2

Discontinuous Finite Element Procedures

The discontinuous finite element method makes use of the same function space as the continuous method, but with relaxed continuity at interelement boundaries. It was first introduced by Reed and Hill [1] for the solution of the neutron transport equation, and its history and recent development have been reviewed by Cockburn *et al*. [2, 3]. The essential idea of the method is derived from the fact that the shape functions can be chosen so that either the field variable, or its derivatives or generally both, are considered discontinuous across the element boundaries, while the computational domain continuity is maintained. From this point of view, the discontinuous finite element method includes, as its subsets, both the finite element method and the finite difference (or finite volume) method. Therefore, it has the advantages of both the finite difference and the finite element methods, in that it can be effectively used in convection-dominant applications, while maintaining geometric flexibility and higher local approximations through the use of higher order elements. This feature makes it uniquely useful for computational dynamics and heat transfer. Because of the local nature of a discontinuous formulation, no global matrix needs to be assembled; and thus, this reduces the demand on the in-core memory. The effects of the boundary conditions on the interior field distributions then gradually propagate through element-by-element connection. This is another important feature that makes this method useful for fluid flow calculations. Computational fluid dyanmics is an evolving subject, and very recent developments in the area are discussed in [4].

In the literature, the discontinuous finite element method is also called the discontinuous Galerkin method, or the discontinuous Galerkin finite element method, or the discontinuous method [1, 2, 3, 5, 6]. These terms will be used interchangeably throughout this book.

This chapter introduces the basic ideas of the discontinuous finite element method through simple and illustrative examples. The keyword here is *discontinuous*. Various views have been adapted to interpret the concept of discontinuity and three widely accepted ones are presented below [5, 7]. The discontinuous finite element formulation for boundary value problems, and overall procedures for numerical solutions are presented. The advantages and disadvantages of the various methods are also discussed, in comparison to the continuous finite element method. Examples used to illustrate the basic features and the solution procedures of the discontinuous finite element formulation are given.

2.1 The Concept of Discontinuous Finite Elements

To illustrate the basic ideas of the discontinuous finite element method, we consider a simple, one-dimensional, first order differential equation with u specified at one of the boundaries:

$$C(u)\frac{du}{dx} + f(u) = 0; \qquad x \in [a,b] \tag{2.1}$$

$$u(x=a) = u_a \tag{2.2}$$

where, without loss of generality, the coefficient $C(u)$ is considered a function of the field variable u. By defining $dF = C(u)\, du$, the above differential equation may be further written as

$$\frac{dF}{dx} + f(u) = 0 \tag{2.3}$$

The domain is discretized such that $\Omega_j = [x_j, x_{j+1}]$ with $j = 1, 2, \ldots, N$. Then, integrating the above equation over the element j, Ω_j, with respect to a weighting function $v(x)$,

$$\int_{x_j}^{x_{j+1}} \left[\frac{\partial F}{\partial x} + f(u)\right] v(x)\, dx = 0 \tag{2.4a}$$

and performing integration by parts on the differential operator, we have

$$F(u(x_{j+1}))\,v(x_{j+1}) - F(u(x_j))\,v(x_j)$$
$$-\int_{x_j}^{x_{j+1}} \left[F\frac{\partial v(x)}{\partial x} - v(x)f(u)\right] dx = 0 \tag{2.4b}$$

On $\Omega_j = [x_j, x_{j+1}]$, u is approximated by $u_h \in H$, H being an appropriate function space of finite dimension, and v by v_h taken from the same function space as u_h, with $j = 1, 2, \ldots, N$. Upon substituting (u_h, v_h) for (u_h, v_h) in Equation 2.4b, we have the discontinuous Galerkin finite element formulation:

$$F(u_h(x_{j+1}))\,v_h(x_{j+1}) - F(u_h(x_j))\,v_h(x_j)$$

$$-\int_{x_j}^{x_{j+1}}\left[F(u_h)\frac{\partial v_h(x)}{\partial x}-v_h(x)f(u_h)\right]dx=0 \tag{2.5}$$

In the continuous finite element approach, the field variable u_h is forced to be continuous across the boundary. As we know, this causes a problem of numerical instability, when $| c(u_h) |$ is large. The essential idea for the discontinuous method is that u_h is allowed to be discontinuous across the boundary. Therefore, across the element, the following two different values are defined at the two sides of the boundary:

$$u_j^+ = \lim_{x\downarrow x_j^+} u_h(x) \quad \text{and} \quad u_j^- = \lim_{x\uparrow x_j^-} u_h(x) \tag{2.6}$$

Furthermore, we note that u_h is discontinuous only at the element boundaries. The solution u and $F(u)$ are smooth within (but excluding) the boundary. By this definition, the above equation contains the variables only within the integral limits Ω_j. There is no direct coupling with other intervals or other elements. The field values at a node, or the interface between two elements, are not unique. They are calculated using the two limiting values approaching the interface from the two adjacent elements. This feature is certainly desirable for problems with internal discontinuities, such as those pertaining to shock waves. We will discuss these specific problems in the chapters to follow.

The discontinuous formulation expressed in Equtation 2.5 may be viewed from different perspectives, which all involve the cross-element treatments either by weakly imposing the continuity at the element interface, or by using numerical fluxes, or by boundary constraint minimization. These views are discussed below, so that the reader can fully appreciate the concept of discontinuity embedded in the formulation.

2.1.1 Weakly Imposed Cross-element Continuity

For the continuous finite element solution of boundary value problems, the consistency condition often requires that the field variable and its derivative be continuous in the computational domain, which implies the cross-element continuity requirement for these variables [8, 9]. In the continuous finite element formulation, the cross-element continuity is strongly enforced. The discontinuous formulation relaxes this continuity requirement, so that the cross-element continuity is weakly imposed. This is accomplished if $F(u)$, at the element boundaries, is chosen as follows [3, 5]:

$$+F(u_h(x_i))=+F(u_i^+) \quad ; \quad -F(u_h(x_i))=-F(u_i^-) \tag{2.7}$$

so that the upstream value outside the element interval Ω_j is used, following the well known treatment for finite difference schemes. With Equation 2.7 substituted into Equation 2.5, the following integral equation is obtained:

$$F(u_h(x_{j+1}))v_h(x_{j+1}) - F(u_j^-)v_h(x_{j2})$$
$$-\int_{x_j}^{x_{j+1}} \left[F(u_h) \frac{\partial v_h(x)}{\partial x} - v_h(x) f(u_h) \right] dx = 0 \tag{2.8}$$

