Chapter 2

Finite Difference Calculus

In this chapter we review the calculus of finite differences. The topic is classic and covered in many places. The Taylor series is fundamental to most analysis. A good reference for beginners is Hornbeck [45].

2.1 1-D Differences on a Uniform Mesh

Our objective is to develop differentiation formulas which deal only with functions U which are sampled at discrete grid points X_i: $U(X_i) \equiv U_i$. The sampling grid is assumed to lay out in the natural way, ordered with X, left to right as below.

$$\bullet_{i-1} \qquad \bullet_{i} \qquad \bullet_{i+1} \qquad \bullet_{i+2} \qquad \bullet_{i+3}$$

Assuming equal mesh spacing $h \equiv X_{i+1} - X_i$ for all i, we have the Taylor series:

$$U_{i+1} = U_i + h\frac{\partial U_i}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 U_i}{\partial x^2} + O(h^3) \tag{2.1}$$

$$\begin{aligned} \frac{\partial U_i}{\partial x} &= \frac{U_{i+1} - U_i}{h} - \frac{h}{2!}\frac{\partial^2 U_i}{\partial x^2} + O(h^2) \\ &= \frac{\Delta U_i}{h} + O(h) \end{aligned} \tag{2.2}$$

where the leading error is "of order h (which can be made arbitrarily small with mesh refinement.)" For $\frac{\partial^2 U_i}{\partial x^2}$, we write another Taylor series

$$U_{i+2} = U_i + 2h\frac{\partial U_i}{\partial x} + \frac{(2h)^2}{2!}\frac{\partial^2 U_i}{\partial x^2} + \frac{(2h)^3}{3!}\frac{\partial^3 U_i}{\partial x^3} + \ldots \tag{2.3}$$

Adding these such that $\frac{\partial U_i}{\partial x}$ cancels gives:

$$\frac{\partial^2 U_i}{\partial x^2} = \frac{U_{i+2} - 2U_{i+1} + U_i}{h^2} - h\frac{\partial^3 U_i}{\partial x^3} + \ldots \equiv \frac{\Delta^2 U_i}{h^2} + O(h) \tag{2.4}$$

Generally, approximations to higher derivatives are obtained by adding one or more points; each additional point permits an $O(h)$ expression to the next derivative. The notation for the result is

$$\frac{\partial^n U_i}{\partial x^n} = \frac{\Delta^n U_i}{h^n} + O(h) \tag{2.5}$$

with $\Delta^n U_i$ indicating a difference expression among $U_i \to U_{i+n}$. These are called the *Forward Differences* and are tabulated in tTable 2.1 below. They have the recursive property

$$\Delta^n U_i = \Delta(\Delta^{n-1} U_i) \tag{2.6}$$

The core operator Δ indicates the "first forward difference".

Backward Differences are defined in the analogous way:

$$U_{i-1} = U_i - h\frac{\partial U_i}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 U_i}{\partial x^2} + \ldots \tag{2.7}$$

$$\frac{\partial^n U_i}{\partial x^n} = \frac{\nabla^n U_i}{h^n} + O(h) \tag{2.8}$$

$$\nabla^n U_i = \nabla(\nabla^{n-1} U_i) \tag{2.9}$$

These are tabulated in Table 2.2.

Both of these approximations are first-order in the mesh spacing h. Higher order approximations are generated by involving more points.

$$U_{i+1} = U_i + h\frac{\partial U_i}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 U_i}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 U_i}{\partial x^3} + \ldots \tag{2.10}$$

$$U_{i+2} = U_i + 2h\frac{\partial U_i}{\partial x} + \frac{(2h)^2}{2!}\frac{\partial^2 U_i}{\partial x^2} + \frac{(2h)^3}{3!}\frac{\partial^3 U_i}{\partial x^3} + \ldots \tag{2.11}$$

Combining equations 2.10 and 2.11 with weights 1 and A, we get

$$\begin{aligned} U_{i+1} + AU_{i+2} = (1+A)U_i \quad &+ \quad (1+2A)h\frac{\partial U_i}{\partial x} \\ &+ \quad (1+4A)\frac{h^2}{2!}\frac{\partial^2 U_i}{\partial x^2} + (1+8A)\frac{h^3}{3!}\frac{\partial^3 U_i}{\partial x^3} + \ldots \end{aligned} \tag{2.12}$$

