y = \operatorname{arsinh} x \iff x = \sinh y$ $y = \operatorname{arcosh} x,\ x \geq 1 \iff x = \cosh y,\ y \geq 0$ $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$ $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right),\ x \geq 1$	Definition of the inverse hyperbolic functions.

Complex numbers

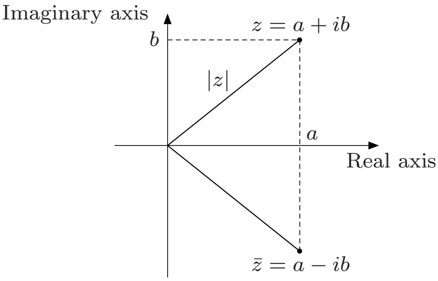
2.75 $z = a + ib, \quad \bar{z} = a - ib$

A *complex number* and its *conjugate*. $a, b \in \mathbb{R}$, and $i^2 = -1$. i is called the *imaginary unit*.

2.76 $|z| = \sqrt{a^2 + b^2}, \quad \mathrm{Re}(z) = a, \quad \mathrm{Im}(z) = b$

$|z|$ is the *modulus* of $z = a + ib$. $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the *real* and *imaginary parts* of z.

2.77



Geometric representation of a complex number and its conjugate.

2.78

- $(a + ib) + (c + id) = (a + c) + i(b + d)$
- $(a + ib) - (c + id) = (a - c) + i(b - d)$
- $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$
- $\dfrac{a + ib}{c + id} = \dfrac{1}{c^2 + d^2}\big((ac + bd) + i(bc - ad)\big)$

Addition, subtraction, multiplication, and *division* of complex numbers.

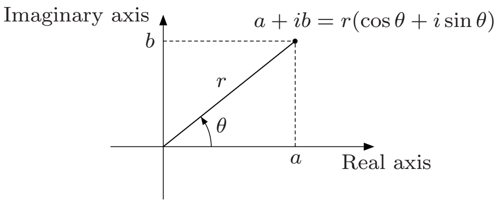
2.79 $|\bar{z}_1| = |z_1|, \; z_1\bar{z}_1 = |z_1|^2, \; \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,$
$|z_1 z_2| = |z_1||z_2|, \; |z_1 + z_2| \le |z_1| + |z_2|$

Basic rules. z_1 and z_2 are complex numbers.

2.80 $z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where
$r = |z| = \sqrt{a^2 + b^2}, \quad \cos\theta = \dfrac{a}{r}, \quad \sin\theta = \dfrac{b}{r}$

The *trigonometric* or *polar* form of a complex number. The angle θ is called the *argument* of z. See (2.84) for $e^{i\theta}$.

2.81



Geometric representation of the trigonometric form of a complex number.

2.82 If $z_k = r_k(\cos\theta_k + i\sin\theta_k)$, $k = 1, 2$, then

$$z_1 z_2 = r_1 r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\big)$$

Multiplication and division on trigonometric form.

2.83 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

De Moivre's formula, $n = 0, 1, \ldots$.

2.84 If $z = x + iy$, then

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

In particular,

$$e^{iy} = \cos y + i\sin y$$

The *complex exponential function*.

2.85 $e^{\pi i} = -1$

A striking relationship.

2.86 $e^{\bar{z}} = \overline{e^z}$, $e^{z+2\pi i} = e^z$, $e^{z_1+z_2} = e^{z_1}e^{z_2}$, $e^{z_1-z_2} = e^{z_1}/e^{z_2}$

Rules for the complex exponential function.

2.87 $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

Euler's formulas.

2.88 If $a = r(\cos\theta + i\sin\theta) \neq 0$, then the equation

$$z^n = a$$

has exactly n roots, namely

$$z_k = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.

nth roots of a complex number, $n = 1, 2, \ldots$.

References

Most of these formulas can be found in any calculus text, e.g. Edwards and Penney (1998) or Sydsæter and Hammond (2005). For (2.3)–(2.12), see e.g. Turnbull (1952).

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