Preface

Since the last century, the postulational method and an abstract point of view have played a vital role in the development of modern mathematics. The experience gained from the earlier concrete studies of analysis point to the importance of passage to the limit. The basis of this operation is the notion of distance between any two points of the line or the complex plane. The algebraic properties of underlying sets often play no role in the development of analysis; this situation naturally leads to the study of metric spaces. The abstraction not only simplifies and elucidates mathematical ideas that recur in different guises, but also helps economize the intellectual effort involved in learning them. However, such an abstract approach is likely to overlook the special features of particular mathematical developments, especially those not taken into account while forming the larger picture. Hence, the study of particular mathematical developments is hard to overemphasize.

The language in which a large body of ideas and results of functional analysis are expressed is that of *metric spaces*. The books on functional analysis seem to go over the preliminaries of this topic far too quickly. The present authors attempt to provide a leisurely approach to the theory of metric spaces. In order to ensure that the ideas take root gradually but firmly, a large number of examples and counterexamples follow each definition. Also included are several worked examples and exercises. Applications of the theory are spread out over the entire book.

The book treats material concerning metric spaces that is crucial for any advanced level course in analysis. Chapter 0 is devoted to a review and systematisation of properties which we shall generalize or use later in the book. It includes the Cantor construction of real numbers. In Chapter 1, we introduce the basic ideas of metric spaces and Cauchy sequences and discuss the completion of a metric space. The topology of metric spaces, Baire's category theorem and its applications, including the existence of a continuous, nowhere differentiable function and an explicit example of such a function, are discussed in Chapter 2. Continuous mappings, uniform convergence of sequences and series of functions, the contraction mapping principle and applications are discussed in Chapter 3. The concepts of connected, locally connected and arcwise connected spaces are explained in Chapter 4. The characterizations of connected subsets of the reals and arcwise connected subsets of the plane are also in Chapter 4. The notion of compactness, together with its equivalent characterisations, is included in Chapter 5. Also contained in this chapter are characterisations of compact subsets of special metric spaces. In Chapter 6, we discuss product metric spaces and provide a proof of Tychonoff's theorem.

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2 Topology of a Metric Space

The real number system has two types of properties. The first type are algebraic properties, dealing with addition, multiplication and so on. The other type, called topological properties, have to do with the notion of distance between numbers and with the concept of limit. In this chapter, we study topological properties in the framework of metric spaces. We begin by looking at the notions of open and closed sets, limit points, closure and interior of a set and some elementary results involving them. The concept of base of a metric topology and related ideas are also discussed. In the final section, we deal with the important concept of category due to Baire and its usefulness in existence proofs. Also included are some theorems due to Baire.

2.1. Open and Closed Sets

There are special types of sets that play a distinguished role in analysis; these are the open and closed sets. To expedite the discussion, it is helpful to have the notion of a neighbourhood in metric spaces.

Definition 2.1.1. Let (X, d) be a metric space. The set

$$S(x_0, r) = \{x \in X : d(x_0, x) < r\}, \qquad \text{where } r > 0 \text{ and } x_0 \in X,$$

is called the **open ball** of radius r and centre x_0. The set

$$\bar{S}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}, \qquad \text{where } r > 0 \text{ and } x_0 \in X,$$

is called the **closed ball** of radius r and centre x_0.

A few concrete examples are in order.

Examples 2.1.2. (i) The open ball $S(x_0, r)$ on the real line is the bounded open interval $(x_0 - r,\ x_0 + r)$ with midpoint x_0 and total length $2r$. Conversely, it is clear that any bounded open interval on the real line is an open ball. So the open balls on the real line are precisely the bounded open intervals. The closed balls $\bar{S}(x_0, r)$ on the real line are precisely the bounded closed intervals but containing more than one point.

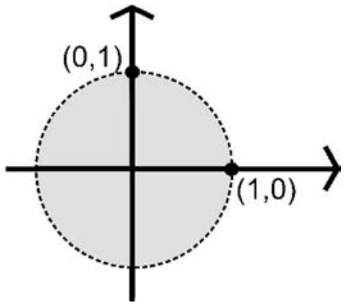

FIGURE 2.1

(ii) The open ball $S(x_0, r)$ in $\mathbf{R}^2$ with metric d_2 (see Example 1.2.2(iii)) is the inside of the circle with centre x_0 and radius r as in Fig. 2.1. Open balls of radius 1 and centre (0,0), when the metric is d_1 or d_∞ (see Example 1.2.2(iv) for the latter) are illustrated in Figs. 2.2 and 2.3.

(iii) If (X, d) denotes the discrete metric space (see Example 1.2.2(v)), then $S(x, r) = \{x\}$ for all $x \in X$ and any positive $r \le 1$, whereas $S(x, r) = X$ for all $x \in X$ and any $r > 1$.

(iv) Consider the metric space $C_{\mathbf{R}}[a, b]$ of Example 1.2.2(ix). The open ball $S(x_0, r)$, where $x_0 \in C_{\mathbf{R}}[a, b]$ and $r > 0$, consists of all continuous functions $x \in C_{\mathbf{R}}[a, b]$ whose graphs lie within a band of vertical width $2r$ and is centred around the graph of x_0. (See Fig. 2.4.)

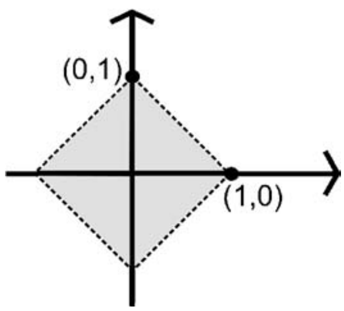

FIGURE 2.2

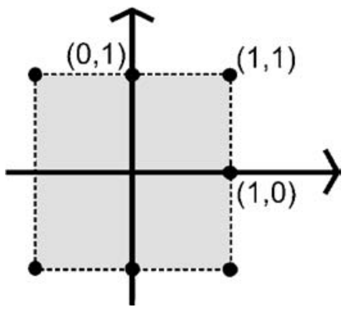

FIGURE 2.3

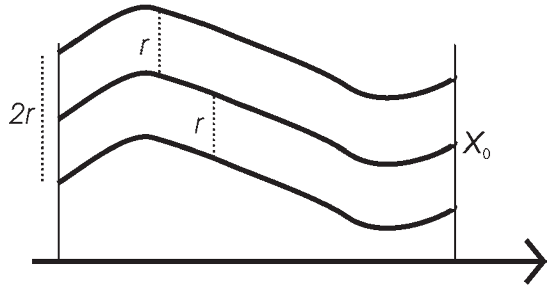

FIGURE 2.4

Definition 2.1.3. Let (X, d) be a metric space. A **neighbourhood** of the point $x_0 \in X$ is any open ball in (X, d) with centre x_0.

Definition 2.1.4. A subset G of a metric space (X, d) is said to be **open** if given any point $x \in G$, there exists $r > 0$ such that $S(x, r) \subseteq G$, i.e., each point of G is the centre of some open ball contained in G. Equivalently, every point of the set has a neighbourhood contained in the set.

Theorem 2.1.5. In any metric space (X, d), each open ball is an open set.

Proof. First observe that $S(x,r)$ is nonempty, since $x \in S(x, r)$. Let $y \in S(x, r)$, so that $d(y, x) < r$, and let $r' = r - d(y, x) > 0$. We shall show that $S(y, r') \subseteq S(x, r)$, as illustrated in Fig. 2.5. Consider any $z \in S(y, r')$. Then we have

$$d(z, x) \leq d(z, y) + d(y, x) < r' + d(y, x) = r,$$

which means $z \in S(x, r)$. Thus, for each $y \in S(x, r)$, there is an open ball $S(y, r') \subseteq S(x, r)$. Therefore $S(x, r)$ is an open subset of X. □

Examples 2.1.6. (i) In **R**, any bounded open interval is an open subset because it is an open ball. It is easy to see that even an unbounded open interval is an open subset.

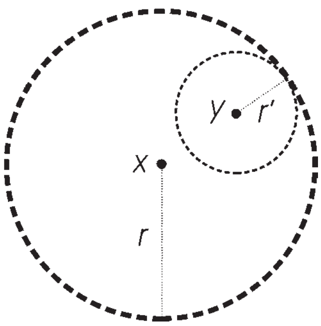

FIGURE 2.5

(ii) In ℓ_2, let $G = \{x = \{x_i\}_{i\geq 1}: \sum_1^\infty |x_i|^2 < 1\}$. Then G is an open subset of ℓ_2. Indeed, $G = S(0,1)$ is the open ball with centre $0 = (0,0,\dots)$ and radius 1. It is now a consequence of Theorem 2.1.5 that G is open.

(iii) In a discrete metric space X, any subset G is open, because any $x \in G$ is the centre of the open ball $S(x, 1/2)$ which is nothing but $\{x\}$.

The following are fundamental properties of open sets.

Theorem 2.1.7. Let (X, d) be a metric space. Then

(i) $\varnothing$ and X are open sets in (X, d);
(ii) the union of any finite, countable or uncountable family of open sets is open;
(iii) the intersection of any finite family of open sets is open.

Proof. (i) As the empty set contains no points, the requirement that each point in $\varnothing$ is the centre of an open ball contained in it is automatically satisfied. The whole space X is open, since every open ball centred at any of its points is contained in X.

(ii) Let $\{G_\alpha: \alpha \in \Lambda\}$ be an arbitrary family of open sets and $H = \cup_{\alpha\in\Lambda} G_\alpha$. If H is empty, then it is open by part (i). So assume H to be nonempty and consider any $x \in H$. Then $x \in G_\alpha$ for some $\alpha \in \Lambda$. Since G_α is open, there exists an $r > 0$ such that $S(x, r) \subseteq G_\alpha \subseteq H$. Thus, for each $x \in H$ there exists an $r > 0$ such that $S(x, r) \subseteq H$. Consequently, H is open.

(iii) Let $\{G_i: 1 \leq i \leq n\}$ be a finite family of open sets in X and let $G = \cap_{i=1}^n G_i$. If G is empty, then it is open by part (i). Suppose G is nonempty and let $x \in G$. Then $x \in G_j, j = 1, \dots, n$. Since G_j is open, there exists $r_j > 0$ such that $S(x, r_j) \subseteq G_j, j = 1, \dots, n$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then $r > 0$ and $S(x, r) \subseteq S(x, r_j), j = 1, \dots, n$. Therefore the ball $S(x,r)$ centred at x satisfies

$$S(x,\ r) \subseteq \bigcap_{j=1}^{n} S(x,\ r_j) \subseteq G.$$

This completes the proof. □

Remark 2.1.8. The intersection of an infinite number of open sets need not be open. To see why, let $S_n = S(0, \frac{1}{n}) \subseteq \mathbf{C},\ n = 1,\ 2, \dots$. Each S_n is an open ball in the complex plane and hence an open set in $\mathbf{C}$. However,

$$\bigcap_{n=1}^{\infty} S_n = \{0\},$$

which is not open, since there exists no open ball in the complex plane with centre 0 that is contained in $\{0\}$.

The following theorem characterises open subsets in a metric space.

Theorem 2.1.9. A subset G in a metric space (X, d) is open if and only if it is the union of all open balls contained in G.

Proof. Suppose that G is open. If G is empty, then there are no open balls contained in it. Thus, the union of all open balls contained in G is a union of an empty class, which is empty and therefore equal to G. If G is nonempty, then since G is open, each of its points is the centre of an open ball contained entirely in G. So, G is the union of all open balls contained in it. The converse follows immediately from Theorem 2.1.5 and Theorem 2.1.7. □

Remark 2.1.10. The above Theorem 2.1.9 describes the structure of open sets in a metric space. This information is the best possible in an arbitrary metric space. For open subsets of **R**, Theorem 2.1.9 can be improved.

Theorem 2.1.11. Each nonempty open subset of **R** is the union of a countable family of disjoint open intervals. Moreover, the endpoints of any open interval in the family lie in the complement of the set and are no less than the infimum and no greater than the supremum of the set.

Proof. Let G be a nonempty open subset of **R** and let $x \in G$. Since G is open, there exists a bounded open interval with centre x and contained in G. So there exists some $y > x$ such that $(x, y) \subseteq G$ and some $z < x$ such that $(z, x) \subseteq G$. Let

$$a = \inf\{z : (z, x) \subseteq G\} \text{ and } b = \sup\{y : (x, y) \subseteq G\}. \qquad (1)$$

Then $a < x < b$ and $I_x = (a, b)$ is an open interval containing x. We shall show that $a \notin G$, $b \notin G$ and $I_x \subseteq G$. This is obvious if $a = -\infty$ or if $b = \infty$. So, assume $-\infty < a$ and $\infty > b$. If a were to be in G, we would have $(a - \varepsilon, a + \varepsilon) \subseteq G$ for some $\varepsilon > 0$, whence we would also have $(a - \varepsilon, x) \subseteq G$, contradicting (1). The argument that $b \notin G$ is similar. Now suppose $w \in I_x$ we shall show that $w \in G$. If $w = x$, then of course $w \in G$. Let $w \neq x$, so that either $a < w < x$ or $x < w < b$. We need consider only the former case: $a < w < x$. Since $a < w$, it follows from (1) that there exists some $z < w$ such that $(z, x) \subseteq G$. Since $w < x$, this implies that $w \in G$.

Consider the collection of open intervals $\{I_x\}$, $x \in G$. Since each $x \in G$ is contained in I_x and each I_x is contained in G, it follows that $G = \bigcup\{I_x : x \in G\}$. We shall next show that any two intervals in the collection $\{I_x : x \in G\}$ are disjoint. Let (a, b) and (c, d) be two intervals in this collection with a point in common. Then we must have $c < b$ and $a < d$. Since c does not belong to G, it does not belong to (a, b) and so $c \leq a$. Since a does not belong to G, and hence also does not belong to (c, d), we also have $a \leq c$. Therefore, $c = a$. Similarly, $b = d$, which shows that (a,b) and (c, d) are actually the same interval. Thus, $\{I_x : x \in G\}$ is a collection of disjoint intervals.

Now we establish that the collection is countable. Each nonempty open interval contains a rational number. Since disjoint intervals cannot contain the same number and the rationals are countable, it follows that the collection $\{I_x : x \in G\}$ is countable.