We want the first derivative in terms of U_i, U_{i+1}, and U_{i+2}. If we choose A such that $(1+4A) = 0$, the second derivative term will vanish and the third derivative term will be the leading error term.

$$\frac{\partial U_i}{\partial x} = \frac{[AU_{i+2} + U_{i+1} - (1+A)U_i] - (1+4A)\frac{h^2}{2!}\frac{\partial^2 U_i}{\partial x^2} - (1+8A)\frac{h^3}{3!}\frac{\partial^3 U_i}{\partial x^3} + \ldots}{(1+2A)h} \tag{2.13}$$

$$1 + 4A = 0; \quad A = -1/4 \tag{2.14}$$

$$\frac{\partial U_i}{\partial x} = \frac{-U_{i+2} + 4U_{i+1} - 3U_i}{2h} - \frac{h^2}{3!}\frac{\partial^3 U_i}{\partial x^3} + O(h^3) \tag{2.15}$$

The leading error is $O(h^2)$. This is the second-order correct, forward difference approximation to the first derivative. Higher derivatives at this accuracy can be obtained by adding extra points, as in the $O(h)$ formulas. Tables 2.3 and 2.4 below record these and their backward difference counterparts.

The obvious supplement to these one-sided differences are the *Central Difference* approximations. Assuming a uniform mesh, these combine the forward and backward formulas such that the leading errors cancel. The result is an extra order of accuracy for the same number of points. For example:

$$\frac{\partial U_i}{\partial x} = \frac{\Delta U_i}{h} - \frac{h}{2!}\frac{\partial^2 U_i}{\partial x^2} + O(h^2) \tag{2.16}$$

$$\frac{\partial U_i}{\partial x} = \frac{\nabla U_i}{h} + \frac{h}{2!}\frac{\partial^2 U_i}{\partial x^2} + O(h^2) \tag{2.17}$$

Combining these,

$$\frac{\partial U_i}{\partial x} = \frac{\nabla U_i + \Delta U_i}{2h} + O(h^2) \tag{2.18}$$

$$\nabla U_i + \Delta U_i = U_{i+1} - U_{i-1} \equiv \delta U_i \tag{2.19}$$

The symbol δ in this context indicates the central difference operator; and the centered approximation to the first derivative is

$$\frac{\partial U_i}{\partial x} = \frac{\delta U_i}{2h} + O(h^2) \tag{2.20}$$

This is accurate to second order in h. Higher derivatives can be obtained by adding more points, symmetrically. The $O(h^2)$ centered differences are summarized in Tables 2.5 and 2.6 below.

Summary - Uniform Mesh

The Taylor Series provides difference formulas and error estimates for derivatives of arbitrary order and precision. The procedure is systematic and, as is shown in the next sections, easily generalized to nonuniform meshes and to multiple dimensions. The 1-D results on a uniform mesh may be summarized as:

Forward difference

$$\frac{d^n U_i}{dx^n} = \frac{\Delta^n U_i}{h^n} + O(h)$$

Backward difference

$$\frac{d^n U_i}{dx^n} = \frac{\nabla^n U_i}{h^n} + O(h)$$

Centered difference

$$\frac{d^n U_i}{dx^n} = \frac{\delta^n U_i}{(1 \ or \ 2)h^n} + O(h^2)$$

All of these have N+1 points with nonzero weights. The centered formulas provide an extra order of accuracy for the same number of points.

To attain *higher-order accuracy*, more points need to be added. In the *uncentered* cases, we have

$$\begin{array}{lllll} n^{th} \text{ derivative} & + & O(h^2) & \Rightarrow & n+2 \text{ pts.} \\ n^{th} \text{ derivative} & + & O(h^3) & \Rightarrow & n+3 \text{ pts.} \\ n^{th} \text{ derivative} & + & O(h^4) & \Rightarrow & n+4 \text{ pts.} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Table 2.1: Forward difference representations, $O(h)$. [45].

	f_i	f_{i+1}	f_{i+2}	f_{i+3}	f_{i+4}
$hf'(x_i)$	-1	1			
$h^2f''(x_i)$	1	-2	1		
$h^3f'''(x_i)$	-1	3	-3	1	
$h^4f^{iv}(x_i)$	1	-4	6	-4	1

