# Preface

The notion of an order plays an important rôle not only throughout mathematics but also in adjacent disciplines such as logic and computer science. The purpose of the present text is to provide a basic introduction to the theory of ordered structures. Taken as a whole, the material is mainly designed for a postgraduate course. However, since prerequisites are minimal, selected parts of it may easily be considered suitable to broaden the horizon of the advanced undergraduate. Indeed, this has been the author's practice over many years.

A basic tool in analysis is the notion of a continuous function, namely a mapping which has the property that the inverse image of an open set is an open set. In the theory of ordered sets there is the corresponding concept of a residuated mapping, this being a mapping which has the property that the inverse image of a principal down-set is a principal down-set. It comes therefore as no surprise that residuated mappings are important as far as ordered structures are concerned. Indeed, albeit beyond the scope of the present exposition, the naturality of residuated mappings can perhaps best be exhibited using categorical concepts. If we regard an ordered set as a small category then an order-preserving mapping $f : A \to B$ becomes a functor. Then $f$ is residuated if and only if there exists a functor $f^+ : B \to A$ such that $(f, f^+)$ is an adjoint pair.

Residuated mappings play a central rôle throughout this exposition, with fundamental concepts being introduced whenever possible in terms of natural properties of them. For example, an order isomorphism is precisely a bijection that is residuated; an ordered set $E$ is a meet semilattice if and only if, for every principal down-set $x^\downarrow$, the canonical embedding of $x^\downarrow$ into $E$ is residuated; and a Heyting algebra can be characterised as a lattice-based algebra in which every translation $\lambda_x : y \mapsto x \wedge y$ is residuated. The important notion of a closure operator, which arises in many situations that concern ordered sets, is intimately related to that of a residuated mapping. Likewise, Galois connections can be described in terms of residuated mappings, and vice versa. Residuated mappings have the added advantage that they can be composed to form new residuated mappings. In particular, the set $\operatorname{Res} E$ of residuated mappings on an ordered set $E$ forms a semigroup, and here we include descriptions of the types of semigroup that arise.

A glance at the list of contents will reveal how the material is marshalled. Roughly speaking, the text may be divided into two parts though it should be stressed that these are not mutually independent. In Chapters 1 to 8 we deal with the essentials of ordered sets and lattices, including boolean algebras, $p$-algebras, Heyting algebras, and their subdirectly irreducible algebras. In Chapters 9 to 14 we provide an introduction to ordered algebraic structures, including ordered groups, rings, fields, and semigroups. In particular, we include a characterisation of the real numbers as, to within isomorphism, the only Dedekind complete totally ordered field, something that is rarely seen by mathematics graduates nowadays. As far as ordered groups are concerned, we develop the theory as far as proving that every archimedean lattice-ordered group is commutative. In dealing with ordered semigroups we concentrate mainly on naturally ordered regular and inverse semigroups and provide a unified account which highlights those that admit an ordered group as an image under a residuated epimorphism, culminating in structure theorems for various types of Dubreil-Jacotin semigroups.

Throughout the text we give many examples of the structures arising, and interspersed with the theorems there are bundles of exercises to whet the reader's appetite. These are of varying degrees of difficulty, some being designed to help the student gain intuition and some serving to provide further examples to supplement the text material. Since this is primarily designed as a non-encyclopaedic introduction to the vast area of ordered structures we also include relevant references.

We are deeply indebted to Professora Doutora Maria Helena Santos and Professor Doutor Herberto Silva for their assistance in the proof-reading. The traditional free copy is small recompense for their labour. Finally, our thanks go to the editorial team at Springer for unparalleled courtesy and efficiency.

St Andrews, Scotland T.S. Blyth

# 2

# Lattices; lattice morphisms

## 2.1 Semilattices and lattices

If $E$ is an ordered set and $x \in E$ then the canonical embedding of $x^{\downarrow}$ into $E$, i.e., the restriction to $x^{\downarrow}$ of the identity mapping on $E$, is clearly isotone. As we shall now see, consideration of when each such embedding is a residuated mapping has important consequences as far as the structure of $E$ is concerned.

**Theorem 2.1** *If $E$ is an ordered set then the following are equivalent*:

(1) *for every $x \in E$ the canonical embedding of $x^{\downarrow}$ into $E$ is residuated*;
(2) *the intersection of any two principal down-sets is a principal down-set.*

*Proof* For each $x \in E$ let $i_x : x^{\downarrow} \to E$ be the canonical embedding. Then (1) holds if and only if, for all $x, y \in E$, there exists $\alpha = \max\{z \in x^{\downarrow} \mid z = i_x(z) \leqslant y\}$. Clearly, this is equivalent to the existence of $\alpha \in E$ such that $x^{\downarrow} \cap y^{\downarrow} = \alpha^{\downarrow}$, which is (2). $\square$

**Definition** If $E$ satisfies either of the equivalent conditions of Theorem 2.1 then we shall denote by $x \wedge y$ the element $\alpha$ such that $x^{\downarrow} \cap y^{\downarrow} = \alpha^{\downarrow}$, and call $x \wedge y$ the **meet** of $x$ and $y$. In this situation we shall say that $E$ is a **meet semilattice**. Equivalent terminology is a **$\wedge$-semilattice**.

**Example 2.1** Every chain is a meet semilattice in which $x \wedge y = \min\{x, y\}$.

**Example 2.2** $(\mathbb{N}\,; |)$ is a meet semilattice in which $m \wedge n = \mathrm{hcf}\{m, n\}$.

**Example 2.3** The set $\operatorname{Equ} E$ of equivalence relations on $E$ is a meet semilattice in which $\vartheta \wedge \varphi$ is given by

$$(x, y) \in \vartheta \wedge \varphi \iff \big((x, y) \in \vartheta \text{ and } (x, y) \in \varphi\big).$$

Meet semilattices can also be characterised in a purely algebraic way which we shall now describe. First we observe that in a meet semilattice $E$ the assignment $(x, y) \mapsto x \wedge y$ defines a law of composition $\wedge$ on $E$. Now since $x^{\downarrow} \cap (y^{\downarrow} \cap z^{\downarrow}) = (x^{\downarrow} \cap y^{\downarrow}) \cap z^{\downarrow}$ we see that $\wedge$ is associative; and since $x^{\downarrow} \cap y^{\downarrow} = y^{\downarrow} \cap x^{\downarrow}$ it is commutative; and, moreover, since $x^{\downarrow} \cap x^{\downarrow} = x^{\downarrow}$ it is idempotent.

Hence $(E;\wedge)$ is a commutative idempotent semigroup. As the following result shows, the converse holds: every commutative idempotent semigroup gives rise in a natural way to a meet semilattice.

**Theorem 2.2** *Every commutative idempotent semigroup can be ordered in such a way that it forms a meet semilattice.*

*Proof* Suppose that $E$ is a commutative idempotent semigroup in which we denote the law of composition by multiplication. Define a relation $R$ on $E$ by $x\,R\,y \iff xy = x$. Then $R$ is an order. In fact, since $x^2 = x$ for every $x \in E$ we have $xRx$, so that $R$ is reflexive; if $xRy$ and $yRx$ then $x = xy = yx = y$, so that $R$ is anti-symmetric; if $xRy$ and $yRz$ then $x = xy$ and $y = yz$ whence $x = xy = xyz = xz$ and therefore $xRz$, so that $R$ is transitive. In what follows we write $\leqslant$ for $R$. If now $x, y \in E$ we have $xy = xxy = xyx$ and so $xy \leqslant x$. Inverting the roles of $x, y$ we also have $xy \leqslant y$ and therefore $xy \in x^{\downarrow} \cap y^{\downarrow}$. Suppose now that $z \in x^{\downarrow} \cap y^{\downarrow}$. Then $z \leqslant x$ and $z \leqslant y$ give $z = zx$ and $z = zy$, whence $z = zy = zxy$ and therefore $z \leqslant xy$. It follows that $x^{\downarrow} \cap y^{\downarrow}$ has a top element, namely $xy$. Thus $E$ is a meet semilattice in which $x \wedge y = xy$. □

## EXERCISES

2.1. Draw the Hasse diagrams for all possible meet semilattices with 4 elements.

2.2. If $P$ and $Q$ are meet semilattices prove that the set of isotone mappings from $P$ to $Q$ forms a meet semilattice with respect to the order described in Example 1.7.

**Definition** If $E$ is an ordered set and $F$ is a subset of $E$ then $x \in E$ is said to be a **lower bound** of $F$ if $(\forall y \in F)\ x \leqslant y$; and an **upper bound** of $F$ if $(\forall y \in F)\ y \leqslant x$.

In what follows we shall denote the set of lower bounds of $F$ in $E$ by $F^{\downarrow}$, and the set of upper bounds of $F$ by $F^{\uparrow}$.

*Remark* We note here that the notation $A^{\downarrow}$ is often used to denote the down-set generated by $A$, namely $\{x \in E \mid (\exists a \in A)\ x \leqslant a\}$, and $A^{\uparrow}$ to denote the up-set generated by $A$. Other commonly used notation for lower, upper bounds include $A^{\ell}$, $A^{u}$ and $A^{\blacktriangledown}$, $A^{\blacktriangle}$.

