5

Homogeneous Tensor Algebra: Tensor Homomorphisms

5.1 Introduction

The chapter starts by presenting the main theorem on tensor contraction, which ensures that a contraction of a tensor product when applied to indices of different valency leads to a tensor.

It continues by presenting the contracted tensor products as homomorphisms and applies them to different tensor products as particular cases. Some tensor criteria motivated by the contraction are also discussed, including the well-known quotient law criterion.

Next, a detailed study of the matrix representation of the permutation tensors and some simple and double contracted homomorphisms is performed.

The chapter ends with a novel theory of eigentensors and generalized multilinear mappings.

5.2 Main theorem on tensor contraction

Though in Section 3.5 the contraction of tensor products has already been mentioned, and in Theorem 3.1 any contraction of mixed tensors has been examined from the homomorphism point of view, that is, of linear mappings of a primary linear space (tensor space) into another secondary linear space, one can have doubts about whether or not the resulting "range" space would be a simple linear space, or would also be a tensor space.

Fortunately, this doubt is positively resolved, because the "homomorphic" image of a tensor space is another tensor space.

Remark: the word "homomorphic" *always* has the sense of a mixed tensor "contracted from two indices of different valency".

Next, we prove this property with the required emphasis, and later it will be enunciated as a theorem.

Consider a mixed homogeneous tensor $\vec{t} \in \left(\overset{3}{\underset{1}{\otimes}} V^n\right) \otimes \left(\overset{2}{\underset{1}{\otimes}} V_*^n\right)(K)$ of order $r = 5$,

$$\vec{t} = t^{\alpha\beta\gamma\circ\circ}_{\circ\circ\circ\lambda\mu} \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}_\gamma \otimes \vec{e}^{*\lambda} \otimes \vec{e}^{*\mu} \tag{5.1}$$

and denote by $S(\alpha, \gamma, \mu)$ the "system of scalars" resulting from the contraction of indices 2 and 4 of different valency (β and λ):

$$s(\alpha, \gamma, \mu) = t^{\alpha\theta\gamma\circ\circ}_{\circ\circ\circ\theta\mu}\,. \tag{5.2}$$

In detail, we have

$$s(\alpha, \gamma, \mu) = t^{\alpha 1\gamma\circ\circ}_{\circ\circ\circ 1\mu} + t^{\alpha 2\gamma\circ\circ}_{\circ\circ\circ 2\mu} + \cdots + t^{\alpha n\gamma\circ\circ}_{\circ\circ\circ n\mu}\,. \tag{5.3}$$

The system $S(\alpha, \gamma, \mu)$ is called a system of scalars because one cannot anticipate if it is a tensor. The power of the set $S(\alpha, \gamma, \mu)$ is n^3, because we have three free indices.

Next, we perform a change-of-basis in the $\left(\overset{3}{\underset{1}{\otimes}} V^n\right) \otimes \left(\overset{2}{\underset{1}{\otimes}} V_*^n\right)(K)$ tensor space. Since its vectors are homogeneous tensors, we have

$$t^{ijk\circ\circ}_{\circ\circ\circ\ell m} = t^{\alpha\beta\gamma\circ\circ}_{\circ\circ\circ\lambda\mu} \gamma^{i\circ}_{\circ\alpha} \gamma^{j\circ}_{\circ\beta} \gamma^{k\circ}_{\circ\gamma} c^{\circ\lambda}_{\ell\circ} c^{\circ\mu}_{m\circ}. \tag{5.4}$$

The indices (j, ℓ) are contracted. Preparing Expression (5.4) and calling the set of scalars in the left-hand side $s(i, k, m)$, we get

$$t^{ijk\circ\circ}_{\circ\circ\circ\ell m} = t^{\alpha\beta\gamma\circ\circ}_{\circ\circ\circ\lambda\mu} \gamma^{i\circ}_{\circ\alpha} \gamma^{k\circ}_{\circ\gamma} (\gamma^{j\circ}_{\circ\beta} c^{\circ\lambda}_{\ell\circ}) c^{\circ\mu}_{m\circ} \tag{5.5}$$

$$s(i, k, m) = t^{\alpha\beta\gamma\circ\circ}_{\circ\circ\circ\lambda\mu} \gamma^{i\circ}_{\circ\alpha} \gamma^{k\circ}_{\circ\gamma} (\gamma^{x\circ}_{\circ\beta} c^{\circ\lambda}_{x\circ}) c^{\circ\mu}_{m\circ}. \tag{5.6}$$

The expression $(\gamma^{x\circ}_{\circ\beta} c^{\circ\lambda}_{x\circ})$ is the "product" of matrices $C^{-1} \odot C^t$ but executed by "*multiplying row by row*" (not by column, due to the position of x); but this is the same as $C^{-1} \cdot C = I_n$. So, that

$$\gamma^{x\circ}_{\circ\beta} c^{\circ\lambda}_{x\circ} = \delta^{\circ\lambda}_{\beta\circ}, \tag{5.7}$$

and replacing (5.7) into (5.6) we obtain

$$s(i, k, m) = (t^{\alpha\beta\gamma\circ\circ}_{\circ\circ\circ\lambda\mu} \delta^{\circ\lambda}_{\beta\circ}) \gamma^{i\circ}_{\circ\alpha} \gamma^{k\circ}_{\circ\gamma} c^{\circ\mu}_{m\circ},$$

where the product in parentheses is the contraction of (β, λ)

$$s(i, k, m) = t^{\alpha\theta\gamma\circ\circ}_{\circ\circ\circ\theta\mu} \gamma^{i\circ}_{\circ\alpha} \gamma^{k\circ}_{\circ\gamma} c^{\circ\mu}_{m\circ} \tag{5.8}$$

and, on account of (5.2), the previous expression can be written as

$$s(i,k,m) = s(\alpha,\gamma,\mu)\gamma_{\circ\alpha}^{i\,\circ}\gamma_{\circ\gamma}^{k\,\circ}c_{m\,\circ}^{\circ\,\mu}, \tag{5.9}$$

which declares that the system of scalars $S(\alpha,\gamma,\mu)$ satisfies the tensor criteria, that is, it is a tensor. Whence

$$s(\alpha,\gamma,\mu) \equiv s_{\circ\,\circ\,\mu}^{\alpha\gamma\,\circ}.$$

This proof can be repeated over other two indices with *different valency.* We leave this for the reader to do.

If, by error, we were to choose two indices with the same valency, when reaching Expression (5.6) products of the type $(\gamma_{\circ\beta}^{x\,\circ}\gamma_{\circ\lambda}^{x\,\circ})$ or $(c_{x\,\circ}^{\circ\beta}c_{x\,\circ}^{\circ\lambda})$ would appear that *are not* the Kronecker delta, making the proof invalid. We understand that this expression can be generalized to tensors of order superior to $r = 5$, and proceed to state the "tensor contraction" general theorem.

Theorem 5.1 (Fundamental theorem of tensor contraction). *The contraction with respect to indices of different valency in mixed tensors of order r, is a sufficient condition for obtaining another homogeneous tensor of order $(r-2)$.* □

5.3 The contracted tensor product and tensor homomorphisms

In Section 3.4 we have dealt with tensor product of tensors, and in Section 3.5 the contracted tensor product concept was defined. Since in that definition the conditions of "tensor contraction" are satisfied, Theorem 3.1 guarantees that the *contracted tensor products* can be considered as simple homomorphisms (Formula (3.16)), that transform tensors from a tensor space into tensors of another space by the action of a *contracted tensor homomorphism.*

This point of view will be exploited at the end of this chapter, more precisely, on tensors of simple order, and it will be executed using the matrix expression

$$T_{\sigma'} = H_{n^{r-2},n^r} \bullet T_\sigma; \quad \text{with } \sigma = n^r; \quad \sigma' = n^{r-2}. \tag{5.10}$$

Nevertheless, before ending, we want to point out the analytical representation of the contracted tensor product, in the classic mode.

Given the tensors $\vec{t} = t_{\circ\,\circ\,\gamma}^{\alpha\beta\,\circ}\vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}^{*\gamma}$ and $\vec{v} = v_{\lambda\mu}^{\circ\,\circ}\vec{e}^{*\lambda} \otimes \vec{e}^{*\mu}$, we look for the *contracted tensor product* tensor $\vec{p} = \mathcal{C}\binom{\alpha}{\lambda}(\vec{t} \otimes \vec{v})$, with $\vec{p} = p_{\circ\gamma\mu}^{\beta\,\circ\,\circ}\vec{e}_\beta \otimes \vec{e}^{*\gamma} \otimes \vec{e}^{*\mu}$.

This can be done in two different forms:

1. We obtain the tensor product tensor

$$\vec{w} = \vec{t} \otimes \vec{v} = w^{\alpha\beta\circ\circ\circ}_{\circ\circ\gamma\lambda\mu}\vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}^{*\gamma} \otimes \vec{e}^{*\lambda} \otimes \vec{e}^{*\mu} \tag{5.11}$$

with the condition

$$w^{\alpha\beta\circ\circ\circ}_{\circ\circ\gamma\lambda\mu} = t^{\alpha\beta\circ}_{\circ\circ\gamma} \cdot v^{\circ\circ}_{\lambda\mu} \tag{5.12}$$

and then we contract

$$\vec{p} = \mathcal{C}\binom{\alpha}{\lambda}\vec{w} = w^{\theta\beta\circ\circ\circ}_{\circ\circ\gamma\theta\mu}\vec{e}_\beta \otimes \vec{e}^{*\gamma} \otimes \vec{e}^{*\mu},$$

where we also have

$$w^{\theta\beta\circ\circ\circ}_{\circ\circ\gamma\theta\mu} = w^{\alpha\beta\circ\circ\circ}_{\circ\circ\gamma\lambda\mu} \cdot \delta^{\circ\lambda}_{\alpha\circ}, \tag{5.13}$$

where $\delta^{\circ\lambda}_{\alpha\circ}$ is the Kronecker tensor.

2. The second form is used by certain authors, who prefer a direct execution of the product and the contraction *simultaneously*, based on matrix representations:

$$p^{\beta\circ\circ}_{\circ\gamma\mu} = (t^{\alpha\beta\circ}_{\circ\circ\gamma}) \cdot \delta^{\circ\lambda}_{\alpha\circ} \cdot (v^{\circ\circ}_{\lambda\mu}). \tag{5.14}$$

Evidently, (5.14) is the result of replacing (5.12) into (5.13), because $\delta^{\circ\lambda}_{\alpha\circ} \equiv \delta^{\lambda\circ}_{\circ\alpha}$ is symmetric, and then, both methods lead to the same result.

Example 5.1 (Matrix associated with an operator). Consider two linear spaces $V^n(K)$ and $W^p(K)$. In the first space we consider a linear operator T_1 with associated matrix A_n in the basis $\{\vec{e}_i\}$ of the given space. Similarly, another linear operator T_2, with associated matrix B_p, transforms the vectors of $W^p(K)$ in the basis $\{\vec{\epsilon}_j\}$. We look for the matrix associated with the operator T defined to transform vectors in the tensor space $V \otimes W(K)$, in such way that

$$T(\vec{V} \otimes \vec{W}) = T_1(\vec{V}) \otimes T_2(\vec{W}).$$

Will $A_n \otimes B_p$ be the T operator matrix? That is, will "the tensor product homomorphism" be the homomorphisms' tensor product?

Solution: For a homomorphism to be correctly defined we need to know the image vectors of all basic vectors that will constitute the columns of the operator associated matrix.

The basis of our tensor space is $\beta = \{\vec{e}_i \otimes \vec{\epsilon}_j\}$, with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p$.

The sought after matrix T is a square matrix of $n \times p$ rows and columns, because in the basis there exist $n \times p$ vectors the images of which are to be studied.

Applying the formula proposed in the statement to an arbitrary basic vector, we have

$$
\begin{aligned}
T(\vec{e}_i \otimes \vec{e}_j) &= T_1(\vec{e}_1) \otimes T_2(\vec{e}_j) \\
&= \left([\,\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n\,] \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \right) \otimes \left([\,\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_p\,] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} \right)^t \\
&= [\,\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n\,] \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \otimes [\,b_{1j} \quad b_{2j} \quad \cdots \quad b_{pj}\,] \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ \vec{e}_p \end{bmatrix} \\
&= \left([\,\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n\,] \begin{bmatrix} a_{1i}b_{1j} & a_{1i}b_{2j} & \cdots & a_{1i}b_{pj} \\ a_{2i}b_{1j} & a_{2i}b_{2j} & \cdots & a_{2i}b_{pj} \\ \cdots & \cdots & \cdots & \cdots \\ a_{ni}b_{1j} & a_{ni}b_{2j} & \cdots & a_{ni}b_{pj} \end{bmatrix} \right) \otimes \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ \vec{e}_p \end{bmatrix} \\
&= a_{1i}b_{1j}\vec{e}_1 \otimes \vec{e}_1 + a_{1i}b_{2j}\vec{e}_1 \otimes \vec{e}_2 + \cdots + a_{hi}b_{kj}\vec{e}_h \otimes \vec{e}_k \\
&\quad + \cdots + a_{ni}b_{pj}\vec{e}_n \otimes \vec{e}_p
\end{aligned}
$$

with $h = 1, 2, \ldots, n;\; k = 1, 2, \ldots, p$.

Assigning now values to the indices (i, j), according to the axiomatic ordering criterion for the basis $\mathcal{B} = \{\vec{e}_i \otimes \vec{e}_j\}$ and placing the image vectors in consecutive columns, the matrix $T_{n \times p}$ is obtained, which is the solution to the problem, and the columns of which correspond to

$$
\begin{array}{cccccc}
T(\vec{e}_1 \otimes \vec{e}_1) & T(\vec{e}_1 \otimes \vec{e}_2) & \cdots & T(\vec{e}_h \otimes \vec{e}_k) & \cdots & T(\vec{e}_n \otimes \vec{e}_p)
\end{array}
$$

$$
T = \begin{bmatrix}
a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{1h}b_{1k} & \cdots & a_{1n}b_{1p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{2h}b_{1k} & \cdots & a_{2n}b_{1p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{h1}b_{11} & a_{h1}b_{12} & \cdots & a_{hh}b_{1k} & \cdots & a_{hn}b_{1p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1}b_{11} & a_{n1}b_{12} & \cdots & a_{nh}b_{1k} & \cdots & a_{nn}b_{1p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
$$

a square matrix of order $n \times p$.

Assigning particular values to n and p (for example $n = 2; p = 3$) we immediately detect the following block construction:

$$
T = \begin{bmatrix}
a_{11}B & \cdots & a_{1h}B & \cdots & a_{1n}B \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{h1}B & \cdots & a_{hh}B & \cdots & a_{hn}B \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1}B & \cdots & a_{nh}B & \cdots & a_{nn}B
\end{bmatrix} = A \otimes B.
$$

The conclusion is that the proposed theorem in our statement: "the tensor product homomorphism (T) is the tensor product of the given homomorphisms $(T_1 \otimes T_2)$" is correct. □

Example 5.2 (Change of basis). In the geometric affine ordinary space $E^2(\mathbb{R})$ we consider two bases: the initial basis of the unit classic vectors of a rectangular system XOY (on the OX axis and on the OY axis), and the new basis of the unit vectors on the OX axis and on the bisectrix of the XOY quadrant.