Finally, we note that it follows from (1) that $a \geq \inf G$ and $b \leq \sup G$. □

Definition 2.1.12. Let A be a subset of a metric space (X, d). A point $x \in X$ is called an **interior point** of A if there exists an open ball with centre x contained in A, i.e.,

$$x \in S(x, r) \subseteq A \text{ for some } r > 0,$$

or equivalently, if x has a neighbourhood contained in A. The set of all interior points of A is called the **interior of** A and is denoted either by Int(A) or A°. Thus

$$\text{Int}(A) = A^\circ = \{x \in A: S(x, r) \subseteq A \text{ for some } r > 0\}.$$

Observe that Int(A) $\subseteq A$.

Example 2.1.13. The interior of the subset $[0,\ 1] \subseteq \mathbf{R}$ can be shown to be (0,1). Let $x \in (0, 1)$. Since (0,1) is open, there exists $r > 0$ such that $(x - r, x + r) \subseteq [0, 1]$. So, x is an interior point of [0,1]. Also, 0 is not an interior point of [0,1], because there exists no $r > 0$ such that $(-\ r,\ r) \subseteq [0,\ 1]$. Similarly, 1 is also not an interior point of $[0, 1]$.

The next theorem relates interiors to open sets and provides a characterisation of open subsets in terms of interiors.

Theorem 2.1.14. Let A be a subset of a metric space (X, d). Then

(i) A° is an open subset of A that contains every open subset of A;
(ii) A is open if and only if $A = A^\circ$.

Proof. (i) Let $x \in A^\circ$ be arbitrary. Then, by definition, there exists an open ball $S(x,\ r) \subseteq A$. But $S(x,r)$ being an open set (see Theorem 2.1.5), each point of it is the centre of some open ball contained in $S(x,r)$ and consequently also contained in A. Therefore each point of $S(x,r)$ is an interior point of A, i.e., $S(x,\ r) \subseteq A^\circ$. Thus, x is the centre of an open ball contained in A°. Since $x \in A^\circ$ is arbitrary, it follows that each $x \in A^\circ$ has the property of being the centre of an open ball contained in A°. Hence, A° is open.

It remains to show that A° contains every open subset $G \subseteq A$. Let $x \in G$. Since G is open, there exists an open ball $S(x,\ r) \subseteq G \subseteq A$. So $x \in A^\circ$. This shows that $x \in G \Rightarrow x \in A^\circ$. In other words, $G \subseteq A^\circ$.

(ii) is immediate from (i). □

The following are basic properties of interiors.

Theorem 2.1.15. Let (X, d) be a metric space and A, B be subsets of X. Then

(i) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$;
(ii) $(A \cap B)^\circ = A^\circ \cap B^\circ$;
(iii) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$.

Proof. (i) Let $x \in A^\circ$. Then there exists an $r > 0$ such that $S(x, r) \subseteq A$. Since $A \subseteq B$, we have $S(x, r) \subseteq B$, i.e., $x \in B^\circ$.

(ii) $A \cap B \subseteq A$ as well as $A \cap B \subseteq B$. It follows from (i) that $(A \cap B)^\circ \subseteq A^\circ$ as well as $(A \cap B)^\circ \subseteq B^\circ$, which implies that $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. On the other hand, let $x \in A^\circ \cap B^\circ$. Then $x \in A^\circ$ and $x \in B^\circ$. Therefore, there exist $r_1 > 0$ and $r_2 > 0$ such that $S(x, r_1) \subseteq A$ and $S(x, r_2) \subseteq B$. Let $r = \min\{r_1, r_2\}$. Clearly, $r > 0$ and $S(x, r) \subseteq A \cap B$, i.e., $x \in (A \cap B)^\circ$.

(iii) $A \subseteq A \cup B$ as well as $B \subseteq A \cup B$. Now apply (i). □

Remark 2.1.16. The following example shows that $(A \cup B)^\circ$ need not be the same as $A^\circ \cup B^\circ$. Indeed, if $A = [0, 1]$ and $B = [1, 2]$, then $A \cup B = [0, 2]$. Since $A^\circ = (0, 1), B^\circ = (1, 2)$ and $(A \cup B)^\circ = (0, 2)$, we have $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

Definition 2.1.17. Let X be a metric space and F a subset of X. A point $x \in X$ is called a **limit point** of F if each open ball with centre x contains at least one point of F different from x, i.e.,

$$(S(x, r) - \{x\}) \cap F \neq \emptyset.$$

The set of all limit points of F is denoted by F' and is called the **derived set** of F.

Examples 2.1.18. (i) The subset $F = \{1,\ 1/2,\ 1/3, \ldots\}$ of the real line has 0 as a limit point; in fact, 0 is its only limit point. Thus the derived set of F is $\{0\}$, i.e., $F' = \{0\}$.

(ii) The subset $\mathbf{Z}$ of integers of the real line, consisting of all the integers, has no limit point. Its derived set $\mathbf{Z}'$ is $\emptyset$.

(iii) Each real number is a limit point of the subset of rationals: $\mathbf{Q}' = \mathbf{R}$.

(iv) If (X, d) is a discrete metric space (see Example 1.2.2(v)) and $F \subseteq X$, then F has no limit points, since every open ball of radius 1 consists only of the centre.

(v) Consider the subset $F = \{(x, y) \in \mathbf{C} : x > 0, y > 0\}$ of the complex plane. Each point of the subset $\{(x, y) \in \mathbf{C} : x \geq 0,\ y \geq 0\}$ is a limit point of F. In fact, the latter set is precisely F'.

(vi) For an interval $I \subseteq \mathbf{R}$, the set I' consists of not only all the points of I but also any endpoints I may have, even if they do not belong to I. Thus $(0, 1)' = (0, 1]' = [0, 1)' = [0, 1]' = [0, 1]$.

Proposition 2.1.19. Let (X, d) be a metric space and $F \subseteq X$. If x_0 is a limit point of F, then every open ball $S(x_0, r),\ r > 0$, contains an infinite number of points of F.

Proof. Suppose that the ball $S(x_0, r)$ contains only a finite number of points of F. Let $y_1, y_2, \ldots, y_n$ denote the points of $S(x_0, r) \cap F$ that are distinct from x_0. Let

$$\delta = \min\{d(y_1, x_0),\ d(y_2, x_0), \ldots, d(y_n, x_0)\}.$$

Then the ball $S(x_0, \delta)$ contains no point of F distinct from x_0, contradicting the assumption that x_0 is a limit point of F. □

The following characterisation of the limit points of a set in a metric space is useful.

Proposition 2.1.20. Let (X, d) be a metric space and $F \subseteq X$. Then a point x_0 is a limit point of F if and only if it is possible to select from the set F a sequence of distinct points $x_1, x_2, \ldots, x_n, \ldots$ such that $\lim_n d(x_n, x_0) = 0$.

Proof. If $\lim_n d(x_n, x_0) = 0$, where $x_1, x_2, \ldots, x_n, \ldots$ is a sequence of distinct points of F, then every ball $S(x_0, r)$ with centre x_0 and radius r contains each of x_n, where $n \geq n_0$ for some suitably chosen n_0. As $x_1, x_2, \ldots, x_n, \ldots$ in F are distinct, it follows that $S(x_0, r)$ contains a point of F different from x_0. So, x_0 is a limit point of F.

On the other hand, assume that x_0 is a limit point of F. Choose a point $x_1 \in F$ in the open ball $S(x_0, 1)$ such that x_1 is different from x_0. Next, choose a point $x_2 \in F$ in the open ball $S(x_0, 1/2)$ different from x_0 as well as from x_1; this is possible by Proposition 2.1.19. Continuing this process in which, at the nth step of the process we choose a point $x_n \in F$ in $S(x_0, 1/n)$ different from $x_1, x_2, \ldots, x_{n-1}$, we have a sequence $\{x_n\}$ of distinct points of the set F such that $\lim_n d(x_n, x_0) = 0$. □

Definition 2.1.21. A subset F of the metric space (X, d) is said to be **closed** if it contains each of its limit points, i.e., $F' \subseteq F$.

Examples 2.1.22. (i) The set **Z** of integers is a closed subset of the real line.

(ii) The set $F = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ is not closed in **R**. In fact, $F' = \{0\}$, which is not contained in F.

(iii) The set $F = \{(x, y) \in \mathbf{C}\colon x \geq 0, y \geq 0\}$ is a closed subset of the complex plane **C**. In this case, the derived set is $F' = F$.

(iv) Each subset of a discrete metric space is closed.

Proposition 2.1.23. Let F be a subset of the metric space (X, d). The set of limit points of F, namely, F' is a closed subset of (X, d), i.e., $(F')' \subseteq F'$.

Proof. If $F' = \varnothing$ or $(F')' = \varnothing$, then there is nothing to prove. Let $F' \neq \varnothing$ and let $x_0 \in (F')'$. Choose an arbitrary open ball $S(x_0, r)$ with centre x_0 and radius r. By the definition of limit point, there exists a point $y \in F'$ such that $y \in S(x_0, r)$. If $r' = r - d(y, x_0)$, then $S(y, r')$ contains infinitely many points of F by Proposition 2.1.19. But $S(y, r') \subseteq S(x_0, r)$ as in the proof of Theorem 2.1.5. So, infinitely many points of F lie in $S(x_0, r)$. Therefore, x_0 is a limit point of F, i.e., $x_0 \in F'$. Thus, F' contains all its limit points and hence F' is closed. □

Theorem 2.1.24. Let (X, d) be a metric space and let F_1, F_2 be subsets of X.

(i) If $F_1 \subseteq F_2$, then $F_1' \subseteq F_2'$.
(ii) $(F_1 \cup F_2)' = F_1' \cup F_2'$.
(iii) $(F_1 \cap F_2)' \subseteq F_1' \cap F_2'$.

Proof. The proofs of (i) and (iii) are obvious. For the proof of (ii), observe that $F_1' \cup F_2' \subseteq (F_1 \cup F_2)'$, which follows from (i). It remains to show that

$(F_1 \cup F_2)' \subseteq F_1' \cup F_2'$. Let $x_0 \in (F_1 \cup F_2)'$. Then there exists a sequence $\{x_n\}_{n \geq 1}$ of distinct points in $F_1 \cup F_2$ such that $d(x_n, x_0) \to 0$ as $n \to \infty$ by Proposition 2.1.20. If an infinite number of points x_n lie in F_1, then $x_0 \in F_1'$, and, consequently, $x_0 \in F_1' \cup F_2'$. If only a finite number of points of $\{x_n\}_{n \geq 1}$ lie in F_1, then $x_0 \in F_2' \subseteq F_1' \cup F_2'$. We therefore have $x_0 \in F_1' \cup F_2'$ in either case. This completes the proof of (ii). □

Definition 2.1.25. Let F be a subset of a metric space (X, d). The set $F \cup F'$ is called the **closure** of F and is denoted by $\bar{F}$.

Corollary 2.1.26. The closure $\bar{F}$ of $F \subseteq X$, where (X, d) is a metric space, is closed.

Proof. In fact, by Proposition 2.1.23 and Theorem 2.1.24(ii),

$$(\bar{F})' = (F \cup F')' = F' \cup (F')' \subseteq F' \cup F' = F' \subseteq \bar{F}.$$

Corollary 2.1.27. (i) Let F be a subset of a metric space(X, d). Then F is closed if and only if $F = \bar{F}$.
(ii) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.
(iii) If $A \subseteq F$ and F is closed, then $\bar{A} \subseteq F$.

Proof. (i) If $F = \bar{F}$, then it follows from Corollary 2.1.26 that F is closed. On the other hand, suppose that F is closed; then

$$\bar{F} = F \cup F' = F \subseteq \bar{F}.$$

It follows from the above relations that $F = \bar{F}$.
(ii) This is an immediate consequence of Theorem 2.1.24(i).
(iii) This is an immediate consequence of (ii) above. □

Proposition 2.1.28. Let (X, d) be a metric space and $F \subseteq X$. Then the following statements are equivalent:

(i) $x \in \bar{F}$;
(ii) $S(x, \varepsilon) \cap F \neq \varnothing$ for every open ball $S(x, \varepsilon)$ centred at x;
(iii) there exists an infinite sequence $\{x_n\}$ of points (not necessarily distinct) of F such that $x_n \to x$.

Proof. (i)⇒(ii). Let $x \in \bar{F}$. If $x \in F$, then obviously $S(x, \varepsilon) \cap F \neq \varnothing$. If $x \notin F$, then by the definition of closure, we have $x \in F'$. By definition of a limit point,

$$(S(x, \varepsilon) \backslash \{x\}) \cap F \neq \varnothing$$

and, a fortiori,

$$S(x, \varepsilon) \cap F \neq \varnothing.$$

(ii)⇒(iii). For each positive integer n, choose $x_n \in S(x, 1/n) \cap F$. Then the sequence $\{x_n\}$ of points in F converges to x. In fact, upon choosing $n_0 > 1/\varepsilon$, where $\varepsilon > 0$ is arbitrary, we have $d(x_n, x) < 1/n < 1/n_0 < \varepsilon$, i.e., $x_n \in S(x, \varepsilon)$ whenever $n \geq n_0$.

(iii)$\Rightarrow$(i) If the sequence $\{x_n\}_{n\geq 1}$ of points in F consists of finitely many distinct points, then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = x$ for all k. So, $x \in F$. If however, $\{x_n\}_{n\geq 1}$ contains infinitely many distinct points, then there exists a subsequence $\{x_{n_k}\}$ consisting of distinct points and $\lim_k d(x_{n_k}, x) = 0$, for $\lim_n d(x_n, x) = 0$ by hypothesis. By Proposition 2.1.20, it follows that $x \in F' \subseteq \bar{F}$. □

Condition (ii) of Definition 1.5.1 of a completion can be rephrased in view of condition (i) and Proposition 2.1.28 (iii) as saying that the closure of metric space X as a subset of its completion X^* must be the whole of X^*.

The following proposition is an easy consequence of Theorem 2.1.24.

Proposition 2.1.29. Let F_1, F_2 be subsets of a metric space (X, d). Then

(i) $\overline{(F_1 \cup F_2)} = \bar{F}_1 \cup \bar{F}_2$;
(ii) $\overline{(F_1 \cap F_2)} \subseteq \bar{F}_1 \cap \bar{F}_2$.

Proof. Using Theorem 2.1.24 (ii), we have

$$\begin{aligned}\overline{(F_1 \cup F_2)} &= (F_1 \cup F_2) \cup (F_1 \cup F_2)' = (F_1 \cup F_2) \cup (F_1' \cup F_2') \\ &= (F_1 \cup F_1') \cup (F_2 \cup F_2') = \bar{F}_1 \cup \bar{F}_2,\end{aligned}$$

which establishes (i). The proof of (ii) is equally simple.