This is one popular formulation for discontinuous finite element solutions. Equation 2.8 may be integrated once again with the result,

$$F(u_h(x_{j+1}))v_h(x_{j+1}) - F(u_j^-)v_h(x_j)$$
$$-(F(u_h(x_{j+1}))v_h(x_j) - F(u_j^+)v_h(x_j))$$
$$+\int_{x_j}^{x_{j+1}} \left[\frac{\partial F(u_h(x))}{\partial x} + f(u) \right] v(x)\, dx = 0 \tag{2.9}$$

Here we stay with the upwinding rule at x_j, because only one boundary condition is available and it is applied at x_j. For this first order equation, $F(u_h^+) = F(u_h(x_{j+1}))$ at x_{j+1}. If one works with a second order equation, a similar rule may be applied at x_{j+1}. This point will be discussed further in Chapter 4. With these choices, the above equation is simplified as:

$$\left(F(u_j^+) - F(u_j^-)\right) v_h(x_j) + \int_{x_j}^{x_{j+1}} \left[\frac{\partial F(u_h(x))}{\partial x} + f(u_h) \right] v_h(x)\, dx = 0 \tag{2.10}$$

or more often, it is written in terms of a jump across the element boundary,

$$\int_{x_j}^{x_{j+1}} \left[\frac{\partial F(u_h(x))}{\partial x} + f(u_h) \right] v_h(x)\, dx + [F(u_j)] v_h(x_j) = 0 \tag{2.11}$$

where the jump is defined by

$$[F(u_j)] = F(u_j^+) - F(u_j^-) \tag{2.12}$$

In deriving the above equations, we have used the upwinding rule: $+F(u_h(x_j)) = +F(u_j^+)$. This procedure is graphically illustrated in Figure 2.1.

We now look at the implication of the above equation, i.e., Equation 2.12. Here, in essence, the continuity condition at x_j is satisfied *weakly* with respect to the weighting function $v(x)$. Note that x_j can be an internal boundary or external boundary. This is in contrast to the continuous finite element formulation, by which the continuity conditions are satisfied *strongly* across the element boundaries, $[F(u_j)] = 0$.

We note that since $v(x)$ is arbitrary, Equation 2.11 is equivalent to the following mathematical statement:

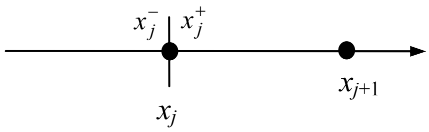

Figure 2.1. An illustration of the jump across x_j of element j: x_j and x_{j+1} mark the boundaries of the element

$$F(u_j^+) - F(u_j^-) = 0 \qquad \text{for} \quad x = x_j \tag{2.13}$$

$$\frac{\partial F(u(x))}{\partial x} + f(u) = 0 \qquad \text{for} \quad x \in (x_j, x_{j+1}) \tag{2.14}$$

Here, $F(u_j^+) - F(u_j^-) = 0$ also implies that $u_j^+ = u_j^-$ for monotone $F(u)$. Thus, Equation 2.11 is the weak form of Equations 2.13 and 2.14.

2.1.2 Numerical Boundary Fluxes for Discontinuity

Another treatment of the cross-element continuity is based on the use of a numerical flux to model $F(u)$. This is demonstrated by Cockburn *et al.* [2, 3]. Using this approach, $F(u)$ is replaced by the following flux expressions:

$$F(u_h(x_{j+1})) = h(u_{j+1}^-, u_{j+1}^+)\ ; \quad F(u_h(x_j)) = h(u_j^-, u_j^+) \tag{2.15}$$

with an imposed consistency condition,

$$h(u, u) = F(u) \tag{2.16}$$

Many different types of flux expressions have been used in the literature for this purpose, and have been reviewed in a recent paper by Arnold *et al.* [10]. To reproduce Equation 2.5, we may use the following definition for the numerical flux:

$$h(u_j^-, u_j^+) = F(u_j^-) \tag{2.17}$$

which basically states that the flux at the element boundary is equal to the flux of the upstream element. With the numerical flux, the discontinuous finite element formulation for the 1-D problem is recast as

$$h(u_{j+1}^-, u_{j+1}^+)\, v_h(x_{j+1}) - h(u_j^-, u_j^+)\, v_h(x_j)$$

$$-\int_{x_j}^{x_{j+1}}\left[F(u_h)\frac{\partial v_h(x)}{\partial x}-v_h(x)f(u_h)\right]dx=0 \tag{2.18}$$

It is apparent from the above discussion that construction of consistent numerical fluxes is important in discontinuous finite element calculations. These fluxes need to be chosen to satisfy numerical stability conditions and various forms of numerical fluxes and their stability conditions are given in [3, 10]. We note that different forms of numerical fluxes may be used to model various types of differential equations, and, as such, Equation 2.18 is more general. Selection of appropriate numerical fluxes for computational fluid dynamics applications is discussed in Chapters 4–7.

2.1.3 Boundary Constraint Minimization

The third view of the discontinuous treatment across the element boundaries is from the element boundary constraint minimization approach. To illustrate this view, we apply the Weight Residuals method to both the elements and their boundaries,

$$\sum_{j=1}^{N}\int_{x_j}^{x_{j+1}}\left[\frac{dF(u(x))}{dx}+f(u)\right]v(x)\,dx+\sum_{j=1}^{N}\int_{x_j^-}^{x_j^+}\frac{dF(u(x))}{dx}v(x)\,dx=0 \tag{2.19}$$

Performing integrating by parts on Equation 2.19 and noting that the test function does not have to be continuous across the boundaries because of the intrinsic assumptions associated with a discontinuous finite element formulation, we have the following expression:

$$\sum_{j=1}^{N}\int_{x_j}^{x_{j+1}}\left[\frac{dF(u_h(x))}{dx}+f(u_h)\right]v_h(x)\,dx$$
$$+\sum_{j=1}^{N}\left(F(u_j^+)-F(u_j^-)\right)\quad v_h(x)\quad dx=0 \tag{2.20}$$

where (u, v) are approximated by (u_h, v_h). Different forms of the weighting function may be used. One of the simple forms uses a linear combination of $v_h(x)$, defined on two adjacent elements as

$$v_h(x_j)=\alpha\, v_j^+ +(1-\alpha)\, v_j^- \tag{2.21}$$

With the above equation substituted into Equation 2.20, one obtains the following formulation:

$$\sum_{j=1}^{N}\alpha\left(F(u_j^+)-F(u_j^-)\right)v_j^+ +(1-\alpha)\left(F(u_j^+)-F(u_j^-)\right)v_j^-$$

$$+\sum_{j=1}^{N}\int_{x_j}^{x_{j+1}}\left[\frac{dF(u_h(x))}{dx}+f(u_h)\right]v_h(x)\,dx=0 \tag{2.22}$$