The new unit basic vectors $||\hat{\vec{e}}_i||$ referred to the initial basic vectors $||\vec{e}_\alpha||$, are

$$\hat{\vec{e}}_1 = \vec{e}_1; \;\; \hat{\vec{e}}_2 = \frac{\sqrt{2}}{2}\vec{e}_1 + \frac{\sqrt{2}}{2}\vec{e}_2.$$

Determine the new components as a function of the initial ones in the following cases:

1. For a tensor of first order, i.e. the vector v^α.
2. For a mixed tensor of second order, i.e. the matrix $t^{\alpha\circ}_{\circ\beta}$.
3. For a mixed tensor of third order, $t^{\alpha\circ\gamma}_{\circ\beta\circ}$.
4. Solve the second question using the homomorphism (contracted product) $y^{\alpha}_{\circ} = t^{\alpha\circ}_{\circ\theta} x^{\theta}_{\circ}$.

Solution:

The change-of-basis can be written in matrix form as

$$||\hat{\vec{e}}_i|| = ||\vec{e}_\alpha|| C \rightarrow [\,\hat{\vec{e}}_1 \quad \hat{\vec{e}}_2\,] = [\,\vec{e}_1 \quad \vec{e}_2\,] \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix},$$

and then

$$C = \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}; \;\; C^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix}; \;\; C^t = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

1. The tensor analytical equation of the vector is $v^{i}_{\circ} = v^{\alpha}_{\circ} \gamma^{i\circ}_{\circ\alpha}$, and in matrix form
$$[v^{i}_{\circ}] = [\gamma^{i\circ}_{\circ\alpha}][v^{\alpha}_{\circ}] \rightarrow \begin{bmatrix} \hat{v}^1 \\ \hat{v}^2 \end{bmatrix} = C^{-1} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \tag{5.15}$$
$$\begin{bmatrix} \hat{v}^1 \\ \hat{v}^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 - v^2 \\ \sqrt{2} v^2 \end{bmatrix}.$$
2. The tensor analytical equation of the vector is $t^{i\circ}_{\circ j} = t^{\alpha\circ}_{\circ\beta} \gamma^{i\circ}_{\circ\alpha} c^{\circ\beta}_{j\circ}$ (classic matrix method), and in matrix form
$$[t^{i\circ}_{\circ j}] = [\gamma^{i\circ}_{\circ\alpha}][t^{\alpha\circ}_{\circ\beta}][c^{\beta\circ}_{\circ j}],$$

that is,

$$\begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} = C^{-1} \begin{bmatrix} t^{1\circ}_{\circ 1} & t^{1\circ}_{\circ 2} \\ t^{2\circ}_{\circ 1} & t^{2\circ}_{\circ 2} \end{bmatrix} (C^t)^t = \begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} t^{1\circ}_{\circ 1} & t^{1\circ}_{\circ 2} \\ t^{2\circ}_{\circ 1} & t^{2\circ}_{\circ 2} \end{bmatrix} \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} (t^{1\circ}_{\circ 1} - t^{2\circ}_{\circ 1}) & \frac{\sqrt{2}}{2}[(t^{1\circ}_{\circ 1} - t^{2\circ}_{\circ 1}) + (t^{1\circ}_{\circ 2} - t^{2\circ}_{\circ 2})] \\ \sqrt{2} t^{2\circ}_{\circ 1} & (t^{2\circ}_{\circ 1} + t^{2\circ}_{\circ 2}) \end{bmatrix}.$$

3. The tensor analytical equation of the vector (direct method) is

$$t^{i \circ k}_{\circ j \circ} = t^{\alpha \circ \gamma}_{\circ \beta \circ} \gamma^{i \circ}_{\circ \alpha} c^{\circ \beta}_{j \circ} \gamma^{k \circ}_{\circ \gamma}; \quad \sigma = n^r = 2^3 = 8.$$

and its "extended" matrix expression

$$\hat{T}_{\sigma,1} = Z^{-1}_{\sigma,\sigma} T_{\sigma,1} \to \hat{T}_{8,1} = (C^{-1} \otimes C^t \otimes C^{-1}) \bullet T_{8,1}$$

$$Z^{-1}_{\sigma} = C^{-1} \otimes C^t \otimes C^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix};$$

$$\hat{T}_{8,1} = \begin{bmatrix} \hat{t}^{1\circ 1}_{\circ 1 \circ} \\ \hat{t}^{1\circ 2}_{\circ 1 \circ} \\ \hat{t}^{1\circ 1}_{\circ 2 \circ} \\ \hat{t}^{1\circ 2}_{\circ 2 \circ} \\ \hat{t}^{2\circ 1}_{\circ 1 \circ} \\ \hat{t}^{2\circ 2}_{\circ 1 \circ} \\ \hat{t}^{2\circ 1}_{\circ 2 \circ} \\ \hat{t}^{2\circ 2}_{\circ 2 \circ} \end{bmatrix} = Z^{-1}_{\sigma,\sigma} \bullet T_{8,1}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \bullet \begin{bmatrix} t^{1\circ 1}_{\circ 1 \circ} \\ t^{1\circ 2}_{\circ 1 \circ} \\ t^{1\circ 1}_{\circ 2 \circ} \\ t^{1\circ 2}_{\circ 2 \circ} \\ t^{2\circ 1}_{\circ 1 \circ} \\ t^{2\circ 2}_{\circ 1 \circ} \\ t^{2\circ 1}_{\circ 2 \circ} \\ t^{2\circ 2}_{\circ 2 \circ} \end{bmatrix},$$

that is,

$$\hat{t}^{1\circ 1}_{\circ 1\circ} = t^{1\circ 1}_{\circ 1\circ} - t^{1\circ 2}_{\circ 1\circ} - t^{2\circ 1}_{\circ 1\circ} + t^{2\circ 2}_{\circ 1\circ}$$

$$\hat{t}^{1\circ 2}_{\circ 1\circ} = \sqrt{2}t^{1\circ 2}_{\circ 1\circ} - \sqrt{2}t^{2\circ 2}_{\circ 1\circ}$$

$$\hat{t}^{1\circ 1}_{\circ 2\circ} = \frac{\sqrt{2}}{2}(t^{1\circ 1}_{\circ 1\circ} - t^{1\circ 2}_{\circ 1\circ} + t^{1\circ 1}_{\circ 2\circ} - t^{1\circ 2}_{\circ 2\circ} - t^{2\circ 1}_{\circ 1\circ} + t^{2\circ 2}_{\circ 1\circ} - t^{2\circ 1}_{\circ 2\circ} + t^{2\circ 2}_{\circ 2\circ})$$

$$\hat{t}^{1\circ 2}_{\circ 2\circ} = t^{1\circ 2}_{\circ 1\circ} + t^{1\circ 2}_{\circ 2\circ} - t^{2\circ 2}_{\circ 1\circ} - t^{2\circ 2}_{\circ 2\circ}$$

$$\hat{t}^{2\circ 1}_{\circ 1\circ} = \sqrt{2}t^{2\circ 1}_{\circ 1\circ} - \sqrt{2}t^{2\circ 2}_{\circ 1\circ}$$

$$\hat{t}^{2\circ 2}_{\circ 1\circ} = 2t^{2\circ 2}_{\circ 1\circ}$$

$$\hat{t}^{2\circ 1}_{\circ 2\circ} = t^{2\circ 1}_{\circ 1\circ} - t^{2\circ 2}_{\circ 1\circ} + t^{2\circ 1}_{\circ 2\circ} - t^{2\circ 2}_{\circ 2\circ}$$

$$\hat{t}^{2\circ 2}_{\circ 1\circ} = \sqrt{2}t^{2\circ 2}_{\circ 1\circ} + \sqrt{2}t^{2\circ 2}_{\circ 2\circ}.$$

4. The given tensor homomorphism can be interpreted in matrix form as In the initial basis $\{\vec{e}_\alpha\}$:

$$\begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} t^{1\circ}_{\circ 1} & t^{1\circ}_{\circ 2} \\ t^{2\circ}_{\circ 1} & t^{2\circ}_{\circ 2} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \tag{5.16}$$

and in the new basis $\{\hat{\vec{e}}_i\}$:

$$\begin{bmatrix} \hat{y}^1 \\ \hat{y}^2 \end{bmatrix} = \begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} \begin{bmatrix} \hat{x}^1 \\ \hat{x}^2 \end{bmatrix}. \tag{5.17}$$

Applying the relation (5.15) to matrices X and Y, we have

$$\begin{bmatrix} \hat{x}^1 \\ \hat{x}^2 \end{bmatrix} = C^{-1} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \tag{5.18}$$

and

$$\begin{bmatrix} \hat{y}^1 \\ \hat{y}^2 \end{bmatrix} = C^{-1} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}, \tag{5.19}$$

and substituting (5.18) and (5.19) into (5.17), we get

$$C^{-1} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} C^{-1} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

and substituting into the left-hand side of (5.16) the result is

$$C^{-1}\begin{bmatrix} t^{1\circ}_{\circ 1} & t^{1\circ}_{\circ 2} \\ t^{2\circ}_{\circ 1} & t^{2\circ}_{\circ 2} \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} C^{-1}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix},$$

and for this to be valid for any matrix X, we must have

$$C^{-1}\begin{bmatrix} t^{1\circ}_{\circ 1} & t^{1\circ}_{\circ 2} \\ t^{2\circ}_{\circ 1} & t^{2\circ}_{\circ 2} \end{bmatrix} = \begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} C^{-1}$$

or

$$\begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} = C^{-1}\begin{bmatrix} t^{1\circ}_{\circ 1} & t^{1\circ}_{\circ 2} \\ t^{2\circ}_{\circ 1} & t^{2\circ}_{\circ 2} \end{bmatrix} C$$

and operating we finally get

$$\begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} t^{1\circ}_{\circ 1} & t^{1\circ}_{\circ 2} \\ t^{2\circ}_{\circ 1} & t^{2\circ}_{\circ 2} \end{bmatrix}\begin{bmatrix} 1 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\begin{bmatrix} \hat{t}^{1\circ}_{\circ 1} & \hat{t}^{1\circ}_{\circ 2} \\ \hat{t}^{2\circ}_{\circ 1} & \hat{t}^{2\circ}_{\circ 2} \end{bmatrix} = \begin{bmatrix} (t^{1\circ}_{\circ 1} - t^{2\circ}_{\circ 1}) & \frac{\sqrt{2}}{2}[(t^{1\circ}_{\circ 1} - t^{2\circ}_{\circ 1}) + (t^{1\circ}_{\circ 2} - t^{2\circ}_{\circ 2})] \\ \sqrt{2}t^{2\circ}_{\circ 1} & (t^{2\circ}_{\circ 1} + t^{2\circ}_{\circ 2}) \end{bmatrix}.$$

□

5.4 Tensor product applications

In this section, some important tensor products applications are discussed.

5.4.1 Common simply contracted tensor products

First, we mention the contracted tensor product of first-order tensors.

Consider the tensors $\vec{x} = x^{\alpha}_{\circ}\vec{e}_{\alpha} \in V^n(K)$, $\vec{y} = y^{\beta}_{\circ}\vec{e}^{*\beta} \in V^n_*(K)$; their contracted tensor product is

$$p = \mathcal{C}\binom{\alpha}{\beta}(\vec{x}\otimes\vec{y}) = \mathcal{C}\binom{\alpha}{\beta}(x^{\alpha}_{\circ}y^{\circ}_{\beta}\vec{e}_{\alpha}\otimes\vec{e}^{*\beta}) = x^{\theta}y_{\theta} = x^1y_1 + x^2y_2 + \cdots + x^ny_n, \tag{5.20}$$

which is the classic dot product for geometric vectors or the classic inner product for first-order matrices.

Second, we mention the contracted tensor product of second-order tensors, known as the “interior product” or “classic product” of matrices.

Consider the tensors $\vec{a} = a^{\alpha\circ}_{\circ\beta}\vec{e}_{\alpha}\otimes\vec{e}^{*\beta}$ and $\vec{b} = b^{\gamma\circ}_{\circ\delta}\vec{e}_{\gamma}\otimes\vec{e}^{*\delta}$. Let $\vec{c}$ be their contracted tensor product of indices 2 and 3:

$$\vec{c} = \mathcal{C}\binom{\gamma}{\beta}(\vec{a} \otimes \vec{b}) = (a^{\alpha\circ}_{\circ\theta} \cdot b^{\theta\circ}_{\circ\delta})\vec{e}_{\alpha} \otimes \vec{e}^{*\delta},$$

where

$$c^{\alpha\circ}_{\circ\delta} = a^{\alpha\circ}_{\circ\theta} \cdot b^{\theta\circ}_{\circ\delta}, \tag{5.21}$$

which is the analytical tensor expression of the classic matrix product, of both matrices, as tensors.

Remark 5.1. In reality, the discovery of this idea occurred in the reverse order; first, Kronecker established the interior and tensor products of matrices, and then, under the name of Einstein's contraction, this concept was extended to tensors. □

5.4.2 Multiply contracted tensor products

It is obvious that when contracting a tensor product of tensors of certain orders, the resulting tensor can be a mixed tensor, with indices *not only of different valency* but coming from *different factors*; we can then continue contracting more indices, following the same criteria as the first time.

If we do this, we will obtain another tensor and we could practice contractions successively when the following two conditions are satisfied: (a) the indices must be of different valency, and (b) of different factor-tensor.

Evidently, this concept can be extended to products of *three or more tensors*, satisfying the associative law by operating the tensors two by two, and satisfying the index conditions.

On the other hand, the result of the contractions can be a zero-order tensor, that is, a scalar, which obviously is invariant under changes of basis. This is the reason why zero-order tensors are called "invariants".

5.4.3 Scalar and inner tensor products

Certain authors use the term "scalar product of tensors" for the totally contracted product of two tensors A and B, which allow it, and denote it by $A \bullet B = k$. The result is a zero-order tensor (a scalar). In this way, but based on a third fundamental tensor, we will later establish the tensor spaces with a interior connection.

It is also convenient to mention that, as a consequence of the concept of contracted tensor product, when selecting a tensor space of mixed tensors which contravariant and covariant indices coincide ($p = q$), the tensor product of *two* arbitrary tensors of this space can be contracted p times, leading to a contracted tensor product, that is, another tensor of the same space.

In such cases, some authors talk about a "tensor space with an interior product". For example, for $p = q = 2$, we would have

$$(t)^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta} \bullet (t')^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta} = (t'')^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}$$

leading to the *interior product of tensors* concept, and the concept of associated linear algebras.

Next, some illustrative examples of contractions will be given.