Remarks 2.1.30. (i) It is not necessarily the case that the closure of an arbitrary union is the union of the closures of the subsets in the union. If $\{A_\alpha\}_{\alpha\in\Lambda}$ is an infinite family of subsets of (X, d), it follows from Corollary 2.1.27 (ii) that

$$\bigcup_{\alpha\in\Lambda} \bar{A}_\alpha \subseteq \overline{\bigcup_{\alpha\in A} A_\alpha}.$$

Equality need not hold, as the following example shows: If $A_n = \{r_n\}$, $n = 1, 2, \ldots$ and $r_1, r_2, \ldots, r_n, \ldots$ is an enumeration of rationals, then $\overline{A_n} = \overline{\{r_n\}} = \{r_n\}$ and $\bigcup_{n=1}^\infty \overline{A_n} = \mathbf{Q}$, whereas $\overline{\bigcup_{n=1}^\infty A_n} = \bar{\mathbf{Q}} = \mathbf{R}$.

(ii) In Proposition 2.1.29 (ii), equality need not hold. For example, if F_1 denotes the set of rationals in $\mathbf{R}$ and F_2 the set of irrationals in $\mathbf{R}$, then $\overline{(F_1 \cap F_2)} = \bar{\emptyset} = \emptyset$ but $\bar{F}_1 = \bar{F}_2 = \mathbf{R}$.

Proposition 2.1.31. Let (X, d) be a metric space. The empty set $\emptyset$ and the whole space X are closed sets.

Proof. Since the empty set has no limit points, the requirement that a closed set contain all its limit points is automatically satisfied by the empty set.

Since the whole space contains all points, it certainly contains all its limit points (if any), and is thus closed. □

The following is a useful characterisation of closed sets in terms of open sets.

Theorem 2.1.32. Let (X, d) be a metric space and F be a subset of X. Then F is closed in X if and only if F^c is open in X.

Proof. Suppose F is closed in X. We show that F^c is open in X. If $F = \varnothing$(respectively, X), then $F^c = X$ (respectively, $\varnothing$) and it is open by Theorem 2.1.7(i); so we may suppose that $F \neq \varnothing \neq F^c$. Let x be a point in F^c. Since F is closed and $x \notin F$, x cannot be a limit point of F. So there exists an $r > 0$ such that $S(x, r) \subseteq F^c$. Thus, each point of F^c is contained in an open ball contained in F^c. This means F^c is open.

For the converse, suppose F^c is open. We show that F is closed. Let $x \in X$ be a limit point of F. Suppose, if possible, that $x \notin F$. Then $x \in F^c$, which is assumed to be open. Therefore, there exists $r > 0$ such that $S(x, r) \subseteq F^c$, i.e.,

$$S(x, r) \cap F = \varnothing.$$

Thus, x cannot be a limit point of F, which is a contradiction. Hence, x belongs to F. □

Theorem 2.1.33. Let (X, d) be a metric space and $\bar{S}(x, r) = \{y \in X: d(y, x) \leq r\}$ be a closed ball in X. Then $\bar{S}(x, r)$ is closed.

Proof. We show that $(\bar{S}(x, r))^c$ is open in X (see Theorem 2.1.32). Let $y \in (\bar{S}(x, r))^c$. Then $d(y, x) > r$. If $r_1 = d(y, x) - r$, then $r_1 > 0$. Moreover, $S(y, r_1) \subseteq (\bar{S}(x, r))^c$. Indeed, if $z \in S(y, r_1)$, then

$$d(z, x) \geq d(y, x) - d(y, z) > d(y, x) - r_1 = r.$$

Thus, $z \notin \bar{S}(x, r)$, i.e., $z \in (\bar{S}(x, r))^c$. □

The following fundamental properties of closed sets are analogues of the properties of open sets formulated in Theorem 2.1.7 and are easy consequences of it along with de Morgan's laws (see Chapter 0, p. 3) and Proposition 2.1.31.

Theorem 2.1.34. Let (X, d) be a metric space. Then

(i) $\varnothing$ and X are closed;
(ii) any intersection of closed sets is closed;
(iii) a finite union of closed sets is closed.

Proof. (i) This is a restatement of Proposition 2.1.31.

(ii) Let $\{F_\alpha\}$ be a family of closed sets in X and $F = \bigcap_\alpha F_\alpha$. Then by Theorem 2.1.32, F is closed if F^c is open. Since $F^c = \bigcup_\alpha F_\alpha^c$ by de Morgan's laws, and since each F_α^c is open (Theorem 2.1.32), $\bigcup_\alpha F_\alpha^c$ is open by Theorem 2.1.7, i.e., F^c is open.

(iii) This proof is similar to (ii). □

Remark 2.1.35. An arbitrary union of closed sets need not be closed. Indeed, $\bar{S}(0, 1 - 1/n),\ n \geq 2$, is a closed subset of the complex plane, but

$$\bigcup_{n=2}^{\infty} \bar{S}\left(0, 1 - \frac{1}{n}\right) = S(0,1)$$

is not closed (because each point z satisfying $|z| = 1$ is a limit point of $S(0,1)$ but is not contained in $S(0,1)$).

An explicit characterisation of open sets on the real line is the content of Theorem 2.1.11. We now turn to the study of closed sets on the real line. Observe that closed intervals and finite unions of closed intervals are closed sets in **R**. Since a set consisting of a single point is a closed interval with identical endpoints, single point sets, and consequently finite sets, are closed sets as well.

Theorem 2.1.36. Let F be a nonempty bounded closed subset of **R** and let $\alpha = \inf F$ and $\beta = \sup F$. Then $\alpha \in F$ and $\beta \in F$.

Proof. We need only show that if $\alpha \notin F$, then α is a limit point of F. By the definition of infimum, for any $\varepsilon > 0$, there exists at least one member $x \in F$ such that $\alpha \leq x < \alpha + \varepsilon$. But $\alpha \notin F$, whereas $x \in F$. So,

$$\alpha < x < \alpha + \varepsilon.$$

Thus, every neighbourhood of α contains at least one member $x \in F$ which is different from α. Hence, α is a limit point of F. □

Definition 2.1.37. Let F be a nonempty bounded subset of **R** and let $\alpha = \inf F$ and $\beta = \sup F$. The closed interval $[\alpha, \beta]$ is called the **smallest closed interval containing** F.

Theorem 2.1.38. If $[\alpha, \beta]$ is the smallest closed interval containing F, where F is a nonempty bounded closed subset of **R**, then

$$[\alpha, \beta] \backslash F = (\alpha, \beta) \cap F^c$$

and so is open in **R**.

Proof. Let $x_0 \in [\alpha, \beta] \backslash F$; this means that $x_0 \in [\alpha, \beta],\ x_0 \notin F$. If $x_0 \notin F$, then $x_0 \neq \alpha$ and $x_0 \neq \beta$, because α and β do belong to F, by Theorem 2.1.36. It follows that $x_0 \in (\alpha, \beta)$. Moreover, it is obvious that $x_0 \in F^c$, so that

$$[\alpha, \beta] \backslash F \subseteq (\alpha, \beta) \cap F^c.$$

The reverse inclusion is obvious. □

The following characterisation of closed subsets of **R** is a direct consequence of Theorems 2.1.11 and 2.1.38.

Theorem 2.1.39. Let F be a nonempty bounded closed subset of $\mathbf{R}$. Then F is either a closed interval or is obtained from some closed interval by removing a countable family of pairwise disjoint open intervals whose endpoints belong to F.

Proof. Let $[\alpha, \beta]$ be the smallest closed interval containing F, where $\alpha = \inf F$ and $\beta = \sup F$. By Theorem 2.1.38,

$$[\alpha, \beta] \backslash F = (\alpha, \beta) \cap F^c$$

is open and hence is a countable union of disjoint open intervals by Theorem 2.1.11. Moreover, the endpoints of the open intervals do not belong to $[\alpha, \beta] \backslash F$ but do belong to $[\alpha, \beta]$. So they belong to F. The set F thus has the desired property. $\square$

This seemingly simple looking process of writing a nonempty bounded closed subset of $\mathbf{R}$ leads to some very interesting and useful examples. The following example, which is of particular importance, is due to Cantor.

Example 2.1.40. (Cantor) Divide the closed interval $I = [0, 1]$ into three equal parts by the points 1/3 and 2/3 and remove the open interval (1/3, 2/3) from I. Divide each of the remaining two closed intervals $[0, 1/3]$ and $[2/3, 1]$ into three equal parts by the points 1/9, 2/9 and by 7/9, 8/9, respectively, and remove the open intervals (1/9, 2/9) and (7/9, 8/9). Now divide each of the remaining four intervals $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$ and $[8/9, 1]$ into three equal parts and remove the middle third open intervals. Continue this process indefinitely. The open set G removed in this way from $I = [0, 1]$ is the union of disjoint open intervals

$$G = \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots.$$

The complement of G in [0,1], denoted by P, is called the **Cantor set**. Important properties of this set are listed in the Exercise 16 and Section 6.4.

The completeness of $\mathbf{R}$ can also be characterised in terms of nested sequences of bounded closed intervals. An analogue of this result for metric spaces is proved in Theorem 2.1.44. We begin with some relevant definitions.

2.1.41. Definition. Let (X, d) be a metric space and let A be a nonempty subset of X. We say that A is **bounded** if there exists $M > 0$ such that

$$d(x, y) \leq M \qquad x, y \in A.$$

If A is bounded, we define the **diameter** of A as

$$\operatorname{diam}(A) = d(A) = \sup\{d(x, y) : x, y \in A\}.$$

If A is not bounded, we write $d(A) = \infty$.

We define the distance between the point $x \in X$ and the subset B of X by

$$d(x, B) = \inf\{d(x, y) : y \in B\},$$

and, in an analogous manner, we define the distance between two nonempty subsets B and C of X by

$$d(B, C) = \inf\{d(x, y) : x \in B, y \in C\}.$$

2.1.42. Examples. (i) Recall that a subset A of $\mathbf{R}$ (respectively, $\mathbf{R}^2$) is bounded if and only if A is contained in an interval (respectively, square) of finite length (respectively, whose edge has finite length). Thus, our definition of bounded set in an arbitrary metric space is consistent with the definition of bounded set of real numbers (respectively, bounded set of pairs of real numbers).

(ii) The interval $(0, \infty)$ is not a bounded subset of $\mathbf{R}$. However, if $\mathbf{R}$ is equipped with the discrete metric, then every subset A of this discrete space (in particular, the set $(0, \infty)$) is bounded, since $d(x, y) \leq 1$ for $x, y \in A$. Indeed, $d(A) = 1$, provided A contains more than one point. Moreover, any subset of any discrete metric space has diameter 1 if it contains more than one point.

(iii) If $\mathbf{R}$ is equipped with the nondiscrete metric $d(x, y) = |x - y|/[1 + |x - y|]$, then every subset is bounded and $d(\mathbf{R}) = 1$.

(iv) In the space (ℓ_2, d) (see Example 1.2.2(vii)), consider the set

$$Y = \{e_1, e_2, \ldots, e_n, \ldots\},$$

where e_n denotes the sequence all of whose terms are equal to 0 except the nth term, which is equal to 1. If $j \neq k$, then $d(e_j, e_k) = \sqrt{2}$. Hence, Y is bounded and $d(Y) = \sqrt{2}$.

2.1.43. Proposition. If A is a subset of the metric space (X, d), then $d(A) = d(\overline{A})$.

Proof. If $x, y \in \overline{A}$, then there exist sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ in A such that $d(x, x_n) < \varepsilon/2$ and $d(y, y_n) < \varepsilon/2$ for $n \geq n_0$, say, where $\varepsilon > 0$ is arbitrary. Now for $n \geq n_0$, we have

$$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \\ &\leq \frac{\varepsilon}{2} + d(x_n, y_n) + \frac{\varepsilon}{2} \\ &\leq d(A) + \varepsilon, \end{aligned}$$

and so $d(\overline{A}) \leq d(A)$, since $\varepsilon > 0$ is arbitrary. Clearly, $d(A) \leq d(\overline{A})$. □

Let $\{I_n\}_{n\geq 1}$ be a sequence of intervals in $\mathbf{R}$. The sequence $\{I_n\}_{n\geq 1}$ is said to be **nested** if $I_{n+1} \subseteq I_n$, $n = 1, 2, \ldots$. The sequence $I_n = (0, 1/n)$, $n \in \mathbf{N}$, is nested. However $\bigcap_{n=1}^{\infty} I_n$ is empty. Similarly, the sequence $J_n = [n, \infty), n \in \mathbf{N}$, is nested with $\bigcap_{n=1}^{\infty} J_n = \varnothing$. In the metric space of rationals, the nested sequence $K_n = \{x \in \mathbf{Q} : |x - \sqrt{2}| < 1/n\}$ is such that $\bigcap_{n=1}^{\infty} K_n = \varnothing$, since $\sqrt{2}$ belongs to K_n for no n. The reader will note that the sequence $\{I_n\}_{n\geq 1}$ consists of intervals that are not closed, the sequence $\{J_n\}_{n\geq 1}$ consists of intervals that are not bounded, whereas the sequence $\{K_n\}_{n\geq 1}$ is in $\mathbf{Q}$, which is not complete. It is a very important property of real numbers that every nested sequence of closed bounded intervals does have a nonempty intersection. An analogue of this result holds in metric spaces.

Theorem 2.1.44. (Cantor) Let (X, d) be a metric space. Then (X, d) is complete if and only if, for every nested sequence $\{F_n\}_{n\geq 1}$ of nonempty closed subsets of X, that is,
(a) $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$ such that (b) $d(F_n) \to 0$ as $n \to \infty$,
the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point.

Proof. First suppose that (X, d) is complete. For each positive integer n, let x_n be any point in F_n. Then by (a),

$$x_n, x_{n+1}, x_{n+2}, \ldots$$

all lie in F_n. Given $\varepsilon > 0$, there exists by (b) some integer n_0 such that $d(F_{n_0}) < \varepsilon$. Now, $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots$ all lie in F_{n_0}. For $m, n \geq n_0$, we then have $d(x_m, x_n) \leq d(F_{n_0}) < \varepsilon$. This shows that the sequence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in the complete metric space X. So, it is convergent. Let $x \in X$ be such that $\lim_{n\to\infty} x_n = x$. Now for any given n, we have the sequence $x_n, x_{n+1}, \ldots \subseteq F_n$. In view of this,

$$x = \lim_{n\to\infty} x_n \in \bar{F}_n = F_n$$

since F_n is closed. Hence,

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

If $y \in X$ and $y \neq x$, then $d(y, x) = \alpha > 0$. There exists n large enough so that $d(F_n) < \alpha = d(y, x)$, which ensures that $y \notin F_n$. Hence, y cannot be in $\bigcap_{n=1}^{\infty} F_n$.