In particular, we have $\{x\}^{\downarrow} = x^{\downarrow}$ and $\{x\}^{\uparrow} = x^{\uparrow}$. Note that $F^{\downarrow}$ and $F^{\uparrow}$ may be empty, but not so when $E$ is **bounded**, in the sense that it has both a top element 1 and a bottom element 0. If $E$ has a top element 1 then $E^{\uparrow} = \{1\}$; otherwise $E^{\uparrow} = \emptyset$. Similarly, if $E$ has a bottom element 0 then $E^{\downarrow} = \{0\}$; otherwise $E^{\downarrow} = \emptyset$. Note that if $F = \emptyset$ then every $x \in E$ satisfies (vacuously) the relation $y \leqslant x$ for every $y \in F$. Thus $\emptyset^{\uparrow} = E$; and similarly $\emptyset^{\downarrow} = E$.

**Definition** If $E$ is an ordered set and $F$ is a subset of $E$ then by the **infimum**, or **greatest lower bound**, of $F$ we mean the top element (when such exists) of the set $F^{\downarrow}$ of lower bounds of $F$. We denote this by $\inf_E F$ or simply $\inf F$ if there is no confusion.

Since $\emptyset^{\downarrow} = E$ we see that $\inf_E \emptyset$ exists if and only if $E$ has a top element 1, in which case $\inf_E \emptyset = 1$.

It is immediate from what has gone before that a meet semilattice can be described as an ordered set in which every pair of elements $x, y$ has a greatest lower bound; here we have $\inf\{x, y\} = x \wedge y$. A simple inductive argument shows that for every finite subset $\{x_1, \dots, x_n\}$ of a meet semilattice we have that $\inf\{x_1, \dots, x_n\}$ exists and is $x_1 \wedge \cdots \wedge x_n$.

We can of course develop the duals of the above, obtaining in this way the notion of a **join semilattice** which is characterised by the intersection of any two principal up-sets being a principal up-set, the element $\beta$ such that $x^{\uparrow} \cap y^{\uparrow} = \beta^{\uparrow}$ being denoted by $x \vee y$ and called the **join** of $x$ and $y$. Equivalent terminology for this is a **$\vee$-semilattice**. Then Theorem 2.2 has an analogue for join semilattices in which the order is defined by $x\,R\,y \iff xy = y$. Likewise by duality we can define the notion of that of **supremum** or **least upper bound** of a subset $F$, denoted by $\sup_E F$. In particular, we see that $\sup_E \emptyset$ exists if and only if $\emptyset^{\uparrow} = E$ has a bottom element 0, in which case $\sup_E \emptyset = 0$. In a join semilattice we have $\sup\{x, y\} = x \vee y$ and, by induction, $\sup\{x_1, \dots, x_n\} = x_1 \vee \cdots \vee x_n$.

**Definition** A **lattice** is an ordered set $(E; \leqslant)$ which, with respect to its order, is both a meet semilattice and a join semilattice.

Thus a lattice is an ordered set in which every pair of elements (and hence every finite subset) has an infimum and a supremum. We often denote a lattice by $(E; \wedge, \vee, \leqslant)$.

**Theorem 2.3** *A set $E$ can be given the structure of a lattice if and only if it can be endowed with two laws of composition $(x, y) \mapsto x \Cap y$ and $(x, y) \mapsto x \Cup y$ such that*

(1) *$(E; \Cap)$ and $(E; \Cup)$ are commutative semigroups;*

(2) *the following* **absorption laws** *hold:*

$$(\forall x, y \in E) \quad x \Cap (x \Cup y) = x = x \Cup (x \Cap y).$$

*Proof* $\Rightarrow$: If $E$ is a lattice then $E$ has two laws of composition that satisfy (1), namely $(x, y) \mapsto x \wedge y$ and $(x, y) \mapsto x \vee y$. To show that (2) holds, we observe that $x \leqslant \sup\{x, y\} = x \vee y$ and so $x \wedge (x \vee y) = \inf\{x, x \vee y\} = x$; and similarly $x \wedge y = \inf\{x, y\} \leqslant x$ gives $x \vee (x \wedge y) = \sup\{x, x \wedge y\} = x$.

$\Leftarrow$: Suppose now that $E$ has two laws of composition $\Cap$ and $\Cup$ that satisfy (1) and (2). Using (2) twice, we see that $x \Cup x = x \Cup [x \Cap (x \Cup x)] = x$, and similarly that $x \Cap x = x$. This, together with Theorem 2.2 and its dual shows that $(E; \Cap)$ and $(E; \Cup)$ are semilattices. In order to show that $(E; \Cup, \Cap)$ is a lattice with (for example) $\Cap$ as $\wedge$, and $\Cup$ as $\vee$, we must show that the orders defined by $\Cap$ and $\Cup$ coincide. In other words, we must show that $x \Cap y = x$ is equivalent to $x \Cup y = y$. But if $x \Cap y = x$ then, using the absorption laws, we have $y = (x \Cap y) \Cup y = x \Cup y$; and if $x \Cup y = y$ then $x = x \Cap (x \Cup y) = x \Cap y$. Thus we see that $E$ is a lattice in which $x \leqslant y$ is described equivalently by $x \Cap y = x$ or by $x \Cup y = y$. $\square$

**Example 2.4** Every chain is a lattice; here we have $\inf\{x,y\} = \min\{x,y\}$ and $\sup\{x,y\} = \max\{x,y\}$.

**Example 2.5** For every set $E$, $\big(\mathbb{P}(E); \cap, \cup, \subseteq\big)$ is a bounded lattice.

**Example 2.6** For every infinite set $E$ let $\mathbb{P}_f(E)$ be the set of finite subsets of $E$. Then $\big(\mathbb{P}_f(E); \cap, \cup, \subseteq\big)$ is a lattice with no top element.

**Example 2.7** $(\mathbb{N}\,;|)$ is a bounded lattice. The bottom element is 1 and the top element is 0. Here we have $\inf\{m,n\} = \operatorname{hcf}\{m,n\}$ and $\sup\{m,n\} = \operatorname{lcm}\{m,n\}$.

**Example 2.8** If $V$ is a vector space and if $\operatorname{Sub} V$ denotes the set of subspaces of $V$ then in the ordered set $(\operatorname{Sub} V; \subseteq)$ we have $\inf\{A,B\} = A \cap B$ since $A \cap B$ is the biggest subspace that is contained in both $A$ and $B$. Also, $\sup\{A,B\}$ exists and is the smallest subspace to contain both $A$ and $B$, namely the subspace $A + B = \{a + b \mid a \in A, b \in B\}$. Thus $(\operatorname{Sub} V; \cap, +, \subseteq)$ is a lattice.

**Example 2.9** If $L, M$ are lattices then the set of isotone mappings $f : L \to M$ forms a lattice in which $f \wedge g$ and $f \vee g$ are given by the prescriptions

$$(f \wedge g)(x) = f(x) \wedge g(x), \quad (f \vee g)(x) = f(x) \vee g(x).$$

The concept of a lattice was introduced by Peirce [91] and Schröder [101] towards the end of the nineteenth century. It derives from pioneering work by Boole [35], [36] on the formalisation of propositional logic. The terms idempotent, commutative, associative, and absorption are mostly due to Boole. The study of lattices became systematic with Birkhoff's first paper [8] in 1933 and his book [13] the first edition of which appeared in 1940 and was for several decades the bible of lattice theorists. Over the years the theory of lattices and its many applications has grown considerably. Notable reference works include books by Abbott [1], Balbes and Dwinger [3], Crawley and Dilworth [40], Davey and Priestley [42], Dubreil-Jacotin, Lesieur and Croisot [46], Freese, Ježek and Nation [50], Ganter and Wille [54], Hermes [63], Maeda and Maeda [83], Rutherford [96], Saliĭ [100], Sikorski [102], and Szász [107]. In recent times the Birkhoff bible has been replaced by that of Grätzer [58].

## EXERCISES

2.3. If $L$ is a lattice and $x, y, z \in L$ prove that

$$[(x \wedge y) \vee (x \wedge z)] \wedge [(x \wedge y) \vee (y \wedge z)] = x \wedge y.$$

2.4. If $x_{ij}$ $(i = 1, \dots, m;\ j = 1, \dots, n)$ are $mn$ elements of a lattice $L$, establish the **minimax inequality**

$$\bigvee_{j=1}^{n} \bigwedge_{i=1}^{m} x_{ij} \leqslant \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n} x_{ij}.$$

A regiment of soldiers, each of a different height, stands at attention in a rectangular array. Of the soldiers who are the tallest in their row, the smallest is Sergeant Mintall; and of the soldiers who are the smallest in their column, the tallest is Corporal Max Small. Which of these two soldiers is the taller?

2.5. If $L$ is a lattice and $a, b \in L$ define $f_{a,b} : L \to L$ by the prescription

$$f_{a,b}(x) = [(a \wedge b) \vee x] \wedge (a \vee b).$$

Prove that $f_{a,b}$ is isotone and idempotent. What is $\operatorname{Im} f_{a,b}$?