Example 5.3 (Multiple contractions). Consider the tensors

$$\vec{t} = t_{\alpha\beta\circ}^{\circ\circ\gamma}\vec{e}^{*\alpha} \otimes \vec{e}^{*\beta} \otimes \vec{e}_\gamma; \quad \vec{x} = x^\lambda \vec{e}_\lambda \text{ and } \vec{u} = u_{\circ\nu}^{\mu\circ}\vec{e}_\mu \otimes \vec{e}^{*\nu}$$

and the tensor $P = \vec{t} \otimes \vec{x} \otimes \vec{u}$ with components

$$p_{\alpha\beta\circ\circ\circ\nu}^{\circ\circ\gamma\lambda\mu\circ} = t_{\alpha\beta\circ}^{\circ\circ\gamma} \cdot x_\circ^\lambda \cdot u_{\circ\nu}^{\mu\circ}.$$

We want to perform the following multiple contractions:

1. *Double*:

$$\vec{p}_1 = \mathcal{C}\left(\begin{array}{c|c} \lambda & \mu \\ \alpha & \beta \end{array}\right)(\vec{t} \otimes \vec{x} \otimes \vec{u}) \tag{5.22}$$

$$\vec{p}_2 = \mathcal{C}\left(\begin{array}{c|c} \lambda & \gamma \\ \alpha & \nu \end{array}\right)(\vec{t} \otimes \vec{x} \otimes \vec{u}) \tag{5.23}$$

$$\vec{p}_3 = \mathcal{C}\left(\begin{array}{c|c} \mu & \gamma \\ \alpha & \nu \end{array}\right)(\vec{t} \otimes \vec{x} \otimes \vec{u}), \tag{5.24}$$

which lead to

$$\vec{p}_1 \to p_{\circ\nu}^{\gamma\circ} = t_{\theta\phi\circ}^{\circ\circ\gamma} x_\circ^\theta a_{\circ\nu}^{\phi\circ} \tag{5.25}$$

$$\vec{p}_2 \to p_{\beta\circ}^{\circ\mu} = t_{\theta\beta\circ}^{\circ\circ\phi} x_\circ^\theta a_{\circ\phi}^{\mu\circ} \tag{5.26}$$

$$\vec{p}_3 \to p_{\beta\circ}^{\circ\lambda} = t_{\theta\beta\circ}^{\circ\circ\phi} x_\circ^\lambda a_{\circ\phi}^{\theta\circ}. \tag{5.27}$$

2. *Triple*:

$$\vec{p}_4 = \mathcal{C}\left(\begin{array}{c|c|c} \lambda & \mu & \gamma \\ \alpha & \beta & \nu \end{array}\right)(\vec{t} \otimes \vec{x} \otimes \vec{u}) \tag{5.28}$$

$$\vec{p}_5 = \mathcal{C}\left(\begin{array}{c|c|c} \mu & \lambda & \gamma \\ \alpha & \beta & \nu \end{array}\right)(\vec{t} \otimes \vec{x} \otimes \vec{u}), \tag{5.29}$$

which lead to the scalars

$$p_4 = t_{\theta\phi\circ}^{\circ\circ w} x_\circ^\theta a_{\circ w}^{\phi\circ} \tag{5.30}$$

$$p_5 = t_{\theta\phi\circ}^{\circ\circ w} x_\circ^\phi a_{\circ w}^{\theta\circ}. \tag{5.31}$$

□

5.5 Criteria for tensor character based on contraction

In Section 4.5 the tensor criteria for homogeneous tensors were established with respect to changes of basis in tensor spaces. However, next we will establish other tensor criteria based on tensor contraction.

We present them as theorems, and in the proof of the third we will examine in detail its necessarity and sufficiency.

Theorem 5.2 (First elemental criterion for tensor character). *The necessary and sufficient condition for a system of scalars $s(\alpha_1, \alpha_2, \cdots, \alpha_r)$ of order r (the α_j are indices) to be a pure homogeneous tensor, of order r, totally contravariant, is that the expression "totally r-contracted product":*

$$s(\alpha_1, \alpha_2, \ldots, \alpha_r) x_{\alpha_1} \cdot x_{\alpha_2} \ldots x_{\alpha_r}; \quad \forall \vec{x} = x_{\alpha_j} \vec{e}^{\,*\alpha_j} \in V_*^n(K),$$
$$j \in I_r = \{1, 2, \ldots, r\}; \;\; \alpha_j \in I_n = \{1, 2, \ldots, n\} \qquad (5.32)$$

be a escalar*, that is, be* invariant *with respect to changes of basis in $V_*^n(K)$.*
□

Theorem 5.3 (Second elemental criterion for tensor character). *The necessary and sufficient condition for a system of scalars $s(\alpha_1, \alpha_2, \ldots, \alpha_r)$ of order r to be a pure homogeneous tensor, of order r, totally covariant, is that the expression "totally r-contracted product":*

$$s(\alpha_1, \alpha_2, \ldots, \alpha_r) x^{\alpha_1} \cdot x^{\alpha_2} \ldots x^{\alpha_r}; \quad \forall \vec{x} = x^{\alpha_j} \vec{e}_{\alpha_j} \in V^n(K),$$
$$j \in I_r = \{1, 2, \ldots, r\}; \;\; \alpha_j \in I_n = \{1, 2, \ldots, n\} \qquad (5.33)$$

be a escalar*, that is, be* invariant *with respect to changes of basis in $V^n(K)$.*
□

Theorem 5.4 (General criterion for homogeneous tensor character). *The necessary and sufficient condition for a system of scalars $s(\alpha_1, \alpha_2, \cdots, \alpha_r)$ of order r to be a mixed homogeneous tensor, of order r, p-contravariant and q-covariant ($p + q = r$), is that the expression "totally r-contracted product":*

$$s(\alpha_1, \alpha_2, \cdots, \alpha_p, \alpha_{p+1}, \ldots, \alpha_{p+q}) x_{\alpha_1} \cdot x_{\alpha_2} \ldots x_{\alpha_p} \cdot x^{\alpha_{p+1}} \cdot x^{\alpha_{p+2}} \cdots x^{\alpha_{p+q}};$$
$$\forall \vec{x} = x_{\alpha_j} \vec{e}^{\,*\alpha_j} \in V_*^n(K); \quad j \in I_p = \{1, 2, \ldots, p\}; \; \alpha_j \in I_n$$
$$\forall \vec{x} = x^{\alpha_k} \vec{e}_{\alpha_k} \in V^n(K) \quad k \in I_q = \{p+1, p+2, \ldots, p+q\}; \; \alpha_k \in I_n \qquad (5.34)$$

be a escalar*, that is, be* invariant *with respect to changes of basis in $V^n(K)$ and the corresponding changes of basis "in dual bases" in $V_*^n(K)$.* □

Proof.

Necessarity:

Let $\vec{t} = t^{\alpha\beta\circ}_{\circ\circ\gamma}\vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}^{*\gamma}$ be a mixed tensor of third order ($r = 3$), $p = 2$ times contravariant and $q = 1$ covariant, with $p + q = 2 + 1 = 3 = r$.

Consider the vectors $\vec{x} = x_\lambda \vec{e}^{*\lambda}, \vec{y} = y_\mu \vec{e}^{*\mu}$ and $\vec{z} = z^\nu \vec{e}_\nu$, where $\vec{x}, \vec{y} \in V^n_*(K)$ and $\vec{z} \in V^n(K)$.

If we execute the r-contracted tensor product:

$$p = \mathcal{C}\left(\begin{array}{c|c|c} \alpha & \beta & \lambda \\ \lambda & \mu & \nu \end{array}\right)(\vec{t} \otimes \vec{x} \otimes \vec{y} \otimes \vec{z}) \tag{5.35}$$

we get

$$p = t^{\theta\phi\circ}_{\circ\circ w} \cdot x_\theta \cdot y_\phi \cdot z^w = \text{scalar (zero-order tensor)}, \tag{5.36}$$

which proves that if $\vec{t}$ is a tensor, the theorem holds.

Sufficiency:

Consider now a system of scalars such that

$$p = s(\alpha, \beta, \gamma) \cdot x_\alpha \cdot y_\beta \cdot z^\gamma \tag{5.37}$$

for any pair of vectors $\vec{x} = x_\alpha \vec{e}^{*\alpha}, \vec{y} = y_\beta \vec{e}^{*\beta} \in V^n_*(K)$ and for all $\vec{z} = z^\gamma \vec{e}_\gamma \in V^n(K)$ where p is a given scalar.

We perform a change-of-basis in the linear space $V^n(K)$, and in the dual space $V^n_*(K)$ in which we choose the dual reciprocal basis of the one selected in $V^n(K)$.

Since p is a fixed scalar, the relation (5.37) is also satisfied in the new basis, that is, the p remains invariant for any new vector:

$$p = s(i, j, k) \cdot x_i \cdot y_j \cdot z^k; \quad \forall \vec{x} = x_i \vec{e}^{*i}, \vec{y} = y_j \vec{e}^{*j} \in V^n_*(K) \text{ and } \forall \vec{z} = z^k \vec{e}_k \in V^n(K). \tag{5.38}$$

Using the change-of-basis relations (3.46) and (3.24):

$$\vec{e}^{*i} = \gamma^{\circ\, i}_{\alpha\circ}\, \vec{e}^{*\alpha} \text{ in } V^n_*(K),$$

$$\vec{e}_k = c^{\gamma\circ}_{\circ k} \vec{e}_\gamma \text{ in } V^n(K)$$

we get the expressions that directly relate the vector components, in the initial and new bases:

$$x_\alpha = \gamma^{\circ\, i}_{\alpha\circ} x_i; \quad y_\beta = \gamma^{\circ\, j}_{\beta\circ} y_j,$$

for vectors of $V^n_*(K)$, and

$$z^\gamma = c^{\gamma\circ}_{\circ k} z^k,$$

for the vector of $V^n(K)$.

Transposing these equalities one gets

$$x_\alpha = x_i \gamma^{i\,\circ}_{\circ\,\alpha}; \quad y_\beta = y_j \gamma^{j\,\circ}_{\circ\,\beta}; \quad z^\gamma = z^k c^{\circ\,\gamma}_{k\,\circ} \tag{5.39}$$

and replacing (5.39) and (5.37) we obtain

$$p = s(\alpha, \beta, \gamma)(x_i \gamma^{i\,\circ}_{\circ\,\alpha})(y_j \gamma^{j\,\circ}_{\circ\,\beta})(z^k c^{\circ\,\gamma}_{k\,\circ}),$$

which is operated as

$$p = (x_i \cdot y_j \cdot z^k)(s(\alpha, \beta, \gamma)\gamma^{i\,\circ}_{\circ\,\alpha}\gamma^{j\,\circ}_{\circ\,\beta}c^{\circ\,\gamma}_{k\,\circ}). \tag{5.40}$$

Equating the constant p in (5.38) and (5.40), we get

$$p = (x_i \cdot y_j \cdot z^k)s(i, j, k) = (x_i \cdot y_j \cdot z^k)\left(s(\alpha, \beta, \gamma)\gamma^{i\,\circ}_{\circ\,\alpha}\gamma^{j\,\circ}_{\circ\,\beta}c^{\circ\,\gamma}_{k\,\circ}\right)$$

and since the previous relation must hold for all x_i, y_j, z^k, it must be

$$s(i, j, k) = s(\alpha, \beta, \gamma)\gamma^{i\,\circ}_{\circ\,\alpha}\gamma^{j\,\circ}_{\circ\,\beta}c^{\circ\,\gamma}_{k\,\circ}, \tag{5.41}$$

which shows that the system of scalars $s(\alpha, \beta, \gamma)$ satisfies the general tensor character criterion, Formula (4.34), so that the system of scalars must be notated as

$$s(\alpha, \beta, \gamma) = t^{\alpha\beta\circ}_{\circ\circ\gamma} \text{ or } s(i, j, k) = t^{ij\circ}_{\circ\circ k},$$

which proves its tensor character. Obviously, the necessity and the sufficiency have been proved only for $r = 3$, but we have preferred this simple case, which clearly reveals the process followed, to the general case with the generic r, which hides the demonstration process under the confused complexity of subindices.

We close this part, dedicated to tensor product contraction, simple or multiple, of homogeneous tensors, by pointing out that its treatment can be considered in the wider frame of absolute tensors, that is, of heterogeneous tensors established on diverse factor linear spaces, studied in Chapters 2 and 3, and in Chapter 4, where the absolute tensor character criteria for them were established, Formulas (4.24), (4.25), (4.34) and (4.35).

However, the most frequent use of contraction occurs in the homogeneous tensor algebra, which justifies the decision made in this chapter.

5.6 The contracted tensor product in the reverse sense: The quotient law

Theorem 5.5 (Quotient law). *Consider the system of scalars $S(\alpha_1, \ldots, \alpha_r)$ of order r. A sufficient condition for such a system to be considered a homogeneous tensor is that its p-contracted tensor product by a generic (arbitrary) homogeneous tensor $\vec{b}$ of order r', called a "test tensor", lead to another tensor of order $(r + r' - 2p)$.* □

Proof. We state the proof for a concrete case.

Let $s(\alpha,\beta,\gamma,\delta)$ be the data system of scalars, of order $r=4$, and let $\vec{b}=b^{\lambda\circ\circ}_{\circ\mu\nu}\vec{e}_\lambda\otimes\vec{e}^{*\mu}\otimes\vec{e}^{*\nu}$ be the "test" tensor, of order $r'=3$. As a consequence of their doubly contracted ($p=2$) product we arrive at the set of scalars $h^{\alpha\circ\circ}_{\circ\delta\mu}$, which is a known tensor, of order $r+r'-2p=4+3-2\times 2=3$.

Since $\vec{h}$ is a tensor, due to the tensor criteria we have

$$h^{i\circ\circ}_{\circ dm}=h^{\alpha\circ\circ}_{\circ\delta\mu}\gamma^{i\circ}_{\circ\alpha}c^{\circ\delta}_{d\circ}c^{\circ\mu}_{m\circ}. \tag{5.42}$$

In addition we have

$$h^{\alpha\circ\circ}_{\circ\delta\mu}=\mathcal{C}\left(\begin{array}{c|c}\lambda&\gamma\\\beta&\nu\end{array}\right)\left(s(\alpha,\beta,\gamma,\delta)\otimes b^{\lambda\circ\circ}_{\circ\mu\nu}\right), \tag{5.43}$$

a relation stated in the initial basis of $V^n(K)$, and also

$$h^{i\circ\circ}_{\circ dm}=\mathcal{C}\left(\begin{array}{c|c}\ell&k\\j&n\end{array}\right)\left(s(i,j,k,d)\otimes b^{\ell\circ\circ}_{\circ mn}\right), \tag{5.44}$$

stated in the final basis of $V^n(K)$.