This expression of the formulation is also general. In fact, when α = 1, one recovers the upwinding approach, as in Equation 2.11. On the other hand, if we carry out the integration once more and define the numerical flux as follows:

$$h(u_j^-,u_j^+)=\alpha F(u_j^-)+(1-\alpha)F(u_j^+) \tag{2.23}$$

then Equation 2.22 reduces to Equation 2.18.

From the examples given above, a discontinuous element formulation can be constructed in three different ways: (1) by weakly imposed boundary conditions across element boundaries (Equation 2.11), (2) by the use of numerical flux expressions at the element boundaries (Equation 2.18), and (3) by the minimization of constraints across element boundaries (Equation 2.22). We note that while these three approaches treat cross-element discontinuities differently, they all fall into the general category of the Weighted Residuals method [6]. The first two involve the integration by parts, while the third one does not. If equations are written in non-conservative form, or if a conservative form does not exist, it is not straightforward to perform partial integration of the equations, because there is no "flux". In this case, the boundary minimization is more convenient for developing a discontinuous finite element formulation for these equations.

2.1.4 Treatment of Discontinuity for Non-conservative Systems

As stated in Section 1.7, a system of differential equations may be written in the "divergence" or "conservation law" form. By the definition given in Section 1.7, Equation 2.3 is in a conservative form, while Equation 2.1 is not.

In numerical analyses, the primitive variable is often solved instead of the flux function $F(u)$, and thus Equation 2.1 needs to be applied directly. In this case, from the definition, $dF(u)=C(u)\,du$, we may write,

$$F(u_i^+)-F(u_i^-)=\int_{u_i^-}^{u_i^+}C(u)\,du=[u]_i\int_{-1/2}^{+1/2}C\left([u]_i t+\tfrac{1}{2}(u_i^+ + u_i^-)\right)dt \tag{2.24}$$

where $[u]_i=u_i^+-u_i^-$ is the jump across the element boundary. Since u is a smooth function, and $[u]_i$ is small, we may numerically approximate the integral by a mid-point rule,

$$\int_{-1/2}^{+1/2} C\left([u]_i t + \tfrac{1}{2}(u_i^+ + u_i^-)\right)dt = C\left(\tfrac{1}{2}(u_i^+ + u_i^-)\right) + O([u]_i^2) \qquad (2.25)$$

The relations given in the above two equations allow us to rewrite Equation 2.22 in the following form:

$$\int_{x_j}^{x_{j+1}} \left[C(u_h)\frac{du_h(x)}{dx} + f(u_h)\right] v_h(x)\,dx + \alpha\, C\left(\tfrac{1}{2}(u_j^+ + u_j^-)\right)[u]_j v_h(x_j)$$

$$+ (1-\alpha) C\left(\tfrac{1}{2}(u_j^+ + u_j^-)\right)[u]_j\, v_h(x_j) + O([u]_j^3) = 0 \qquad (2.26)$$

The last term, however, can be discarded without affecting the accuracy [3]. Thus, for the non-conservative equation stated in Equation 2.21, the discontinuous formulation is: find $u_h(x) \in P_l(\Omega_j)$ such that

$$\int_{x_j}^{x_{j+1}} \left[C(u_h)\frac{du_h(x)}{dx} + f(u_h)\right] v_h(x)\,dx$$

$$+(1-\alpha) C\left(\tfrac{1}{2}(u_j^+ + u_j^-)\right)[u]_j\, v(x_j) + \alpha\, C\left(\tfrac{1}{2}(u_j^+ + u_j^-)\right)[u]_j\, v_h(x_j)$$

$$+O([u_h]_j^3) = 0\,, \qquad \forall v_h(x) \in P_l(V_j) \qquad (2.27)$$

where $P_l(\Omega_j)$ is a piecewise polynomial of degree l defined over the interval $\Omega_j = [x_j, x_{j+1}]$. The boundary terms are set at $u_0^1 = u_0$ and $u_{N+1}^+ = u_{N+1}^-$.

2.1.5 Transient Problems

The discussion thus far has been limited to steady state problems. As with other methods, the treatment of the cross-element discontinuities can be readily extended to develop discontinuous finite element formulations for transient problems. Let us illustrate this point by considering a 1-D transient problem of hyperbolic type, sometimes referred to as convective wave equation, or convection equation, which is mathematically stated as

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0\,, \qquad c > 0, x \in [a,b],\ \ t > 0 \qquad (2.28)$$

where c is a constant. Any of the above formulations can be applied to develop the needed integral formulation for a discontinuous finite element solution. Here we take the boundary constraint minimization approach and integrate the above partial differential equation with respect to a weighting function $v(x)$, whence we have the following result:

$$\sum_{j=1}^{N}\int_{x_j}^{x_{j+1}}\left[\frac{\partial u_h}{\partial t}+c\frac{\partial u_h}{\partial x}\right]v_h(x)\,dx+\sum_{j=1}^{N}\int_{x_j^-}^{x_j^+}\frac{\partial u_h(x)}{\partial x}v_h(x)\,dx=0 \quad (2.29)$$

For a typical interval, $\Omega_j = [x_j, x_{j+1}]$, $j = 1, \ldots, N$, the above equation reduces to the following form after integrating the second term:

$$\int_{x_j}^{x_{j+1}}\left[\frac{\partial u_h}{\partial t}+c\frac{\partial u_h}{\partial x}\right]v_h(x)\,dx+\alpha\, c[u]_j\, v_h(x_j)+(1-\alpha)c[u]_j\, v_h(x_j)=0 \quad (2.30)$$

Comparing this with Equation 2.22, and noticing that $F(u) = cu$ for this problem, we see that the transient term enters the integral description directly, as in the continuous finite element method.