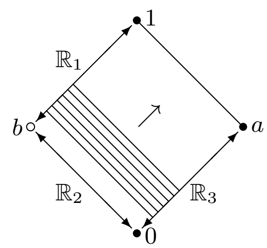
2.6. For $p \leqslant q$ in a lattice $L$ let $[p, q] = \{x \in L \mid p \leqslant x \leqslant q\}$. Given any $a, b \in L$, prove that the mapping $f : [a \wedge b, b] \to [a, a \vee b]$ defined by $f(x) = x \vee a$ is residuated and determine $f^+$.

2.7. Prove that the set $N(G)$ of normal subgroups of a group $G$ forms a lattice in which $\sup\{H, K\} = \{hk \mid h \in H,\, k \in K\}$.

2.8. Draw the Hasse diagram of the lattice of subgroups of the alternating group $\mathcal{A}_4$.

2.9. Let $L, M$ be lattices and let $\operatorname{Res}(L, M)$ be the set of residuated mappings from $L$ to $M$. Prove that if $f, g \in \operatorname{Res}(L, M)$ then $f \vee g \in \operatorname{Res}(L, M)$.

2.10. Consider the lattice $L$ described by the following Hasse diagram:




in which each $\mathbb{R}_i$ is a copy of the chain of real numbers. Let $L^\star$ be the lattice $L \backslash \{b\}$. Prove that the mapping $f_a : L^\star \to L^\star$ given by

$$f_a(x) = \begin{cases} x & \text{if } x \leqslant a; \\ a & \text{otherwise,} \end{cases}$$

is residuated. If $g \in \operatorname{Res} L^\star$ is such that $g \leqslant f_a$ and $g \leqslant \mathrm{id}_{L^\star}$ prove that there exists $c \in \mathbb{R}_1$ such that $c < g^+(0)$. Show further that if $h_c : L^\star \to L^\star$ is given by

$$h_c(x) = \begin{cases} 0 & \text{if } x \leqslant c; \\ x \wedge a & \text{otherwise,} \end{cases}$$

then $h_c \in \operatorname{Res} L^\star$ with $h_c \leqslant f_a$ and $h_c \leqslant \mathrm{id}_{L^\star}$. Moreover, show that $g < h_c$. Conclude from this that $\operatorname{Res} L^\star$ is not a $\wedge$-semilattice.

2.11. Given a lattice $L$, let $f$ be a residuated closure on $L$ and let $g$ be a residuated dual closure on $\operatorname{Im} f$. Prove that if $\alpha : L \to L$ is given by the prescription

$$(\forall x \in L)\;\; \alpha(x) = g[f(x)]$$

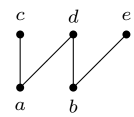
then $\alpha$ is an idempotent element of the semigroup $\operatorname{Res} L$. Prove further that every idempotent of $\operatorname{Res} L$ arises in this way.

## 2.2 Down-set lattices

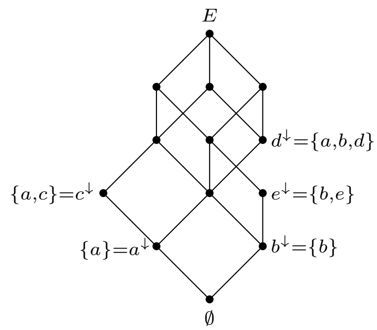
If $E$ is an ordered set and $A, B$ are down-sets of $E$ then clearly so also are $A \cap B$ and $A \cup B$. Thus the set of down-sets of $E$ is a lattice in which $\inf\{A, B\} = A \cap B$ and $\sup\{A, B\} = A \cup B$. We shall denote this lattice by $\mathcal{O}(E)$.

We recall from the definition of a down-set that we include the empty subset as such. Thus the lattice $\mathcal{O}(E)$ is bounded with top element $E$ and bottom element $\emptyset$.

**Example 2.10**




$E$




$\mathcal{O}(E)$

Down-set lattices will be of considerable interest to us later. For the moment we shall consider how to compute the cardinality of $\mathcal{O}(E)$ when the ordered set $E$ is finite. Upper and lower bounds for this are provided by the following result.

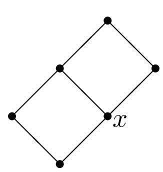
**Theorem 2.4** *If $E$ is a finite ordered set with $|E| = n$ then*

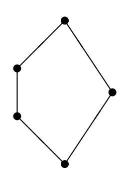
$$n + 1 \leqslant |\mathcal{O}(E)| \leqslant 2^n.$$

*Proof* Clearly, $E$ has the least number of down-sets when it is a chain, in which case $\mathcal{O}(E)$ is also a chain, of cardinality $n + 1$. Correspondingly, $E$ has the greatest number of down-sets when it is an anti-chain, in which case $\mathcal{O}(E) = \mathbb{P}(E)$ which is of cardinality $2^n$. □

In certain cases $|\mathcal{O}(E)|$ can be calculated using an ingenious algorithm that we shall describe. For this purpose, we shall denote by $E\backslash x$ the ordered set obtained from $E$ by deleting the element $x$ and related comparabilities whilst retaining all comparabilities resulting from transitivity through $x$.

**Example 2.11**



$L$ $L\backslash x$

We shall also use the notation $x^{\updownarrow}$ to denote the **cone** through $x$, namely the set of elements that are comparable to $x$; formally,

$$x^{\updownarrow} = x^{\downarrow} \cup x^{\uparrow} = \{y \in E \mid y \nparallel x\}.$$

**Example 2.12** If $L$ is as in Example 2.11 then $L\backslash x^{\updownarrow}$ is a singleton.

Finally, we shall say that $x \in E$ is **maximal** if there is no $y \in E$ such that $y > x$. The dual notion is that of a **minimal** element. Clearly, a top (bottom) element can be characterised as a unique maximal (minimal) element.

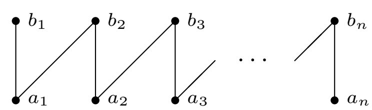
**Theorem 2.5** (Berman–Köhler [4]) *If $E$ is a finite ordered set then*

$$|\mathcal{O}(E)| = |\mathcal{O}(E \setminus x)| + |\mathcal{O}(E \setminus x^{\updownarrow})|.$$

*Proof* Observe first that every non-empty down-set $X$ of $E$ is determined by a unique antichain in $E$, namely the set of maximal elements of $X$. Counting $\emptyset$ as an antichain, we thus see that $|\mathcal{O}(E)|$ is the number of antichains in $E$. For any given element $x$ of $E$ this can be expressed as the number of antichains that contain $x$ plus the number that do not contain $x$.

Now if an antichain $A$ contains a particular element $x$ of $E$ then $A$ contains no other elements of the cone $x^{\updownarrow}$. Thus every antichain that contains $x$ determines a down-set of $\mathcal{O}(E \setminus x^{\updownarrow})$, and conversely. Hence we see that the number of antichains that contain $x$ is precisely $|\mathcal{O}(E \setminus x^{\updownarrow})|$. Since likewise the number of antichains that do not contain $x$ is precisely $|\mathcal{O}(E \setminus x)|$, the result follows. □

**Example 2.13** By an **even fence** we shall mean an ordered set $F_{2n}$ of the form




it being assumed that all the elements are distinct.

We can also define two non-isomorphic **odd fences**, namely by setting $F_{2n+1} = F_{2n} \cup \{b_{n+1}\}$ with the single extra relation $a_n < b_{n+1}$; and its dual $F^d_{2n+1} = F_{2n} \cup \{a_0\}$ with the single extra relation $a_0 < b_1$.

If we apply Theorem 2.5 to $F_{2n}$ with $x = a_n$, we obtain

$$|\mathcal{O}(F_{2n})| = |\mathcal{O}(F_{2n} \setminus a_n)| + |\mathcal{O}(F_{2n-2})| = |\mathcal{O}(F_{2n-1})| + |\mathcal{O}(F_{2n-2})|;$$

and then to $F_{2n-1}$ with $x = b_n$, we obtain

$$|\mathcal{O}(F_{2n-1})| = |\mathcal{O}(F_{2n-2})| + |\mathcal{O}(F_{2n-3})|.$$

Writing $\alpha_k = |\mathcal{O}(F_k)|$, we thus see that $\alpha_k$ satisfies the recurrence relation

$$\alpha_k = \alpha_{k-1} + \alpha_{k-2}.$$

Now in recognising this recurrence relation the reader will recall that the **Fibonacci sequence** $(f_n)_{n \geqslant 0}$ is defined by

$$f_0 = 0, \quad f_1 = 1, \quad (n \geqslant 2)\ f_n = f_{n-1} + f_{n-2}.$$

Furthermore, as is readily computed, we have $\alpha_2 = |\mathcal{O}(F_2)| = 3 = f_4$ and $\alpha_3 = |\mathcal{O}(F_3)| = 5 = f_5$. We therefore conclude from the above that $\alpha_k$, the cardinality of $\mathcal{O}(F_k)$, is the Fibonacci number $f_{k+2}$.

