Introduction

The last thing one knows when writing a book is what to put first.

Blaise Pascal (1623–1662)

It is tempting to think that there are just three special mathematical constants: π, e and i. In fact there are many, each with its own definition, each originating in some natural way in its own area of mathematics, each given a special symbol and a name too. They need symbols to represent them because they are awkward; that is, they have no convenient, finite numeric representation and no patterned infinite one: the ratio of the circumference to the diameter of any circle is not 3.142 or $\frac{22}{7}$, it is 3.141 59..., which is as mysterious as $(2.718\,28\ldots)^x$ essentially being the only function equal to its own derivative; in each case the trailing dots suggest the irrationality (let alone transcendence) of the numbers. Compared with these, writing i for $\sqrt{-1}$ is a small convenience. The number, now universally known as Gamma, is generally accepted to be the most significant of the 'constants obscura' and as such is the fourth important special constant of mathematics; its symbol is the Greek letter γ and the constant it represents is forever associated with the name of the Swiss genius, Leonhard Euler (1707–1783). Its value is the unprepossessing 0.577 215 6..., with its own trailing dots making the same suggestions about its character—but unlike its illustrious colleagues, so far they remain no more than suggestions.

This book is an exploration of γ and inescapably this means that it is also an exploration of logarithms and the harmonic series, since it is the interrelationship between them that Euler exploited to define his constant as

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n\right),$$

where the ln is the ubiquitous log to the base e, derived from the French expression 'logarithmic natural'; the harmonic series, which occupies a less publicized place in mathematical literature, is its discrete counterpart:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

The mid 1970s brought with it the hand-held, microchip-centred, battery-powered, comparatively cheap calculator, thereby bringing to an end the role of logarithms and the slide rule as calculative aids. Yet the appearance of them in a piece of mathematics is seldom a cause for surprise. Anyone who has studied calculus would see them materialize time and again, quite probably in the expression for the integral of some function or in their role as the inverse of the exponential function, with e vying with π for constant supremacy. They can also arise without warning in situations that seem remote from their influence, and when they do so they exercise a surprising control in unexpected places—as we shall see: we will also see that the harmonic series, and others related to it, enjoy an important existence of their own.

The book naturally separates into two parts: Chapters 1–11 might be described as 'theory', and the remainder as 'practice'.

In the 'theory' part we are concerned with definitions and some consequences of them, methods to approximate, and to some extent, with preparation for the remaining chapters. We start by looking at the peculiar way in which logarithms were initially defined, a way which reveals the immense intellectual effort that must have been invested to turn multiplication into addition, to utilize an idea from the old world that helped to usher in the new. The harmonic series, with its three peculiar properties, is discussed and then its specializations and generalizations, before looking more closely at that definition of γ and having done that, and having convinced ourselves that the number actually exists, at ways of approximating its value, using both decimal and fractional methods. Among all of this we prove a barely credible result about co-prime integers and establish an identity (of Euler's) that holds the key to the modern study of prime numbers.

The later chapters, which are devoted to 'practice', look at some of the ways in which the three objects of our attention can appear in mathematics, and to some extent, in applications of it. Gamma's varied roles in analysis and number theory are mentioned, some surprising appearances of the harmonic series are discussed, and three such of logarithms. The finale is really just another application of logarithms, but since the application is the Prime Number Theorem, leading to the Riemann Hypothesis (neither of which we prove!), it is deservedly singled out. It is inevitable that our journey reaches mathematics that is 'worthy of serious consideration', as Euler himself said of γ, but none is more worthy than that celebrated Prime Number Theorem and that awesome Riemann Hypothesis; the first harnesses the wayward behaviour of the primes, the second adds finesse to that control by asking about the zeros of a function that seems to have none, but which stands alone as the greatest problem in mathematics today.

How difficult is the mathematics? That of course is a subjective matter. Certainly, we have not shied away from the use of symbols, since to do so would have condemned us merely to talking about mathematics rather than actually doing it. Yet, there are few really advanced techniques used, it is more that in some places simple ideas have been used in advanced ways. Mathematics makes a nice distinction between the usually synonymous terms 'elementary' and 'simple', with 'elementary' taken to mean that not very much mathematical knowledge is needed to read the work and 'simple' to mean that not very much mathematical ability is needed to understand it. In these terms we think the content is often elementary but in places not so very simple. The reader should expect to make use of a pen and paper in many places; mathematics is not a spectator sport! The approach is reasonably rigorous but informal, as this is no textbook, it is more a context book of mathematics in which the reader is asked to take time out from studying the mathematics to read a little around it and about the mathematicians who produced it or of the times in which they lived; sometimes in detail but other times just a few lines and then not always, as this is no history of mathematics book either; it merely acknowledges that mathematics comes from mathematicians, not books, and seeks to bring a sometimes shadowy figure forward to share the prominence of his ideas, and to give some sort of feel for the way in which those ideas developed over time.