2.2 Discontinuous Finite Element Formulation

We may extend the discussions on the 1-D examples to consider a more general class of problems and formally introduce the discontinuous finite element formulation for boundary value problems.

2.2.1 Integral Formulation

Let us consider a partial differential equation, written in the form of the conservation law for a scalar u,

$$\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{F}(u)+b=0\ ;\quad u(0,\mathbf{x})=u_0(\mathbf{x})\ ,\ \mathbf{x}\in\Omega,\ t>0 \quad (2.31)$$

To start, the computational domain is broken into a tessellation of finite elements $\Omega=\bigcup_{j=1}^{N}\Omega_j$. The field variable u is approximated by the interpolation function u_h, defined on each element Ω_j. Since the function u_h is allowed to be discontinuous across the element boundaries for discontinuous formulations, the finite element space, over which u_h is defined, is sometimes referred to as finite element broken space, to differentiate it from continuous finite element space [11]. The broken space is denoted by V_h^j and $V_h^j \subset L^2(\Omega)$, where $L^2(\Omega)$ is the Lebesgue space of square integrable functions, defined over Ω [12].

If the above equation is integrated over Ω_j with respect to a weighting function v, one has the weak form expression:

$$\int_{\Omega_j} v(\mathbf{x})\left[\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{F}(u)+b\right]dV=0 \quad (2.32)$$

We now perform integration by parts on the second term involving the divergence of flux and obtain the nornal fluxes along the boundary. This procedure yields the following result:

$$\int_{\Omega_j}\left[v_h\frac{\partial u_h}{\partial t}-\mathbf{F}(u_h)\cdot\nabla v_h+bv_h\right]dV+\int_{\partial\Omega_j}v_h\mathbf{F}(u_h)\cdot\mathbf{n}\,dS=0\,,$$

$$\forall v_h\in V_h^j \qquad (2.33)$$

where $\mathbf{n}$ is the local outnormal vector on the element boundary $\partial\Omega_j$. By substituting numerical fluxes along the element boundaries,

$$\mathbf{F}\cdot\mathbf{n}=F_n(u^-,\ u^+) \qquad (2.34)$$

Equation 2.33 can be integrated numerically. The construction of numerical fluxes is important, and there are many different fluxes for popular fluid flow and heat transfer problems. These fluxes will be discussed in subsequent chapters for specific applications.

The integration of Equation 2.33 with an appropriate choice of numerical fluxes will result in a set of ordinary differential equations,

$$\mathbf{M}\frac{d\mathbf{U}_{(j)}}{dt}+\mathbf{K}\mathbf{U}_{(j)}=\mathbf{F}_{(j)} \qquad (2.35)$$

where $\mathbf{U}_{(j)}$ is the vector of nodal values of variable u associated with element j, $\mathbf{K}$ the stiffness matrix, $\mathbf{M}$ the mass matrix, and $\mathbf{F}_{(j)}$ the force vector consisting of contributions from the source and boundary terms.

2.2.2 Time Integration

Time integration can proceed, in theory, by using the general approaches for the solution of initial value problems. Two important points, however, are noted when the time integration is carried out for Equation 2.35. First, since the discontinuous formulation is a local formulation, it often leads to standard explicit structures. Thus, the explicit methods for time integration are preferred with discontinuous finite element formulations, whenever possible. Of course, this does not mean that the implicit method is not possible. In practice, both explicit and implicit integrators can be applied, though the latter is much less frequently used with discontinuous formulations. Second, since the explicit methods are prone to numerical instability, appropriate stability analysis is needed for the time integration schemes [13–15]. Fortunately, stability criteria have been established for the most commonly used time integration methods for the fluid flow and heat transfer applications.

The following equations show some of the commonly used time integration schemes for the discontinuous finite element applications.

(1) First order Euler forward:

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \Delta t\, H(\mathbf{U}_j^n, t^n) \tag{2.36}$$

(2) Second order scheme:

$$f_1 = \Delta t H(\mathbf{U}_j^n, t^n)\,; \qquad f_2 = \Delta t H(\mathbf{U}_j^n + f_1, t^n + \Delta t)\;;$$

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + 0.5(f_1 + f_2) \tag{2.37}$$

(3) Third order scheme:

$$f_1 = \Delta t H(\mathbf{U}_j^n, t^n)\,; \quad f_2 = \Delta t H(\mathbf{U}_j^n + 0.5 f_1, t^n + 0.5\Delta t)\,;$$

$$f_3 = \Delta t H(\mathbf{U}_j^n - f_1 + 2 f_2,\, t^n + 0.5\Delta t)\,;$$

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \tfrac{1}{6}(f_1 + 4 f_2 + f_3) \tag{2.38}$$

(4) Fourth order scheme:

$$f_1 = \Delta t H(\mathbf{U}_j^n, t^n)\,; \quad f_2 = \Delta t H(\mathbf{U}_j^n + 0.5 f_1, t^n + 0.5\Delta t)\,;$$

$$f_3 = \Delta t H(\mathbf{U}_j^n 0.5 f_2, t^n + 0.5\Delta t)\,; \;\; f_4 = \Delta t H(\mathbf{U}_j^n + f_3, t^n + \Delta t)\,;$$

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \tfrac{1}{6}(f_1 + 2 f_2 + 2 f_3 + f_4) \tag{2.39}$$

In the above schemes, $H(\mathbf{U}_j^n, t^n) = \mathbf{M}^{-1}(\mathbf{F}_j(\mathbf{U}_j^n, t^n) - \mathbf{K}(\mathbf{U}_j^n, t^n)\mathbf{U}_j^n)$. Since these schemes are explicit, the time step has to satisfy the *CFL* (Courant–Friedrich–Lewy) condition for stability. While they represent some of the popular choices, other schemes are also possible. For example, a Total Variation Diminishing (*TVD*) scheme has been used for oscillation-free shock wave simulations [15]. It is noted that time integration can be most efficiently calculated if the mass matrix is diagonalized when an explicit scheme is used. For this purpose, the orthogonal hierarchical shape functions presented in Chapter 3 have been proven to be extremely useful. The use of these transient schemes will be discussed in subsequent chapters for the numerical solution of specific problems of fluid dynamics and heat transfer.