To prove the converse, let $\{x_n\}_{n\geq 1}$ be any Cauchy sequence in X. For each natural number n, let

$$F_n = \overline{\{x_m : m \geq n\}}.$$

Then $\{F_n\}_{n\geq 1}$ is a nested sequence of closed sets and since $\{x_n\}_{n\geq 1}$ is a Cauchy sequence,

$$\lim_{n\to\infty} d(F_n) = 0,$$

using Proposition 2.1.43.

Let

$$\bigcap_{n=1}^{\infty} F_n = \{x\}.$$

If $\varepsilon > 0$, then there exists a natural number n_0 such that $d(F_{n_0}) < \varepsilon$. But $x \in F_{n_0}$ and thus $n \geq n_0$ implies $d(x_n, x) < \varepsilon$. □

2.2. Relativisation and Subspaces

Let (X, d) be a metric space and Y a nonempty subset of X. If d_Y denotes the restriction of the function d to the set $Y \times Y$, then d_Y is a metric for $Y$and (Y, d_Y) is a metric space (see Section 1.2). If $Z \subseteq Y \subseteq X$, we may speak of Z being open (respectively, closed) relative to Y as well as open (respectively, closed) relative to X. It may happen that Z is an open (respectively, closed) subset of Y but not of X. For example, let X be $\mathbf{R}^2$ with metric d_2 and $Y = \{(x,0) \colon x \in \mathbf{R}\}$ with the induced metric. Then Y is a closed subset of X (for $Y^c = \{(x,y) \in \mathbf{R}^2 \colon y \neq 0\}$ is open in X). If $Z = \{(x,0) \colon 0 < x < 1\}$, then Z considered as a subset of Y is open in Y. However, Z considered as a subset of X is not open in X. In fact, no point $(x,0) \in Z$ is an interior point of Z (Z considered as a subset of X) because any neighbourhood of $(x,0)$ in X is the ball $S((x,0),r), r > 0$, which is not contained in Z. Thus, $Z = \{(x,0) : 0 < x < 1\}$ is an open subset of $Y = \{(x,0) : x \in \mathbf{R}\}$ but not of $X = (\mathbf{R}^2, d_2)$.

The above examples illustrate that the property of a set being open (respectively closed) depends on the metric space of which it is regarded a subset. The following theorem characterises open (respectively closed) sets in a subspace Y in terms of open (respectively closed) subsets in the space X. First we shall need a lemma.

Lemma 2.2.1. Let (X,d) be a metric space and Y a subspace of X. Let $z \in Y$ and $r > 0$. Then

$$S_Y(z,r) = S_X(z,r) \cap Y,$$

where $S_Y(z,r)$ (respectively $S_X(z,r)$) denotes the ball with centre z and radius r in Y (respectively X).

Proof. We have

$$\begin{aligned} S_X(z,r) \cap Y &= \{x \in X : d(x,z) < r\} \cap Y \\ &= \{x \in Y : d(x,z) < r\} \\ &= S_Y(z,r) \quad \text{since } Y \subseteq X. \end{aligned}$$ □

Let $X = \mathbf{R}^2$ and $Y = \{(x_1,x_2) \colon 0 < x_1 \leq 1,\ 0 \leq x_2 < 1,\ x_1^2 + x_2^2 \geq 1\}$. Here, the open ball in Y with centre $(1,0)$ and radius $\sqrt{2}$ is the entire space Y. (See Figure 2.6.)

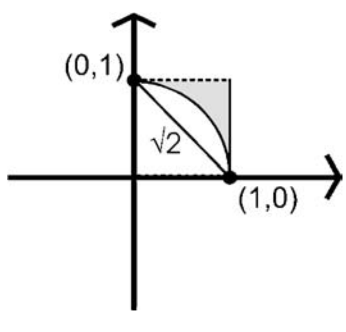

FIGURE 2.6

Theorem 2.2.2. Let (X, d) be a metric space and Y a subspace of X. Let Z be a subset of Y. Then

(i) Z is open in Y if and only if there exists an open set $G \subseteq X$ such that $Z = G \cap Y$;
(ii) Z is closed in Y if and only if there exists a closed set $F \subseteq X$ such that $Z = F \cap Y$.

Proof. (i) Let Z be open in Y. Then if z is any point of Z, there exists an open ball $S_Y(z, r)$ contained in Z. Observe that the radius r of the ball $S_Y(z, r)$ depends on the point $z \in Z$. We then have

$$\begin{aligned} Z &= \bigcup_{z\in Z} S_Y(z, r) \\ &= \bigcup_{z\in Z} (S_X(z, r) \cap Y) \quad \text{using Lemma 2.2.1} \\ &= \left(\bigcup_{z\in Z} S_X(z, r) \right) \cap Y \\ &= G \cap Y, \end{aligned}$$

where $G = \bigcup_{z\in Z} S_X(z, r)$ is open in X.

On the other hand, suppose that $Z = G \cap Y$, where G is open in X. If $z \in Z$, then z is a point of G and so there exists an open ball $S_X(z, r)$ such that $S_X(z, r) \subseteq G$. Hence,

$$\begin{aligned} S_Y(z, r) &= S_X(z, r) \cap Y \quad \text{by Lemma 2.2.1} \\ &\subseteq G \cap Y = Z, \end{aligned}$$

so that z is an interior point of the subset Z of Y. As z is an arbitrary point of Z, it follows that Z is open in Y.

(ii) Z is closed in Y if and only if $(X\backslash Z) \cap Y$ is open in Y. Hence, Z is closed in Y if and only if there exists an open set G in X such that

$$(X\backslash Z) \cap Y = G \cap Y \quad \text{using (i) above.}$$

On taking complements in X on both sides, we have

$$Z \cup (X\backslash Y) = (X\backslash G) \cup (X\backslash Y).$$

Hence

$$\begin{aligned} Z = Z \cap Y &= (Z \cup (X\backslash Y)) \cap Y \\ &= ((X\backslash G) \cup (X\backslash Y)) \cap Y\,. \\ &= (X\backslash G) \cap Y \end{aligned}$$

So, Z is the intersection of the closed set $X\backslash G$ and Y.

Conversely, let $Z = F \cap Y$, where F is closed in X. Then $X \backslash Z = (X \backslash F) \cup (X \backslash Y)$ and so

$$(X \backslash Z) \cap Y = ((X \backslash F) \cup (X \backslash Y)) \cap Y = (X \backslash F) \cap Y,$$

where $X \backslash F$ is open in X. Hence $(X \backslash Z) \cap Y$ is open in Y, i.e., Z is closed in Y. □

Proposition 2.2.3. Let Y be a subspace of a metric space (X, d).

(i) Every subset of Y that is open in Y is also open in X if and only if Y is open in X.
(ii) Every subset of Y that is closed in Y is also closed in X if and only if Y is closed in X.

Proof. (i) Suppose every subset of Y open in Y is also open in X. We want to show that Y is open in X. Since Y is an open subset of Y, it must be open in X. Conversely, suppose Y is open in X. Let Z be an open subset of Y. By Theorem 2.2.2(i), there exists an open subset G of X such that $Z = G \cap Y$. Since G and Y are both open subsets of X, their intersection must be open in X, i.e., Z must be open in X.

(ii) The proof is equally easy and is, therefore, not included. □

Proposition 2.2.4. Let (X, d) be a metric space and $Z \subseteq Y \subseteq X$. If $\text{cl}_X Z$ and $\text{cl}_Y Z$ denote, respectively, the closures of Z in the metric spaces X and Y, then

$$\text{cl}_Y Z = Y \cap \text{cl}_X Z.$$

Proof. Obviously, $Z \subseteq Y \cap \text{cl}_X Z$. Since $Y \cap \text{cl}_X Z$ is closed in Y (see Theorem 2.2.2(ii)), it follows that $\text{cl}_Y Z \subseteq Y \cap \text{cl}_X Z$. On the other hand, by Theorem 2.2.2(ii), $\text{cl}_Y Z = Y \cap F$, where F is a closed subset of X. But then

$$Z \subseteq \text{cl}_Y Z \subseteq F,$$

and hence, by Corollary 2.1.27(ii),

$$\text{cl}_X Z \subseteq F.$$

Therefore,

$$\text{cl}_Y Z = Y \cap F \supseteq Y \cap \text{cl}_X Z.$$

This completes the proof. □

In contrast to the relative properties discussed above, there are some properties that are intrinsic. In fact, the property of x being a limit point of F holds in any subspace containing x and F as soon as it holds in the whole space, and conversely. Another such property is that of being complete. The following propositions describe relations between closed sets and complete sets.

Proposition 2.2.5. If Y is a nonempty subset of a metric space (X, d), and (Y, d_Y) is complete, then Y is closed in X.

Proof. Let x be any limit point of Y. Then x is the limit of a sequence $\{y_n\}_{n\geq 1}$ in Y. In view of Proposition 1.4.3, the sequence $\{y_n\}_{n\geq 1}$ is Cauchy, and hence, by assumption, converges to a point y of Y. But by Remark 3 following Definition 1.3.2, $y = x$. Therefore, $x \in Y$. This shows that Y is closed in X. □

Proposition 2.2.6. Let (X, d) be a complete metric space and Y a closed subset of X. Then (Y, d_Y) is a complete space.

Proof. Let $\{y_n\}_{n\geq 1}$ be a Cauchy sequence in (Y, d_Y). Then $\{y_n\}_{n\geq 1}$ is also a Cauchy sequence in (X, d); so there exists an $x \in X$ such that $\lim_{n\to\infty} y_n = x$. If follows (see Proposition 2.1.28) that $x \in \bar{Y}$, which is the same set as Y by Corollary 2.1.27(i). □

2.3. Countability Axioms and Separability

Definition 2.3.1. Let (X, d) be a metric space and $x \in X$. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a family of open sets, each containing x. The family $\{G_\lambda\}_{\lambda\in\Lambda}$ is said to be a **local base at *x*** if, for every nonempty open set G containing x, there exists a set G_μ in the family $\{G_\lambda\}_{\lambda\in\Lambda}$ such that $x \in G_\mu \subseteq G$.

Examples 2.3.2. (i) In the metric space $\mathbf{R}^2$ with the Euclidean metric, let $G_\lambda = S(x, \lambda)$, where $x = (x_1, x_2) \in \mathbf{R}^2$ and $0 < \lambda \in \mathbf{R}$. The family $\{G_\lambda : 0 < \lambda \in \mathbf{R}\} = \{S(x, \lambda) : 0 < \lambda \in \mathbf{R}\}$ is a family of balls and is a local base at x. Note that $S(x, \lambda)$, where $x = (x_1, x_2)$, can also be described as $\{(y_1, y_2) \in \mathbf{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 < \lambda^2\}$.

(ii) Let $x = (x_1, x_2) \in \mathbf{R}^2$ and $G_\lambda = \{(y_1, y_2) \in \mathbf{R}^2 : (y_1 - x_1)^2 + 2(y_2 - x_2)^2 < \lambda\}$, where $0 < \lambda \in \mathbf{R}$. Then the family $\{G_\lambda : 0 < \lambda \in \mathbf{R}\}$ is a local base at x. To see why, consider any open set $G \subseteq \mathbf{R}^2$ such that $x \in G$. Since G is open, there exists $r > 0$ such that $S(x, r) \subseteq G$. Now $S(x, r) = \{(y_1, y_2) \in \mathbf{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\}$. Let $\lambda = r^2$. Then $y \in G_\lambda \Rightarrow (y_1 - x_1)^2 + 2(y_2 - x_2)^2 < \lambda \Rightarrow (y_1 - x_1)^2 + (y_2 - x_2)^2 < \lambda \Rightarrow (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2 \Rightarrow y \in S(x, r)$, so that $G_\lambda \subseteq S(x, r) \subseteq G$. In this example, the sets G_λ are ellipses.

(iii) Let $x \in \mathbf{R}$. Consider the family of all open intervals (r,s) containing x and having rational endpoints r and s. This family is a local base at x. It consists of open balls, not necessarily centred at x. Moreover, the family is countable and thus constitutes what is called a **countable base at *x***.

Proposition 2.3.3. In any metric space, there is a countable base at each point.

Proof. Let (X, d) be a metric space and $x \in X$. The family of open balls centred at x and having rational radii, i.e., $\{S(x, \rho) : \rho$ rational and positive$\}$ is a countable base at x. In fact, if G is an open set and $x \in G$, then by the definition of an open set, there exists an $\varepsilon > 0$ (ε depending on x) such that $x \in S(x, \varepsilon) \subseteq G$. Let ρ be a positive rational number less than ε. Then

$$x \in S(x, \rho) \subseteq S(x, \varepsilon) \subseteq G.$$

□

Definition 2.3.4. A family $\{G_\lambda\}_{\lambda\in\Lambda}$ of nonempty open sets is called a **base for the open sets** of (X, d) if every open subset of X is a union of a subfamily of the family $\{G_\lambda\}_{\lambda\in\Lambda}$.

The condition of the above definition can be expressed in the following equivalent form: If G is an arbitrary nonempty open set and $x \in G$, then there exists a set G_μ in the family such that $x \in G_\mu \subseteq G$.

Proposition 2.3.5. The collection $\{S(x, \varepsilon): x \in X,\ \varepsilon > 0\}$ of all open balls in X is a base for the open sets of X.

Proof. Let G be a nonempty open subset of X and let $x \in G$. By the definition of an open subset, there exists a positive $\varepsilon(x)$ (depending upon x) such that

$$x \in S(x, \varepsilon(x)) \subseteq G.$$

This completes the proof. □

Generally speaking, an open base is useful if its sets are simple in form. A space that has a countable base for the open sets has pleasant properties and goes by the name of "second countable".

Definition 2.3.6. A metric space is said to be **second countable** (or satisfy **the second axiom of countability**) if it has a countable base for its open sets.

The reason for the name *second* countable is that the property of having a countable base at each point, as in Proposition 2.3.3, is usually called *first* countability.