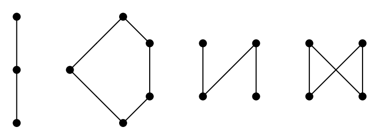
## EXERCISES

2.12. If $E$ and $F$ are finite ordered sets prove that

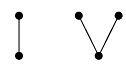
$$\mathcal{O}(E \cup F) \simeq \mathcal{O}(E) \times \mathcal{O}(F).$$

2.13. Let **2** denote the 2-element chain $0 < 1$ and for every ordered set $E$ let $\mathrm{Isomap}(E, \mathbf{2})$ be the set of isotone mappings $f : E \to \mathbf{2}$. Prove that the ordered sets $\mathcal{O}(E)$ and $\mathrm{Isomap}(E, \mathbf{2})$ are dually isomorphic.
[*Hint.* Consider $\alpha : \mathrm{Isomap}(E, \mathbf{2}) \to \mathcal{O}(E)$ given by $\alpha(f) = f^{\leftarrow}\{0\}$.]

2.14. Draw the Hasse diagram of the lattice of down-sets of each of the following ordered sets:



2.15. If $P_1$ and $P_2$ are the ordered sets



draw the Hasse diagram of the lattice of down-sets of $P_1 \cup P_2$.

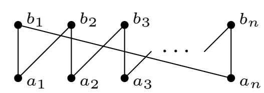
2.16. The **Lucas sequence** $(\ell_n)_{n\geqslant 0}$ is defined by

$$\ell_0 = 1,\ \ \ell_1 = 1,\ \ (n \geqslant 2)\ \ell_n = \ell_{n-1} + \ell_{n-2}.$$

If $f_i$ denotes the $i$-th Fibonacci number, establish the identity

$$\ell_{2n} = f_{2n+2} - f_{2n-2}.$$

By a **crown** we mean an ordered set $C_{2n}$ of the form




it being assumed that all the elements are distinct. Prove that $|\mathcal{O}(C_{2n})| = \ell_{2n}$.

2.17. Let $E_{2n}$ be the ordered set obtained from the crown $C_{2n}$ by adjoining comparabilities in such a way that $a_i < b_j$ for all $i, j$. Determine $|\mathcal{O}(E_{2n})|$.

## 2.3 Sublattices

As we have seen, important substructures of an ordered set are the down-sets and the principal down-sets. We now consider substructures of (semi)lattices.

**Definition** By a **$\wedge$-subsemilattice** of a $\wedge$-semilattice $L$ we mean a non-empty subset $E$ of $L$ that is closed under the meet operation, in the sense that if $x, y \in E$ then $x \wedge y \in E$. A **$\vee$-subsemilattice** of a $\vee$-semilattice is defined dually. By a **sublattice** of a lattice we mean a subset that is both a $\wedge$-subsemilattice and a $\vee$-subsemilattice.

**Example 2.14** If $V$ is a vector space then, by Example 2.8, the set $\operatorname{Sub} V$ of subspaces of $V$ is a $\cap$-subsemilattice of the lattice $\mathbb{P}(V)$.

**Example 2.15** For every ordered set $E$ the lattice $\mathcal{O}(E)$ of down-sets of $E$ is a sublattice of the lattice $\mathbb{P}(E)$.

Particularly important sublattices of a lattice are the following.

**Definition** By an **ideal** of a lattice $L$ we shall mean a sublattice of $L$ that is also a down-set; dually, by a **filter** of $L$ we mean a sublattice that is also an up-set.

**Theorem 2.6** *If $L$ is a lattice then, ordered by set inclusion, the set $\mathcal{I}(L)$ of ideals of $L$ forms a lattice in which the lattice operations are given by*

$$\begin{cases} \inf\{J,K\} = J \cap K; \\ \sup\{J,K\} = \{x \in L \mid (\exists j \in J)(\exists k \in K)\ x \leqslant j \vee k\}. \end{cases}$$

*Proof* It is clear that if $J$ and $K$ are ideals of $L$ then so is $J \cap K$, and that this is the biggest ideal of $L$ that is contained in both $J$ and $K$. Hence $\inf\{J,K\}$ exists in $\mathcal{I}(L)$ and is $J \cap K$.

Now any ideal that contains both $J$ and $K$ must clearly contain all the elements $x$ such that $x \leqslant j \vee k$ where $j \in J$ and $k \in K$. Conversely, the set of all such $x$ clearly contains both $J$ and $K$, and is contained in every ideal of $L$ that contains both $J$ and $K$. Moreover, this set is also an ideal of $L$. Thus we see that $\sup\{J,K\}$ exists in $\mathcal{I}(L)$ and is as described above. $\square$

Note from Theorem 2.6 that although $\mathcal{I}(L)$ is a $\cap$-subsemilattice of $\mathcal{O}(L)$ it is not a sublattice since suprema are not the same. This situation, in which a subsemilattice of a given lattice $L$ that is not a sublattice of $L$ can also form a lattice with respect to the same order as $L$, is quite common in lattice theory. Another instance of this has been seen before in Example 2.8 where the set $\mathrm{Sub}\, V$ of subspaces of a vector space $V$ forms a lattice in which $\inf\{A,B\} = A \cap B$ and $\sup\{A,B\} = A + B$, so that $(\mathrm{Sub}\, V; \subseteq)$ forms a lattice that is a $\cap$-subsemilattice, but not a sublattice, of $\big(\mathbb{P}(V); \subseteq\big)$. As we shall now see, a further instance is provided by a closure mapping on a lattice.

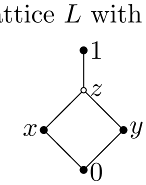
**Theorem 2.7** *Let $L$ be a lattice and let $f : L \to L$ be a closure. Then $\mathrm{Im}\, f$ is a lattice in which the lattice operations are given by*

$$\inf\{a,b\} = a \wedge b, \quad \sup\{a,b\} = f(a \vee b).$$

*Proof* Recall that for a closure $f$ on $L$ we have $\mathrm{Im}\, f = \{x \in L \mid x = f(x)\}$. If then $a, b \in \mathrm{Im}\, f$ we have, since $f$ is isotone with $f \geqslant \mathrm{id}_L$, $f(a) \wedge f(b) = a \wedge b \leqslant f(a \wedge b) \leqslant f(a) \wedge f(b)$ and the resulting equality gives $a \wedge b \in \mathrm{Im}\, f$. It follows that $\mathrm{Im}\, f$ is a $\wedge$-subsemilattice of $L$.

As for the supremum in $\mathrm{Im}\, f$ of $a, b \in \mathrm{Im}\, f$, we observe first that $a \vee b \leqslant f(a \vee b)$ and so $f(a \vee b) \in \mathrm{Im}\, f$ is an upper bound of $\{a,b\}$. Suppose now that $c = f(c) \in \mathrm{Im}\, f$ is any upper bound of $\{a,b\}$ in $\mathrm{Im}\, f$. Then from $a \vee b \leqslant c$ we obtain $f(a \vee b) \leqslant f(c) = c$. Thus, in the subset $\mathrm{Im}\, f$, the upper bound $f(a \vee b)$ is less than or equal to every upper bound of $\{a,b\}$. Consequently, $\sup\{a,b\}$ exists in $\mathrm{Im}\, f$ and is $f(a \vee b)$. $\square$

**Example 2.16** Consider the lattice $L$ with Hasse diagram




Let $f : L \to L$ be given by

$$f(t) = \begin{cases} 1 & \text{if } t = z; \\ t & \text{otherwise.} \end{cases}$$

It is readily seen that $f$ is a closure with $\operatorname{Im} f = \{0, x, y, 1\}$. In the corresponding lattice (the elements of which are denoted by $\bullet$) we have $\sup\{x, y\} = f(x \vee y) = f(z) = 1$.

**EXERCISES**

2.18. If $L$ is a lattice prove that, ordered by set inclusion, the set $\mathcal{F}(L)$ of filters of $L$ forms a lattice and determine the lattice operations.

2.19. Let $L, M$ be lattices and let $f : L \to M$ be residuated. Prove that $\operatorname{Im} f$ is a lattice in which $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = ff^+(x \wedge y)$.

2.20. Prove that in the ordered set $\operatorname{Equ} E$ of equivalence relations on $E$ the supremum of $\vartheta$ and $\varphi$ is the relation $\psi$ given by $(x, y) \in \psi$ if and only if there exist $a_1, \ldots, a_n \in E$ such that

$$x \equiv a_1 \equiv a_2 \equiv \cdots \equiv a_{n-1} \equiv a_n \equiv y$$

where each $\equiv$ denotes $\vartheta$ or $\varphi$.

## 2.4 Lattice morphisms

We now consider isotone mappings that preserve lattice operations.