Executing the contraction indicated in (5.43) and (5.44) and using the Kronecker deltas, we get the relations

$$h^{\alpha\circ\circ}_{\circ\delta\mu}=s(\alpha,\beta,\gamma,\delta)\delta^{\beta\circ}_{\circ\lambda}\delta^{\circ\nu}_{\gamma\circ}b^{\lambda\circ\circ}_{\circ\mu\nu} \tag{5.45}$$

$$h^{i\circ\circ}_{\circ dm}=s(i,j,k,d)\delta^{j\circ}_{\circ\ell}\delta^{\circ n}_{k\circ}b^{\ell\circ\circ}_{\circ mn} \tag{5.46}$$

and since $\vec{b}$ is a tensor (the "test" tensor), we state its tensor character criterion in the form (4.35), leading to

$$b^{\lambda\circ\circ}_{\circ\mu\nu}=b^{\ell\circ\circ}_{\circ mn}c^{\lambda\circ}_{\circ\ell}\gamma^{\circ m}_{\mu\circ}\gamma^{\circ n}_{\nu\circ} \tag{5.47}$$

and replacing (5.47) into (5.45), we get

$$h^{\alpha\circ\circ}_{\circ\delta\mu}=s(\alpha,\beta,\gamma,\delta)\delta^{\beta\circ}_{\circ\lambda}\delta^{\circ\nu}_{\gamma\circ}b^{\ell\circ\circ}_{\circ mn}c^{\lambda\circ}_{\circ\ell}\gamma^{\circ m}_{\mu\circ}\gamma^{\circ n}_{\nu\circ}. \tag{5.48}$$

Finally, substituting (5.46) and (5.48) into the left- and right-hand sides of (5.42), respectively, we get

$$s(i,j,k,d)\delta^{j\circ}_{\circ\ell}\delta^{\circ n}_{k\circ}b^{\ell\circ\circ}_{\circ mn}=\left[s(\alpha,\beta,\gamma,\delta)\delta^{\beta\circ}_{\circ\lambda}\delta^{\circ\nu}_{\gamma\circ}b^{\ell\circ\circ}_{\circ mn}c^{\lambda\circ}_{\circ\ell}\gamma^{\circ m}_{\mu\circ}\gamma^{\circ n}_{\nu\circ}\right]\gamma^{i\circ}_{\circ\alpha}c^{\circ\delta}_{d\circ}c^{\circ\mu}_{m\circ},$$

and conveniently grouping the factors we obtain

$$\left[s(i,j,k,d)\delta^{j\circ}_{\circ\ell}\delta^{\circ n}_{k\circ}\right]b^{\ell\circ\circ}_{\circ mn}=\left[s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}(\delta^{\beta\circ}_{\circ\lambda}c^{\lambda\circ}_{\circ\ell})(\gamma^{\circ m}_{\mu\circ}c^{\circ\mu}_{m\circ})(\delta^{\circ\nu}_{\gamma\circ}\gamma^{\circ n}_{\nu\circ}),c^{\circ\delta}_{d\circ}\right]b^{\ell\circ\circ}_{\circ mn}$$

and executing the indicated contractions:

$$\left[s(i,j,k,d)\delta^{j\circ}_{\circ\ell}\delta^{\circ n}_{k\circ}\right] b^{\ell\circ\circ}_{\circ mn} = \left[s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}(c^{\beta\circ}_{\circ\ell})\cdot 1\cdot(\gamma^{\circ n}_{\gamma\circ})c^{\circ\delta}_{d\circ}\right] b^{\ell\circ\circ}_{\circ mn}.$$

Finally, passing everything to the left-hand side and taking common factors, the result is

$$\left[s(i,j,k,d)\delta^{j\circ}_{\circ\ell}\delta^{\circ n}_{k\circ} - s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}c^{\beta\circ}_{\circ\ell}\gamma^{\circ n}_{\gamma\circ}c^{\circ\delta}_{d\circ}\right] b^{\ell\circ\circ}_{\circ mn} = 0. \tag{5.49}$$

Since the "test" tensor $\vec{b} \neq \vec{\Omega}$ (it is not the null tensor), their components $b^{\ell\circ\circ}_{\circ mn} \neq 0$, which forces the null factor to be the bracketed term in (5.49)

$$s(i,j,k,d)\delta^{j\circ}_{\circ\ell}\delta^{\circ n}_{k\circ} = s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}c^{\beta\circ}_{\circ\ell}\gamma^{\circ n}_{\gamma\circ}c^{\circ\delta}_{d\circ}. \tag{5.50}$$

Next, we isolate the factor $s(i,j,k,d)$ on the left-hand side of (5.50). To this end, we multiply both members by the Kronecker delta $\delta^{\ell\circ}_{\circ j}$, inverse of $\delta^{j\circ}_{\circ\ell}$:

$$s(i,j,k,d)(\delta^{\ell\circ}_{\circ j}\delta^{j\circ}_{\circ\ell})\delta^{\circ n}_{k\circ} = s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}(\delta^{\ell\circ}_{\circ j}c^{\beta\circ}_{\circ\ell})\gamma^{\circ n}_{\gamma\circ}c^{\circ\delta}_{d\circ}$$

or

$$s(i,j,k,d)(1)\delta^{\circ n}_{k\circ} = s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}(c^{\circ\beta}_{\ell\circ}\delta^{\ell\circ}_{\circ j})\gamma^{\circ n}_{\gamma\circ}c^{\circ\delta}_{d\circ}$$

contracting the grouped product, and multiplying both members by $\delta^{\circ k}_{n\circ}$, the inverse of $\delta^{\circ n}_{k\circ}$, we get

$$s(i,j,k,d)(\delta^{\circ k}_{n\circ}\delta^{\circ n}_{k\circ}) = s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}c^{\circ\beta}_{\ell\circ}(\delta^{\circ k}_{n\circ}\gamma^{\circ n}_{\gamma\circ})c^{\circ\delta}_{d\circ}$$

or

$$s(i,j,k,d)(1) = s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}c^{\circ\beta}_{\ell\circ}(\gamma^{n\circ}_{\circ\gamma}\delta^{\circ k}_{n\circ})c^{\circ\delta}_{d\circ},$$

and contracting the grouped product, we finally get

$$s(i,j,k,d) = s(\alpha,\beta,\gamma,\delta)\gamma^{i\circ}_{\circ\alpha}c^{\circ\beta}_{\ell\circ}\gamma^{k\circ}_{\circ\gamma}c^{\circ\delta}_{d\circ}. \tag{5.51}$$

This last expression indicates that the set of scalars $s(\alpha,\beta,\gamma,\delta)$ is a tensor, since it satisfies a concrete tensor criterion. In addition, it shows us *its whole nature*. In reality it is

$$s(\alpha,\beta,\gamma,\delta) = s^{\alpha\circ\gamma\circ}_{\circ\beta\circ\delta}. \tag{5.52}$$

The theorem that has been proved is called the "quotient law", a disputed title, that some impute to a simple conception of this relation among tensors, such as

If $X\cdot T_1 = T_2 \rightarrow X = \frac{T_2}{T_1} \rightarrow X = T_2\cdot T_1^{-1}$, which is certainly simple, and at least justifies its name.

This theorem, which is frequently used in solving tensor analysis theoretical problems and also in practical exercises, to detect whether or not a system of scalars is a tensor, has severe limitations that it is convenient to point out.

On one hand, one must be lucky when choosing the test tensor because, if after an unfortunate selection, the contraction does not lead to a tensor, no conclusion can be drawn, because of the sufficient character of the theorem. So, another test tensor must be selected and so on.

On the other hand, frequently, after the contraction is performed with the selected "test" tensor, we have great difficulties in proving that the result is *another* tensor, arriving at a new problem that can be even more complex than the initial one.

Consequently, the most frequent applications of the "quotient law" are those in which the contracted product is an *invariant*, which it is well known to be a zero-order tensor.

5.7 Matrix representation of permutation homomorphisms

We say that a tensor is the "permutation tensor of a given tensor" if it has the same associated scalars as the given tensor, but in different *positions*; one possibility of building a permutation tensor of a given tensor is to create with a different name a tensor with at least a changed index but with the same scalars:

$$\forall t^{\alpha\circ\gamma\circ}_{\circ\beta\circ\delta} \equiv u^{\alpha'\circ\gamma'\circ}_{\circ\beta'\circ\delta'},$$

where $(\alpha', \beta', \gamma', \delta')$ is one of the possible permutations of $(\alpha, \beta, \gamma, \delta)$.

Consider the linear space $K^\sigma, \sigma = n^r$, i.e., the linear space of matrices $T_{\sigma,1} \in K^\sigma$, "extensions" of the homogeneous tensors of a generic type $t^{\alpha_1\circ\alpha_3}_{\circ\alpha_2\circ}\cdots{}^{\circ}_{\alpha_r}$, defined over the "factor" linear space $V^n(K)$. We will study the permutation homomorphisms $P : K^\sigma \to K^\sigma$, the associated square matrix of which, P_{n^r}, is a permutation of the unit matrix I_{n^r} and which transforms by means of the following matrix equation:

$$P_{n^r} \bullet T_{\sigma,1} = T'_{\sigma,1}. \tag{5.53}$$

These transformations maintain the tensor dimension σ, together with its scalars, though obviously they change them in position. We will study two different types of homomorphisms P.

5.7.1 Permutation matrix tensor product types in K^n

Consider the tensor

$$T = \left[t^{\alpha\beta}_{\circ\circ}\right] = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ m & n & p & q \\ r & s & t & u \end{bmatrix}$$

and the tensor

$$T' = \left[u^{\gamma\delta}_{\circ\circ}\right] = \begin{bmatrix} d & c & b & a \\ q & p & n & m \\ h & g & f & e \\ u & t & s & r \end{bmatrix},$$

which obviously is a permutation of T, where $\sigma = 4^2 = 16$.

We build the corresponding matrix extensions of T and T' ($T_{\sigma,1} = T_{16,1}$ and $T'_{\sigma,1} = T'_{16,1}$), and we observe that the permutation matrix that relates both is

$$T'_{16,1} = P \cdot T_{16,1},$$

where

$$T'_{16,1} = \begin{bmatrix} d \\ c \\ b \\ a \\ q \\ p \\ n \\ m \\ h \\ g \\ f \\ e \\ u \\ t \\ s \\ r \end{bmatrix}; \quad T_{16,1} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ m \\ n \\ p \\ q \\ r \\ s \\ t \\ u \end{bmatrix}$$

$$P \equiv P_{16,16} = \left[\begin{array}{c|c|c|c} \begin{smallmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{smallmatrix} & \Omega & \Omega & \Omega \\ \hline \Omega & \Omega & \begin{smallmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{smallmatrix} & \Omega \\ \hline \Omega & \begin{smallmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{smallmatrix} & \Omega & \Omega \\ \hline \Omega & \Omega & \Omega & \begin{smallmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{smallmatrix} \end{array}\right].$$

An analysis of P discovers that in this case

$$P = \begin{bmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{bmatrix} \otimes \begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix},$$

i.e., the permutation matrix is the tensor product of two permutation matrices that operate in the linear space K^4, which reveals that some P have this type of construction.

5.7.2 Linear span of precedent types

In Example 4.5 of Chapter 4, we considered the five permutation tensors U, V, W, R and S of a given tensor $T = [t^{\alpha\circ\gamma}_{\circ\beta\circ}]$ of third order ($r = 3, n = 3$ and $\sigma = n^r = 27$). We will examine what type of construction has the permutation homomorphism matrix that applies $P_{(1)} : [t^{\alpha\circ\gamma}_{\circ\beta\circ}] \to [u^{\gamma\alpha\circ}_{\circ\circ\beta}]$, that is, $U_{27,1} = P_{(1)} \bullet T_{27,1}$. The solution matrices in this case are (see Example 4.5)

$$T_{27,1} = \left[\begin{array}{r} 1\\ 0\\ -1\\ 2\\ 3\\ 0\\ -1\\ 2\\ 0\\ \hline 2\\ -1\\ 1\\ 0\\ 1\\ 0\\ 2\\ 0\\ 1\\ \hline 0\\ 0\\ 1\\ 5\\ 1\\ 2\\ 1\\ 0\\ 0 \end{array}\right]; \quad U_{27,1} = \left[\begin{array}{r} 1\\ 2\\ -1\\ 2\\ 0\\ 2\\ 0\\ 5\\ 1\\ \hline 0\\ 3\\ 2\\ -1\\ 1\\ 0\\ 0\\ 1\\ 0\\ \hline -1\\ 0\\ 0\\ 1\\ 0\\ 1\\ 1\\ 2\\ 0 \end{array}\right]; \quad \text{where } \beta = \{E_i\} \equiv \left\{ \begin{bmatrix} 1\\0\\0 \end{bmatrix} \begin{bmatrix} 0\\1\\0 \end{bmatrix} \begin{bmatrix} 0\\0\\1 \end{bmatrix} \right\}.$$

We have that

$$P_{(1)} \equiv P_{27} = E_1 \otimes I_3 \otimes I_3 \otimes E_1^t + E_2 \otimes I_3 \otimes I_3 \otimes E_2^t + E_3 \otimes I_3 \otimes I_3 \otimes E_3^t, \quad (5.54)$$

that is, a matrix written as a linear combination of tensor products.

With respect to the permutation tensor V:

$$P_{(2)} : [t^{\alpha\circ\gamma}_{\circ\beta\circ}] \to [v^{\circ\gamma\alpha}_{\beta\circ\circ}],$$

that is,

$$V_{27,1} = P_{(2)} \bullet T_{27,1}.$$

The solution matrices are in this case, the $T_{27,1}$ matrix previously cited, and matrices

$$V_{27,1} = \begin{bmatrix} 1\\ 2\\ 0\\ 0\\ -1\\ 0\\ -1\\ 1\\ 1\\ --\\ 2\\ 0\\ 5\\ 3\\ 1\\ 1\\ 0\\ 0\\ 2\\ --\\ -1\\ 2\\ 1\\ 2\\ 0\\ 0\\ 0\\ 1\\ 0 \end{bmatrix}$$

$$P_{(2)} \equiv P'_{27} = E_1^t \otimes I_3 \otimes I_3 \otimes E_1 + E_2^t \otimes I_3 \otimes I_3 \otimes E_2 + E_3^t \otimes I_3 \otimes I_3 \otimes E_3. \quad (5.55)$$

The permutation matrix $P_{(2)}$ is of the type $P_{(1)}$, that is, a linear combination of tensor products, and $P_{(2)}$ is $P_{(1)}^t$.

For the permutation tensor W:

$$P_{(3)} : [t^{\alpha\circ\gamma}_{\circ\beta\circ}] \to [w^{\alpha\gamma\circ}_{\circ\circ\beta}],$$

that is

$$W_{27,1} = P_{(3)} \bullet T_{27,1}$$

the solution is given by the matrices

$$W_{27,1} = \begin{bmatrix} 1\\ 2\\ -1\\ 0\\ 3\\ 2\\ -1\\ 0\\ 0\\ --\\ 2\\ 0\\ 2\\ -1\\ 1\\ 0\\ 1\\ 0\\ 1\\ --\\ 0\\ 5\\ 1\\ 0\\ 1\\ 0\\ 1\\ 2\\ 0 \end{bmatrix}$$

$$P_{(3)} = I_3 \otimes \left[E_1 \otimes I_3 \otimes E_1^t + E_2 \otimes I_3 \otimes E_2^t + E_3 \otimes I_3 \otimes E_3^t\right], \qquad (5.56)$$

also a linear combination of tensor products. $P_{(3)}$ is symmetric.