Examples 2.3.7. (i) Let $(\mathbf{R}, d)$ be the real line with the usual metric. The collection $\{(x, y) : x, y \text{ rational}\}$ of all open intervals with rational endpoints form a countable base for the open sets of $\mathbf{R}$.

(ii) The collection

$\{S(x, r): x = (x_1, x_2, \ldots, x_n), x_i \text{ rationals}, 1 \le i \le n, \text{ and } r \text{ positive rational}\}$ of all r-balls with rational centres and rational radii is a countable base for the open sets of the metric space $(\mathbf{R}^n, d)$, where d may be any of the metrics on $\mathbf{R}^n$ described in Example 1.2.2(iii).

(iii) Let X have the discrete metric. Then any set $\{x\}$ containing a single point x is also the open ball $S(x, 1/2)$ and therefore must be a union of nonempty sets of any base. So any base has to contain each set $\{x\}$ as one of the sets in it. If X is nondenumerable, then the sets $\{x\}$ are also nondenumerable, forcing every base to be nondenumerable as well. Consequently, X does not satisfy the second axiom of countability when it is nondenumerable.

It is easy to see that any subspace of a second countable space is also a second countable space. In fact, the class of all intersections with the subspace of the sets of a base form a base for the open sets of the subspace.

Definition 2.3.8. Let (X, d) be a metric space and $\mathcal{G}$ be a collection of open sets in X. If for each $x \in X$ there is a member $G \in \mathcal{G}$ such that $x \in G$, then $\mathcal{G}$ is called an **open cover** (or **open covering**) of X. A subcollection of $\mathcal{G}$ which is itself an open cover of X is called a **subcover** (or **subcovering**).

Examples 2.3.9. (i) The union of the family $\{\dots, (-3, -1), (-2, 0), (-1, 1), (0, 2), \dots\}$ of open intervals is $\mathbf{R}$. The family is therefore an open covering of $\mathbf{R}$. However, the family of open intervals $\{\dots, (-2, -1), (-1, 0), (0, 1), (1, 2), \dots\}$ is not an open covering, because the intervals' union does not contain the integers. The aforementioned cover contains no subcovering besides itself, because, if we delete any interval from the family, the midpoint of the deleted interval will not belong to the union of the remaining intervals.

(ii) Let X be the discrete metric space consisting of the five elements a, b, c, d, e. The union of the family of subsets $\{\{a\}, \{b, c\}, \{c, d\}, \{a, d, e\}\}$ is X and all subsets are open. Therefore the family is an open cover. The family $\{\{b, c\}, \{c, d\}, \{a, d, e\}\}$ is a proper subcover.

(iii) Consider the set $\mathbf{Z}$ of all integers with the discrete metric. As in any discrete metric space, all subsets are open. Consider the family consisting of the three subsets

$$\{3n : n \in \mathbf{Z}\}, \{3n + 1 : n \in \mathbf{Z}\} \text{ and } \{3n + 2 : n \in \mathbf{Z}\}.$$

Since every integer must be of the form $3n$, $3n + 1$ or $3n + 2$, the above three subsets form an open cover of $\mathbf{Z}$. There is no proper subcover.

(iv) The family of intervals $\{(-n, n) : n \in \mathbf{N}\}$ is an open cover of $\mathbf{R}$ and the family consisting of the open balls $\{z \in \mathbf{C} : |z + 17| < n^{3/2}, n \in \mathbf{N}\}$ is an open cover of $\mathbf{C}$. If we extract a subfamily by restricting n to be greater than some integer n_0, the subfamily is also an open cover. Indeed, if we delete a finite number of sets in the family, the remaining subfamily is an open cover. Thus, there are infinitely many open subcovers.

Definition 2.3.10. A metric space is said to be **Lindelöf** if each open covering of X contains a countable subcovering.

Proposition 2.3.11. Let (X, d) be a metric space. If X satisfies the second axiom of countability, then every open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of X contains a countable subcovering. In other words, a second countable metric space is Lindelöf.

Proof. Let $\{G_i : i = 1, 2, \dots\}$ be a countable base of open sets for X. Since each U_α is a union of sets G_i, it follows that a subfamily $\{G_{i_j} : j = 1, 2, \dots\}$ of the base $\{G_i : i = 1, 2, \dots\}$ is a covering of X. Choose $U_{i_j} \supseteq G_{i_j}$ for each j. Then $\{U_{i_j} : j = 1, 2, \dots\}$ is the required countable subcovering. □

Definition 2.3.12. A subset X_0 of a metric space (X, d) is said to be **everywhere dense** or simply **dense** if $\overline{X_0} = X$, i.e., if every point of X is either a point or a limit point of X_0. This means that, given any point x of X, there exists a sequence of points of X_0 that converges to x.

It follows easily from this definition and the definition of interior (see Definition 2.1.12) that a subset of X_0 is dense if and only if X_0^c has empty interior.

It may be noted that X is always a dense subset of itself; interest centres around what *proper* subsets of a metric space are dense.

Examples 2.3.13. (i) The set of rationals is a dense subset of $\mathbf{R}$ (usual metric) and so is the set of irrationals. Note that the former is countable whereas the latter is not.

(ii) Consider the metric space $(\mathbf{R}^n, d)$ with any of the metrics described in Example 1.2.2(iii). Within any neighbourhood of any point in $\mathbf{R}^n$, there is a point with rational coordinates. Thus,

$$\mathbf{Q}^n = \mathbf{Q} \times \mathbf{Q} \times \ldots \times \mathbf{Q}$$

is dense in $\mathbf{R}^n$.

(iii) In the space $C[0, 1]$ of Example 1.2.2(ix), we consider the set C_0 consisting of all polynomials with rational coefficients. We shall check that C_0 is dense in $C[0,1]$. Let $x(t) \in C[0,1]$. By Weierstrass' theorem (Theorem 0.8.4), there exists a polynomial $P(t)$ such that

$$\sup\{|x(t) - P(t)| : 0 \leq t \leq 1\} < \frac{\varepsilon}{2},$$

where $\varepsilon > 0$ is given. Corresponding to $P(t)$ there is a polynomial $P_0(t)$ with rational coefficients such that

$$\sup\{|P(t) - P_0(t)| : 0 \leq t \leq 1\} < \frac{\varepsilon}{2}.$$

So,

$$\sup\{|x(t) - P_0(t)| : 0 \leq t \leq 1\} < \varepsilon.$$

It is easy to see that C_0 is countable. In fact, if $\mathcal{P}_n$ denotes the set of all polynomials of degree n and having rational coefficients, then the cardinality of $\mathcal{P}_n$ is the same as that of $\mathbf{Q}^{n+1} = \mathbf{Q} \times \mathbf{Q} \times \ldots \times \mathbf{Q}$, which is countable. The assertion now follows from the fact that a countable union of countable sets is countable.

(iv) Let (X, d) be a discrete metric space. Since every subset is closed, the only dense subset is X itself.

(v) Let $X = \ell_p$ of Example 1.2.2(vii). Recall that the metric is given by

$$d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}.$$

Let E denote the set of all elements of the form $(r_1, r_2, \ldots, r_n, 0, 0 \ldots)$, where r_i are rational numbers and n is an arbitrary natural number. We shall show that E is dense in ℓ_p. Let $x = (x_1, x_2, \dots)$ be an element in ℓ_p and let $\varepsilon > 0$ be given. There exists a natural number n_0 such that

$$\sum_{j=n_0+1}^{\infty} |x_j|^p < \frac{\varepsilon^p}{2}.$$

Choose an element $x_0 = (r_1, r_2, \dots, r_{n_0}, 0, 0, \dots)$ in E such that

$$\sum_{j=1}^{n_0} |x_j - r_j|^p < \frac{\varepsilon^p}{2}.$$

We then obtain

$$(d(x, x_0))^p = \sum_{j=1}^{n_0} |x_j - r_j|^p + \sum_{j=n_0+1}^{\infty} |x_j|^p < \varepsilon^p,$$

and this implies

$$d(x, x_0) < \varepsilon.$$

Thus, E is dense in (ℓ_p, d). Also, E is countable (in fact, if E_n denotes the subset of all those elements $x = \{r_i\}_{i \geq 1}$ such that $r_j = 0$ for $j \geq n+1$, then E_n is countable and $E = \bigcup_{n=1}^{\infty} E_n$).

(vi) By Definition 1.5.1, any metric space is dense in its completion.

Definition 2.3.14. The metric space X is said to be **separable** if there exists a countable, everywhere dense set in X. In other words, X is said to be separable if there exists in X a sequence

$$\{x_1, x_2, \dots\} \tag{2.1}$$

such that for every $x \in X$, some sequence in the range of (2.1) converges to x.

Examples 2.3.15. In Examples 2.3.13(i)–(iii) and (v), we saw dense sets that are countable. Therefore, the spaces concerned are separable. In (iv) however, the space is separable if and only if the set X is countable.

There are metric spaces other than the discrete metric space mentioned above which fail to satisfy the separability criterion. The next example is one such case. Let X denote the set of all bounded sequences of real numbers with metric

$$d(x, y) = \sup\{|x_i - y_i| : i = 1, 2, 3, \dots\},$$

as in Example 1.2.2(vi). We shall show that X is **inseparable**.

First we consider the set A of elements $x = (x_1, x_2, \dots)$ of X for which each x_i is either 0 or 1 and show that it is uncountable. If E is any countable subset of A, then the elements of E can be arranged in a sequence $s_1, s_2, \dots$. We construct a sequence s as follows. If the m^{th} element of s_m is 1, then the m^{th} element of s is 0, and vice versa. Then the element s of X differs from each s_m in the m^{th} place and is therefore equal to none of them. So, $s \notin E$ although $s \in A$. This shows that any countable subset of A must be a proper subset of A. It follows that A is uncountable, for if it were to be countable, then it would have to be a proper subset of itself, which is absurd. We proceed to use the uncountability of the subset A to argue that X must be inseparable.

The distance between two distinct elements $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ of A is $d(x, y) = \sup\{|x_i - y_i| : i = 1, 2, 3, \dots\} = 1$. Suppose, if possible, that E_0 is a countable, everywhere dense subset of X. Consider the balls of radii 1/3 whose centres are the points of E_0. Their union is the entire space X, because E_0 is everywhere dense, and in particular contains A. Since the balls are countable in number while A is not, in at least one ball there must be two distinct elements x and y of A. Let x_0 denote the centre of such a ball. Then

$$1 = d(x, y) \leq d(x, x_0) + d(x_0, y) < \frac{1}{3} + \frac{1}{3} < 1,$$

which is, however, impossible. Consequently, (X, d) cannot be separable.

Proposition 2.3.16. Let (X, d) be a metric space and $Y \subseteq X$. If X is separable, then Y with the induced metric is separable, too.

Proof. Let $E = \{x_i : i = 1, 2, \dots\}$ be a countable dense subset of X. If E is contained in Y, then there is nothing to prove. Otherwise, we construct a countable dense subset of Y whose points are arbitrarily close to those of E. For positive integers n and m, let $S_{n,m} = S(x_n, 1/m)$ and choose $y_{n,m} \in S_{n,m} \cap Y$ whenever this set is nonempty. We show that the countable set $\{y_{n,m}$: n and m positive integers$\}$ of Y is dense in Y.

For this purpose, let $y \in Y$ and $\varepsilon > 0$. Let m be so large that $1/m < \varepsilon/2$ and find $x_n \in S(y, 1/m)$. Then $y \in S_{n,m} \cap Y$ and

$$d(y, y_{n,m}) \leq d(y, x_n) + d(x_n, y_{n,m}) < \frac{1}{m} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $y_{n,m} \in S(y, \varepsilon)$. Since $y \in Y$ and $\varepsilon > 0$ are arbitrary, the assertion is proved. □

The main result of this section is the following.

Theorem 2.3.17. Let (X, d) be a metric space. The following statements are equivalent:

(i) (X, d) is separable;
(ii) (X, d) satisfies the second axiom of countability;
(iii) (X, d) is Lindelöf.

Proof. (i)⇒(ii). Let $E = \{x_i : i = 1, 2, \dots\}$ be a countable, dense subset of X and let $\{r_j : j = 1, 2, \dots\}$ be an enumeration of positive rationals. Consider the countable collection of balls with centres at x_i, $i = 1, 2, \dots$ and radii r_j, $j = 1, 2, \dots$; i.e.,

$$\{S(x_i, r_j) : x_i \in E \text{ for } i = 1, 2, \dots \text{ and } r_j, \text{ is rational } j = 1, 2, \dots\}.$$

If G is any open set and $x \in G$, we want to show that for some i and some j, $x \in S(x_i, r_j) \subseteq G$. Since G is open, there is a ball $S(x, \delta)$ such that $S(x, \delta) \subseteq G$. Let $r_k > 0$ be a rational number such that $0 < r_k < \delta$. Since x is a point of closure of E, there is a point $x_i \in E$ such that $d(x, x_i) < 1/2r_k$. Hence,

$$x \in S(x_i, \frac{1}{2} r_k) \subseteq S(x, r_k) \subseteq G.$$

In fact, if $y \in S(x_i, 1/2r_k)$ then $d(y, x) \leq d(y, x_i) + d(x_i, x) < 1/2r_k + 1/2r_k = r_k$.

(ii)$\Rightarrow$(iii). See Proposition 2.3.11.

(iii)$\Rightarrow$(i). From each open covering $\{S(x, \varepsilon) : x \in X\}$, we extract a countable subcovering $\{S(x_i, \varepsilon) : x_i \in X,\ i = 1, 2, \ldots\}$ and let $A(\varepsilon) = \{x_1, x_2, \ldots\}$. Define $E = \bigcup_{n=1}^{\infty} A(\frac{1}{n})$. Then E is a countable, dense subset of X. □

2.4. Baire's Category Theorem

Definition 2.4.1. Let (X, d) be a metric space. A subset $Y \subseteq X$ is said to be **nowhere dense** if $(\bar{Y})^o$ is empty, i.e., $(\bar{Y})^o$ contains no interior point. A subset $F \subseteq X$ is said to be of **category I** if it is a countable union of nowhere dense subsets. Subsets that are not of category I are said to be of **category II**.

Remarks 2.4.2. (i) A subset Y of X is nowhere dense if and only if the complement $(\bar{Y})^c$ is dense in X, or $\overline{(X - \bar{Y})} = X$. This follows easily from the remark immediately after Definition 2.3.12.

(ii) If d denotes the discrete metric, the only nowhere dense set is the null set.