**Definition** If $L$ and $M$ are $\vee$-semilattices then $f : L \to M$ is said to be a **$\vee$-morphism** if $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$. The notion of a **$\wedge$-morphism** is defined dually. If $L$ and $M$ are lattices then $f : L \to M$ is a **lattice morphism** if it is both a $\vee$-morphism and a $\wedge$-morphism. If $L$ and $M$ are $\vee$-semilattices then a mapping $f : L \to M$ is said to be a **complete $\vee$-morphism** if, for every family $(x_\alpha)_{\alpha \in I}$ of elements of $L$ such that $\bigvee_{\alpha \in I} x_\alpha$ exists in $L$, $\bigvee_{\alpha \in I} f(x_\alpha)$ exists in $M$ and $f\big(\bigvee_{\alpha \in I} x_\alpha\big) = \bigvee_{\alpha \in I} f(x_\alpha)$. The notion of a **complete $\wedge$-morphism** is defined dually.

**Theorem 2.8** *If $L$ and $M$ are $\vee$-semilattices then every residuated mapping $f : L \to M$ is a complete $\vee$-morphism.*

*Proof* Suppose that $(x_\alpha)_{\alpha \in I}$ is a family of elements of $L$ such that $x = \bigvee_{\alpha \in I} x_\alpha$ exists in $L$. Clearly, for each $\alpha \in I$ we have $f(x) \geqslant f(x_\alpha)$. Now if $y \geqslant f(x_\alpha)$ for each $\alpha \in I$ then $f^+(y) \geqslant f^+[f(x_\alpha)] \geqslant x_\alpha$ and so $f^+(y) \geqslant \bigvee_{\alpha \in I} x_\alpha = x$. But then $y \geqslant f[f^+(y)] \geqslant f(x)$. Thus we see that $\bigvee_{\alpha \in I} f(x_\alpha)$ exists and is $f(x)$. $\square$

**Definition** We shall say that lattices $L$ and $M$ are **isomorphic** if they are isomorphic as ordered sets.

**Theorem 2.9** *Lattices $L, M$ are isomorphic if and only if there is a bijection $f : L \to M$ that is a $\vee$-morphism.*

*Proof* $\Rightarrow$: If $L \simeq M$ then there is a residuated bijection $f : L \to M$, and by Theorem 2.8 this is a $\vee$-morphism.

$\Leftarrow$: If $f : L \to M$ is a bijection and a $\vee$-morphism then we have

$$\begin{aligned} x \leqslant y \iff y = x \vee y &\iff f(y) = f(x \vee y) = f(x) \vee f(y) \\ &\iff f(x) \leqslant f(y), \end{aligned}$$

whence, by Theorem 1.10, $L \simeq M$. □

**EXERCISES**

2.21. Let $L$ be a lattice. Prove that every isotone mapping from $L$ to an arbitrary lattice $M$ is a lattice morphism if and only if $L$ is a chain.
[*Hint*. If $L$ is not a chain then there exist $a, b \in L$ with $a \parallel b$. Construct a lattice $M$ by substituting a chain $a \wedge b < \alpha < \beta < a \vee b$ for the sublattice $[a, b] = \{x \in L \mid a \leqslant x \leqslant b\}$. Consider the mapping $f : L \to M$ given by

$$f(x) = \begin{cases} \alpha & \text{if } a \wedge b < x < a; \\ \beta & \text{if } a \wedge b < x < a \vee b \text{ and } x \not\leqslant a; \\ x & \text{otherwise.} \end{cases}]$$

2.22. If $L$ is a lattice and $a, b \in L$ let

$$X_{a,b} = \{x \in L \mid x = (x \vee b) \wedge a\}, \quad Y_{a,b} = \{y \in L \mid y = (y \wedge a) \vee b\}.$$

Prove that $X_{a,b}$ and $Y_{a,b}$ are isomorphic lattices.

## 2.5 Complete lattices

We have seen that in a meet semilattice the infimum of every finite subset exists. We now extend this concept to arbitrary subsets.

**Definition** A $\wedge$-semilattice $L$ is said to be **$\wedge$-complete** if every subset $E = \{x_\alpha \mid \alpha \in A\}$ of $L$ has an infimum which we denote by $\inf_L E$ or by $\bigwedge_{\alpha \in A} x_\alpha$. In a dual manner we define the notion of a **$\vee$-complete $\vee$-semilattice**, in which we use the notation $\sup_L E$ or $\bigvee_{\alpha \in A} x_\alpha$. A lattice is said to be **complete** if it is both $\wedge$-complete and $\vee$-complete.

**Theorem 2.10** *Every complete lattice has a top and a bottom element.*

*Proof* Clearly, if $L$ is complete then $\sup_L L$ is the top element of $L$, and $\inf_L L$ is the bottom element. □

**Example 2.17** For every non-empty set $E$ the power set lattice $\mathbb{P}(E)$ is complete. The top element is $E$ and the bottom element is $\emptyset$.

**Example 2.18** Let $L$ be the lattice that is formed by adding to the chain $\mathbb{Q}$ of rationals a top element $\infty$ and a bottom element $-\infty$. Then $L$ is bounded but is not complete; for example $\sup_L\{x \in \mathbb{Q} \mid x^2 \leqslant 2\}$ does not exist.

**Example 2.19** For every non-empty set $E$ the set $\operatorname{Equ} E$ of equivalence relations on $E$ is a complete lattice. In fact, if $F = (R_\alpha)_{\alpha\in A}$ is a family of equivalence relations on $E$, then $\inf_{\alpha\in A} R_\alpha$ clearly exists in $\operatorname{Equ} E$ and is the relation $\bigwedge_{\alpha\in A} R_\alpha$ given by

$$(x,y) \in \bigwedge_{\alpha\in A} R_\alpha \iff (\forall\alpha\in A)\ (x,y)\in R_\alpha.$$

As for the supremum of this family, consider the relation $\vartheta$ defined by $(x,y) \in \vartheta$ if and only if there exist $z_1,\dots,z_n$ and $R_{\alpha_1},\dots,R_{\alpha_{n+1}}$ such that

$$x \overset{R_{\alpha_1}}{\equiv} z_1 \overset{R_{\alpha_2}}{\equiv} z_2 \overset{R_{\alpha_3}}{\equiv} \cdots \overset{R_{\alpha_n}}{\equiv} z_n \overset{R_{\alpha_{n+1}}}{\equiv} y.$$

It is clear that $\vartheta \in \operatorname{Equ} E$. If $x \overset{R_\alpha}{\equiv} y$ for any $R_\alpha \in F$ then since this is a trivial example of such a display it is clear that $R_\alpha \subseteq \vartheta$. Thus $\vartheta$ is an upper bound of $F$. Observe now that, by the transitivity of $\vartheta$, every relation on $E$ that is implied by every $R_{\alpha_i}$ (i.e. every upper bound of $F$) is also implied by $\vartheta$. We therefore conclude that $\vartheta = \sup_{\alpha\in A} R_\alpha = \bigvee_{\alpha\in A} R_\alpha$. Hence $\operatorname{Equ} E$ forms a complete lattice. The relation $\vartheta$ so described is called the **transitive product** of the family $(R_\alpha)_{\alpha\in A}$.

The relationship between complete semilattices and complete lattices is highlighted by the following useful result (and its dual).

**Theorem 2.11** *A $\wedge$-complete $\wedge$-semilattice is a complete lattice if and only if it has a top element.*

*Proof* The condition is clearly necessary. To show that it is also sufficient, let $L$ be a $\wedge$-complete $\wedge$-semilattice with top element 1. Let $X = \{x_\alpha \mid \alpha \in A\}$ be a non-empty subset of $L$. We show as follows that $\sup_L X$ exists.

Observe first that the set $X^\uparrow$ of upper bounds of $X$ is not empty since it contains the top element 1. Let $X^\uparrow = \{m_\beta \mid \beta\in B\}$. Then, since $L$ is $\wedge$-complete, $\bigwedge_{\beta\in B} m_\beta$ exists. Now clearly we have $x_\alpha \leqslant m_\beta$ for all $\alpha$ and $\beta$. It follows that $x_\alpha \leqslant \bigwedge_{\beta\in B} m_\beta$ for every $x_\alpha \in X$, whence $\bigwedge_{\beta\in B} m_\beta \in X^\uparrow$. By its very definition, $\bigwedge_{\beta\in B} m_\beta$ is then the supremum $X$ in $L$. Hence $L$ is a complete lattice. □

**Example 2.20** Let $E$ be an infinite set and let $\mathbb{P}_f(E)$ be the set of all finite subsets of $E$. Ordered by set inclusion, $\mathbb{P}_f(E)$ is a lattice which is clearly $\cap$-complete. By Theorem 2.11, $\mathbb{P}_f(E) \cup \{E\}$ is then a complete lattice.

**Example 2.21** If $G$ is a group let $\operatorname{Sub} G$ be the set of all subgroups of $G$. Ordered by set inclusion, $\operatorname{Sub} G$ is clearly a $\cap$-semilattice that is $\cap$-complete. By Theorem 2.11, $\operatorname{Sub} G$ is a complete lattice. In this, joins are given as follows. If $X$ is any subset of $G$ then the subgroup generated by $X$ (i.e. the smallest subgroup of $G$ that contains $X$) is the set

$$\langle X\rangle = \Big\{\prod_{i=1}^n a_i \;\Big|\; a_i \in X \text{ or } a_i^{-1} \in X\Big\}.$$

Thus, if $(H_i)_{i\in I}$ is a family of subgroups of $G$ then the subgroup $\bigvee\limits_{i\in I} H_i$ generated by $\bigcup\limits_{i\in I} H_i$ is the set of all finite products $\prod\limits_{i=1}^n h_i$ where each $h_i \in \bigcup\limits_{i\in I} H_i$. Consequently $(\operatorname{Sub} G; \subseteq, \cap, \vee)$ is a complete lattice.

