ALGEBRA AND GEOMETRY

This text gives a basic introduction and a unified approach to algebra and geometry. It covers the ideas of complex numbers, scalar and vector products, determinants, linear algebra, group theory, permutation groups, symmetry groups and various aspects of geometry including groups of isometries, rotations and spherical geometry. The emphasis is always on the interaction between these topics, and each one is constantly illustrated by using it to describe and discuss the others. Many of the ideas are developed gradually throughout the book. For example, the definition of a group is given in Chapter 1 so that it can be used in a discussion of the arithmetic of real and complex numbers; however, many of the properties of groups are given later, and at a time when the importance of the concept has become clear. The text is divided into short sections, with exercises at the end of each one.
ALGEBRA AND GEOMETRY

ALAN F. BEARDON
To Dylan, Harry, Fionn and Fenella
## Contents

### Preface

- Groups and permutations  
  - 1.1 Introduction  
  - 1.2 Groups  
  - 1.3 Permutations of a finite set  
  - 1.4 The sign of a permutation  
  - 1.5 Permutations of an arbitrary set  

### 2 The real numbers

- 2.1 The integers  
- 2.2 The real numbers  
- 2.3 Fields  
- 2.4 Modular arithmetic  

### 3 The complex plane

- 3.1 Complex numbers  
- 3.2 Polar coordinates  
- 3.3 Lines and circles  
- 3.4 Isometries of the plane  
- 3.5 Roots of unity  
- 3.6 Cubic and quartic equations  
- 3.7 The Fundamental Theorem of Algebra  

### 4 Vectors in three-dimensional space

- 4.1 Vectors  
- 4.2 The scalar product  
- 4.3 The vector product  
- 4.4 The scalar triple product  

---

© Cambridge University Press  
www.cambridge.org
## Contents

4.5 The vector triple product 62  
4.6 Orientation and determinants 63  
4.7 Applications to geometry 68  
4.8 Vector equations 72  

5 Spherical geometry 74  
5.1 Spherical distance 74  
5.2 Spherical trigonometry 75  
5.3 Area on the sphere 77  
5.4 Euler’s formula 79  
5.5 Regular polyhedra 83  
5.6 General polyhedra 85  

6 Quaternions and isometries 89  
6.1 Isometries of Euclidean space 89  
6.2 Quaternions 95  
6.3 Reflections and rotations 99  

7 Vector spaces 102  
7.1 Vector spaces 102  
7.2 Dimension 106  
7.3 Subspaces 111  
7.4 The direct sum of two subspaces 115  
7.5 Linear difference equations 118  
7.6 The vector space of polynomials 120  
7.7 Linear transformations 124  
7.8 The kernel of a linear transformation 127  
7.9 Isomorphisms 130  
7.10 The space of linear maps 132  

8 Linear equations 135  
8.1 Hyperplanes 135  
8.2 Homogeneous linear equations 136  
8.3 Row rank and column rank 139  
8.4 Inhomogeneous linear equations 141  
8.5 Determinants and linear equations 143  
8.6 Determinants 144  

9 Matrices 149  
9.1 The vector space of matrices 149  
9.2 A matrix as a linear transformation 154  
9.3 The matrix of a linear transformation 158
Contents

9.4 Inverse maps and matrices 163
9.5 Change of bases 167
9.6 The resultant of two polynomials 170
9.7 The number of surjections 173

10 Eigenvectors 175
10.1 Eigenvalues and eigenvectors 175
10.2 Eigenvalues and matrices 180
10.3 Diagonalizable matrices 184
10.4 The Cayley–Hamilton theorem 189
10.5 Invariant planes 193

11 Linear maps of Euclidean space 197
11.1 Distance in Euclidean space 197
11.2 Orthogonal maps 198
11.3 Isometries of Euclidean n-space 204
11.4 Symmetric matrices 206
11.5 The field axioms 211
11.6 Vector products in higher dimensions 212

12 Groups 215
12.1 Groups 215
12.2 Subgroups and cosets 218
12.3 Lagrange’s theorem 223
12.4 Isomorphisms 225
12.5 Cyclic groups 230
12.6 Applications to arithmetic 232
12.7 Product groups 235
12.8 Dihedral groups 237
12.9 Groups of small order 240
12.10 Conjugation 242
12.11 Homomorphisms 246
12.12 Quotient groups 249