(iii) The notion of being nowhere dense is not the opposite of being everywhere dense, i.e., not being nowhere dense does not imply that the set is everywhere dense. For an example of a set which is neither, let $\mathbf{R}$ denote the real line with the usual metric and consider the set $Y = \{x \in \mathbf{R} : 1 < x < 2\}$. Then

$$(\bar{Y})^o = Y \neq \varnothing \text{ and } (\bar{Y})^c = \{x \in \mathbf{R} : x < 1 \text{ or } x > 2\} = (-\infty, 1) \cup (2, \infty),$$

which is not dense in $\mathbf{R}$.

(iv) Every subset must be either of category I or of category II.

(v) It is clear that the null set is of category I. Also, the subset $\mathbf{Q}$ of rationals in $\mathbf{R}$ is a set of category I. Indeed, if $x_1, x_2, \ldots$ is an enumeration of the rationals, each $\{x_i\}$ is closed and $\{x_i\}^o = \varnothing$; it follows that $\cup\{x_i\}$, the set of all rationals in $\mathbf{R}$, is of category I.

(vi) Since a denumerable union of denumerable sets is again a denumerable set, it follows that, if $Y_1, Y_2, \ldots$ are each of category I, then so must be $\bigcup_i Y_i$.

(vii) If $X = Y_1 \cup Y_2$ and it is known that Y_1 is of category I while X is of category II, then Y_2 must be category II. For, if Y_2 is of category I, then it follows from (vi) above that X, too, is of category I, which is a contradiction.

(viii) A subset of a nowhere dense set is nowhere dense and, therefore, a subset of a set of category I is again of category I.

Theorem 2.4.3. (Baire Category Theorem) Any complete metric space is of category II.

Proof. We assume the contrary, i.e., we suppose that (X, d) is a complete metric space and

$$X = \bigcup_{n=1}^{\infty} E_n,$$

where each of the E_n is nowhere dense. Since each E_n is nowhere dense, each $(\overline{E_n})^c$ is everywhere dense. So we can assert the existence of points in each of these sets $(\overline{E_n})^c$ (i.e., none of them can be empty). In the case of $(\overline{E_1})^c$, let $x_1 \in (\overline{E_1})^c$. Since $(\overline{E_1})^c$ is open, there exists $r > 0$ such that $S(x_1, r) \subseteq (\overline{E_1})^c$. For $\varepsilon_1 < r$, we have

$$\bar{S}(x_1, \varepsilon_1) \subseteq S(x_1, r) \subseteq (\overline{E_1})^c \subseteq E_1^c.$$

This, in turn, implies

$$\bar{S}(x_1, \varepsilon_1) \cap E_1 = \varnothing.$$

We make the following induction hypothesis: There exist balls $S(x_k, \varepsilon_k)$ for $k = 1, 2, \ldots, n-1$ such that

$$\bar{S}(x_k, \varepsilon_k) \cap E_k = \varnothing, \qquad \text{where } x_k \in (\overline{E_k})^c$$

and

$$\varepsilon_k \leq \frac{1}{2}\varepsilon_{k-1} \qquad \text{for } k = 2, \ldots, n-1.$$

Using this information, we can construct the nth ball with the above properties. To this end, choose

$$x_n \in S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{E_n})^c.$$

Such an element must exist, because otherwise

$$S(x_{n-1}, \varepsilon_{n-1}) \subseteq \overline{E_n}$$

and this implies $x_{n-1} \in (\overline{E_n})^o$, contradicting the fact that $(\overline{E_n})^o$ is empty. Since the intersection $S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{E_n})^c$ is open, there exists $\varepsilon > 0$ such that

$$S(x_n, \varepsilon) \subseteq S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{E_n})^c.$$

Now we choose a positive $\varepsilon_n < \min\{\varepsilon, (1/2)\varepsilon_{n-1}\}$. Then

$$\bar{S}(x_n, \varepsilon_n) \subseteq S(x_n, \varepsilon) \subseteq S(x_{n-1}, \varepsilon_{n-1}) \cap (\overline{E_n})^c,$$

which says

$$\bar{S}(x_n, \varepsilon_n) \cap E_n = \varnothing.$$

As we also have

$$\varepsilon_n \leq \frac{1}{2}\varepsilon_{n-1},$$

the nth ball with the requisite properties has been constructed.

As $\bar{S}(x_n, \varepsilon_n) \subseteq \bar{S}(x_{n-1}, \varepsilon_{n-1})$, the balls $\{\bar{S}(x_n, \varepsilon_n)\}_{n\geq 1}$ form a nested sequence of nonempty closed balls in a complete metric space with diameters tending to zero. By Theorem 2.1.44, there exists $x_0 \in \bigcap_1^\infty \bar{S}(x_n, \varepsilon_n)$. Since $\bar{S}(x_n, \varepsilon_n) \cap E_n = \varnothing$ for every n, we have $x_0 \notin E_n$ for any n, i.e., $x_0 \in E_n^c$ for all n. However, $\bigcap_1^\infty E_n^c = \varnothing$. This contradiction shows that X is not of category I. This completes the proof. □

Corollary 2.4.4. The irrationals in **R** are of category II.

Proof. Since **R** is a complete metric space, it follows from Remarks (v) and (vii) prior to the Baire category theorem (Theorem 2.4.3) that the irrationals are of category II in **R**. □

Corollary 2.4.5. A nonempty open interval is of category II.

Proof. If a nonempty open interval is of category I, then so is each of its translates. Since **R** is a countable union of such translates, it follows that **R** is of category I, contradicting the Baire category theorem (Theorem 2.4.3). □

We next take up some applications of the Baire category theorem.

Theorem 2.4.6. (Osgood) Let $\mathcal{F}$ be a collection of continuous real-valued functions on **R** such that for each $x \in \mathbf{R}$, there exists $M_x > 0$ for which $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Then there exists an $M > 0$ and a nonempty open subset $Y \subseteq \mathbf{R}$ such that

$$|f(x)| \leq M \text{ for each } x \in Y \text{ and for each } f \in \mathcal{F}.$$

Proof. For each integer n, let $E_{n,f} = \{x \in X: |f(x)| \leq n\} = f^{-1}([-n, n])$. This set $E_{n,f}$ is closed for the following reason: Let x_0 be a limit point of it. Then there exists a sequence $\{x_m\}_{m\geq 1}$ in $E_{n,f}$ such that $\lim_{m\to\infty} x_m = x_0$. For each m, we have $-n \leq f(x_m) \leq n$, from which it follows that $-n \leq f(x_0) \leq n$, using the continuity of f. Therefore, $x_0 \in E_{n,f}$, showing that the set is closed. It now follows that the intersection $E_n = \bigcap_{f\in\mathcal{F}} E_{n,f}$ is a closed subset of **R**. Observe that $\mathbf{R} = \bigcup_{n=1}^\infty E_n$. Indeed, if $x \in \mathbf{R}$, by hypothesis there exists $M_x > 0$ such that $|f(x)| \leq M_x$ for each $f \in \mathcal{F}$, which shows that $x \in E_{n_0}$ for any integer $n_0 > M_x$. Since **R** is complete, there exists an integer $M > 0$ such that E_M is not nowhere dense (Baire category theorem). Since E_M is closed, it must contain some nonempty open set Y. Then, for each $x \in Y$, we have $|f(x)| \leq M$ for all $f \in \mathcal{F}$. □

Another illuminating application of Baire's category theorem is the following. To begin with, we make an observation regarding a continuous real-valued function f defined on [0,1]. Let f_1 be an integral of f, that is,

$$f_1'(x) = f(x) \qquad \text{for all } x \in [0,\ 1].$$

Let f_2 be an integral of f_1 and so on. If $f_k \equiv 0$ for some k, then obviously the same is true of f. The proof of the following generalisation of this observation uses the Baire category theorem.

Theorem 2.4.7. Let f be a continuous real-valued function on [0, 1]. Let f_1 be an integral of f, that is, $f_1'(x) = f(x)$ for all $x \in [0,\ 1]$. Let f_2 be an integral of f_1 and so on. If for each $x \in [0,\ 1]$, there is an integer k depending on x such that $f_k(x) = 0$, then f is identically 0 on [0,1].

Proof. Let $Z_n = \{x \in [0,\ 1]: f_n(x) = 0\}$. Observe that Z_n is closed. Indeed, if $x \in [0,\ 1]$ is the limit of a sequence $\{x_m\}$ in Z_n, then $f_n(x) = f_n(\lim_m x_m) = \lim_m f_n(x_m) = 0$, so that $x \in Z_n$. Also, by hypothesis,

$$\bigcup_{n=1}^{\infty} Z_n = [0, 1].$$

Since [0,1] is a complete metric space, there exists a positive integer n such that Z_n is not nowhere dense and so $Z_n^\circ \neq \emptyset$. Let $x_0 \in Z_n^\circ$. Then there exists an $\varepsilon > 0$ such that $[x_0 - \varepsilon,\ x_0 + \varepsilon] \subseteq Z_n^\circ$. Since $f_n(x) = 0$ on $[x_0 - \varepsilon,\ x_0 + \varepsilon]$, it follows that $f(x) = 0$ on $[x_0 - \varepsilon,\ x_0 + \varepsilon]$; in particular, $f(x_0) = 0$, and, hence, $f(x) = 0$ for all $x \in Z_n^\circ$.

Let $Y_n = Z_n \backslash Z_n^\circ = Z_n \cap ([0,\ 1] \backslash Z_n^\circ)$. Now Y_n, being the intersection of closed sets, is itself closed. Moreover, $(\overline{Y_n})^\circ = Y_n^\circ = \emptyset$. So, Y_n is nowhere dense. Thus, $f(x) = 0$ for all $x \in [0,\ 1]$ except possibly for a set of category I. Since f is continuous, we shall argue that $f(x) = 0$ for all $x \in [0,\ 1]$. Let $x_0 \in [0,\ 1]$ be such that $f(x_0) \neq 0$. Since f is continuous, there exists a nonempty open interval I_{x_0} containing x_0 such that $f(x) \neq 0$ for $x \in I_{x_0}$. By the argument above, I_{x_0} is contained in a set of category I and hence is itself a set of category I (see Remark (vii) after Definition 2.4.1), which contradicts Corollary 2.4.5 of the Baire category theorem. □

That a continuous function may fail to have a derivative at any point of its domain of definition, though surprising, is nevertheless true. It turns out that "most" continuous functions have this property. More specifically, the set of continuous functions that have a finite derivative even on one side constitute a set of category I in the metric space C[0,1]. Thus, the functions that one deals with in calculus form a subset of a set of category I. In what follows, we shall show that functions in C[0,1] that are nowhere differentiable form a set of category II.

Consider the metric space C[0,1] equipped (as usual) with the metric

$$d(f, g) = \sup\{|f(x) - g(x)| : 0 \le x \le 1\},\ f, g \in C[0, 1].$$

It is a complete metric space. (See Proposition 1.4.13.)

Let A denote the subset of $C[0,1]$ such that, for some $x \in [0,1]$, f has a finite right hand derivative, i.e., there exists $\ell \in \mathbf{R}$ such that, given any $\varepsilon > 0$, there exists a $\delta > 0$ for which

$$\left| \frac{f(x+h) - f(x)}{h} - \ell \right| < \varepsilon$$

for all h satisfying $x + h \in [0,1]$ and $0 < h < \delta$.

For each positive integer n, let E_n denote the set of all $f \in C[0,\ 1]$ such that for some $x \in [0, 1 - 1/n]$,

$$\left| \frac{f(x+h) - f(x)}{h} \right| \le n$$

whenever $0 < h < 1/n$. It is clear that $E_n \subseteq E_{n+1}$. Moreover, if f has a finite right hand derivative at x, then $f \in E_n$ for some n. So, $A \subseteq \bigcup_{n=1}^{\infty} E_n$.

We shall show that each E_n is nowhere dense; then the union of the E_n is of category I and, hence, so is A. The space $C[0,1]$ with metric d, being complete, is of category II. Consequently, A^c, which consists of those functions in $C[0,1]$ that do not possess a right hand derivative at any point, is of category II. Since A^c is a subset of those $f \in C[0,1]$ that do not possess a derivative anywhere, it follows that there exist continuous functions that are nowhere differentiable and that the collection of these functions is a subset of category II.

In order to prove that each E_n is nowhere dense, we proceed by showing: (i) $\overline{E_n} = E_n$, and (ii) E_n° is empty.

Let $g \in \overline{E_n}$ and $\{f_j\}_{j \ge 1}$ be a sequence of functions in E_n such that $\lim_{j\to\infty} d(f_j, g) = 0$. Since each of the f_j is in E_n, there exists some point x_j (depending on f_j) such that

$$\left| \frac{f_j(x_j + h) - f_j(x_j)}{h} \right| \le n \qquad \text{for } 0 < h < 1/n,\ x_j \in [0, 1 - 1/n].$$

The points $\{x_{j_k}\}_{j \ge 1}$ constitute a bounded sequence of real numbers and so, by the Bolzano-Weierstrass theorem (Theorem 0.4.2), there exists a subsequence $\{x_{j_k}\}_{k \ge 1}$ such that $x_{j_k} \to x_0$. Since any subsequence of a convergent sequence converges to the same limit, it follows that $\lim_{k\to\infty} d(f_{j_k}, g) = 0$. Now,

$$\begin{aligned} |g(x_0 + h) - g(x_0)| \le{}& |g(x_0 + h) - g(x_{j_k} + h)| + |g(x_{j_k} + h) - f_{j_k}(x_{j_k} + h)| \\ &+ |f_{j_k}(x_{j_k} + h) - f_{j_k}(x_{j_k})| + |f_{j_k}(x_{j_k}) - g(x_{j_k})| + |g(x_{j_k}) - g(x_0)| \end{aligned} \tag{2.2}$$

By continuity of g, there exists m_1 such that $j_k \ge m_1$ implies

$$|g(x_{j_k}) - g(x_0)| < \frac{1}{4}\varepsilon h \qquad \text{and} \qquad |g(x_0 + h) - g(x_{j_k} + h)| < \frac{1}{4}\varepsilon h,$$

in view of the fact that $x_{j_k} \to x_0$. Since $\lim_{k\to\infty} d(f_{j_k}, g) = 0$, there exists m_2 such that $j_k \geq m_2$ implies

$$\sup\{|f_{j_k}(x) - g(x)| : x \in [0,1]\} < \frac{1}{4}\varepsilon h.$$

Choosing $j_k > \max\{m_1, m_2\}$, we obtain from (2.2) that

$$\left|\frac{g(x_0+h) - g(x_0)}{h}\right| \leq \left|\frac{f_{j_k}(x_{j_k}+h) - f_{j_k}(x_{j_k})}{h}\right| + \varepsilon \leq n + \varepsilon.$$

Since $x_0 \in [0, 1 - 1/n]$ and $\varepsilon > 0$ is arbitrary, it follows that $g \in E_n$. This establishes (i).