**Example 2.22** Consider the lattice $(\mathbb{N};\, |)$. This is bounded above by 0 and bounded below by 1. If $X$ is any non-empty subset of $\mathbb{N}$ then $\inf_{\mathbb{N}} X$ exists, being the greatest common divisor of the elements of $X$. It follows by Theorem 2.11 that $(\mathbb{N};\, |)$ is a complete lattice. For $X$ finite $\sup_{\mathbb{N}} X$ is the least common multiple of the elements of $X$; and for $X$ infinite the supremum is the top element 0, this following from the observation that an infinite subset of $\mathbb{N}$ contains integers that are greater than any fixed positive integer.

Concerning complete lattices we have the following remarkable result.

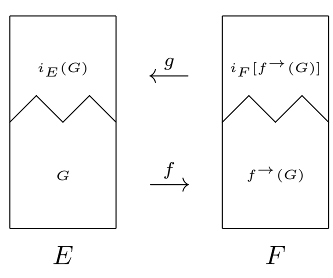
**Theorem 2.12** (Knaster [74]) *If $L$ is a complete lattice and if $f : L \to L$ is an isotone mapping then $f$ has a fixed point.*

*Proof* Consider the set $A = \{x \in L \mid x \leqslant f(x)\}$. Observe that $A \neq \emptyset$ since $L$ has a bottom element 0, and $0 \in A$. By completeness, there exists $\alpha = \sup_L A$. Now for every $x \in A$ we have $x \leqslant \alpha$ and therefore $x \leqslant f(x) \leqslant f(\alpha)$. It follows from this that $\alpha = \sup_L A \leqslant f(\alpha)$. Consequently $f(\alpha) \leqslant f[f(\alpha)]$, which gives $f(\alpha) \in A$ and therefore $f(\alpha) \leqslant \sup_L A = \alpha$. Thus we have $f(\alpha) = \alpha$. $\square$

An interesting application of Theorem 2.12 is to a proof of the following important set-theoretic result.

**Theorem 2.13** (Bernstein [5]) *If $E$ and $F$ are sets and if there are injections $f : E \to F$ and $g : F \to E$ then $E$ and $F$ are equipotent.*

*Proof* We use the notation $i_X : \mathbb{P}(X) \to \mathbb{P}(X)$ to denote the antitone mapping that sends every subset of $X$ to its complement in $X$. Consider the mapping $\zeta : \mathbb{P}(E) \to \mathbb{P}(E)$ given by $\zeta = i_E \circ g^{\to} \circ i_F \circ f^{\to}$. Since $f^{\to}$ and $g^{\to}$ are isotone, so also is $\zeta$. By Theorem 2.12, there exists $G \subseteq E$ such that $\zeta(G) = G$, and therefore $i_E(G) = (g^{\to} \circ i_F \circ f^{\to})(G)$. The situation may be summarised pictorially:




Now since $f$ and $g$ are injective by hypothesis this configuration shows that we can define a bijection $h : E \to F$ by the prescription

$$h(x) = \begin{cases} f(x) & \text{if } x \in G; \\ \text{the unique element of } g^{\leftarrow}\{x\} & \text{if } x \notin G. \end{cases}$$

Hence $E$ and $F$ are equipotent. $\square$

**EXERCISES**

2.23. If $L$ is a lattice prove that the ideal lattice $\mathcal{I}(L)$ is complete if and only if $L$ has a bottom element.

2.24. If $V$ is a vector space prove that the lattice $\mathrm{Sub}\,V$ of subspaces of $V$ is complete.

2.25. If $E$ is an ordered set prove that the set of closure mappings on $E$ is a complete lattice.

2.26. Let $T$ be the subset of $\mathrm{Rel}\,E$ consisting of the transitive relations on $E$. Prove that $T$ is a $\cap$-complete $\cap$-semilattice. Given $R \in \mathrm{Rel}\,E$ let $T(R)$ be the set of transitive relations on $E$ that contain $R$, and let $\overline{R} = \inf T(R)$. Show that

$$(x, y) \in \overline{R} \iff (\exists a_0, \dots, a_n \in E)\;\; x = a_0 \overset{R}{\equiv} a_1 \overset{R}{\equiv} \cdots \overset{R}{\equiv} a_n = y.$$

In the complete lattice $\mathrm{Equ}\,E$ prove that $\sup_{\alpha \in A} R_\alpha = \overline{\bigcup_{\alpha \in A} R_\alpha}$.

2.27. Let $L$ be a complete lattice and let $f : L \to L$ be an isotone mapping. If $\omega$ is a fixed point of $f$ and $a = \bigvee\limits_{n \geqslant 0} f^n(0)$ prove that $a \leqslant \omega$. Hence show that $f$ has a smallest fixed point.

2.28. Let $L$ be a complete lattice with top element 1 and bottom element 0. If $f : L \to L$ is a closure mapping prove that $f$ is residuated if and only if $\mathrm{Im}\,f$ is a complete sublattice of $L$ containing 0 and 1.

2.29. Prove that if $L$ and $M$ are complete lattices then a mapping $f : L \to M$ is residuated if and only if it is a complete $\vee$-morphism and $f(0_L) = 0_M$.

Using Theorem 2.11 we can extend as follows the result of Theorem 2.7 to complete lattices.

**Theorem 2.14** (Ward [111]) *Let $L$ be a complete lattice. If $f$ is a closure on $L$ then* $\mathrm{Im}\,f$ *is a complete lattice. Moreover, for every non-empty subset $A$ of* $\mathrm{Im}\,f$*,*

$$\inf_{\mathrm{Im}\,f} A = \inf_L A \quad \textit{and} \quad \sup_{\mathrm{Im}\,f} A = f(\sup_L A).$$

*Proof* First we observe that $\mathrm{Im}\,f$ is a $\wedge$-complete $\wedge$-semilattice. To see this, recall that $\mathrm{Im}\,f$ is the set of fixed points of $f$. Given $C \subseteq \mathrm{Im}\,f$ let $a = \inf_L C$. Then for every $x \in C$ we have $a \leqslant x$ and so $f(a) \leqslant f(x) = x$. Thus $f(a) \leqslant \inf_L C = a$ and consequently $f(a) = a$, whence $a \in \mathrm{Im}\,f$ and $\mathrm{Im}\,f$ is $\wedge$-complete. Now since $L$ is complete it has a top element 1; and since $f \geqslant \mathrm{id}_L$ we have necessarily $1 = f(1) \in \mathrm{Im}\,f$. It now follows by Theorem 2.11 that $\mathrm{Im}\,f$ is a complete lattice.

Suppose now that $A \subseteq \mathrm{Im}\,f$. If $a = \inf_L A$ then, from the above, we have $a = f(a) \in \mathrm{Im}\,f$. If now $y \in \mathrm{Im}\,f$ is such that $y \leqslant x$ for every $x \in A$ then $y \leqslant a$. Consequently we have $a = \inf_{\mathrm{Im}\,f} A$.

Now let $b = \sup_L A$ and $b^\star = \sup_{\mathrm{Im}\,f} A$. Since $\mathrm{Im}\,f$ is complete we have $b^\star \in \mathrm{Im}\,f$; and since $b^\star \geqslant x$ for every $x \in A$ we have $b^\star \geqslant \sup_L A = b$. Thus $b^\star = f(b^\star) \geqslant f(b)$. But $f(b) \geqslant f(x) = x$ for every $x \in A$, and so we also have $f(b) \geqslant \sup_{\mathrm{Im}\,f} A = b^\star$. Thus $b^\star = f(b)$ as asserted. $\square$

We now proceed to describe an important application of Theorem 2.14. For this purpose, given an ordered set $E$, consider the mapping $\vartheta : \mathbb{P}(E) \to \mathbb{P}(E)$ given by $\vartheta(A) = A^{\downarrow}$ and the mapping $\varphi : \mathbb{P}(E) \to \mathbb{P}(E)$ given by $\varphi(A) = A^{\uparrow}$. If $A \subseteq B$ then clearly every lower bound of $B$ is a lower bound of $A$, whence $B^{\downarrow} \subseteq A^{\downarrow}$. Hence $\vartheta$ is antitone. Dually, so is $\varphi$. Now every element of $A$ is clearly a lower bound of the set of upper bounds of $A$, whence $A \subseteq A^{\uparrow\downarrow}$ and therefore $\mathrm{id}_{\mathbb{P}(E)} \leqslant \vartheta\varphi$. Dually, every element of $A$ is an upper bound of the set of lower bounds of $A$, so $A \subseteq A^{\downarrow\uparrow}$ and therefore $\mathrm{id}_{\mathbb{P}(E)} \leqslant \varphi\vartheta$. Consequently we see that $(\vartheta, \varphi)$ establish a Galois connection on $\mathbb{P}(E)$. We shall focus on the associated closure $A \mapsto A^{\uparrow\downarrow}$. For this purpose we shall also require the following facts.