13 Möbius transformations 254
13.1 Möbius transformations 254
13.2 Fixed points and uniqueness 259
13.3 Circles and lines 261
13.4 Cross-ratios 265
13.5 Möbius maps and permutations 268
13.6 Complex lines 271
13.7 Fixed points and eigenvectors 273
13.8 A geometric view of infinity 276
13.9 Rotations of the sphere 279

14 Group actions 284
14.1 Groups of permutations 284
14.2 Symmetries of a regular polyhedron 290
14.3 Finite rotation groups in space 295
14.4 Groups of isometries of the plane 297
14.5 Group actions 303

15 Hyperbolic geometry 307
15.1 The hyperbolic plane 307
15.2 The hyperbolic distance 310
15.3 Hyperbolic circles 313
15.4 Hyperbolic trigonometry 315
15.5 Hyperbolic three-dimensional space 317
15.6 Finite Möbius groups 319

Index 320
Nothing can permanently please, which does not contain in itself the reason why it is so, and not otherwise

S.T. Coleridge, 1772–1834

The idea for this text came after I had given a lecture to undergraduates on the symmetry groups of regular solids. It is a beautiful subject, so why was I unhappy with the outcome? I had covered the subject in a more or less standard way, but as I came away I became aware that I had assumed Euler’s theorem on polyhedra, I had assumed that every symmetry of a polyhedron extended to an isometry of space, and that such an isometry was necessarily a rotation or a reflection (again due to Euler), and finally, I had not given any convincing reason why such polyhedra did actually exist. Surely these ideas are at least as important (or perhaps more so) than the mere identification of the symmetry groups of the polyhedra?

The primary aim of this text is to present many of the ideas and results that are typically given in a university course in mathematics in a way that emphasizes the coherence and mutual interaction within the subject as a whole. We believe that by taking this approach, students will be able to support the parts of the subject that they find most difficult with ideas that they can grasp, and that the unity of the subject will lead to a better understanding of mathematics as a whole. Inevitably, this approach will not take the reader as far down any particular road as a single course in, say, group theory might, but we believe that this is the right approach for a student who is beginning a university course in mathematics. Increasingly, students will be taking more and more courses outside mathematics, and the pressure to include a wide spread of mathematics within a limited time scale will increase. We believe that the route advocated above will, in addition to being educationally desirable, help solve this problem.
To illustrate our approach, consider once again the symmetries of the five (regular) Platonic solids. These symmetries may be viewed as examples of permutations (acting on the vertices, or the faces, or even on the diagonals) of the solid, but they can also be viewed as finite groups of rotations of Euclidean 3-space. This latter point of view suggests that the discussion should lead into, or away from, a discussion of the nature of isometries of 3-space, for this is fundamental to the very definition of the symmetry groups. From a different point of view, probably the easiest way to identify the Platonic solids is by means of Euler’s formula for the sphere. Now Euler’s formula can be (and here is) proved by means of spherical geometry and trigonometry, and the requisite formulae here are simple (and important) applications of the standard scalar and vector product of the ‘usual’ vectors in 3-space (as studied in applied mathematics). Next, by studying rotation groups acting on the unit sphere in 3-space one can prove that the symmetry groups of the regular solids are the only finite groups of rotations of 3-space, a fact that it not immediately apparent from the geometry. Finally, by using stereographic projection (as appears in any complex analysis course that acknowledges the point at infinity) the symmetry groups of the regular solids appear as the only finite groups of Möbius transformations acting in hyperbolic space. Moreover in this guise one can also introduce rotations of 3-space in terms of quaternions which then appear as 2-by-2 complex matrices.

The author firmly believes that this is the way mathematics should be introduced, and moreover that it can be so introduced at a reasonably elementary level. In many cases, students find mathematics difficult because they fail to grasp the initial concepts properly, and in this approach preference is given to understanding and reinforcing these basic concepts from a variety of different points of view rather than moving on in the traditional way to provide yet more theorems that the student has to try to cope with from a sometimes uncertain base.

This text includes the basic definitions, and some early results, on, for example, groups, vector spaces, quaternions, eigenvectors, the diagonalization of matrices, orthogonal groups, isometries of the complex plane and of Euclidean space, scalar and vector products in 3-space, Euclidean, spherical and (briefly) hyperbolic geometries, complex numbers and Möbius transformations. Above all, it is these basic concepts and their mutual interaction which is the main theme of this text.

Finally an earlier version of this book can be freely downloaded as an html file from http://www.cambridge.org/0521890497. This file is under development and the aim is to create a fully linked electronic textbook.