We next establish (ii), i.e., that E_n is nowhere dense. Since E_n is closed by (i), it is enough to show that E_n^c is everywhere dense (see Remark (i) after Definition 2.4.1). Let $f \in C[0,1]$. Since f is uniformly continuous on [0,1], there exists $\delta > 0$ such that

$$|f(x) - f(x')| < \frac{1}{2}\varepsilon \qquad \text{whenever } |x - x'| < \delta.$$

Choose a positive integer n such that $(1/n)\varepsilon < \delta$. Let

$$0 = x_0 < x_1 < \ldots < x_n = 1$$

be the partition of $[0,1]$ that divides the interval into n equal parts. Consider the rectangle with vertices

$$(x_{k-1}, f(x_{k-1}) - \frac{1}{2}\varepsilon),\ (x_k, f(x_k) - \frac{1}{2}\varepsilon),\ (x_k, f(x_k) + \frac{1}{2}\varepsilon),\ (x_{k-1}, f(x_{k-1}) + \frac{1}{2}\varepsilon).$$

Join the points $(x_{k-1}, f(x_{k-1}))$, $(x_k,\ f(x_k))$ by a sawtooth function that remains inside the rectangle and whose line segments have slopes greater than n in absolute value. Carrying out this process for each subinterval (x_{k-1}, x_k), $k = 1, 2, \ldots, n$, we obtain a function g in C[0,1] such that $|f(x) - g(x)| < \varepsilon$ for all $x \in$ [0,1]. Moreover, $g \in E_n^c$. This completes the proof. □

The above proof of the existence of a continuous function that is nowhere differentiable is nonconstructive in the sense that it does not provide a concrete example of such a function. The first known example, namely, $\sum_{n=0}^{\infty} \frac{\cos 3^n x}{2^n}$, is due to Weierstrass. The following example, due to van der Waerden, of a continuous nowhere differentiable function is the easiest to work with. Although the proof of its continuity uses a result to be proved in the next chapter, we prefer to present it here because of its immediate relevance to the foregoing discussion.

Example 2.4.8. Let $h : \mathbf{R} \to \mathbf{R}$ be defined as

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

extended to all of **R** by requiring that

$$h(x+1) = h(x);$$

in other words, h is periodic of period 1. (See Figure 2.7.) It is easily verified that h is continuous on $\mathbf{R}$. Define

$$f(x) = \sum_{n=0}^{\infty} \frac{h(10^n x)}{10^n}.$$

Since this series is dominated by the convergent series $(1/2)\sum_{n=0}^{\infty} 1/10^n$, it follows by the Weierstrass M-test (see Theorem 3.6.12) that the series converges uniformly. Its sum is therefore a continuous function, as argued in Chapter 1. We shall show that this function is nowhere differentiable. As the function is periodic, we may restrict ourselves to the case when $0 \leq x < 1$. Let $a \in [0, 1)$ have the decimal representation $a = .\,a_1 a_2 \ldots a_n \ldots$.

For $n \in \mathbf{N}$, let

$$x_n = .\,a_1 a_2 \ldots a_{n-1} b_n a_{n+1} \ldots,$$

where $b_n = a_n + 1$ if $a_n \neq 4$ or 9, while $b_n = a_n - 1$ if $a_n = 4$ or 9. Thus $x_n - a = \pm 10^{-n}$ and so $\lim_{n\to\infty} x_n = a$. To complete the proof, it will be suficient to show that

$$\lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a}$$

does not exist.

Now,

$$h(10^m x_n) - h(10^m a) = \begin{cases} 0 & \text{if } m \geq n, \\ \pm 10^{m-n} & \text{if } m < n. \end{cases}$$

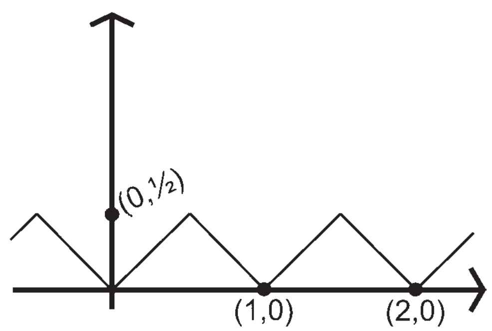

Figure 2.7

Thus,

$$\begin{aligned}\frac{f(x_n)-f(a)}{x_n-a} &= \sum_{m=0}^{\infty}\frac{h(10^m x_n)-h(10^m a)}{10^m(x_n-a)}\\ &= \sum_{m=0}^{n-1}\pm\frac{10^{m-n}}{10^m(\pm 10^{-n})}\\ &= \sum_{m=0}^{n-1}\pm 1. \qquad (2.3)\end{aligned}$$

Thus, for each n, the difference quotient on the left of (2.3) is the sum of n terms, each of which is either 1 or -1, so that the sum is an odd integer when n is odd and an even integer when n is even. It follows that

$$\lim_{n\to\infty}\frac{f(x_n)-f(a)}{x_n-a}$$

does not exist.

Finally, we show that the set of points of discontinuity of an arbitrary real-valued function defined on **R** is of a special kind. We begin with the following definition.

Definition 2.4.9. A subset S of **R** is said to be of type $\boldsymbol{F_\sigma}$ if $S=\bigcup_{n=1}^{\infty}S_n$, where each S_n is a closed subset of **R**.

Examples 2.4.10. (i) If F is a closed subset of **R**, then F is of type F_σ, since $F=\bigcup_{n=1}^{\infty}F_n$, where $F_1=F$ and $F_2=F_3=\ldots=\varnothing$.

(ii) The set **Q** of rationals in **R** is of type F_σ. Indeed, if $x_1, x_2, \ldots$ is an enumeration of **Q**, then each set $\{x_i\}$ is closed and we have $\mathbf{Q}=\bigcup_{i=1}^{\infty}\{x_i\}$.

(iii) Each open interval (a,b) is of type F_σ. This is because, if m is a positive integer such that $2/m<b-a$, then

$$(a,b)=\bigcup_{n=m}^{\infty}\left[a+\frac{1}{n},b-\frac{1}{n}\right].$$

The statement now follows, as $[a+1/n,\ b-1/n]$ is closed for each n.

Let f be a real-valued function defined on **R**. We shall show that the set of points of **R** at which f is discontinuous is always of type F_σ.

Definition 2.4.11. Let $f\colon\mathbf{R}\to\mathbf{R}$. If I is any bounded open interval of **R**, we define $\boldsymbol{\omega(f,I)}$, called the **oscillation over** I of the function f, as

$$\omega(f,I)=\sup_{x\in I}f(x)-\inf_{x\in I}f(x).$$

If $a\in\mathbf{R}$ is arbitrary, the **oscillation at** $\boldsymbol{a}$ of the function f, $\boldsymbol{\omega(f,a)}$, is defined as

$$\omega(f,a)=\inf\omega(f,I),$$

where the inf is taken over all bounded open intervals containing a.

Clearly, $\omega(f, I)$ and $\omega(f, a)$ are both nonnegative.

The following criterion of continuity is well known from real analysis:

Proposition 2.4.12. Let f be a real-valued function defined on $\mathbf{R}$. Then $\omega(f, a) = 0$ if and only if f is continuous at a.

Proof. Suppose f is continuous at a. Let $\varepsilon > 0$ be arbitrary. There exists a $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon/2$. If $I = (a - \delta, a + \delta)$, then for $x \in I$, $f(a) - \varepsilon/2 < f(x) < f(a) + \varepsilon/2$. So, $\omega(f, I) = \sup_{x\in I} f(x) - \inf_{x\in I} f(x) < \varepsilon$ and consequently,

$$\omega(f, a) = \inf \omega(f, I) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, and $\omega(f, a) \geq 0$, it follows that $\omega(f, a) = 0$.

On the other hand, suppose that $\omega(f, a) = 0$. If f is not continuous at a, there exists $\varepsilon > 0$ such that in every bounded open interval containing a, there exists an x for which $|f(x) - f(a)| \geq \varepsilon$, that is,

$$f(x) \geq f(a) + \varepsilon \text{ or } f(x) \leq f(a) - \varepsilon.$$

So, for every bounded open interval I containing a,

$$\omega(f, I) = \sup_{x\in I} f(x) - \inf_{x\in I} f(x) \geq 2\varepsilon,$$

which, in turn, implies

$$\omega(f, a) \geq 2\varepsilon,$$

and this contradicts the supposition that $\omega(f, a) = 0$. □

Theorem 2.4.13. Let $f: \mathbf{R} \to \mathbf{R}$ and $S_n = \{x \in \mathbf{R}: \omega(f, x) \geq 1/n\}$. Denote by S the set of points of $\mathbf{R}$ at which f is not continuous. Then, for each n the set S_n is closed. Moreover,

$$S = \bigcup_{n=1}^{\infty} S_n.$$

Thus, the points of $\mathbf{R}$ at which f is not continuous form a set of type F_σ.

Proof. Let x be a limit point of S_n. We need to show that $x \in S_n$. Let I be a bounded open interval containing x. Then I contains a point $y \in S_n$. But then $\omega(f, I) \geq \omega(f, y) \geq 1/n$. As I is any bounded open interval containing x, we have $\omega(f, x) \geq 1/n$, that is, $x \in S_n$.

It remains to show that $S = \bigcup_{n=1}^{\infty} S_n$. Let $x \in S$. Then by Proposition 2.4.12, $\omega(f, x) > 0$. So, there exists a positive integer n such that $\omega(f, x) \geq 1/n$. Hence $x \in S_n$. On the other hand, if $x \in S_n$, then clearly, $x \in S$. □

The irrational numbers in **R** form a set A of category II (see Corollary 2.4.4). We shall show that A is not of type F_σ. Suppose that, on the contrary,

$$A = \bigcup_{i=1}^{\infty} F_i,$$

where each F_i is closed. Since each closed set F_i contains only irrational numbers, it cannot contain an interval. Thus, F_i is nowhere dense and so A is of category I. This contradicts the fact that A is of category II. We have thus proved the following theorem:

Theorem 2.4.14. There is no real-valued function defined on **R** that is continuous at each rational point and is discontinuous at each irrational point.

We give an example of a function that is continuous at every irrational number and discontinuous at every rational number.

Example 2.4.15. The function f defined as

$$f(x) = \begin{cases} 1/n & \text{where } n \text{ is least in } \mathbf{N} \text{ such that } x = m/n\ , \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

has the required property, as we shall argue.

Let $c \in \mathbf{R}$ be rational, so that $f(c) = 1/n$, where n is the least integer in **N** such that $c = m/n$ and $m \in \mathbf{Z}$. Choose $\varepsilon = 1/2n$. For any $\delta > 0$, the interval $(c - \delta, c + \delta)$ contains an irrational number x, so that $|f(x) - f(c)| = |0 - 1/n| = 1/n > \varepsilon$. Therefore, when $\varepsilon = 1/2n$, no positive number δ can have the property that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

On the other hand, if $c \in \mathbf{R}$ is an irrational number and ε any positive number whatsoever, there exists (by the Archimedean property of **R**) some $n_0 \in \mathbf{N}$ such that $1/n_0 < \varepsilon$. Now consider the interval $(c - 1/2n_0^2, c + 1/2n_0^2)$. For any p and q in this interval, $|p - q| < 1/n_0^2$. It follows that this interval can contain at most one rational number of the form $r = m/n$ with $n \leq n_0$ because, when $m_1 n_2 - m_2 n_1 \neq 0$, we have

$$n_1 \leq n_0,\ n_2 \leq n_0 \Rightarrow \left|\frac{m_1}{n_1} - \frac{m_2}{n_2}\right| = \frac{|m_1 n_2 - m_2 n_1|}{n_1 n_2} \geq \frac{1}{n_1 n_2} \geq \frac{1}{n_0^2}.$$

If there is any such r in the interval $(c - 1/2n_0^2,\ c + 1/2n_0^2)$, let $\delta = |c - r|$. If there is no such r, let $\delta = 1/2n_0^2$. In both cases, no number x in the interval $(c - \delta, c + \delta)$ can be of the form $x = m/n$ with $n \leq n_0$. Thus, every number in this interval is either irrational or is a rational number of the form m/n with $n > n_0$. Therefore,

$$|x-c|<\delta \Rightarrow \text{either } f(x)=0 \text{ or } f(x)=\frac{1}{n} \text{ with } n>n_0$$
$$\Rightarrow \text{either } |f(x)-f(c)|=0 \text{ or } |f(x)-f(c)|=\frac{1}{n} \text{ with } n>n_0$$
$$\Rightarrow |f(x)-f(c)|<\frac{1}{n_0}<\varepsilon.$$

2.5. Exercises

1. Let $S(x, \delta)$ be a ball with centre x and radius δ in a metric space (X, d). Prove that if $0<\varepsilon<\delta-d(x,z)$, then $S(z,\varepsilon)\subseteq S(x,\delta)$.
Hint: If $y\in S(z,\varepsilon)$, then $d(x,y)\leq d(x,z)+d(z,y)<d(x,z)+\varepsilon<\delta$.

2. Prove that $\overline{S(x,\varepsilon)}\subseteq\{y: d(x,y)\leq\varepsilon\}$ and give an example of a metric space containing a ball for which the inclusion is proper.
Hint: $\{y: d(x,y)\leq\varepsilon\}$ is closed and contains $S(x,\varepsilon)$; use Corollary 2.1.27(iii). Let (X, d) be discrete, X contain more than one point, and let $\varepsilon=1$. Then $\overline{S(x,1)}=\overline{\{x\}}=\{x\}$, whereas $\{y: d(x,y)\leq 1\}=X$.

3. Show that for any two points x and y of a metric space there exist disjoint open balls such that one is centred at x and the other at y.
Hint: Let $r=d(x,y)$. Then $r>0$, and $S(x,r/2)$ and $S(y,r/2)$ are the desired balls.