**Theorem 2.15** *Let $E$ be an ordered set. If $(A_\alpha)_{\alpha\in I}$ is a family of subsets of $E$ then*

$$\Big(\bigcup_{\alpha\in I} A_\alpha\Big)^{\uparrow} = \bigcap_{\alpha\in I} A_\alpha^{\uparrow} \quad \textit{and} \quad \Big(\bigcup_{\alpha\in I} A_\alpha\Big)^{\downarrow} = \bigcap_{\alpha\in I} A_\alpha^{\downarrow}.$$

*Proof* Since each $A_\alpha$ is contained in $\bigcup_{\alpha\in I} A_\alpha$ and $A \mapsto A^{\uparrow}$ is antitone, we have that $\Big(\bigcup_{\alpha\in I} A_\alpha\Big)^{\uparrow} \subseteq \bigcap_{\alpha\in I} A_\alpha^{\uparrow}$. To obtain the reverse inclusion, observe that if $x \in \bigcap_{\alpha\in I} A_\alpha^{\uparrow}$ then $x$ is an upper bound of $A_\alpha$ for every $\alpha \in I$, whence $x$ is an upper bound of $\bigcup_{\alpha\in I} A_\alpha$ and therefore belongs to $\Big(\bigcup_{\alpha\in I} A_\alpha\Big)^{\uparrow}$. The second statement is proved similarly. □

**Definition** By an **embedding** of an ordered set $E$ into a lattice $L$ we mean a mapping $f : E \to L$ such that, for all $x, y \in E$,

$$x \leqslant y \iff f(x) \leqslant f(y).$$

**Theorem 2.16** (Dedekind–MacNeille [79]) *Every ordered set $E$ can be embedded in a complete lattice $L$ in such a way that meets and joins that exist in $E$ are preserved in $L$.*

*Proof* If $E$ does not have a top element or a bottom element we begin by adjoining whichever of these bounds is missing. Then, without loss of generality we may asume that $E$ is a bounded ordered set.

Let $f : \mathbb{P}(E) \to \mathbb{P}(E)$ be the closure mapping given by $f(A) = A^{\uparrow\downarrow}$. Then, by Theorem 2.14, $L = \operatorname{Im} f$ is a complete lattice. Observe that $f(\{x\}) = \{x\}^{\uparrow\downarrow} = x^{\downarrow}$ for all $x \in E$ and hence that $x \leqslant y \iff f(\{x\}) \subseteq f(\{y\})$. It follows that $f$ induces an embedding $f^\star : E \to L$, namely that given by $f^\star(x) = f(\{x\}) = \{x\}^{\uparrow\downarrow} = x^{\downarrow}$. Suppose now that $A = \{x_\alpha \mid \alpha \in I\} \subseteq E$. If $a = \bigwedge_{\alpha\in I} x_\alpha$ exists then clearly $a^{\downarrow} = \bigcap_{\alpha\in I} x_\alpha^{\downarrow}$ so that $f^\star(a) = \bigcap_{\alpha\in I} f^\star(x_\alpha)$, i.e. existing infima are preserved.

Suppose now that $b = \bigvee_{\alpha\in I} x_\alpha$ exists. Since

$$\begin{aligned} y \geqslant b &\iff (\forall \alpha \in I)\ \ y \in x_\alpha^{\uparrow} = \{x_\alpha\}^{\uparrow\downarrow\uparrow} \\ &\iff y \in \bigcap_{\alpha\in I} \{x_\alpha\}^{\uparrow\downarrow\uparrow} = \Big(\bigcup_{\alpha\in I}\{x_\alpha\}^{\uparrow\downarrow}\Big)^{\uparrow} \quad \text{(Theorem 2.15)}, \end{aligned}$$

we see that $b^{\uparrow} = \Big(\bigcup_{\alpha\in I}\{x_\alpha\}^{\uparrow\downarrow}\Big)^{\uparrow}$. Consequently,

$$\begin{aligned} f^\star(b) = \{b\}^{\uparrow\downarrow} = \Big(\bigcup_{\alpha\in I}\{x_\alpha\}^{\uparrow\downarrow}\Big)^{\uparrow\downarrow} &= f\big(\sup\nolimits_{\mathbb{P}(E)}\big\{\{x_\alpha\}^{\uparrow\downarrow} \ \big|\ \alpha \in I\big\}\big) \\ &= \sup\nolimits_{\operatorname{Im} f}\big\{\{x_\alpha\}^{\uparrow\downarrow} \ \big|\ \alpha \in I\big\} \ \ \text{(Theorem 2.14)} \\ &= \sup\nolimits_{\operatorname{Im} f}\big\{f^\star(x_\alpha) \ \big|\ \alpha \in I\big\}, \end{aligned}$$

so that existing suprema are also preserved. □

**Definition** The complete lattice $L = \operatorname{Im} f = \{A^{\uparrow\downarrow} \mid A \in \mathbb{P}(E)\}$ in the above is called the **Dedekind–MacNeille completion** of $E$.

This construction is also known as the **completion by cuts** of $E$ since it generalises the method of constructing $\mathbb{R}$ from $\mathbb{Q}$ by Dedekind sections.

The Dedekind–MacNeille completion of $E$ has the following property.

**Theorem 2.17** *Let $E$ be an ordered set and let $\operatorname{DMac} E$ together with the embedding $f^\star : E \to \operatorname{DMac} E$ be the Dedekind–MacNeille completion of $E$. If $g : E \to M$ is any embedding of $E$ into a complete lattice $M$ then there is an embedding $\zeta : \operatorname{DMac} E \to M$ such that $\zeta \circ f^\star = g$.*

*Proof* We have the situation

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "f^\star"'] & M \\ \operatorname{DMac} E & \end{tikzcd}$$

in which $f^\star : x \mapsto x^{\downarrow}$, and the requirement is to produce an embedding $\zeta : \operatorname{DMac} E \to M$ such that $\zeta \circ f^\star = g$. For this purpose consider the mapping $\zeta : \operatorname{DMac} E \to M$ defined by the prescription

$$\zeta(X) = \sup\nolimits_M\{g(x) \mid x \in X\}.$$

It is clear that $\zeta$ is isotone. Suppose now that $\zeta(X) \leqslant \zeta(Y)$. Then for every $x \in X$ we have

$$g(x) \leqslant \zeta(X) \leqslant \zeta(Y) = \sup\nolimits_M\{g(y) \mid y \in Y\}$$

and so $g(x) \leqslant g(z)$ for all $z \in Y^{\uparrow}$. Since $g$ is an embedding we deduce that $x \leqslant z$ for all $z \in Y^{\uparrow}$. Hence $x \in Y^{\uparrow\downarrow} = Y$ and so $X \subseteq Y$. Hence $\zeta$ is an embedding.

Now for every $x \in E$ we have

$$\zeta[f^\star(x)] = \zeta(x^{\downarrow}) = \sup\nolimits_M\{g(t) \mid t \in x^{\downarrow}\} = g(x).$$

Consequently we have $\zeta \circ f^\star = g$. □

**EXERCISE**

2.30. Construct the Dedekind–MacNeille completion of each of the following:
(1) a finite chain;
(2) a finite antichain;
(3) the 4-element fence;
(4) the 4-element crown.

## 2.6 Baer semigroups

We now show how the coordinatisation of a bounded ordered set can be extended to that of a bounded lattice. For this purpose we require the notion of a Baer semigroup, which pre-dates that of a generalised Baer semigroup (hence the terminology for the latter).

**Definition** Let $S$ be a semigroup with a zero element. Then we say that $S$ is a **Baer semigroup** if the Galois connection $(L, R)$ of Example 1.27 has the property that for each $x \in S$ there are idempotents $e, f \in S$ such that $R(x) = eS$ and $L(x) = Sf$.

Thus a semigroup $S$ with a 0 is a Baer semigroup if and only if the right annihilator of every $x \in S$ is an idempotent-generated principal right ideal, and the left annihilator of every $x \in S$ is an idempotent-generated principal left ideal. In particular, we note that since $S = R(0) = L(0)$ there exist idempotents $e, f$ such that $S = eS = Sf$. Then $e$ is a left identity and $f$ is a right identity, whence $e = f$ and so $S$ has an identity element.

**Example 2.23** Let $V$ be a vector space. Consider the ring $A$ of endomorphisms on $V$ as a semigroup under composition. Given $\vartheta \in A$ let $e$ be the projection onto $\operatorname{Ker}\vartheta$, and let $f$ be the projection onto $\operatorname{Im}\vartheta$. Then $R(\vartheta) = eA$ and $L(\vartheta) = A(\mathrm{id}_V - f)$. Hence $A$ is a Baer semigroup.

**Example 2.24** The set of square matrices over a field is a Baer semigroup.

**Example 2.25** If $S$ is a Baer semigroup and if $T$ is a full subsemigroup of $S$ (i.e., $T$ contains all the idempotents of $S$) then $T$ is a Baer semigroup.