Next, we analyze the permutation tensor R:

$$P_{(4)} : [t^{\alpha\circ\gamma}_{\circ\beta\circ}] \to [r^{\gamma\circ\alpha}_{\circ\beta\circ}],$$

that is,

$$R_{27,1} = P_{(4)} \bullet T_{27,1},$$

the solution of which is

$$R_{27} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \\ 5 \\ -1 \\ 2 \\ 1 \\ -- \\ 0 \\ -1 \\ 0 \\ 3 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \\ -- \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \beta \equiv \text{ basis of } \mathbb{R}^3 \equiv \{E_1, E_2, E_3\} \equiv \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$P_{(4)} = \sum_{1\le i,j\le 3} (E_i \otimes E_j^t) \otimes I_3 \otimes (E_i^t \otimes E_j), \qquad (5.57)$$

also a sum of tensor products. $P_{(4)}$ is symmetric.

We arrive at the last permutation S of Example 4.5, i.e.,

$$P_{(5)} : [t^{\alpha\circ\gamma}_{\circ\beta\circ}] \to [s^{\circ\alpha\gamma}_{\beta\circ\circ}],$$

that is,

$$S_{27,1} = P_{(5)} \bullet T_{27,1},$$

with solution matrices

$$S_{27,1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -- \\ 2 \\ 3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 5 \\ 1 \\ 2 \\ -- \\ -1 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

with

$$P_{(5)} = \left[E_1 \otimes I_3 \otimes E_1^t + E_2 \otimes I_3 \otimes E_2^t + E_3 \otimes I_3 \otimes E_3^t\right] \otimes I_3 = P_{(5)}^t. \quad (5.58)$$

We end the section dedicated to permutation homomorphisms by citing the model of this type of matrices which will be called "transposer" since it operates over second-order homogeneous tensors, the square matrices $a^{\alpha\beta}_{\circ\circ}, b^{\alpha\circ}_{\circ\beta}, c^{\circ\beta}_{\alpha\circ}$ or $d^{\circ\circ}_{\alpha\beta}$, with $r = 2; n = n; \sigma = n^2$, *transposing them.*

Whence

$$H(a^{\alpha\beta}_{\circ\circ}) = a^{\beta\alpha}_{\circ\circ}; \quad H(b^{\alpha\circ}_{\circ\beta}) = b^{\circ\alpha}_{\beta\circ}, \ldots, \text{ etc.}$$

is the matrix called a "transposition matrix" in Section 1.3.7, Formula (1.38). Here we present a generalization, in its usual mode of permutation homomorphism:

$$T'_{\sigma,1} = P_{n^2} \bullet T_{\sigma,1},$$

where $T_{\sigma,1}$ is the extension matrix that is to be transposed.

The permutation "transposer" is the block matrix:

$$P_{n^2} = P = \begin{bmatrix} E_{11} & | & E_{21} & | & \cdots & | & E_{n1} \\ -- & + & -- & + & -- & + & -- \\ E_{12} & | & E_{22} & | & \cdots & | & E_{n2} \\ -- & + & -- & + & -- & + & -- \\ \cdots & | & \cdots & | & \cdots & | & \cdots \\ -- & + & -- & + & -- & + & -- \\ E_{1n} & | & E_{2n} & | & \cdots & | & E_{nn} \end{bmatrix}, \quad (5.59)$$

where $\mathcal{B} = \{E_{ij}\}$ is the canonical basis of the tensor space $K^{n\times n}$ of square matrices of order n (noting the block ordering inside P_{n^2})

The reader can test its effect using it in the exercises.

With respect to the permutation type "transposer", responds to the expression

$$P_{n^2} = \sum E_{ii} \otimes E_{ii} + \sum_{i<j;i,j\in\{1,2,\cdots,n\}} (E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij}). \tag{5.60}$$

Example 5.4 (Permutation homomorphisms). Consider the linear space $\tau^{27}(\mathbb{R})$ as a tensor product of $\mathbb{R}^3 \otimes \mathbb{R}^{*3} \otimes \mathbb{R}^3$. Let $T \in \tau$ be a tensor of components

$$t^{\alpha\circ\gamma}_{\circ\beta\circ} = \left[\begin{array}{rrr|rrr|rrr} 1 & 2 & 0 & 2 & 0 & 5 & -1 & 2 & 1 \\ 0 & -1 & 0 & 3 & 1 & 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right],$$

where α is the row, β is the column, and γ is the matrix.

Let $\widehat{\vec{e}}_1(-1,0,-1), \widehat{\vec{e}}_2(1,1,0), \widehat{\vec{e}}_3(0,0,3)$ be a change of the canonical basis of $\mathbb{R}^3$ that produces the corresponding change-of-basis of tensor nature in τ.

Determine the new components of tensor T, using the permutation homomorphisms, to execute the change-of-basis on the tensor ordered according to the axiom.

Solution: It is evident that the assigning of subindices in the statement does not correspond to the *axiomatic* order for the canonical basis of $\mathbb{R}^3 \otimes \mathbb{R}^{*3} \otimes \mathbb{R}^3$, which requires (see the theory and Example 2.1, question 4) that the matrix index (γ) must be the first and the column index (β) must be the last. So, before executing the change-of-basis, we must find the *fundamental* tensor $(t')^{\gamma\alpha\circ}_{\circ\circ\beta}$, which, subject to adequate permutation, provides the given data.

Tensor $(t')^{\gamma\alpha\circ}_{\circ\circ\beta}$ is the one that must be subject to the change-of-basis, given by the theory, and obviously the permutation must be undone in order to find the sought after tensor $t^{i\circ k}_{\circ j\circ}$.

Let $T'_{27,1}$, be the stretched version of $(t')^{\gamma\alpha\circ}_{\circ\circ\beta}$, and $T_{27,1}$, the stretched version of $t^{\alpha\circ\gamma}_{\circ\beta\circ}$ (data).

The permutation relation between them is

$$P_{(2)} \cdot T'_{27,1} = T_{27,1},$$

where $P_{(2)}$ (Formula (5.55)) is

$$\begin{aligned} P_{(2)} &= E_1^t \otimes I_9 \otimes E_1 + E_2^t \otimes I_9 \otimes E_2 + E_3^t \otimes I_9 \otimes E_3 \\ &= [\,1\ 0\ 0\,] \otimes I_9 \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + [\,0\ 1\ 0\,] \otimes I_9 \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + [\,0\ 0\ 1\,] \otimes I_9 \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Then, $P_{(2)}$ becomes

$$P_{(2)} = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
\begin{matrix}100\\000\\000\\010\\000\\000\\001\\000\\000\end{matrix} & \Omega & \Omega & \begin{matrix}000\\100\\000\\000\\010\\000\\000\\001\\000\end{matrix} & \Omega & \Omega & \begin{matrix}000\\000\\100\\000\\000\\010\\000\\000\\001\end{matrix} & \Omega & \Omega \\
\hline
\Omega & \begin{matrix}100\\000\\000\\010\\000\\000\\001\\000\\000\end{matrix} & \Omega & \Omega & \begin{matrix}000\\100\\000\\000\\010\\000\\000\\001\\000\end{matrix} & \Omega & \Omega & \begin{matrix}000\\000\\100\\000\\000\\010\\000\\000\\001\end{matrix} & \Omega \\
\hline
\Omega & \Omega & \begin{matrix}100\\000\\000\\010\\000\\000\\001\\000\\000\end{matrix} & \Omega & \Omega & \begin{matrix}000\\100\\000\\000\\010\\000\\000\\001\\000\end{matrix} & \Omega & \Omega & \begin{matrix}000\\000\\100\\000\\000\\010\\000\\000\\001\end{matrix}
\end{array}\right].$$

Since $P_{(2)}$ is orthogonal, $P_{(2)}^{-1} \equiv P_{(2)}^t$,

$$P_{(2)}^{-1} = \left[\begin{array}{c|c|c}
\begin{matrix}100000000\\000100000\\000000100\end{matrix} & \Omega & \Omega \\
\hline
\Omega & \begin{matrix}100000000\\000100000\\000000100\end{matrix} & \Omega \\
\hline
\Omega & \Omega & \begin{matrix}100000000\\000100000\\000000100\end{matrix} \\
\hline
\begin{matrix}010000000\\000010000\\000000010\end{matrix} & \Omega & \Omega \\
\hline
\Omega & \begin{matrix}010000000\\000010000\\000000010\end{matrix} & \Omega \\
\hline
\Omega & \Omega & \begin{matrix}010000000\\000010000\\000000010\end{matrix} \\
\hline
\begin{matrix}001000000\\000001000\\000000001\end{matrix} & \Omega & \Omega \\
\hline
\Omega & \begin{matrix}001000000\\000001000\\000000001\end{matrix} & \Omega \\
\hline
\Omega & \Omega & \begin{matrix}001000000\\000001000\\000000001\end{matrix} \\
\hline
\end{array}\right].$$

Returning to the initial permutation relations, we get

$$T'_{27,1} = P_{(2)}^{-1} \cdot T_{27,1} = P_{(2)}^{-1} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \\ -- \\ 2 \\ 0 \\ 5 \\ 3 \\ 1 \\ 1 \\ 0 \\ 0 \\ 2 \\ -- \\ -1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 3 \\ 0 \\ -1 \\ 2 \\ 0 \\ -- \\ 2 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ -- \\ 0 \\ 0 \\ 1 \\ 5 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since $T'_{27,1}$ is the stretched version of the fundamental tensor $(t')^{\gamma\alpha\circ}_{\circ\circ\beta}$, according to the Formula (4.36) the corresponding is $\widehat{T}'_{27,1}$, the stretched version of $(t')^{ki\circ}_{\circ\circ j}$, with expression

$$\widehat{T}'_{27,1} = Z^{-1} \cdot T'_{27,1}; \text{ with } Z^{-1} = C^{-1} \otimes C^{-1} \otimes C^t.$$

In our case is

$$C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}; \quad C^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}; \quad C^t = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then, the matrix associated with the indicated change-of-basis is

$$Z^{-1} = (C^{-1} \otimes C^{-1}) \otimes C^t = \left[\begin{array}{ccc|ccc|ccc} 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & -1/3 & -1/3 & -1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ \hline 1/3 & -1/3 & 0 & -1/3 & 1/3 & 0 & -1/3 & 1/3 & 0 \\ 0 & -1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/9 & -1/9 & -1/9 & -1/9 & 1/9 & 1/9 & -1/9 & 1/9 & 1/9 \end{array}\right] \otimes C^t$$

$$
Z^{-1} = \left[\begin{array}{ccccccccc|ccccccccc|ccccccccc}
-1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & & & & & & & & & \\
1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 3 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & & & & & \Omega & & & & \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & & & & & & & & & \\
-\frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & & & & & & & & & \\
\frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & & & & & & & & & \\
\hline
& & & & & & & & & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & & & & & & & & & \\
& & & & & & & & & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & & & & & & & & & \\
& & & & & & & & & 0 & 0 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & & & & & & & & & \\
& & & & & & & & & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & & & & & & & & & \\
& & & & \Omega & & & & & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & & & & & \Omega & & & & \\
& & & & & & & & & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & & & & & & & & & \\
& & & & & & & & & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & & & & & & & & & \\
& & & & & & & & & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & & & & & & & & & \\
& & & & & & & & & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & & & & & & & & & \\
\hline
-\frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{1}{9} & 0 & -\frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & -\frac{1}{9} & 0 & -\frac{1}{9} & -\frac{1}{9} & 0 & -\frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & -\frac{1}{9} & 0 & -\frac{1}{9} & -\frac{1}{9} & 0 & -\frac{1}{9} \\
\frac{1}{9} & \frac{1}{9} & 0 & -\frac{1}{9} & -\frac{1}{9} & 0 & -\frac{1}{9} & -\frac{1}{9} & 0 & -\frac{1}{9} & -\frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & -\frac{1}{9} & -\frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3}
\end{array}\right]
$$

$$
\hat{T}'_{27,1} = Z^{-1} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 3 \\ 0 \\ -1 \\ 2 \\ 0 \\ \hline 2 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 1 \\ 5 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -6 \\ 2 \\ -4 \\ 0 \\ 1/3 \\ -1 \\ -1 \\ \hline 3 \\ 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2/3 \\ 0 \\ \hline -1/3 \\ 2/3 \\ -1 \\ -5/3 \\ 2/3 \\ 2 \\ -2/3 \\ 4/9 \\ 0 \end{bmatrix}.
$$

Once the change-of-basis has been performed, we must return to the data permutation:

$$P_{(2)} \cdot \hat{T}'_{27,1} = \hat{T}_{27,1}$$

yielding

$$\hat{T}_{27} = \begin{bmatrix} 5 \\ 3 \\ -1/3 \\ -4 \\ 0 \\ 2/3 \\ -6 \\ -3 \\ -1 \\ -- \\ 2 \\ 0 \\ -5/3 \\ -4 \\ 1 \\ 2/3 \\ 0 \\ 0 \\ 2 \\ -- \\ 1/3 \\ 0 \\ -2/3 \\ -1 \\ 2/3 \\ 4/9 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

which after its condensation leads to

$$t^{i\circ k}_{\circ j \circ} = \left[\begin{array}{rrr|rrr|rrr} 5 & 3 & -1/3 & 2 & 0 & -5/3 & 1/3 & 0 & -2/3 \\ -4 & 0 & 2/3 & -4 & 1 & 2/3 & -1 & 2/3 & 4/9 \\ -6 & -3 & -1 & 0 & 0 & 2 & -1 & 0 & 0 \end{array}\right],$$

where i is the row index, j is the column index, and k is the matrix index. □

5.7.3 The isomers of a tensor

We give the name "isomers" to certain tensors that come from permutations of a given tensor; they are the isomeric tensors of such a tensor.

For pure tensors (totally contravariant or covariant) the permutation of a partial number (or all) of its indices, strictly between them, leads to an isomer.

In other words, not all permutation tensors coming from a *pure* tensor are isomers of such a tensor, since some of them do not come from altering the indices. If the tensor is a mixed tensor, they are the tensors coming from permuting partially or totally: (a) only the contravariant indices among them, without altering the covariant indices, and (b) only the covariant indices among them, without altering the contravariant indices.

In Example 4.5, which was examined in Section 5.7.2, the tensor $R = [r^{\gamma\circ\alpha}_{\circ\beta\circ}]$ is an isomer of tensor $T = [t^{\alpha\circ\gamma}_{\circ\beta\circ}]$. Similarly, the tensor $U = [u^{\gamma\alpha\circ}_{\circ\circ\beta}]$ is an isomer of tensor $W = [w^{\alpha\gamma\circ}_{\circ\circ\beta}]$.

Example 5.5 (Rotation tensor). Consider a given pure tensor (totally contravariant or covariant) of order r: $t^{\alpha\beta\gamma\delta\cdots\rho}_{\circ\circ\circ\circ\cdots\circ}$. We define as the "rotation tensor" of the given tensor any of its isomers that *do not* maintain indices in the same positions as the initial one.