Table 2.2: Backward difference representations, $O(h)$. [45].

	f_{i-4}	f_{i-3}	f_{i-2}	f_{i-1}	f_i
$hf'(x_i)$				-1	1
$h^2f''(x_i)$			1	-2	1
$h^3f'''(x_i)$		-1	3	-3	1
$h^4f^{iv}(x_i)$	1	-4	6	-4	1

Table 2.3: Forward difference representations, $O(h^2)$. [45].

	f_i	f_{i+1}	f_{i+2}	f_{i+3}	f_{i+4}	f_{i+5}
$2hf'(x_i)$	-3	4	-1			
$h^2f''(x_i)$	2	-5	4	-1		
$2h^3f'''(x_i)$	-5	18	-24	14	-3	
$h^4f^{iv}(x_i)$	3	-14	26	-24	11	-2

Table 2.4: Backward difference representations, $O(h^2)$. [45].

	f_{i-5}	f_{i-4}	f_{i-3}	f_{i-2}	f_{i-1}	f_i
$2hf'(x_i)$				1	-4	3
$h^2f''(x_i)$			-1	4	-5	2
$2h^3f'''(x_i)$		3	-14	24	-18	5
$h^4f^{iv}(x_i)$	-2	11	-24	26	-14	3

Table 2.5: Central difference representations, $O(h^2)$. [45].

	f_{i-2}	f_{i-1}	f_i	f_{i+1}	f_{i+2}
$2hf'(x_i)$		-1	0	1	
$h^2f''(x_i)$		1	-2	1	
$2h^3f'''(x_i)$	-1	2	0	-2	1
$h^4f^4(x_i)$	1	-4	6	-4	1

Table 2.6: Central difference representations, $O(h^4)$. [45]

	f_{i-3}	f_{i-2}	f_{i-1}	f_i	f_{i+1}	f_{i+2}	f_{i+3}
$12hf'(x_i)$		1	−8	0	8	−1	
$12h^2f''(x_i)$		−1	16	−30	16	−1	
$8h^3f'''(x_i)$	1	−8	13	0	−13	8	−1
$6h^4f^4(x_i)$	−1	12	−39	56	−39	12	−1

and so on; whereas for the *centered* cases, we have extra accuracy for the same number of points:

$$\begin{array}{lcccl} n^{th} \text{ derivative} & + & O(h^4) & \Rightarrow & n+3 \text{ pts.} \\ n^{th} \text{ derivative} & + & O(h^6) & \Rightarrow & n+5 \text{ pts.} \\ n^{th} \text{ derivative} & + & O(h^8) & \Rightarrow & n+7 \text{ pts.} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

For the same number of points, the centered formulas always stay one order ahead of the uncentered formulas. Points are added alternately at the center of the formula, or in symmetric pairs.

These rules apply equally well to 1-D differences on nonuniform meshes (see below), with the exception that the special accuracy of Centered Differences is lost.

2.2 Use of the Error Term

The leading error terms are important; since they may interact, they should be kept in detail in all derivations. For example, we may construct a higher-order approximation from two lower-order approximations as follows:

$$\frac{\partial U_i}{\partial x} = \frac{\Delta U_i}{h} - \frac{h}{2}\frac{\partial^2 U_i}{\partial x^2} + O(h^2) \tag{2.21}$$

Substitute a difference formula for the leading error term $\frac{\partial^2 U_i}{\partial x^2}$

$$\frac{\partial^2 U_i}{\partial x^2} = \frac{\Delta^2 U_i}{h^2} + O(h) \tag{2.22}$$

This will push the error term to $O(h) \cdot O(h)$:

$$\begin{aligned} \frac{\partial U_i}{\partial x} &= \frac{\Delta U_i}{h} - \frac{h}{2}[\frac{\Delta^2 U_i}{h^2} + O(h)] + O(h^2) \\ &= \frac{\Delta U_i}{h} - \frac{1}{2}\frac{\Delta^2 U_i}{h} + O(h^2) \\ &= [\frac{(U_{i+1} - U_i)}{h} - \frac{1}{2}\frac{(U_{i+2} - 2U_{i+1} + U_i)}{h}] + O(h^2) \qquad (2.23) \\ \frac{\partial U_i}{\partial x} &= \frac{[-3U_i + 4U_{i+1} - U_{i+2}]}{2h} + O(h^2) \qquad (2.24) \end{aligned}$$

This is the same as in the forward difference Tables derived directly from Taylor series. This procedure has obvious generality; it will produce a difference expression whose order is the product of its two parts.