4. Let (X, d) be a metric space and let $S(x,r_1)$ and $S(y,r_2)$ be two intersecting balls containing a common point z. Show that there exists an $r_3>0$ such that $S(z,r_3)\subseteq S(x,r_1)\cap S(y,r_2)$.
Hint: Since $z\in S(x,r_1)$ and $S(x,r_1)$ is open, there exists an open ball $S(z,r_1')$ centred at z and with radius r_1' such that $S(z,r_1')\subseteq S(x,r_1)$. Similarly, there exists an open ball $S(z,r_2')$ centred at z and with radius r_2' such that $S(z,r_2')\subseteq S(y,r_2)$. Let $r_3=\min\{r_1',r_2'\}$. Then $S(z,r_3)\subseteq S(x,r_1)\cap S(y,r_2)$ since $S(z,r_3)\subseteq S(z,r_1')$ as well as $S(z,r_3)\subseteq S(z,r_2')$.

5. Let $S(x,r)$ be an open ball in a metric space (X, d) and A be closed subset of X such that $d(A)\leq r$ and $A\cap S(x,r)\neq\varnothing$. Show that $A\subseteq S(x,2r)$.
Hint: Let $y\in A\cap S(x,r)$. For $z\in A$,
$$d(z,x)\leq d(z,y)+d(y,x)<r+r=2r.$$

6. Let $A\subseteq[0,1]$ and $F=\{f\in C[0,1]: f(t)=0 \text{ for every } t\in A\}$. Show that F is a closed subset of $C[0,1]$ equipped with the uniform metric.
Hint: Let $t\in A$ be fixed. The set $\{f\in C[0,1]: f(t)=0\}$ can be shown to be a closed subset of $C[0,1]$ as follows: If f is a limit point of the set, then there exists a sequence $\{f_n\}_{n\geq 1}$ in the set such that $\lim_{n\to\infty} f_n=f$ uniformly. Since uniform convergence implies pointwise convergence, $\lim_{n\to\infty} f_n(t)=f(t)$. Since F is an intersection of such sets, Theorem 2.1.34(i) applies.

7. Let $C[0,1]$ be equipped with the metric defined by

$$d(f,g) = \int_0^1 |f(t) - g(t)|dt, \; f,g \in C[0,1].$$

With notations as in Exercise 6, show that F is not necessarily closed.
Hint: Let $A = \{0\}$. Consider the sequence

$$f_n(t) = \begin{cases} nt & 0 \le t \le \frac{1}{n} \\ 1 & t > \frac{1}{n}. \end{cases}$$

If $f \equiv 1$, then

$$d(f_n, f) = \int_0^{1/n} (1 - nt)dt = \frac{1}{2n} \to 0 \text{ as } n \to \infty.$$

The functions of the sequence $\{f_n\}_{n \ge 1}$ are in F, but $f \notin F$.

8. Let X denote the space of all bounded sequences with

$$d(x,y) = \sup_i |x_i - y_i|,$$

where $x = \{x_i\}_{i \ge 1}$ and $y = \{y_i\}_{i \ge 1}$ are in X. Show that the subset Y of convergent sequences is closed in X.
Hint: Let $z \in X$ be a limit point of Y. Then there exists a sequence $\{y^{(n)}\}_{n \ge 1}$ in Y satisfying the following condition: For every $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that $n \ge n_0(\varepsilon)$ implies

$$\sup_k |y_k^{(n)} - z_k| < \frac{\varepsilon}{3}.$$

The sequence $\{y_j^{(n_0)} j\}_{j \ge 1}$, being convergent, is Cauchy. So there exists l such that $i, j \ge l$ implies

$$|y_i^{(n_0)} - y_j^{(n_0)}| < \frac{\varepsilon}{3}.$$

Now,

$$|z_i - z_j| \le |z_i - y_i^{(n_0)}| + |y_i^{(n_0)} - y_j^{(n_0)}| + |y_j^{(n_0)} - z_j| < \varepsilon \qquad \text{for } i, j \ge l.$$

The bounded sequence $\{z_i\}_{i \ge 1}$ is Cauchy and is, therefore, convergent and, hence, belongs to Y.

9. Let A be a subset of a metric space (X, d). Show that

$$\bar{A} = \bigcap \{F \subseteq X : F \text{ is closed and } F \supseteq A\}$$

Hint: $\bar{A}$ is a closed set and $\bar{A} \supseteq A$. Therefore, on the one hand, $\bar{A}$ is one of the sets in the intersection, while on the other hand, by Corollary 2.1.27(iii), it is a subset of every set in the intersection.

10. Let $X = \mathbf{C}$ with the usual metric and $A = \{(x, y): y = \sin(1/x),\ 0 < x \le 1\}$. Show that

$$\bar{A} = A \cup \{(0, y) : -1 \le y \le 1\}$$

Hint: Each open ball centred at $(0, y)$, $-1 \le y \le 1$, has nonempty intersection with A. It may be seen that every point outside $A \cup \{(0, y) : -1 \le y \le 1\}$ is the centre of a ball having an empty intersection with A.

11. Let $X = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| < 2 \text{ and } |x_2| < 1\}$ be equipped with the metric induced from $\mathbf{R}^2$. For any $x = (x_1, x_2) \in X$ and $r \ge 2\sqrt{5}$, show that

$$S_X(x, r) = X.$$

Hint: $S_X(x, r) = S(x, r) \cap X$, where

$$S(x, r) = \{(y_1, y_2) \in \mathbf{R}^2 : \sqrt{[(y_1 - x_1)^2 + (y_2 - x_2)^2]} < r\}.$$

It is enough to show that $X \subseteq S(x, r)$. For $x \in X$ as well as $y \in X$, we have $d(y, x) \le d(X) = 2\sqrt{5} \le r$.

12. Let $A = \{z \in \mathbf{C}: |z + 1|^2 \le 1\}$ and $B = \{z \in \mathbf{C}: |z - 1|^2 < 1\}$, and let $A \cup B$ be equipped with the metric induced from $\mathbf{C}$. Identify $\mathrm{cl}_{A\cup B}(B)$.
Hint: $\mathrm{cl}_{A\cup B}(B) = (A \cup B) \cap \mathrm{cl}_{\mathbf{C}}(B) = (A \cup B) \cap \{z \in \mathbf{C}: |z - 1|^2 \le 1\} = B \cup \{0\}$.

13. Let (X, d) be a metric space and A be a subset of X. Show that (i) $X \backslash \bar{A} = (X \backslash A)^o$; (ii) $X \backslash A^o = \overline{(X \backslash A)}$.
Hint: (i) $x \in (X \backslash A)^o$ iff there exists a ball $S(x, \varepsilon))$ centred at x with suitable radius ε such that $S(x, \varepsilon) \subseteq X \backslash A$ iff $S(x, \varepsilon) \cap A = \varnothing$ iff $x \notin \bar{A}$.
(ii) Replace A by $X \backslash A$ in (i) and take complements.

14. Give an example of a subset Y of a metric space (X, d) for which $(\bar{Y})^o \ne \overline{(Y^o)}$.
Hint: Let $(\mathbf{R}, d)$ be the usual real line and Y denote the set of rationals in $\mathbf{R}$. Then

$$(\bar{Y})^o = (\mathbf{R})^o = \mathbf{R} \text{ whereas } \overline{(Y^o)} = \overline{\varnothing} = \varnothing.$$

15. For a subset Y of a metric space (X, d), $(X \backslash Y)^o$ is called the **exterior** of Y and is denoted by **ext(Y)**. The **boundary** of Y is defined to be $\bar{Y} \cap \overline{(X \backslash Y)}$ and is denoted by $\partial(\boldsymbol{Y})$. Show that
   (i) $\partial(Y) = \partial(X \backslash Y)$;
   (ii) $\bar{Y} = Y^o \cup \partial(Y)$;
   (iii) $Y^o \cap \partial(Y) = \varnothing$;
   (iv) $(X \backslash Y)^o \cap \partial(Y) = \varnothing$;
   (v) $X = Y^o \cup \partial(Y) \cup (X \backslash Y)^o$;
   (vi) $Y \backslash \partial(Y) = Y^o$.

Hint: (i) $\partial(X\backslash Y) = \overline{(X\backslash Y)} \cap \overline{(X\backslash(X\backslash Y))} = \overline{(X\backslash Y)} \cap \bar{Y} = \partial(Y)$.

(ii) $\partial(Y) \subseteq \bar{Y}$, $Y^o \subseteq Y \subseteq \bar{Y}$. So $Y^o \cup \partial(Y) \subseteq \bar{Y}$. Let $y \in \bar{Y}$. If $y \in Y^o$, then $y \in Y^o \cup \partial(Y)$. If $y \notin Y^o$, then for all $\varepsilon > 0$, $S(y, \varepsilon) \nsubseteq Y$, i.e., $S(y, \varepsilon) \cap (X\backslash Y) \neq \emptyset$. Hence, $y \in \overline{(X\backslash Y)}$. So $y \in \bar{Y} \cap \overline{(X\backslash Y)} = \partial(Y)$. Consequently, $y \in Y^o \cup \partial(Y)$.

(iii) $Y^o \cap \partial(Y) = Y^o \cap (\bar{Y} \cap \overline{(X\backslash Y)}) = (Y^o \cap \bar{Y}) \cap (Y^o \cap \overline{(X\backslash Y)}) = Y^o \cap (Y^o \cap (X\backslash Y^o)) = Y^o \cap \emptyset = \emptyset$, using Exercise 13(ii).

(iv) Replace Y by $X\backslash Y$ in (iii) and use (i).

(v) $Y^o \cup \partial(Y) \cup (X\backslash Y)^o = \bar{Y} \cup (X\backslash Y)^o = \bar{Y} \cup (X\backslash \bar{Y}) = X$, using (ii) and Exercise 13(i).

(vi) $Y\backslash\partial(Y) = Y \cap (X\backslash\partial(Y)) = Y \cap (Y^o \cup (X\backslash Y)^o) = (Y \cap Y^o) \cup (Y \cap (X\backslash Y)^o) = Y^o$, using (iii), (iv) and (v) above.

16. Show that the Cantor set P is nowhere dense.
Hint: No segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right), \tag{2.3}$$

where k and m are positive integers, has a point in common with P. Since every interval (α, β) contains an interval of the form (2.3) whenever

$$3^{-m} < \frac{\beta - \alpha}{6},$$

it follows that P contains no interval.

17. Consider the rationals $\mathbf{Q}$ as a subset of the complete metric space $\mathbf{R}$. Prove that $\mathbf{Q}$ cannot be expressed as the intersection of a countable collection of open sets.
Hint: Suppose $\mathbf{Q} = G_1 \cap G_2 \cap \ldots$, where each G_i is open in $\mathbf{R}$. Then the set of irrationals is $\bigcup_{i=1}^{\infty} G_i^c$, where each G_i^c is closed. Since each G_i^c contains only irrationals, no G_i^c contains a nonempty interval. Thus, G_i^c is closed and nowhere dense for each $i = 1, 2 \ldots$.

18. Consider a real valued function f on $[0, 1]$. If f has an nth derivative that is identically zero, it easily follows by using the mean value theorem that f coincides on $[0, 1]$ with a polynomial of degree at most $n - 1$. The following generalisation is valid: If f has derivatives of all orders on $[0, 1]$, and if at each x there is an integer $n(x)$ such that $f^{(n(x))}(x) = 0$, then f coincides on $[0, 1]$ with some polynomial.
Hint: See [3; p. 58].

19. Let A be either an open subset or a closed subset of (X, d). Then $(\partial(A))^o = \emptyset$, so that $\partial(A)$ is nowhere dense. Is this true if we drop the requirement that either A is open or A is closed?
Hint: One need prove only the case of a closed set, because A is open iff A^c is open and $\partial(A) = \partial(A^c)$ by Exercise 15(i) above. If A is closed, then $\partial(A) = A \cap \overline{(X\backslash A)}$. Let G be an open subset of (X, d) such that $G \subseteq \partial(A)$.

Then $G \subseteq A \cap \overline{(X\backslash A)}$. This implies not only that $G \subseteq A$, so that $G \subseteq A^o$ (because A^o is the largest open set contained in A), but also that (see Exercise 13(ii)) $G \subseteq \overline{(X\backslash A)} = X\backslash A^o$. Hence, $G = \varnothing$. When A is neither open nor closed, $\partial(A)$ need not be nowhere dense: Consider the set A of rational numbers in the metric space $(\mathbf{R},d)$. For this set, $\partial(A) = \bar{A} \cap \overline{(X\backslash A)} = \mathbf{R} \cap \mathbf{R} = \mathbf{R}$, and hence $(\partial(A))^o = \mathbf{R} \neq \varnothing$.

20. Let $G_1, \ G_2, \ldots$ be a sequence of open subsets of $\mathbf{R}$, each of which is dense. Prove that $\bigcap_{n=1}^{\infty} G_n$ is dense.
Hint: Suppose not. Then there exists $x \in \mathbf{R}$ and an open interval I_x containing x such that $I_x \cap \bigcap_{n=1}^{\infty} G_n = \varnothing$. Thus, $x \in I_x \subseteq \bigcup_{n=1}^{\infty} G_n^c$. But each G_n^c is nowhere dense, and, hence, $\bigcup_{n=1}^{\infty} G_n^c$ is of category I (see Definition 2.4.1). But I_x is of category II by Corollary 2.4.4. Since a subset of a set of category I must be of category I (see Remark (viii) just before Theorem 2.4.3), we arrive at a contradiction. (The reader may note that the argument is valid in any complete metric space.)

21. Let E be a closed subset of a metric space (X, d). Prove that E is nowhere dense if and only if for every open subset G there is a ball contained in $G\backslash E$.
Hint: Suppose E is nowhere dense. Then $G\backslash E \neq \varnothing$ because, otherwise, $G \subseteq E$, and this contradicts the supposition that E is nowhere dense. Let $x \in G\backslash E = G \cap E^c$. Since G and E^c are both open, there exists an $r > 0$ such that $S(x, r) \subseteq G\backslash E$. For the converse, the hypothesis implies that every open set has nonempty intersection with E^c. It follows that $E^c = (\bar{E})^c$ is dense in X, so that E is nowhere dense.

22. Let $(\mathbf{R}, d_1)$ be the metric space where

$$d_1(x, y) = \begin{cases} |x| + |x - y| + |y| & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Show that the ε-ball about 0 with the metric d_1 is the same as the $(\varepsilon/2)$-ball about 0 with the usual metric. Also, if $0 < \varepsilon < |y|$, then the ε-ball about any nonzero element y with the metric d_1 consists of y alone. Describe a base for the open sets of $(\mathbf{R}, d_1)$.