We will denote such a tensor $(t^{\alpha\beta\gamma\delta\cdots\rho}_{\circ\circ\circ\circ\cdots\circ})^{R(k)}$ where $k \in Z; k \neq 0; |k| < r$ is the "rotation index".

By extension, we define as the rotation tensor of a given mixed tensor all those isomers that do not maintain dummy indices of the *same valency*, in the same positions.

These rotation tensors carry the notation $(t^{\alpha\circ\gamma\delta\circ\cdots\rho}_{\circ\beta\circ\circ\epsilon\cdots\circ})^{R(k,k')}$ with two index-parameters k, k', where $k, k' \in Z; k, k' \neq 0; |k| < p; |k'| < q$ with $p + q = r$, where p and q are the contravariant and covariant orders of the given tensor, respectively.

1. Determine the rotation tensor associated with a tensor of order $(r = 2)$ over the linear space $V^2(\mathbb{R})$, and do the same over the linear space $V^3(\mathbb{R})$.

2. Determine the rotation tensors associated with a tensor of order $(r = 3)$ over the linear space $V^2(\mathbb{R})$, and do the same over the linear space $V^3(\mathbb{R})$.

3. Determine the rotation tensors associated with a tensor of order $(r = 4)$ over the linear space $V^2(\mathbb{R})$.

Solution:

1. Case $r = 2, n = 2$. Let $\vec{t} = t^{\circ\circ}_{\alpha\beta}\vec{e}^{*\alpha} \otimes \vec{e}^{*\beta} \Rightarrow (t^{\circ\circ}_{\alpha\beta})^{R(1)} = t^{\circ\circ}_{\beta\alpha}\vec{e}^{*\alpha} \otimes \vec{e}^{*\beta}$.

   Then, $[t^{\circ\circ}_{\alpha\beta}] = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \Rightarrow [t^{\circ\circ}_{\alpha\beta}]^{R(1)} = \begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix}$, which is known as the "transposed" matrix of the given matrix.

   Case $r = 2, n = 3$. In this case we have

   $$[t^{\circ\circ}_{\alpha\beta}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{bmatrix} \Rightarrow [t^{\circ\circ}_{\alpha\beta}]^{R(1)} = \begin{bmatrix} a_1 & d_1 & g_1 \\ b_1 & e_1 & h_1 \\ c_1 & f_1 & i_1 \end{bmatrix}$$

   which also is the transposed matrix of the given matrix.

2. Case $r = 3, n = 2$. Let $\vec{t} = t^{\circ\circ\circ}_{\alpha\beta\gamma}\vec{e}^{*\alpha}\otimes\vec{e}^{*\beta}\otimes\vec{e}^{*\gamma}$. Since there are *two* rotations $\beta\gamma\alpha$ and $\gamma\alpha\beta$ we have $[t^{\circ\circ\circ}_{\alpha\beta\gamma}]^{R(1)}$ and $[t^{\circ\circ\circ}_{\alpha\beta\gamma}]^{R(2)}$, or, if one prefers the notation $[t^{\circ\circ\circ}_{\alpha\beta\gamma}]^{R(1)}$ and $[t^{\circ\circ\circ}_{\alpha\beta\gamma}]^{R(-1)}$, the "first rotation" and its opposite.

Let $[t^{\circ\circ\circ}_{\alpha\beta\gamma}] = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \\ - & - \\ a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be the data tensor, with α the row submatrix index, β the row index of each submatrix, and γ the column index of each submatrix (axiomatic order).
The correspondences in both rotations are

	First rotation	Second rotation
Initial	Transformed	Transformed
$t^{\circ\circ\circ}_{\alpha\beta\gamma}$	$t^{\circ\circ\circ}_{\beta\gamma\alpha}$	$t^{\circ\circ\circ}_{\gamma\alpha\beta}$
$t^{\circ\circ\circ}_{112}$	$t^{\circ\circ\circ}_{121}$	$t^{\circ\circ\circ}_{211}$
$t^{\circ\circ\circ}_{121}$	$t^{\circ\circ\circ}_{211}$	$t^{\circ\circ\circ}_{112}$
$t^{\circ\circ\circ}_{122}$	$t^{\circ\circ\circ}_{221}$	$t^{\circ\circ\circ}_{212}$
$t^{\circ\circ\circ}_{211}$	$t^{\circ\circ\circ}_{112}$	$t^{\circ\circ\circ}_{121}$
$t^{\circ\circ\circ}_{212}$	$t^{\circ\circ\circ}_{122}$	$t^{\circ\circ\circ}_{221}$
$t^{\circ\circ\circ}_{221}$	$t^{\circ\circ\circ}_{212}$	$t^{\circ\circ\circ}_{122}$

and then

$$[t^{\circ\circ\circ}_{\alpha\beta\gamma}]^{R(1)} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ - & - \\ c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}$$

and

$$[t^{\circ\circ\circ}_{\alpha\beta\gamma}]^{R(2)} = \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \\ - & - \\ b_1 & d_1 \\ b_2 & d_2 \end{bmatrix},$$

which are the "transposed" (beware of the word) tensors of tensors $(r = 3, n = 2)$.
Case $r = 3, n = 3$. Let

$$[t^{\circ\circ\circ}_{\alpha\beta\gamma}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \\ - & - & - \\ a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \\ - & - & - \\ a_3 & b_3 & c_3 \\ d_3 & e_3 & f_3 \\ g_3 & h_3 & i_3 \end{bmatrix}$$

be the data tensor. In this case we have

$$[t_{\alpha\beta\gamma}^{\circ\circ\circ}]^{R(1)} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ - & - & - \\ d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ - & - & - \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \\ i_1 & i_2 & i_3 \end{bmatrix}$$

and

$$[t_{\alpha\beta\gamma}^{\circ\circ\circ}]^{R(2)} = \begin{bmatrix} a_1 & d_1 & g_1 \\ a_2 & d_2 & g_2 \\ a_3 & d_3 & g_3 \\ - & - & - \\ b_1 & e_1 & h_1 \\ b_2 & e_2 & h_2 \\ b_3 & e_3 & h_3 \\ - & - & - \\ c_1 & f_1 & i_1 \\ c_2 & f_2 & i_2 \\ c_3 & f_3 & i_3 \end{bmatrix}.$$

3. Case $r = 4, n = 2$. The data tensor is $\vec{t} = t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ}\vec{e}^{*\alpha} \otimes \vec{e}^{*\beta} \otimes \vec{e}^{*\gamma} \otimes \vec{e}^{*\delta}$, where

$$[t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ}] = \left[\begin{array}{cc|cc} a_1 & b_1 & a_2 & b_2 \\ c_1 & d_1 & c_2 & d_2 \\ \hline a_3 & b_3 & a_4 & b_4 \\ c_3 & d_3 & c_4 & d_4 \end{array}\right].$$

In this case $\alpha\beta\gamma\delta$ has the three rotations: $\beta\gamma\delta\alpha$, $\gamma\delta\alpha\beta$ and $\delta\alpha\beta\gamma$. Then, for the rotation $R(1)$, since $t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ} \to t_{\beta\gamma\delta\alpha}^{\circ\circ\circ\circ}$, we have

$$[t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ}]^{R(1)} = \left[\begin{array}{cc|cc} a_1 & a_3 & c_1 & c_3 \\ b_1 & b_3 & d_1 & d_3 \\ \hline a_2 & a_4 & c_2 & c_4 \\ b_2 & b_4 & d_2 & d_4 \end{array}\right].$$

For the rotation $R(2)$, since $t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ} \to t_{\gamma\delta\alpha\beta}^{\circ\circ\circ\circ}$, we have

$$[t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ}]^{R(2)} = \left[\begin{array}{cc|cc} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ \hline c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{array}\right].$$

and, finally, for the rotation $R(3)$, since $t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ} \to t_{\delta\alpha\beta\gamma}^{\circ\circ\circ\circ}$, we obtain

$$[t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ}]^{R(3)} = \left[\begin{array}{cc|cc} a_1 & c_1 & a_3 & c_3 \\ a_2 & c_2 & a_4 & c_4 \\ \hline b_1 & d_1 & b_3 & d_3 \\ b_2 & d_2 & b_4 & d_4 \end{array}\right].$$

□

5.8 Matrices associated with simply contraction homomorphisms

We give the name "simply contraction homomorphism" to a homomorphism that operates according to the equation

$$T'_{\sigma'} = H \bullet T_{\sigma,1} \tag{5.61}$$

and that apply the mixed tensor (because the contraction is assumed to be of indices of different valency), into another tensor of *smaller* dimension $\sigma' < \sigma$, by a tensor contraction of *two* indices.

We will construct the matrices H for the usual cases, that is, for tensors of orders 2, 3, 4 and 5.

5.8.1 Mixed tensors of second order ($r = 2$): Matrices.

This is the case of $T = [t^{\circ\,\beta}_{\alpha\,\circ}]$ or $T = [t^{\alpha\,\circ}_{\circ\,\beta}]$, with $r = 2; \sigma = n^2; n = \dim V^n(K); \sigma' = n^0 = 0$.

Thus, the result of the contraction is a scalar, which is called the "matrix trace".

Assuming that $\{E_{ij}\}$ is the canonical basis of the matrices of order n, the fundamental equation (5.61) is in this case

$$\begin{aligned}\rho &= H_{1,n^2}(\alpha,\beta) \bullet T_{\sigma,1} = ([1\ 1\ \cdots\ 1]_{1,n} \bullet [\,E_{11} \mid E_{22} \mid \cdots \mid E_{nn}\,]) \bullet T_{\sigma,1} \\ &= \left[\, E_1^t \mid E_2^t \mid \cdots \mid E_n^t \,\right] \bullet T_{\sigma,1}; \rho \in K; T'_{\sigma'} = \rho.\end{aligned} \tag{5.62}$$

The notation of H declares its number of rows and columns, together with the indices to be contracted.

$\mathcal{B} = \{E_i\}$ is the canonical basis of the linear space $V^n(K)$:

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix} \quad \cdots \quad \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} \right\}.$$

5.8.2 Mixed tensors of third order ($r = 3$)

These are tensors of the type $T = t^{\alpha\,\circ\,\circ}_{\circ\,\beta\,\gamma}$; $T = t^{\alpha\,\circ\,\gamma}_{\circ\,\beta\,\circ} \cdots$, etc. with $dim\ V^n(K) = n; r = 3; \sigma = n^3; \sigma' = n$.

There are three possible models:

Model 1. $T = t^{\alpha\,\beta\,\circ}_{\circ\,\circ\,\gamma}$ or $T = t^{\alpha\,\circ\,\gamma}_{\circ\,\beta\,\circ}$

$$T'_{\sigma'} = u^\alpha = H_{n,n^3}(\beta,\gamma) \bullet T_{\sigma,1} = (I_n \otimes ([1\,1 \cdots 1]_{1,n} \bullet [E_{11} \,|\, E_{22} \,|\cdots|\, E_{nn}])) \bullet T_{\sigma,1}. \tag{5.63}$$

Model 2. $T = t^{\alpha\circ\gamma}_{\circ\beta\circ}$ or $T = t^{\circ\beta\gamma}_{\alpha\circ\circ}$

$$T'_{\sigma'} = v^{\gamma} = H_{n,n^3}(\alpha,\beta) \bullet T_{\sigma,1} = (([1\,1\cdots1]_{1,n} \bullet [E_{11}\,|\,E_{22}\,|\cdots|\,E_{nn}]) \otimes I_n) \bullet T_{\sigma,1}. \tag{5.64}$$

Model 3. $T = t^{\alpha\beta\circ}_{\circ\circ\gamma}$ or $T = t^{\circ\beta\gamma}_{\alpha\circ\circ}$

$$\begin{aligned} T'_{\sigma'} &= z^{\beta} = H_{n,n^3}(\alpha,\gamma) \bullet T_{\sigma,1} \\ &= [I_n \otimes ([1\ 1\ \cdots\ 1]_{1,n} \bullet E_{11}) | I_n \otimes ([1\ 1\ \cdots\ 1]_{1,n} \\ &\quad \bullet E_{22}) | \dots | I_n \otimes ([1\ 1\ \cdots\ 1]_{1,n} \bullet E_{nn})] \bullet T_{\sigma,1}. \end{aligned} \tag{5.65}$$

Formulas (5.63) to (5.65) can be written in a simpler form:

$$u^{\alpha} = H_{n,n^3}(\beta,\gamma) \bullet T_{\sigma,1} = \left(I_n \otimes [\,E_1^t \mid E_2^t \mid \cdots \mid E_n^t\,]\right) \bullet T_{\sigma,1} \tag{5.66}$$

$$v^{\gamma} = H_{n,n^3}(\alpha,\beta) \bullet T_{\sigma,1} = \left([\,E_1^t \mid E_2^t \mid \cdots \mid E_n^t\,] \otimes I_n\right) \bullet T_{\sigma,1} \tag{5.67}$$

$$z^{\beta} = H_{n,n^3}(\alpha,\gamma) \bullet T_{\sigma,1} = \left[\,I_n \otimes E_1^t |\ I_n \otimes E_2^t | \cdots | I_n \otimes E_n^t\,\right] \bullet T_{\sigma,1}. \tag{5.68}$$

5.8.3 Mixed tensors of fourth order ($r = 4$)

These are tensors of the type

$$T = t^{\alpha\circ\gamma\delta}_{\circ\beta\circ\circ}; \quad T = t^{\alpha\beta\gamma\circ}_{\circ\circ\circ\delta}, \cdots \text{ etc. }; \sigma \equiv n^r = n^4; \sigma' \equiv n^{r-2} = n^{4-2} = n^2.$$

Possibilities for the contraction: $\binom{4}{2} = 6$ models.