2.3 1-D Differences on Nonuniform Meshes

The Taylor Series procedure outlined above is not restricted to uniform meshes. Consider the following 5-point grid:

$$\begin{array}{ccccccccc} & h & & \alpha h & & \beta h & & \gamma h & \\ \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\ i-1 & & i & & i+1 & & i+2 & & i+3 \end{array}$$

Suppose we wish to find difference formulas for derivatives at node i. We proceed to express all the other nodal values in Taylor series about i:

$$\begin{Bmatrix} U_{i-1} \\ U_{i+1} \\ U_{i+2} \\ U_{i+3} \end{Bmatrix} = U_i \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} + \frac{h}{1!}\frac{dU_i}{dx} \begin{Bmatrix} -1 \\ \alpha \\ \alpha+\beta \\ \alpha+\beta+\gamma \end{Bmatrix} + \frac{h^2}{2!}\frac{d^2U_i}{dx^2} \begin{Bmatrix} 1 \\ \alpha^2 \\ (\alpha+\beta)^2 \\ (\alpha+\beta+\gamma)^2 \end{Bmatrix}$$
$$+ \frac{h^3}{3!}\frac{d^3U_i}{dx^3} \begin{Bmatrix} -1 \\ \alpha^3 \\ (\alpha+\beta)^3 \\ (\alpha+\beta+\gamma)^3 \end{Bmatrix} + \frac{h^4}{4!}\frac{d^4U_i}{dx^4} \begin{Bmatrix} 1 \\ \alpha^4 \\ (\alpha+\beta)^4 \\ (\alpha+\beta+\gamma)^4 \end{Bmatrix} + \ldots \quad (2.25)$$

Now form a weighted sum of the four equations; let the weights be (1,A,B,C) (the first one is arbitrary since we can always multiply the result by a constant). The result:

$$\begin{aligned} U_{i-1} + AU_{i+1} + BU_{i+2} + CU_{i+3} &= U_i(1+A+B+C) \quad (2.26) \\ &+ h\frac{dU_i}{dx}(-1+\alpha A+(\alpha+\beta)B+(\alpha+\beta+\gamma)C) \\ &+ \frac{h^2}{2!}\frac{d^2U_i}{dx^2}(1+\alpha^2 A+(\alpha+\beta)^2 B+(\alpha+\beta+\gamma)^2 C) \\ &+ \frac{h^3}{3!}\frac{d^3U_i}{dx^3}(-1+\alpha^3 A+(\alpha+\beta)^3 B+(\alpha+\beta+\gamma)^3 C) \\ &+ \frac{h^4}{4!}\frac{d^4U_i}{dx^4}(1+\alpha^4 A+(\alpha+\beta)^4 B+(\alpha+\beta+\gamma)^4 C) \end{aligned}$$

Now we have at our disposal the three parameters (A,B,C). Suppose we want a difference formula for $\frac{dU_i}{dx}$ which only involves u_{i-1}, u_i, u_{i+1}. Clearly then, B and C must be zero; and if we select A such that the coefficient of $\frac{d^2U_i}{dx^2}$ vanishes, then we will create an O(h^2) approximation for $\frac{dU_i}{dx}$:

$$1+\alpha^2 A = 0 \Rightarrow A = -\frac{1}{\alpha^2} \quad (2.27)$$

and thus

$$U_{i-1} + AU_{i+1} - U_i(1+A) = h\frac{dU_i}{dx}(-1+\alpha A) + \frac{h^3}{3!}\frac{d^3U_i}{dx^3}(-1+\alpha^3 A) + \ldots \quad (2.28)$$

or

$$\frac{dU_i}{dx} = \frac{U_{i-1} + AU_{i+1} - U_i(1+A)}{h(\alpha A - 1)} - \frac{h^2}{3!}\frac{d^3U_i}{dx^3}(\frac{\alpha^3 A - 1}{\alpha A - 1}) + \ldots \quad (2.29)$$

Substituting for A we get

$$\frac{dU_i}{dx} = \frac{-\alpha^2 U_{i-1} + U_i(\alpha^2 - 1) + U_{i+1}}{h(\alpha+1)\alpha} - \alpha \frac{h^2}{3!}\frac{d^3U_i}{dx^3} + ... \tag{2.30}$$

It is readily checked that when $\alpha = 1$ we obtain the familiar central difference formula.