**Example 2.26** Let $X$ be a non-empty set and let $\operatorname{Rel} X$ be the set of binary relations on $X$. If $S \in \operatorname{Rel} X$ and $M \subseteq X$ define

$$S(M) = \{y \in X \mid (\exists x \in M)\ (x, y) \in S\}.$$

Then the **image** of $S$ is the set $\operatorname{Im} S = S(X)$, and the **domain** of $S$ is the set $\operatorname{Dom} S = S^d(X)$ where $S^d$ denotes the dual of $S$. Given $S, T \in \operatorname{Rel} X$, define the composite $ST$ by

$$(x, y) \in ST \iff (\exists z \in X)\ (x, z) \in T \text{ and } (z, y) \in S.$$

Then, with respect to this law of composition, $\operatorname{Rel} X$ becomes a semigroup in which the empty relation $\emptyset$ acts as a zero element. We show as follows that $\operatorname{Rel} X$ is in fact a Baer semigroup.

For this purpose, suppose first that $ST = \emptyset$. If $z \in \operatorname{Im} T$ then we cannot have $z \in \operatorname{Dom} S$ and so $\operatorname{Im} T \subseteq [\operatorname{Dom} S]'$, the complement of $\operatorname{Dom} S$. On the other hand, if $\operatorname{Im} T \subseteq [\operatorname{Dom} S]'$ then clearly there can be no elements $x, y \in X$ such that $(x, y) \in ST$, and therefore $ST = \emptyset$. Thus we see that

$$ST = \emptyset \iff \operatorname{Im} T \subseteq (\operatorname{Dom} S)'.$$

For each subset $M$ of $X$ define the relation $I_M$ by

$$(x, y) \in I_M \iff x = y \in M.$$

Observe that for every subset $M$ of $X$ the relation $I_M$ is an idempotent of $\operatorname{Rel} X$. If now $T \in \operatorname{Rel} X$ is such that $T = I_A T$ then $\operatorname{Im} T = T(X) = I_A[T(X)] \subseteq I_A(X) = A$; and conversely, if $\operatorname{Im} T \subseteq A$ then $T = I_A T$. Thus we have

$$T \in I_A \cdot \operatorname{Rel} X \iff \operatorname{Im} T \subseteq A.$$

It follows from these observations that, for every $S \in \operatorname{Rel} X$,

$$R(S) = I_A \cdot \operatorname{Rel} X \ \text{ where } A = (\operatorname{Dom} S)'.$$

Since $TS = \emptyset \iff S^d T^d = \emptyset$, a dual argument produces the fact that

$$L(S) = \operatorname{Rel} X \cdot I_B \ \text{ where } B = (\operatorname{Dom} S^d)' = (\operatorname{Im} S)'.$$

Hence $\operatorname{Rel} X$ is a Baer semigroup.

We observe that if $S$ is a Baer semigroup then since $S$ has an identity element we have $R(x) = R(Sx)$ and $L(x) = L(xS)$ for every $x \in S$. Thus, if $R(x) = eS$ and $L(x) = Sf$ then $LR(x) = L(eS) = L(e)$ and $RL(x) = R(Sf) = R(f)$. Consequently, every Baer semigroup is a generalised Baer semigroup.

**Theorem 2.18** *If $S$ is a Baer semigroup then $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are dually isomorphic bounded lattices.*

*Proof* Since the restriction to $\mathcal{R}(S)$ of $RL$ is the identity and since the restriction to $\mathcal{L}(S)$ of $LR$ is the identity, the ordered sets $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are dually isomorphic. Note that

$$xS \in \mathcal{R}(S) \iff xS = RL(x).$$

In fact, if $xS \in \mathcal{R}(S)$ then $xS = R(y)$ for some $y \in S$ whence $RL(x) = RL(xS) = RLR(y) = R(y) = xS$. Conversely, if $RL(x) = xS$ let $L(x) = Sf$. Then $xS = R(Sf) = R(f) \in \mathcal{R}(S)$.

Suppose then that $eS, fS \in \mathcal{R}(S)$. Then $eS = RL(eS) = RL(e) = R(e_l)$ and $fS = RL(fS) = RL(f) = R(f_l)$. Let $R(f_l e) = gS$. Then we have $eg \in R(f_l e) = gS$ whence $eg = geg$ and so $eg$ is idempotent. Observe that

(1) $egS = R\{e_l, f_l\}$.

In fact, if $x \in egS$ then $x = egx$ whence on the one hand $e_l x = e_l egx = 0gx = 0$, and on the other hand $f_l x = f_l egx = 0x = 0$. Thus $x \in R\{e_l, f_l\}$. Conversely, if $x \in R\{e_l, f_l\}$ then $x \in R(e_l) = eS$ and so $x = ex$ whence $f_l ex = f_l x = 0$. Thus $x \in R(f_l e) = gS$ and therefore $x = gx = egx \in egS$.

It follows from (1) that we have $RL(eg) = RL(egS) = egS$ and consequently $egS \in \mathcal{R}(S)$. We now see that

$$eS \cap fS = R(e_l) \cap R(f_l) = R\{e_l, f_l\} = egS.$$

We deduce from this that $R(S)$ is a $\cap$-semilattice with bottom element 0. In a dual manner we have that $\mathcal{L}(S)$ is a $\cap$-semilattice with bottom element 0. As $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are dually isomorphic, each is then a bounded lattice. □

The coordinatisation theorem for bounded lattices is the following.

**Theorem 2.19** (Janowitz [68],[69]) *For a bounded ordered set $E$ the following statements are equivalent*:

(1) *$E$ is a lattice*;
(2) $\operatorname{Res} E$ *is a Baer semigroup*;
(3) *$E$ can be coordinatised by a Baer semigroup.*

*Proof* (1) $\Rightarrow$ (2): For each $e \in E$ consider the mappings $\vartheta_e, \psi_e : E \to E$ given by the prescriptions

$$\vartheta_e(x) = \begin{cases} x & \text{if } x \leqslant e; \\ e & \text{otherwise,} \end{cases} \qquad \psi_e(x) = \begin{cases} 0 & \text{if } x \leqslant e; \\ x \vee e & \text{otherwise.} \end{cases}$$

It is clear that $\vartheta_e$ and $\psi_e$ are isotone and idempotent. They are also residuated; simple calculations show that

$$\vartheta_e^+(x) = \begin{cases} 1 & \text{if } x \geqslant e; \\ x \wedge e & \text{otherwise,} \end{cases} \qquad \psi_e^+(x) = \begin{cases} x & \text{if } x \geqslant e; \\ e & \text{otherwise.} \end{cases}$$

Given $f \in \operatorname{Res} E$, observe that $g \in R(f)$ if and only if $g(1) \leqslant f^+(0)$. Thus, if $g \in R(f)$ we have $g = \vartheta_{f^+(0)} g \in \vartheta_{f^+(0)} \circ \operatorname{Res} E$. Conversely, if $g = \vartheta_{f^+(0)} g$ then $g(1) \leqslant \vartheta_{f^+(0)}(1) = f^+(0)$. Thus we see that $R(f) = \vartheta_{f^+(0)} \circ \operatorname{Res} E$. Now a dual argument in the semigroup $\operatorname{Res}^+ E$ shows that $R(f^+) = \psi^+_{f(1)} \circ \operatorname{Res}^+ E$, so in $\operatorname{Res} E$ we have $L(f) = \operatorname{Res} E \circ \psi_{f(1)}$. Hence $\operatorname{Res} E$ is a Baer semigroup.

(2) $\Rightarrow$ (3): This follows exactly as in the proof of Theorem 1.12 with $\vartheta : \mathcal{R}(\operatorname{Res} E) \to E$ given by $\vartheta(\varphi S) = \varphi(1)$.

(3) $\Rightarrow$ (1): This is immediate from Theorem 2.18. □

We note here that a more general definition of a Baer semigroup was developed by Blyth and Janowitz [24] in which the existence of a zero element is replaced by that of a principal ideal $K$ that is generated by a central idempotent $k$.

## EXERCISES

2.31. If $S$ is a Baer semigroup prove that the join operation in the lattice $\mathcal{R}(S)$ is given by

$$eS \vee fS = R(Se_l \cap Sf_l).$$

2.32. If $S$ is a Baer semigroup and $e \in S$ is idempotent prove that $eSe$ is a Baer semigroup. Show also that $\mathcal{R}(eSe)$ is isomorphic to the set of fixed points of $\varphi_e \in \operatorname{Res} \mathcal{R}(E)$.

2.33. A Baer semigroup $S$ is said to be **complete** if the right annihilator of every *subset* of $S$ is a principal right ideal generated by an idempotent. Show that if $S$ is a complete Baer semigroup then the left annihilator of every subset of $S$ is a principal left ideal generated by an idempotent. Show further that the following statements are equivalent:

(1) $E$ is a complete lattice;
(2) $\operatorname{Res} E$ is a complete Baer semigroup;
(3) $E$ can be coordinatised by a complete Baer semigroup.