Model 1. $T = t^{\alpha\circ\gamma\delta}_{\circ\beta\circ\circ}$; The fundamental equation (5.61) in this case is

$$T'_{\sigma'} = H_{n^2,n^4}(\alpha,\beta) \bullet T_{\sigma,1} = \left([\,E_1^t \mid E_2^t \mid \cdots \mid E_n^t\,] \otimes I_n \otimes I_n\right) \bullet T_{\sigma,1} \tag{5.69}$$

Model 2. $T = t^{\alpha\beta\circ\delta}_{\circ\circ\gamma\circ}$

$$T'_{\sigma'} = H_{n^2,n^4}(\alpha,\gamma) \bullet T_{\sigma,1} = \left([\,I_n \otimes E_1^t \mid I_n \otimes E_2^t \mid \cdots \mid I_n \otimes E_n^t\,] \otimes I_n\right) \bullet T_{\sigma,1}. \tag{5.70}$$

Model 3. $T = t^{\alpha\beta\gamma\circ}_{\circ\circ\circ\delta}$

$$T'_{\sigma'} = H_{n^2,n^4}(\alpha,\delta) \bullet T_{\sigma,1} = [I_n \otimes I_n \otimes E_1^t \,|\, I_n \otimes I_n \otimes E_2^t \,|\cdots|\, I_n \otimes I_n \otimes E_n^t] \bullet T_{\sigma,1}. \tag{5.71}$$

Model 4. $T = t^{\alpha\beta\circ\delta}_{\circ\circ\gamma\circ}$

$$T'_{\sigma'} = H_{n^2,n^4}(\beta,\gamma) \bullet T_{\sigma,1} = \left(I_n \otimes [\,E_1^t \mid E_2^t \mid \cdots \mid E_n^t\,] \otimes I_n)\right) \bullet T_{\sigma,1}. \tag{5.72}$$

Model 5. $T = t^{\alpha\beta\gamma\circ}_{\circ\circ\circ\delta}$

$$T'_{\sigma'} = H_{n^2,n^4}(\beta,\delta) \bullet T_{\sigma,1} = \left(I_n \otimes [\,I_n \otimes E_1^t \mid I_n \otimes E_2^t \mid \cdots \mid I_n \otimes E_n^t\,]\right) \bullet T_{\sigma,1}. \tag{5.73}$$

Model 6. $T = t^{\alpha\beta\gamma\circ}_{\circ\circ\circ\delta}$

$$T'_{\sigma'} = H_{n^2,n^4}(\gamma,\delta) \bullet T_{\sigma,1} = \left(I_n \otimes I_n \otimes [\,E_1^t \mid E_2^t \mid \cdots \mid E_n^t\,]\right) \bullet T_{\sigma,1}. \tag{5.74}$$

$T'_{\sigma',1}$ must be given in "condensed" form (as a square matrix).

5.8.4 Mixed tensors of fifth order ($r = 5$)

We present rules for the sequence of formation of matrices representing contractions in tensors of order 5 ($r = 5$) associated with linear spaces $\mathbb{R}^n$ of basis

$$\{E_i\} \equiv \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}.$$

We notate the morphism matrix using power indices and parentheses that declare the indices to be contracted.

Contractions of two indices, resulting tensors of order $r = 3$.

Tensor dimensions of the "stretched" tensors $T_{\sigma,1}$ and $T_{\sigma'}$: $\sigma = n^5$; $\sigma' = n^3$.

There exist $\binom{5}{2} = \frac{5\times 4}{2} = 10$ models. Operation: $T'_{\sigma',1} = H_{\sigma',\sigma} \bullet T_{\sigma,1}$

Model 1. $t^{\alpha\circ\gamma\delta\epsilon}_{\circ\beta\circ\circ\circ}$

$$H_{\sigma',\sigma} \equiv H_{n^3,n^5}(\alpha,\beta) \equiv \left[E_1^t \mid E_2^t \mid \cdots \mid E_n^t \right] \otimes I_n \otimes I_n \otimes I_n.$$

Model 2. $t^{\alpha\beta\circ\delta\epsilon}_{\circ\circ\gamma\circ\circ}$

$$H_{\sigma',\sigma} \equiv H_{n^3,n^5}(\alpha,\gamma) \equiv [I_n \otimes E_1^t \,|\, I_n \otimes E_2^t \,|\cdots|\, I_n \otimes E_n^t] \otimes I_n \otimes I_n.$$

Model 3. $t^{\alpha\beta\gamma\circ\epsilon}_{\circ\circ\circ\delta\circ}$

$$\begin{aligned} H_{\sigma',\sigma} &\equiv H_{n^3,n^5}(\alpha,\delta) \\ &\equiv [I_n \otimes I_n \otimes E_1^t \,|\, I_n \otimes I_n \otimes E_2^t \,|\cdots|\, I_n \otimes I_n \otimes E_n^t] \otimes I_n. \end{aligned}$$

Model 4. $t^{\alpha\beta\gamma\delta\circ}_{\circ\circ\circ\circ\epsilon}$

$$\begin{aligned} H_{\sigma',\sigma} &\equiv H_{n^3,n^5}(\alpha,\epsilon) \\ &\equiv [I_n \otimes I_n \otimes I_n \otimes E_1^t \,|\, I_n \otimes I_n \otimes I_n \otimes E_2^t \,|\cdots|\, I_n \otimes I_n \otimes I_n \otimes E_n^t] \,. \end{aligned}$$

Model 5. $t^{\alpha\beta\circ\delta\epsilon}_{\circ\circ\gamma\circ\circ}$

$$H_{\sigma',\sigma} \equiv H_{n^3,n^5}(\beta,\gamma) \equiv I_n \otimes [E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t] \otimes I_n \otimes I_n.$$

Model 6. $t^{\alpha\beta\gamma\circ\epsilon}_{\circ\circ\circ\delta\circ}$

$$H_{\sigma',\sigma} \equiv H_{n^3,n^5}(\beta,\delta) \equiv I_n \otimes [I_n \otimes E_1^t \,|\, I_n \otimes E_2^t \,|\cdots|\, I_n \otimes E_n^t] \otimes I_n.$$

Model 7. $t^{\alpha\beta\gamma\delta\circ}_{\circ\circ\circ\circ\epsilon}$

$$\begin{aligned} H_{\sigma',\sigma} &\equiv H_{n^3,n^5}(\beta,\epsilon) \\ &\equiv I_n \otimes [I_n \otimes I_n \otimes E_1^t \,|\, I_n \otimes I_n \otimes E_2^t \,|\cdots|\, I_n \otimes I_n \otimes E_n^t]\,. \end{aligned}$$

Model 8. $t^{\alpha\beta\gamma\circ\epsilon}_{\circ\circ\circ\delta\circ}$

$$H_{\sigma',\sigma} \equiv H_{n^3,n^5}(\gamma,\delta) \equiv I_n \otimes I_n \otimes \left[E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t\right] \otimes I_n.$$

Model 9. $t^{\alpha\beta\gamma\delta\circ}_{\circ\circ\circ\circ\epsilon}$

$$H_{\sigma',\sigma} \equiv H_{n^3,n^5}(\gamma,\epsilon) \equiv I_n \otimes I_n \otimes \left[I_n \otimes E_1^t \,|\, I_n \otimes E_2^t \,|\cdots|\, I_n \otimes E_n^t\right].$$

Model 10. $t^{\alpha\beta\gamma\delta\circ}_{\circ\circ\circ\circ\epsilon}$

$$H_{\sigma',\sigma} \equiv H_{n^3,n^5}(\delta,\epsilon) \equiv I_n \otimes I_n \otimes I_n \otimes \left[E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t\right].$$

$T'_{\sigma',1}$ must be given in "condensed" form (as a column-matrix of submatrices).

5.9 Matrices associated with doubly contracted homomorphisms

5.9.1 Mixed tensors of fourth order ($r = 4$)

We look for the tensor resulting from a homogeneous mixed tensor that accepts a double contraction, that is, has at least two contravariant indices and other two covariant indices; $\sigma = n^4; \sigma' = n^0 = 1$. The resulting tensor after the double contraction always is a scalar.

The possibilities for the contraction are: $\binom{4}{2} = 6$ models. Let $\rho \in K$.

Model 1. $H_{1,n^4}(\alpha,\beta|\gamma,\delta)$ means that we contract first indices (α,β) and then, indices (γ,δ):

$$\rho = H_{1,n^4}(\alpha,\beta|\gamma,\delta) \bullet T_{\sigma,1} = ([E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t] \otimes [E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t]) \bullet T_{\sigma,1}. \tag{5.75}$$

Model 2. $H_{1,n^4}(\alpha,\gamma|\beta,\delta)$.

$$\begin{aligned} &\rho = H_{1,n^4}(\alpha,\gamma|\beta,\delta) \bullet T_{\sigma,1} \\ &= [E_1^t \otimes E_1^t | E_1^t \otimes E_2^t | \cdots | E_1^t \otimes E_n^t | \cdots | E_n^t \otimes E_1^t | E_n^t \otimes E_2^t | \cdots | E_n^t \otimes E_n^t] \bullet T_{\sigma,1}. \end{aligned} \tag{5.76}$$

Model 3. $H_{1,n^4}(\alpha, \delta|\beta, \gamma)$.

$$\rho = H_{1,n^4}(\alpha, \delta|\beta, \gamma) \bullet T_{\sigma,1}$$
$$= [[E_1^t|E_2^t|\cdots|E_n^t] \otimes E_1^t|[E_1^t|E_2^t|\cdots|E_n^t] \otimes E_2^t|\cdots|[E_1^t|E_2^t|\cdots|E_n^t] \otimes E_n^t] \bullet T_{\sigma,1}. \tag{5.77}$$

Model 4. $H_{1,n^4}(\beta, \gamma|\alpha, \delta)$

$$\rho = H_{1,n^4}(\beta, \gamma|\alpha, \delta) \bullet T_{\sigma,1} \equiv H_{1,n^4}(\alpha, \delta|\beta, \gamma) \bullet T_{\sigma,1}. \tag{5.78}$$

Model 5. $H_{1,n^4}(\beta, \delta|\alpha, \gamma)$

$$\rho = H_{1,n^4}(\beta, \delta|\alpha, \gamma) \bullet T_{\sigma,1} \equiv H_{1,n^4}(\alpha, \gamma|\beta, \delta) \bullet T_{\sigma,1}. \tag{5.79}$$

Model 6. $H_{1,n^4}(\gamma, \delta|\alpha, \beta)$

$$\rho = H_{1,n^4}(\gamma, \delta|\alpha, \beta) \bullet T_{\sigma,1} \equiv H_{1,n^4}(\alpha, \beta|\gamma, \delta) \bullet T_{\sigma,1}. \tag{5.80}$$

As a mapping of the simple contraction formulas (5.72) and those of extension and condensation (1.30) and (1.32), respectively, we propose that reader establish the direct relation between the classic product of matrices ($A \bullet B$) where $A = [a^{\alpha\circ}_{\circ\beta}]$ and $B = [b^{\gamma\circ}_{\circ\delta}]$) and its tensor product ($A \otimes B$), simplifying the resulting expression.

We remind the reader that the classic product of matrices ($A \bullet B$) is a contracted tensor product.

5.9.2 Mixed tensors of fifth order ($r = 5$)

The contraction of four indices leads to tensors of order ($r = 1$), that is, vectors. The dimensions of the "extended" tensors $T_{\sigma,1}$ and $T_{\sigma'}$ are $\sigma = n^5$ and $\sigma' = n$.

There exist $\binom{5}{2} \times \binom{3}{2} = \frac{5\times 4}{2} \times 3 = 30$ models of double contraction.

Model 1. $t^{\alpha\circ\gamma\circ\epsilon}_{\circ\beta\circ\delta\circ}$

$$\begin{aligned} H_{\sigma',\sigma} &\equiv H_{n,n^5}(\alpha, \beta|\gamma, \delta) = H_{n,n^3}(\gamma, \delta) \bullet H_{n^3,n^5}(\alpha, \beta) \\ &= \left(\left[E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t\right] \otimes I_n\right) \bullet \left(\left[E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t\right] \otimes I_n \otimes I_n \otimes I_n\right). \end{aligned}$$

Model 2. $t^{\alpha\circ\gamma\delta\circ}_{\circ\beta\circ\circ\epsilon}$

$$H_{\sigma',\sigma} \equiv H_{n,n^5}(\alpha, \beta|\gamma, \epsilon) = H_{n,n^3}(\gamma, \epsilon) \bullet H_{n^3,n^5}(\alpha, \beta)$$
$$= \left(\left[I_n \otimes E_1^t \,|\, I_n \otimes E_2^t \,|\cdots|\, I_n \otimes E_n^t\right]\right) \bullet \left(\left[E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t\right] \otimes I_n \otimes I_n \otimes I_n\right).$$

Model 3. $t^{\alpha\circ\gamma\delta\circ}_{\circ\beta\circ\circ\epsilon}$

$$\begin{aligned} H_{\sigma',\sigma} &\equiv H_{n,n^5}(\alpha, \beta|\delta, \epsilon) = H_{n,n^3}(\delta, \epsilon) \bullet H_{n^3,n^5}(\alpha, \beta) \\ &= \left(I_n \otimes \left[E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t\right]\right) \bullet \left(\left[E_1^t \,|\, E_2^t \,|\cdots|\, E_n^t\right] \otimes I_n \otimes I_n \otimes I_n\right). \end{aligned}$$

In a similar form the remaining models can be obtained.

Example 5.6 (Tensor contraction). Contract all indices of the tensor

$$\vec{t} = (2\vec{e}_1 - 3\vec{e}_2) \otimes (5\vec{e}_1 + \vec{e}_2) \otimes (4\vec{e}^{*1} + \vec{e}^{*2}) \otimes (\vec{e}^{*1} - 2\vec{e}^{*2}).$$

Solution: We solve the problem using four different methods.

First method:

We decide to execute the contractions of each pair of contravariant factors with the corresponding covariant factors. There exist two possibilities:

1. We contract factor 1 with factor 3 and factor 2 with factor 4. The connection Gram matrix is I_2, because they are in dual bases:

$$\rho = [\,2 \quad -3\,] I_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \otimes [\,5 \quad 1\,] I_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = (8-3) \times (5-2) = 5 \times 3 = 15.$$

2. We contract factor 1 with factor 4 and factor 2 with factor 3.

$$\rho = [\,2 \quad -3\,] I_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \otimes [\,5 \;\; 1\,] I_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = (2+6) \times (20+1) = 8 \times 21 = 168.$$

Second method:

We decide to associate the contravariant indices between them, and also the covariant indices between them; then, we execute the contraction, to obtain the unique result:

$$\begin{aligned}
\vec{t} = &\left([\,\vec{e}_1 \quad \vec{e}_2\,] \begin{bmatrix} 2 \\ -3 \end{bmatrix} \otimes \left([\,\vec{e}_1 \quad \vec{e}_2\,] \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right)^t \right) \\
&\otimes \left([\,\vec{e}^{*1} \quad \vec{e}^{*2}\,] \begin{bmatrix} 4 \\ 1 \end{bmatrix} \otimes \left([\,\vec{e}^{*1} \quad \vec{e}^{*2}\,] \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)^t \right) \\
= &\left([\,\vec{e}_1 \quad \vec{e}_2\,] \begin{bmatrix} 2 \\ -3 \end{bmatrix} \otimes [\,5 \quad 1\,] \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \end{bmatrix} \right) \otimes \left([\,\vec{e}^{*1} \quad \vec{e}^{*2}\,] \begin{bmatrix} 4 \\ 1 \end{bmatrix} \otimes [\,1 \quad -2\,] \begin{bmatrix} \vec{e}^{*1} \\ \vec{e}^{*2} \end{bmatrix} \right) \\
= &\left([\,\vec{e}_1 \quad \vec{e}_2\,] \begin{bmatrix} 10 & 2 \\ -15 & -3 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \end{bmatrix} \right) \otimes \left([\,\vec{e}^{*1} \quad \vec{e}^{*2}\,] \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \vec{e}^{*1} \\ \vec{e}^{*2} \end{bmatrix} \right) \\
= &\left([\,\vec{e}_1 \otimes \vec{e}_1 \quad \vec{e}_1 \otimes \vec{e}_2 \quad \vec{e}_2 \otimes \vec{e}_1 \quad \vec{e}_2 \otimes \vec{e}_2\,] \begin{bmatrix} 10 \\ 2 \\ -15 \\ -3 \end{bmatrix} \right) \\
&\otimes \left([\,\vec{e}^{*1} \otimes \vec{e}^{*1} \quad \vec{e}^{*1} \otimes \vec{e}^{*2} \quad \vec{e}^{*2} \otimes \vec{e}^{*1} \quad \vec{e}^{*2} \otimes \vec{e}^{*2}\,] \begin{bmatrix} 4 \\ -8 \\ 1 \\ -2 \end{bmatrix} \right).
\end{aligned}$$

The Gram matrix G between both dual tensor spaces is $G \equiv I_4$, because they are in dual bases:

$$\rho = [10 \quad 2 \quad -15 \quad -3] I_4 \begin{bmatrix} 4 \\ -8 \\ 1 \\ -2 \end{bmatrix} = 40 - 16 - 15 + 6 = 15.$$

Third method:

We decide to execute the contraction using its definition. To this end, we need to know the tensor, with the axiomatic ordering of its components.