If we wish to achieve higher accuracy, we must involve another point. Retaining $B \neq 0$, for example, we may set the coefficients of the second and third derivatives equal to zero in 2.27:

$$1 + \alpha^2 A + (\alpha+\beta)^2 B = 0 \Rightarrow A = \frac{-(1+\alpha+\beta)}{\alpha^2\beta} \tag{2.31}$$

$$-1 + \alpha^3 A + (\alpha+\beta)^3 B = 0 \Rightarrow B = \frac{(1+\alpha)}{(\alpha+\beta)^2\beta} \tag{2.32}$$

The resulting difference formula will have a leading error term

$$-\frac{h^3}{4!}\frac{d^4U_i}{dx^4}\left[\frac{1+\alpha^4 A + (\alpha+\beta)^4 B}{-1+\alpha A + (\alpha+\beta)B}\right] \tag{2.33}$$

The above procedure may be expressed in more generality as follows. Suppose we want expressions for derivatives at some grid point. Without loss of generality we take this (temporarily) to be the origin of the coordinate system. Then denoting by μ_i the difference of U_i - U_0, the Taylor series is

$$\mu_i = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^nU_0}{dx^n} x_i^n \tag{2.34}$$

If we invent weights W_i, then

$$\sum_i W_i\mu_i = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^nU_0}{dx^n} \sum_i W_i x_i^n = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^nU_0}{dx^n}\omega_n \tag{2.35}$$

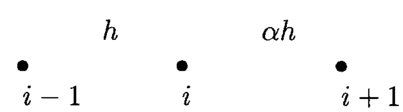
where the index i runs over all grid points, and ω_n is the n^{th} moment of the weights about the origin:

$$\omega_n = \sum_i W_i x_i^n \tag{2.36}$$

A first order derivative difference expression for $\frac{d^N U_0}{dx^N}$ can thus be obtained by setting the first $N-1$ moments of $\mathbf{W}$ equal to zero, which can be achieved with exactly N nonzero weights. Recalling that $\mu_i = U_i$ - U_0, this yields $N+1$ node points in the expression for the N^{th} derivative. Higher order expressions can be obtained by making ω_{N+1} and progressively higher moments equal to zero.

2.4 Alternative to Taylor Series: Polynomial Fit

A different procedure is to fit a polynomial or other interpolant to discrete samples; and differentiate the result. For example, consider the 3-point mesh shown below.



We will fit a second-order polynomial $\widehat{U}(x)$ to samples of U at the three mesh points:

$$\widehat{U} = ax^2 + bx + c \tag{2.37}$$
$$U_{i-1} = ah^2 + b(-h) + c \tag{2.38}$$
$$U_i = c \tag{2.39}$$
$$U_{i+1} = a(\alpha h)^2 + b(\alpha h) + c \tag{2.40}$$

The fit is obtained by solving for the three coefficients

$$\begin{bmatrix} h^2 & -h & 1 \\ 0 & 0 & 1 \\ (\alpha h)^2 & (\alpha h) & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} U_{i-1} \\ U_i \\ U_{i+1} \end{Bmatrix} \tag{2.41}$$

Inverting this gives

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} \dfrac{(\alpha[U_{i-1} - U_i] + [U_{i+1} - U_i])}{(\alpha^2 + \alpha)h^2} \\ \dfrac{(-\alpha^2[U_{i-1} - U_i] + [U_{i+1} - U_i])}{(\alpha^2 + \alpha)h} \\ U_i \end{Bmatrix} \tag{2.42}$$

and the polynomial is differentiated to provide the difference formulas at any point x:

$$\frac{\partial \widehat{U}}{\partial x} = 2ax + b \tag{2.43}$$

$$\frac{\partial^2 \widehat{U}}{\partial x^2} = 2a = 2\left[\frac{U_{i+1} - U_i(1+\alpha) + \alpha U_{i-1}}{\alpha(\alpha+1)h^2}\right] \tag{2.44}$$

These results are identical to those obtained from Taylor Series estimates at the three grid points. (Student should verify this.) This procedure has the advantage of estimating derivatives everywhere, not just at the mesh points; but it lacks the truncation error estimate.