An implicit time integration scheme may also be used with a discontinuous finite element formulation. However, the use of an implicit scheme results in an even larger global matrix than a conventional finite element formulation, thereby eliminating the advantage of localized formulation associated with discontinuous finite elements. Consequently, almost all discontinuous finite element formulations presented thus far use the explicit time integration scheme for the solution of transient problems, for the purpose of facilitating the parallel computation associated with a local formulation.

2.3 Solution Procedures

We now consider the general computational procedure by which the discontinuous finite element method is used to obtain numerical solutions. From the above discontinuous formulations, it is clear that this method is local, in that the weakly imposed across-element boundary conditions permit the element-wise solutions. For each element, the elemental calculation is required, and is essentially the same as that used in the continuous finite element method. Also, as for the continuous counterpart, the discontinuous Galerkin formulation is obtained when the same interpolation functions are used for both unknowns and the trial functions.

One important implication of the above discontinuous formulation is that, because of a weakly imposed boundary condition across adjacent elements, a variety of elements or shape functions, including the discontinuous shape functions, can be chosen for computations. As a result, the discontinuous formulation embeds the continuous finite element and the finite volume/finite difference formulations. If, in particular, a constant element is chosen, then the formulation boils down to the traditional finite difference method. On the other hand, if the continuous function is chosen, and the cross-boundary continuity is enforced, one implements the continuous finite element method.

We note that the discontinuous finite element method falls also into the general category of the Weighted Residuals method for the solution of partial differential equations. Various familiar forms of domain- and boundary-based numerical methods can be derived from this general integral formulation, depending upon the choice of the weighting functions. For the Galerkin formulation, the weighting functions are chosen the same as the shape functions. The weighting functions, however, may be chosen differently from the shape functions. For example, if Green's functions are chosen as weighting functions, then the well known boundary element formulation of boundary value problems is obtained [16].

2.4 Advantages and Disadvantages of Discontinuous Finite Element Formulation

In comparison with the other numerical methods (finite difference and finite elements), the discontinuous finite element formulations have both advantages and disadvantages. It is important to understand these issues for developing specific applications.

2.4.1 Advantages

In discontinuous formulations, the interelement boundary continuity constraints are relaxed. Various upwinding schemes, proven successful for convection-dominant flows, can be easily incorporated through element boundary integrals that only involve the spatial derivative terms in the equations. Inside the elements, all terms are treated by the standard Galerkin method, leading to classical symmetric mass matrices and standard treatment of source terms.

Higher order approximations are obtained simply by increasing the order of the polynomials or other basis functions. The decoupling of the upwinding convection terms, and the other terms, yields a very attractive feature of the discontinuous method, especially in the case of convection-dominant problems. This method performs very well, and in fact it is often better than the SUPG method, for advection type problems. Even for linear elements, this method performs remarkably well [6].

The coupling between element variables is achieved through the boundary integrals only. This means that $\partial u/\partial t$ and the source terms are fully decoupled between elements. The mass matrices can be inverted at the element level, rendering the $\partial \mathbf{U}_{(j)}/\partial t$ explicit. With an appropriate choice of orthogonal shape functions, a diagonal mass matrix can be obtained, thereby resulting in a very efficient time marching algorithm.

The discontinuous finite element formulation is a local formulation and the action is focused on the element and its boundaries. Whatever the space dimension or the number of unknowns, the formulation remains basically the same and no special features need to be introduced.

Because of the local formulation, a discontinuous finite element algorithm will not result in an assembled global matrix and thus the in-core memory demand is not as strong. Also, the local formulation makes it very easy to parallelize the algorithm, taking advantage of either shared memory parallel computing or distributed parallel computing.

Also, because of the local formulation, both the *h*- and *p*-adaptive refinements are made easy and convenient. Compared with the continuous finite element method, the *hp*-adaptive algorithm based on the discontinuous formulation requires no additional cost associated with node renumbering.

2.4.2 Disadvantages

Like any other numerical methods, the discontinuous finite element method has its drawbacks. The blind use of this method would certainly result in a very inefficient algorithm. In comparison with finite elements using continuous basis functions, the number of variables is larger for an identical number of elements [7, 17]. This is obvious from the formulations given above, and is a natural consequence of relaxing the continuity requirements across the element boundaries.

Since the basis and test functions are discontinuous across element boundaries, second order spatial terms (diffusion) need to be handled by mixed methods, which enlarge the number of unknowns, or other special treatments. This is a serious drawback, when compared to the continuous methods where elliptic operators are handled relatively easily. Also, our experience with the heat conduction or diffusion problems indicates that if stabilization parameters are not used, the element matrix may become singular and thus pollute the numerical solution. The solution algorithm, based on the discontinuous formulation in general, is inferior to the continuous finite element method in its execution speed for pure conduction or diffusion problems, in particular steady state heat conduction and diffusion problems. Thus, for these problems, if memory is not a constraint in applications, the discontinuous formulation should be avoided.

In computer-aided thermal and fluids engineering design applications, complex numerical models are often required to represent a wide range of thermal and fluid flow phenomena. It is, therefore, unlikely that one single method would be best suited for modeling all the physical phenomena in a thermal/fluid system. Thus, a combination of methods, best suited for modeling certain types of phenomena, would be required, in order to develop the most efficient algorithms for specific applications. These issues are explored further in subsequent chapters of this book.

2.5 Examples

The examples in this chapter are selected for the purpose of illustrating the basic concepts of the discontinuous finite element formulation, and the general solution procedures for the numerical solution to boundary value problems. As a result, very simple problems are considered.

Example 2.1. Apply the discontinuous Galerkin finite element method to obtain the numerical solution of the following initial value problem:

$$\frac{du}{dx}=1 \qquad \text{with} \quad u(0)=0; \qquad\qquad x\in[0,2] \tag{2.1e}$$

and compare the numerical results with the analytical solution with the domain discretized by two linear elements.

Solution. The analytic solution to the problem is simple, $u = x$. Now, following the procedure in Section 2.1 leading to the element-wise formulation, we have the following integral equation with F replaced by u_h and v by ϕ_i:

$$\int_{x_j}^{x_{j+1}}\left[\frac{du_h(x)}{dx}+f\left(u_h\right)\right]\phi_i(x)\,dx+\left(u_j^+-u_j^-\right)\phi_i(x_j)=0 \tag{2.2e}$$

where ϕ_i is the shape function. Now the domain is discretized into two elements, as shown in Figure 2.1e.

