Preface

Based on the seminal work of Naum Zuselevich Shor (Institute of Cybernetics, Kiev) in the 1980s [1, 2, 3], the theory of positive polynomials lends new theoretical insights into a wide range of control and optimization problems. Positive polynomials can be used to formulate a large number of problems in robust control, non-linear control and non-convex optimization. Only very recently it has been realized that polynomial positivity conditions can be formulated efficiently in terms of Linear Matrix Inequality (LMI) and Semidefinite Programming (SDP) problems. In turn, it is now recognized that LMI and SDP techniques play a fundamental role in convex optimization, see e.g. the plenary talk by Stephen Boyd at the 2002 IEEE Conference on Decision and Control or the successful Workshop on SDP and robust optimization organized in March 2003 by the Institute of Mathematics and its Applications at the University of Minnesota in Minneapolis. For the above reasons, the joint use of positive polynomials and LMI optimization provides an extremely promising approach to difficult control problems.

In the last years, several sessions at major control conferences as well as specialized workshops have been dedicated to these research topics. The invited session *Positive Polynomials in Control* at the 2003 IEEE Conference on Decision and Control, organized by the editors of this volume, has shown that new research directions are quickly emerging, thus pointing out the need for a more detailed overview of the current activity in this research area. This is the main aim of the present book. Another important objective of the book is to collect contributions from several fields (control, optimization, mathematics), in order to show different views and approaches to the topics outlined above.

The book is organized in three parts.

The first part collects a number of articles on applications of positive polynomials and LMI optimization to solve various *control problems*, starting with a contribution by Jarvis-Wloszek, Feeley, Tan, Sun and Packard on the *sum-of-squares (SOS)* decomposition of positive polynomials for nonlinear polynomial systems analysis and design [I.1]. SOS techniques are also used by Papachristodoulou and Prajna to cope with nonlinear non-polynomial systems, using algebraic reformulation techniques [I.2]. Hol and Scherer in [I.3] describe several results on the use of SOS polynomial matrices to derive bounds on the global optima of non-convex bilinear matrix inequality (BMI) problems, in particular those arising in fixed-order H_∞ design. This latter problem, traditionally deemed as difficult in the control community, is approached differently by Henrion [I.4]: with the help of matrix polynomial positivity conditions, sufficient LMI conditions for scalar fixed-order H_∞ design are obtained. Gram-matrix representation of homogeneous forms, similar to the SOS representation, are used by Chesi, Garulli, Tesi and Vicino in [I.5] to construct less conservative quadratic-in-the-state but polynomial-in-the-parameter Lyapunov functions for assessing robust stability of polytopic linear systems. Finally, positivity conditions for multivariate polynomial matrices are obtained by Bliman [I.6] via the Kalman-Yakubovich-Popov (KYP) lemma, and an application to the design of linear-parameter-varying (LPV) gain-scheduled state-feedback control laws is described.

The second part of the book is more mathematical, and gives an overview of different *algebraic techniques* used to cope with polynomial positivity. Results of semi-algebraic geometry by Hilbert and Pólya led Parrilo [4, 5] to construct converging hierarchies of LMI relaxations for optimization over semi-algebraic sets, based on the *theory of SOS representations* of positive polynomials. Independently, results by Schmüdgen and Putinar were used by Lasserre [6] to construct similar converging LMI hierarchies, with the help of the *theory of moments*. Both Parrilo's and Lasserre's approaches can be viewed as *dual to each other*. The paper by De Klerk, Laurent and Parrilo [II.1] shows equivalence between these two approaches in the special case of minimization of forms on the simplex. In [II.2], Lasserre applies the theory of moments to characterize the set of zeros of triangular sets of polynomial equations. Namely, it is shown that the particular structure of the problem allows for the derivation of a simple LMI formulation. Lasserre's hierarchy of LMI relaxations has proved asymptotic convergence under some constraint qualification assumptions, and in particular if the semi-algebraic feasible set is compact: Powers and Reznick [II.3] investigate what happens with the positivity condition of Schmüdgen-Putinar if this compactness assumption is not satisfied. Finally, in [II.4] Šiljak and Stipanović follow a different approach to ensure polynomial positivity. Based on Bernstein's polynomials, they derive criteria for stability analysis and robust stability analysis of two-indeterminate polynomials.

Finally, the third part of the book is dedicated to *numerical aspects* of positivity of polynomials, and recently developed software tools which can be employed to solve the problems discussed in the book. Parrilo in [III.1] surveys a collection of algebraic results (sparse polynomials and Newton polytopes, ideal structure with equality constraints, structural symmetries) to reduce the size of the LMI formulation of SOS decomposition of positive polynomials. Vandenberghe, Balakrishnan, Wallin, Hansson and Roh [III.2] discuss implementations of primal-dual interior-point methods for LMI problems derived from the KYP lemma (positivity conditions on one-indeterminate matrix polynomials). It is shown that the overall cost can be reduced to $O(n^4)$, or even $O(n^3)$, as opposed to the $O(n^6)$ of conventional methods, where n is the size of the Lyapunov matrix. In their paper [III.3], Hachez and Nesterov use the theory of conic duality to study in considerable detail optimization problems over positive polynomials with additional interpolation conditions. As a striking result, they show that the complexity of solving the dual LMI formulation is almost independent of the number of interpolation constraints, which has obvious applications in designing more efficient tailored primal-dual interior-point algorithms. The book winds up with descriptions of recent developments in two alternative Matlab software currently available to handle positive multivariate polynomials, using either the SOS decomposition (SOSTOOLS) or the dual moment approach (GloptiPoly). Prajna, Papachristodoulou, Seiler, and Parrilo survey in [III.4] the main features of SOSTOOLS along with its control applications, whereas Henrion and Lasserre in [III.5] describe the global optimality certificate and solution extraction mechanism implemented in GloptiPoly.

We believe that the organization of the book into three parts reflects the current trends in the area, with interplay between control engineers, mathematicians, optimizers and software developers.

October 2004

Didier Henrion
Andrea Garulli

References

1. N. Z. Shor (1987). Quadratic optimization problems. Soviet J. Comput. Syst. Sci. 25:1-11.
2. N. Z. Shor (1987). Class of global minimum bounds of polynomial functions. Cybernetics, 23(6):731–734. Russian orig.: Kibernetika, 6:9-11, 1987.
3. N. Z. Shor (1998). Nondifferentiable optimization and polynomial problems. Kluwer, Dordrecht.
4. P. A. Parrilo (2000). Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. PhD Thesis, Calif. Inst. Tech, Pasadena.
5. P. A. Parrilo (2003). Semidefinite programming relaxations for semialgebraic problems. Math. Prog. Ser. B, 96(2):293–320.
6. J. B. Lasserre (2001). Global optimization with polynomials and the problem of moments. SIAM J. Opt. 11(3):796–817.