2.5 Difference Formulas with Cross-Derivatives

Generally, the 1-D formulas can be used in higher dimensions (although there are other options). The special case is the mixed derivative with respect to 2 or more dimensions. There are two approaches. First, we can operate with the 2-D Taylor series:

$$\begin{aligned} U(x+\Delta x, y+\Delta y) &= U|_{x,y} + (\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})U|_{x,y} \\ &+ \frac{1}{2!}(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})^2 U|_{x,y} \\ &+ \frac{1}{3!}(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})^3 U|_{x,y} + \dots \end{aligned} \tag{2.45}$$

where

$$(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})^2 = \Delta x^2\frac{\partial^2}{\partial x^2} + 2\Delta x\Delta y\frac{\partial^2}{\partial x\partial y} + \Delta y^2\frac{\partial^2}{\partial y^2} \tag{2.46}$$

and so on. From here, the procedure is generally the same as in 1-D case: write Taylor series for all points in terms of $U, \partial U, \dots$ at point where ∂U is wanted; mix together to get desired accuracy.

The alternative approach is to operate with the 1-D formula already in hand. For example: on an (i,j) mesh with uniform mesh spacing (h,k):

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) \simeq \frac{\left[\frac{U_{j+1}-U_{j-1}}{2k}\right]_{i+1} - \left[\frac{U_{j+1}-U_{j-1}}{2k}\right]_{i-1}}{2h} \tag{2.47}$$

This should be intuitively correct to second order, since centered differences are being invoked. But so far we lack the leading error term. We can get this from the 1-D formula,

$$\frac{\partial U}{\partial x}|_i = \frac{U_{i+1} - U_{i-1}}{2h} - \frac{h^2}{6}\frac{\partial^3 U}{\partial x^3}|_i + \ldots \tag{2.48}$$

Differentiating this,

$$\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)|_{ij} = \frac{\frac{\partial U}{\partial y}|_{i+1,j} - \frac{\partial U}{\partial y}|_{i-1,j}}{2h} - \frac{h^2}{6}\frac{\partial^4 U}{\partial x^3 \partial y}|_{i,j} + \ldots \tag{2.49}$$

By the same formula:

$$\frac{\partial U}{\partial y}|_i = \frac{U_{j+1} - U_{j-1}}{2k} - \frac{k^2}{6}\frac{\partial^3 U}{\partial y^3}|_j + \ldots \tag{2.50}$$

$$\begin{aligned} \frac{\partial^2 U}{\partial y \partial x}|_{ij} &= \frac{1}{2h}\left[\left(\frac{U_{i+1,j+1} - U_{i+1,j-1}}{2k}\right) - \left(\frac{U_{i-1,j+1} - U_{i-1,j-1}}{2k}\right)\right] \\ &- \frac{1}{2h}\frac{k^2}{6}\left[\frac{\partial^3 U}{\partial y^3}|_{i+1,j} - \frac{\partial^3 U}{\partial y^3}|_{i-1,j}\right] - \frac{h^2}{6}\frac{\partial^4 U}{\partial x^3 \partial y}|_{i,j} \end{aligned} \tag{2.51}$$

There is an apparent asymmetry in the error terms. Also, the $\frac{k^2}{h}$ would in general be a fatal problem, reducing the accuracy to first-order. But

$$\frac{\left[\frac{\partial^3 U}{\partial y^3}|_{i+1,j} - \frac{\partial^3 U}{\partial y^3}|_{i-1,j}\right]}{2h} = \frac{\partial}{\partial x}\left(\frac{\partial^3 U}{\partial y^3}\right) + O(h^2) \tag{2.52}$$

So the leading error terms are

$$-\frac{k^2}{6}\frac{\partial^4 U}{\partial x \partial y^3} - \frac{h^2}{6}\frac{\partial^4 U}{\partial y \partial x^3} \tag{2.53}$$

Now we have symmetry, as expected from the form of the difference expression. And the accuracy is second-order in h and k, independently.