For simplicity, a linear interpolation is used for each of the elements. When an isoparametric shape function is used, we have the following relations:

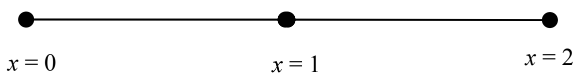

Figure 2.1e. Discretization of the domain into two elements

$$u_h = \phi_1 u_1 + \phi_2 u_2 \ , \quad \phi_1(\xi) = 0.5(1-\xi) \ , \quad \phi_2(\xi) = 0.5(1+\xi)$$

$$x = \phi_1 x_1 + \phi_2 x_2 \ ; \quad dx = d\phi_1 x_1 + d\phi_2 x_2 = 0.5(x_2 - x_1) d\xi$$

$$\frac{du_h}{dx} = \frac{d\phi_1}{dx} u_1 + \frac{d\phi_2}{dx} u_2 \ ; \ \frac{d\phi_1}{dx} = \frac{d\phi_1}{d\xi}\frac{d\xi}{dx} = -0.5 \times 2 = -1 \ ; \ \frac{d\phi_2}{dx} = 1$$

Applying Equation 2.11 to the first element $x \in [0, 1]$, and making use of the condition $u_0^- = u(0^-) = 0$, one has

$$\int_0^1 \left[\frac{du_h}{dx} - 1\right] \phi_i(x)\, dx + u_0^+ \phi_i(0) = 0 \tag{2.3e}$$

Now with $u_0^+ = u_1$, $\phi_1(x=0) = \phi_1(\xi = -1) = 1$, and $u(x^- = 1) = u_2$ and with the unknown variable replaced by its local approximation using interpolation functions in Equation 2.3e, the following expression is obtained for the first element:

$$\int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \left(\frac{d\phi_1}{dx}, \quad \frac{d\phi_2}{dx} \right) dx \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} dx \tag{2.4e}$$

For this problem, the integration can be carried out analytically, whence we have the results,

$$\begin{aligned} \int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \left(\frac{d\phi_1}{dx}, \quad \frac{d\phi_2}{dx} \right) dx &= \frac{1}{4} \int_{-1}^1 \begin{pmatrix} 1-\xi \\ 1+\xi \end{pmatrix} (-1, \quad 1)\, d\xi \\ &= \frac{1}{4} \int_{-1}^1 \begin{pmatrix} -(1-\xi), (1-\xi) \\ -(1+\xi), \ (1+\xi) \end{pmatrix} d\xi = \frac{1}{8} \begin{pmatrix} -4, \ 4 \\ -4, \ 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1, \ 1 \\ -1, \ 1 \end{pmatrix} \end{aligned} \tag{2.5e}$$

$$\int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} dx = \frac{1}{2} \int_{-1}^1 \begin{pmatrix} (1-\xi) \\ (1+\xi) \end{pmatrix} \frac{dx}{d\xi} d\xi = \frac{1}{8} \begin{pmatrix} -(1-\xi)^2 \Big|_{-1}^{1} \\ (1+\xi)^2 \Big|_{-1}^{1} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.6e}$$

Substituting Equations 2.5e–2.6e into Equation 2.4e yields the follwing matrix equation,

$$\frac{1}{2} \begin{pmatrix} -1, \ 1 \\ -1, \ 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.7e}$$

which can be solved for u_1 and u_2,

$$\begin{pmatrix} 1,1 \\ -1,1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.8e}$$

where $u_1 = u(0^+)$ and $u_2 = u(1^-)$.

Now the same procedure is applied to the second element $x \in [1, 2]$ with the result,

$$\int_1^2 \left[\frac{du_h(x)}{dx} - 1\right]\phi_i(x)\,dx + \left(u(1^+) - u(1^-)\right)\phi_i(x_j) = 0 \tag{2.9e}$$

At this point, $u(1^-) = u_2$ is known from Equation 2.8e. Furthermore, if the upwinding scheme is used, the following matrix equation is then obtained:

$$\int_1^2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\begin{pmatrix} \frac{d\phi_1}{dx}, & \frac{d\phi_2}{dx} \end{pmatrix} dx \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$= \int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} dx + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u(1^-) \\ u_2 \end{pmatrix} \tag{2.10e}$$

The detailed integration is almost the same as for the first element,

$$\frac{1}{2}\begin{pmatrix} -1, 1 \\ -1, 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ u_2 \end{pmatrix} \tag{2.11e}$$

Rearranging, we have the solution for the second element,

$$\begin{pmatrix} +1, 1 \\ -1, 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.12e}$$

where $u_1 = u(1^+)$ and $u_2 = u(2^-)$. The numerical results for this elementary example are: $u_0 = 0$, $u_1 = 0.5(u(1^+) + u(1^-)) = 1$ and $u_2 = 2$.

As discussed above, the discontinuous shape functions may be used because the field variable is considered discontinuous across the boundary. The use of discontinuous shape functions to obtain the same numerical results is illustrated in the following example.

Example 2.2. Re-solve the problem defined in Equation (2.1e) using geometrically discontinuous linear elements.

Solution. For the purpose of demonstration only, we consider the discontinuous shape function for the first element that is normalized at $x_1 = 0.2$ and $x_2 = 0.8$, which corresponds to ξ=0 and ξ=1 respectively, as shown in Figure 2.2e. Thus the following expressions are obtained:

$$u_h = \phi_1\, u_1 + \phi_2\, u_2\,; \quad \phi_1(\xi) = 0.5(1 - \xi / 0.8)$$

$$x = \phi_1\, x_1 + \phi_2\, x_2\,; \qquad \phi_2(\xi) = 0.5(1 + \xi / 0.8)$$

$$dx = d\phi_1\, x_1 + d\phi_2\, x_2 = 0.5\,(x_2 - x_1)\, d\xi / 0.8 = d\xi 0.5 \times 3/4$$

$$\frac{\partial u_h}{\partial x} = \frac{\partial \phi_1}{\partial x} u_1 + \frac{\partial \phi_2}{\partial x} u_2\,; \frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_1}{\partial \xi}\frac{\partial \xi}{\partial x} = -\frac{1}{2\times 0.8} \times \frac{2\times 0.8}{(0.8-0.2)} = -\frac{1}{0.6}$$

$$\frac{\partial \phi_2}{\partial x} = \frac{\partial \phi_2}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{2\times 0.8} \times \frac{2\times 0.8}{(0.8-0.2)} = \frac{1}{0.6}$$