The exception to the 'elementary' classification is some of the content of the final chapter on the Riemann Hypothesis; necessarily, this involves some complex function theory and in particular complex differentiation and integration. To those who have met these ideas the work should present few problems, but to those who have not they will look rather frightening; if so, simply ignore them or better still try to find out about them since they are a most glorious and powerful construction; a 'crash course' in some elements of complex function theory is included in Appendix D. The Riemann Hypothesis really is the greatest unsolved problem in mathematics, so it shouldn't be surprising that it is neither 'elementary' nor 'simple'; if the chapter entices hunger in some to get to grips with Cauchy's great invention it will have justified itself on that ground alone.

We hope that the material will appeal to a variety of people who have a little probability and statistics and a good calculus course behind them, and before that a rigorous course in algebra, if such a thing still exists: the motivated senior secondary student, who may well be seeing many of the ideas for the first time, the college student for whom the text may put flesh on what can sometimes be dry bones, the teacher for whom it might be a convenient synthesis of some nice ideas (and maybe the makings of a talk or two), and also those who may have left mathematics behind and who wish to remind themselves why they used to find it so fascinating. The reader will judge to what extent this book achieves its aim: to explain interesting mathematics interestingly.

The names of many mathematicians appear, names that should bring wonder to anyone interested in the subject and its history, but it is that name Euler that will force itself onto the page more than any other. It is not that we happen to pass through the mathematical territory to which he holds title, but more that it would be difficult, if not impossible, to go far in any mathematical direction without feeling his influence. For example, much of the notation that we now take for granted originates from him; in particular, e, i, $f(x)$, $\sum$, Δ, $\sin x$, $\cos x$, etc., as well as the standard manner of labelling a triangle, with the vertex the capital letter corresponding to the opposite side's small letter. It can be hard to appreciate, or easy to forget, just how many important ideas his name is associated with or perhaps even attached to; he invented many vastly important concepts and touched every known area of the subject—and everything he touched he adorned. According to R. Calinger, 'Euler's books and memoirs, of which 873 have so far been listed, comprise approximately a third of the entire corpus of research on mathematics and mechanics, both rational and engineering, published from 1726 to 1800.' The *Opera Omnia*, his collected works, has reached 74 volumes of 300 to 600 pages each; the final part has still to be finished and will comprise at least another seven volumes. Looking up 'Euler' in the index of a mathematics or history of mathematics book can be a frustrating experience, as the eye is routinely confronted with a block of page references, sometimes unspecified, at other times separated into a list, which might begin,

> Euler angles, Euler triangle, Euler characteristic, Euler's identity, Euler circle, Euler circuit, Euler–Mascheroni constant, Euler line, Euler numbers, Euler's first integral, Euler's second integral, Euler polynomials, Euler's Totient function, etc.,

and continue for dozens more entries.

And perhaps all that was needed was to know how to pronounce his name: 'Oiler'.

The noun 'genius' has been defined as 'exalted intellectual power, instinctive and extraordinarily imaginative and creative capacity'. Extravagant use of the word serves only to dilute its meaning or to bring into question the judgement of the author, but we have used it already and will risk employing it on a number of other occasions, no more fittingly than with Euler, safe in the conviction that if he was not a genius and these people were not geniuses then none have yet been born. Yet, to the majority, his name is probably as mysterious as his constant. He breathed life into γ through his Zeta functions (the generalizations of H_n), the summation of one of which was to become a long-standing problem—described as 'the despair of analysts'—until Euler's outrageous solution put an end to it.

With Euler and with those who preceded him and to some extent those who followed him we will deal with times remote from the modern years of 'publish or perish' and in consequence primacy over an initiative is often far from easy to establish; it might depend on a note to a contemporary or a recorded comment more often than an article in a learned journal, and even then that article might appear years after the actual breakthrough (the controversy surrounding the discovery of the calculus by Newton and Leibnitz stands as an infamous example of the problems that can arise). We hope that the reader will understand if the story is not always complete, and agree that where it is not complete it is at least representative.

Dr Urs Burckhardt, President of the Euler Commission, has written, 'Indeed, through his books, which are consistently characterized by the highest striving for clarity and simplicity and which represent the first actual textbooks in the modern sense, Euler became the premier teacher of Europe not only of his time but well into the 19th century.' Euler, as ever, provides a target too distant to reach, or even clearly to see, yet the pleasures (and frustrations) of achieving a fresh understanding of old ideas and realizations of new ones has proved marvellously invigorating and has brought with it the reminder that the best way of learning is by teaching, whether it be by the spoken or written word. We hope that the reader will share our enthusiasm as we take brief excursions though countries, centuries, lives and works, unfolding the stories of some remarkable mathematics from some remarkable mathematicians.