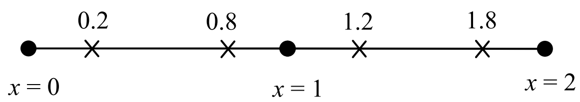

Figure 2.2e. An illustration of two linear discontinuous elements

To calculate the integration limits for the normalized coordinate ξ that correspond to $x = 0$ and $x = 1$, we make use of the isoparametric element to obtain the integration limits:

$$0 = x = \phi_1\, x_1 + \phi_2\, x_2 = 0.5(1 - \xi / 0.8)0.2 + 0.5(1 + \xi / 0.8)0.8 \;=>\xi = -4/3$$

$$1 = x = \phi_1\, x_1 + \phi_2\, x_2 = 0.5(1 - \xi / 0.8)0.2 + 0.5(1 + \xi / 0.8)0.8 \;=>\; \xi = 4/3$$

These expressions and integration limits are now substituted into Equation 2.11 for the first element and the resultant equation can be integrated analytically, whence we have

$$\int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \left(\frac{d\phi_1}{dx},\; \frac{d\phi_2}{dx} \right) dx \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$+ \begin{pmatrix} \phi_1(-4/3) \\ \phi_2(-4/3) \end{pmatrix} \left(\phi_1(-4/3),\; \phi_2(-4/3) \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} dx \quad (2.13e)$$

$$\int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} dx = \frac{3}{16}\int_{-4/3}^{4/3} \begin{pmatrix} 1-\xi/0.8 \\ 1+\xi/0.8 \end{pmatrix} d\xi = \frac{2.4}{32} \begin{pmatrix} -(1-\xi/0.8)^2 \\ (1+\xi/0.8)^2 \end{pmatrix}_{-4/3}^{4/3}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.14e}$$

$$\int_0^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \left(\frac{d\phi_1}{dx}, \quad \frac{d\phi_2}{dx} \right) dx = \frac{1}{3.2}\int_{-4/3}^{4/3} \begin{pmatrix} 1-\xi/0.8 \\ 1+\xi/0.8 \end{pmatrix} (-1, \quad 1)\, d\xi$$

$$= \frac{1}{3.2}\int_{-4/3}^{4/3} \begin{pmatrix} -(1-\xi/0.8), (1-\xi/0.8) \\ -(1+\xi/0.8), \ (1+\xi/0.8) \end{pmatrix} d\xi$$

$$= \frac{1}{8}\begin{pmatrix} (1-\xi/0.8)^2, -(1-\xi/0.8)^2 \\ -(1+\xi/0.8)^2, \ (1+\xi/0.8)^2 \end{pmatrix}_{-4/3}^{4/3} = \frac{1}{1.2}\begin{pmatrix} -1, 1 \\ -1, \ 1 \end{pmatrix} \tag{2.15e}$$

$$\begin{pmatrix} \phi_1(-4/3) \\ \phi_2(-4/3) \end{pmatrix} (\phi_1(-4/3), \quad \phi_2(-4/3)) = \begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix} (4/3, \quad -1/3)$$

$$= \frac{1}{9}\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix} \tag{2.16e}$$

The above results are combined to yield a matrix equation for the first element,

$$\frac{1}{1.2}\begin{pmatrix} -1, 1 \\ -1, \ 1 \end{pmatrix} + \frac{1}{9}\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix} = \frac{1}{3.6}\begin{pmatrix} -3, 3 \\ -3, \ 3 \end{pmatrix} + \frac{1}{3.6}\begin{pmatrix} 6.4, -1.6 \\ -1.6, \ 0.4 \end{pmatrix}$$

$$= \frac{1}{3.6}\begin{pmatrix} 3.4, 1.4 \\ -4.6, \ 3.4 \end{pmatrix} \tag{2.18e}$$

$$\frac{1}{3.6}\begin{pmatrix} 3.4, 1.4 \\ -4.6, \ 3.4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.17e}$$

Equation 2.17 is then solved to obtain the numerical solution,

$$\begin{pmatrix} 3.4 & 1.4 \\ -4.6 & 3.4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 1.8 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} \tag{2.19e}$$

We see that $u_1 = u(x = 0.2) = 0.2$ and $u_2 = u(x = 0.8) = 0.8$, which match with the exact solutions. It is a simple exercise to show that the calculations for the second element yield the same results as in Example 2.1.

Example 2.3. Consider this internal radiation problem, defined by the following differential equation and boundary condition:

$$\frac{du}{dx}+u=1\,,\quad \text{with} \quad u(x=0)=0 \tag{2.20e}$$

Obtain the numerical solution using the discontinuous Galerkin finite element formulation with two linear discontinuous finite elements, and compare the results with the exact solution.

Solution. We first obtain the analytical solution to the problem above. The problem is solved by direct integration and the solution is $u(x) = 1- e^{-x}$. Application of the discontinuous finite element formulation for the first element gives the result,

$$\int_0^1 \left[\frac{du_h}{dx}+u_h-1\right]\phi_i(x)\,dx+u_0^+\,\phi_i(0)=0 \tag{2.21e}$$

where we have applied $u_0^- = u(0^-) = 0$. Now with $u_0^+ = u_1 = 0$, $\phi_1(x = 0) = \phi_1(\xi = -1)$, $u(x^- = 1) = u_2$ substituted, one has

$$\int_0^1 \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\left(\frac{\partial\phi_1}{\partial x},\frac{\partial\phi_2}{\partial x}\right)dx\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\int_0^1\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}(\phi_1,\phi_2)\,dx\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$
$$=\int_0^1\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}dx \tag{2.22e}$$

The detailed calculations are the same as before, whence we have the following expressions:

$$\int_0^1\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}dx=\frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix};\qquad \int_0^1\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\left(\frac{\partial\phi_1}{\partial x},\ \frac{\partial\phi_2}{\partial x}\right)dx=\frac{1}{2}\begin{pmatrix}-1, & 1\\ -1, & 1\end{pmatrix} \tag{2.23e}$$

The additional term comes from the treatment of $u(x)$,

$$\int_{x_j}^{x_{j+1}}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}(\phi_1,\phi_2)\,dx=\frac{1}{8}\int_{-1}^{1}\begin{pmatrix}1-\xi\\ 1+\xi\end{pmatrix}(1-\xi,\ 1+\xi)\,d\xi$$
$$=\frac{1}{8}\int_{-1}^{1}\begin{pmatrix}(1-\xi)^2, & (1-\xi^2)\\ (1-\xi^2), & (1+\xi)^2\end{pmatrix}d\xi=\frac{1}{8}\begin{pmatrix}-\frac{1}{3}(1-\xi)^3, & \left(\xi-\frac{1}{3}\xi^3\right)\\ \left(\xi-\frac{1}{3}\xi^3\right), & \frac{1}{3}(1+\xi)^3\end{pmatrix}_{-1}^{1}$$
$$=\frac{1}{8}\begin{pmatrix}\frac{8}{3}, & \frac{4}{3}\\ \frac{4}{3}, & \frac{8}{3}\end{pmatrix}=\frac{1}{6}\begin{pmatrix}2, & 1\\ 1, & 2\end{pmatrix} \tag{2.24e}$$

Assembling these expressions into Equation 2.33, one has the matrix equation,

$$\frac{1}{2}\begin{bmatrix} -1, 1 \\ -1,\ 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{1}{6}\begin{bmatrix} 2, 1 \\ 1,\ 2 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.25e}$$

Simplifying, we have the following numerical results for the first element:

$$\begin{pmatrix} 5, 4 \\ -2, 5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad => \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u(0^+) \\ u(1^-) \end{pmatrix} = \begin{pmatrix} 0.091 \\ 0.636 \end{pmatrix} \tag{2.26e}$$

where $u_1 = u(0^+)$ and $u_2 = u(1^-)$. This compares with the analytical solution: $u(0) = 0$ and $u(1) = 0.632$.

For the second element, the same procedure is applied with the result,

$$\frac{1}{2}\begin{pmatrix} -1, 1 \\ -1,\ 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{1}{6}\begin{bmatrix} 2, 1 \\ 1,\ 2 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_1^- \\ u_2 \end{pmatrix} \tag{2.27e}$$

Rearranging and setting $u_1^- = u(1^-)$, we have the numerical values for the second element,

$$\begin{pmatrix} 5, 4 \\ -2, 5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 75/11 \\ 3 \end{pmatrix} \quad => \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u(1^+) \\ u(2^-) \end{pmatrix} = \begin{pmatrix} 0.669 \\ 0.868 \end{pmatrix} \tag{2.28e}$$

which compares with the analytical solution: $u(2) = 0.865$.

For this simple example, the solutions can be readily obtained using the continuous finite element method or the finite difference method. For comparison, the numerical results using different methods are listed in Table 2.1e, and compared with those calculated using the analytic solution.

Table 2.1e. Comparison of numeric results with analytical solution

x	0	1	2
DFEM $u(x)$	0.04545	0.653	0.868
Analytic $(1-e^{-x})$	0	0.632	0.865
DFEM $u(x^-)$	0	0.636	0.868
FEM	0	0.643	0.857
FD	0	0.500	0.750

In Table 2.1e, the values of DFEM $u(x)$ are obtained using the averaged quantities across the element boundary: that is, $u(x) = 0.5(u(x^-) + u(x^+))$. The solution is better approximated if we take $u(x) = u(x^-)$, as shown by those given in the row associated with DFEM $u(x^{-})$. The standard continuous finite element solution (FEM) is reasonably good, although not as good as the discontinuous finite element solution (DFEM $u(x^{-})$). The standard finite difference approximation (FD), with upwinding, seems to be least accurate for this problem.

Example 2.4. Consider a two-dimensional convection problem defined by the following differential equation and boundary condition:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \quad x\in[-\pi,\pi]\times[-\pi,\pi]\times(0,T] \tag{2.29e}$$

with periodic boundary conditions and initial data

$$u(x,y,t=0)=\sin(\pi x)\sin(\pi y) \tag{2.30e}$$

Obtain the numerical solutions using the discontinuous finite element method and discuss parallel computing performance.

Solution. This problem was solved by Biswas *et al.* [18], and is used here as an example to demonstrate the parallel performance of the discontinuous finite element method. Their algorithm employed a discontinuous Galerkin finite element discretization, with a basis of piecewise Legendre polynomials. Temporal discretization employes a Runge–Kutta method. Dissipative fluxes and projection limiting prevent oscillations near the solution discontinuities. Parallel computing used from 1 to 256 processors. The computed results are given in Table 2.2e. It is seen from the results that, as the number of processors increases while keeping the work per processor constant, the discontinuous finite element method achieves a very impressive parallel computing performance.

Table 2.2e. Scaled parallel efficiency: solution times (without I/O) and total execution times, measured on the nCUEE/2

Number of processors	Work (W)	Solution time (s)	Solution parallel efficiency	Total time (s)	Total parallel efficiency
1	18432	926.92	–	927.16	–
2	36864	927.06	99.98%	927.31	99.98%
4	73728	927.13	99.97%	927.45	99.96%
8	147456	927.17	99.97%	927.58	99.95%
16	294912	927.38	99.95%	928.13	99.89%
32	589824	927.89	99.89%	929.90	99.70%
64	1179648	928.63	99.81%	931.28	99.55%
128	2359296	930.14	99.65%	937.67	98.88%
256	4718592	933.97	99.24%	950.25	97.57%

Exercises

1. Show that when a delta function is chosen as the weighting function, the Weighted Residuals formulation gives the finite volume scheme.
2. Solve the problem defined by Equation 2.1e using five linear elements and compare with the analytic solution.
3. Solve Example 2.1e using five linear continuous finite elements and five finite volume cells. Compare the results with the results in Exercise 1 and the analytical solution.
4. Complete the calculations in Example 2.2e for the second element and compare with the analytic solution.
5. Apply a discontinuous finite element formulation to solve the problem defined by Equation 2.20e when the domain is discretized into six linear elements.
6. Solve Equation 2.20e using six linear continuous finite elements and six finite volume cells respectively and compare the results with those obtained in Exercise 5.
7. Use discontinuous finite element formulation and three quadratic elements to solve Equation 2.20e and compare with the results obtained in Exercise 5.
8. Develop a computer code for a discontinuous finite element solution to Equation 2.20e, and compare the results obtained from the code with those calculated in Exercises 6 and 7.

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