Executing the last tensor product indicated in the previous method, we obtain

$$\vec{t} = [\, \vec{e}_1 \otimes \vec{e}_1 \;\; \vec{e}_1 \otimes \vec{e}_2 \;\; \vec{e}_2 \otimes \vec{e}_1 \;\; \vec{e}_2 \otimes \vec{e}_2 \,] \begin{bmatrix} 10 \\ 2 \\ -15 \\ -3 \end{bmatrix} \otimes [\, 4 \; -8 \; 1 \; 2 \,] \begin{bmatrix} \vec{e}^{*1} \otimes \vec{e}^{*1} \\ \vec{e}^{*1} \otimes \vec{e}^{*2} \\ \vec{e}^{*2} \otimes \vec{e}^{*1} \\ \vec{e}^{*2} \otimes \vec{e}^{*2} \end{bmatrix}$$

$$= [\, \vec{e}_1 \otimes \vec{e}_1 \;\; \vec{e}_1 \otimes \vec{e}_2 \;\; \vec{e}_2 \otimes \vec{e}_1 \;\; \vec{e}_2 \otimes \vec{e}_2 \,] \begin{bmatrix} 40 & -80 & 10 & -20 \\ 8 & -16 & 2 & -4 \\ -60 & 120 & -15 & 30 \\ -12 & 24 & -3 & 6 \end{bmatrix} \begin{bmatrix} \vec{e}^{*1} \otimes \vec{e}^{*1} \\ \vec{e}^{*1} \otimes \vec{e}^{*2} \\ \vec{e}^{*2} \otimes \vec{e}^{*1} \\ \vec{e}^{*2} \otimes \vec{e}^{*2} \end{bmatrix}.$$

This matrix expression leads to the desired fourth-order tensor $T = [t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}]$. We develop it by rows, in order to get the axiomatic ordering:

$$\begin{aligned} \vec{t} = {} & (\vec{e}_1 \otimes \vec{e}_1) \otimes (40\vec{e}^{*1} \otimes \vec{e}^{*1} - 80\vec{e}^{*1} \otimes \vec{e}^{*2} + 10\vec{e}^{*2} \otimes \vec{e}^{*1} - 20\vec{e}^{*2} \otimes \vec{e}^{*2}) \\ & + (\vec{e}_1 \otimes \vec{e}_2) \otimes (8\vec{e}^{*1} \otimes \vec{e}^{*1} - 16\vec{e}^{*1} \otimes \vec{e}^{*2} + 2\vec{e}^{*2} \otimes \vec{e}^{*1} - 4\vec{e}^{*2} \otimes \vec{e}^{*2}) \\ & + (\vec{e}_2 \otimes \vec{e}_1) \otimes (-60\vec{e}^{*1} \otimes \vec{e}^{*1} + 120\vec{e}^{*1} \otimes \vec{e}^{*2} - 15\vec{e}^{*2} \otimes \vec{e}^{*1} + 30\vec{e}^{*2} \otimes \vec{e}^{*2}) \\ & + (\vec{e}_2 \otimes \vec{e}_2) \otimes (-12\vec{e}^{*1} \otimes \vec{e}^{*1} + 24\vec{e}^{*1} \otimes \vec{e}^{*2} - 3\vec{e}^{*2} \otimes \vec{e}^{*1} + 6\vec{e}^{*2} \otimes \vec{e}^{*2}). \end{aligned}$$

Since the first two factors refer to row and column of each submatrix (first and second tensor indices), we finally get the tensor matrix expression, with the correct ordering

$$[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = \left[\begin{array}{cc|cc} 40 & -80 & 8 & -16 \\ 10 & -20 & 2 & -4 \\ \hline -60 & 120 & -12 & 24 \\ -15 & 30 & -3 & 6 \end{array}\right],$$

where α is the row of submatrices, β the column of submatrices, γ the row of each submatrix, and δ the column of each submatrix.

Next, we start with the contractions. There exist two possibilities:

1. We first contract α with γ, and then β with δ:

$$[u^{\beta\circ}_{\circ\delta}] = \mathcal{C}\binom{\alpha}{\gamma} [t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{\theta\beta\circ\circ}_{\circ\circ\theta\delta}] = t^{1\beta\circ\circ}_{\circ\circ 1\delta} + t^{2\beta\circ\circ}_{\circ\circ 2\delta}$$

$$\rho = \mathcal{C}\binom{\beta}{\delta}[u^{\beta\circ}_{\circ\delta}] = [t^{1\theta\circ\circ}_{\circ\circ1\theta}] + [t^{2\theta\circ\circ}_{\circ\circ2\theta}] = (t^{11\circ\circ}_{\circ\circ11} + t^{12\circ\circ}_{\circ\circ12}) + (t^{21\circ\circ}_{\circ\circ21} + t^{22\circ\circ}_{\circ\circ22})$$

$$\rho = 40 - 16 - 15 + 6 = 46 - 31 = 15.$$

2. We first contract α with δ, and then β with γ:

$$[v^{\beta\circ}_{\circ\gamma}] = \mathcal{C}\binom{\alpha}{\delta}[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{\theta\beta\circ\circ}_{\circ\circ\gamma\theta}] = t^{1\beta\circ\circ}_{\circ\circ\gamma1} + t^{2\beta\circ\circ}_{\circ\circ\gamma2}$$

$$\rho' = \mathcal{C}\binom{\beta}{\gamma}[v^{\beta\circ}_{\circ\gamma}] = [v^{\theta\circ}_{\circ\theta}] = [t^{1\theta\circ\circ}_{\circ\circ\theta1}] + [t^{2\theta\circ\circ}_{\circ\circ\theta2}] = (t^{11\circ\circ}_{\circ\circ11} + t^{12\circ\circ}_{\circ\circ21}) + (t^{21\circ\circ}_{\circ\circ12} + t^{22\circ\circ}_{\circ\circ22})$$

$$\rho' = 40 + 2 + 120 + 6 = 168.$$

Fourth method:

We use the direct homomorphism on the components $T_\sigma(\sigma = 2\times2\times2\times2 = 16)$, that is, the tensor components in a "column matrix".

There are two models to be considered:

1. The homomorphism model (2) of double contraction, Formula (5.76):

$$\begin{aligned}\rho &= H_{1,16}(\alpha,\gamma|\beta,\delta) \bullet T_{16,1} = [E^t_1 \otimes E^t_1 | E^t_1 \otimes E^t_2 | E^t_2 \otimes E^t_1 | E^t_2 \otimes E^t_2 |] \bullet T_{16,1}\\ &= [[\,1\ 0\,] \otimes [\,1\ 0\,] | [\,1\ 0\,] \otimes [\,0\ 1\,] | [\,0\ 1\,] \otimes [\,1\ 0\,] | [\,0\ 1\,] \otimes [\,0\ 1\,]] \bullet T_{16,1}\end{aligned}$$

$$= [\,1\ 0\ 0\ 0\ 0\ 1\ 0\ 0 \mid 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\,] \bullet \begin{bmatrix} 40\\ -80\\ 10\\ -20\\ 8\\ -16\\ 2\\ -4\\ -60\\ 120\\ -15\\ 30\\ -12\\ 24\\ -3\\ 6 \end{bmatrix}$$

$$= 40 - 16 - 15 + 6 = 15.$$

2. The homomorphism model (3), Formula (5.77):

$$\rho' = H_{1,16}(\alpha,\delta|\beta,\gamma) \bullet T_{16,1} = \left[[E^t_1 | E^t_2] \otimes E^t_1 | [E^t_1 | E^t_2] \otimes E^t_2\right] \bullet T_{16,1}$$

$$= [[1 \quad 0 \quad 0 \quad 1\,] \otimes [1 \quad 0] | [1 \quad 0 \quad 0 \quad 1\,] \otimes [0 \quad 1]] \bullet T_{16,1}$$

$$= [1\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \mid 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1] \bullet \begin{bmatrix} 40 \\ -80 \\ 10 \\ -20 \\ 8 \\ -16 \\ 2 \\ -4 \\ -60 \\ 120 \\ -15 \\ 30 \\ -12 \\ 24 \\ -3 \\ 6 \end{bmatrix}$$

$$= 40 + 2 + 120 + 6 = 168.$$

□

Example 5.7 (Contractions). Consider the tensor $\vec{t} = \vec{v}_1 \otimes \vec{v}_2 \otimes \vec{f}^3 \otimes \vec{f}^4$ defined over $\mathbb{R}^2$, where the factor vectors are

$$\vec{v}_1 = \vec{e}_1 + \vec{e}_2; \ \ \vec{v}_2 = 2\vec{e}_1 - \vec{e}_2; \ \ \vec{f}^3 = 2\vec{e}^{*1} + \vec{e}^{*2}; \ \ \vec{f}^4 = 3\vec{e}^{*1};$$

1. Obtain the totally developed analytical expression of the tensor expressed in its corresponding tensor basis.
2. Execute all possible simple contractions, indicating which of the obtained systems of scalars have tensor character.
3. Express the resulting tensors in the previous question, as a function of the vectors $\vec{v}_1, \vec{v}_2, \vec{f}^3, \vec{f}^4$.

Solution:

1. We develop the tensor product

$$\begin{aligned}
\vec{t} &= (\vec{v}_1 \otimes \vec{v}_2) \otimes (\vec{f}^3 \otimes \vec{f}^4) \\
&= (2\vec{e}_1 \otimes \vec{e}_1 - \vec{e}_1 \otimes \vec{e}_2 + 2\vec{e}_2 \otimes \vec{e}_1 - \vec{e}_2 \otimes \vec{e}_2) \otimes (6\vec{e}^{*1} \otimes \vec{e}^{*1} + 3\vec{e}^{*2} \otimes \vec{e}^{*1}) \\
&= 12\vec{e}_1 \otimes \vec{e}_1 \otimes \vec{e}^{*1} \otimes \vec{e}^{*1} + 6\vec{e}_1 \otimes \vec{e}_1 \otimes \vec{e}^{*2} \otimes \vec{e}^{*1} - 6\vec{e}_1 \otimes \vec{e}_2 \otimes \vec{e}^{*1} \otimes \vec{e}^{*1} \\
&\quad -3\vec{e}_1 \otimes \vec{e}_2 \otimes \vec{e}^{*2} \otimes \vec{e}^{*1} + 12\vec{e}_2 \otimes \vec{e}_1 \otimes \vec{e}^{*1} \otimes \vec{e}^{*1} + 6\vec{e}_2 \otimes \vec{e}_1 \otimes \vec{e}^{*2} \otimes \vec{e}^{*1} \\
&\quad -6\vec{e}_2 \otimes \vec{e}_2 \otimes \vec{e}^{*1} \otimes \vec{e}^{*1} - 3\vec{e}_2 \otimes \vec{e}_2 \otimes \vec{e}^{*2} \otimes \vec{e}^{*1}
\end{aligned}$$

and in matrix form

$$[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = \left[\begin{array}{cc|cc} 12 & 0 & -6 & 0 \\ 6 & 0 & -3 & 0 \\ \hline 12 & 0 & -6 & 0 \\ 6 & 0 & -3 & 0 \end{array}\right],$$

where α is the matrix row indicator, β is the matrix column indicator, γ the row indicator of each submatrix, and δ the column indicator of each submatrix, that is, according to the basis axiomatic ordering.

2. Contraction is an operation that can be applied to any *system of scalars* of r indices and n^r components, with $n, r \geq 2$. The result is another system of scalars of order $(r-2)$ and n^{r-2} components, so that the cited operation is defined for such sets independently of whether they are or not tensors. According to this, we separate those contractions over tensor indices of the same valency that do not guarantee a resulting tensor from contractions executed over indices of different valency, in which case it is guaranteed that the contracted system is a tensor.
The contractions of tensor $\vec{t}$, *with no tensor character*, are (be aware of the special notation used for these type of non-tensor contractions)

$$\mathcal{C}(\alpha,\beta) - \mathcal{C}(\gamma,\delta)-, \text{ that is}$$

$$[a^{\circ\circ}_{\gamma\delta}] = \mathcal{C}(\alpha,\beta)[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{11\circ\circ}_{\circ\circ\gamma\delta} + t^{22\circ\circ}_{\circ\circ\gamma\delta}] :$$

$$\left.\begin{aligned} a^{\circ\circ}_{11} &= t^{11\circ\circ}_{\circ\circ11} + t^{22\circ\circ}_{\circ\circ11} = 12-6=6 \\ a^{\circ\circ}_{12} &= t^{11\circ\circ}_{\circ\circ12} + t^{22\circ\circ}_{\circ\circ12} = 0+0=0 \\ a^{\circ\circ}_{21} &= t^{11\circ\circ}_{\circ\circ21} + t^{22\circ\circ}_{\circ\circ21} = 6-3=3 \\ a^{\circ\circ}_{22} &= t^{11\circ\circ}_{\circ\circ22} + t^{22\circ\circ}_{\circ\circ22} = 0+0=0 \end{aligned}\right\} \Rightarrow \; [a^{\circ\circ}_{\gamma\delta}] = \begin{bmatrix} 6 & 0 \\ 3 & 0 \end{bmatrix}$$

$$[b^{\circ\circ}_{\alpha\beta}] = \mathcal{C}(\gamma,\delta)[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{\alpha\beta\circ\circ}_{\circ\circ11} + t^{\alpha\beta\circ\circ}_{\circ\circ22}] :$$

$$\left.\begin{aligned} b^{\circ\circ}_{11} &= t^{11\circ\circ}_{\circ\circ11} + t^{11\circ\circ}_{\circ\circ22} = 12+0=12 \\ b^{\circ\circ}_{12} &= t^{12\circ\circ}_{\circ\circ21} + t^{12\circ\circ}_{\circ\circ22} = -6+0=-6 \\ b^{\circ\circ}_{21} &= t^{21\circ\circ}_{\circ\circ11} + t^{21\circ\circ}_{\circ\circ22} = 12+0=12 \\ b^{\circ\circ}_{22} &= t^{22\circ\circ}_{\circ\circ11} + t^{22\circ\circ}_{\circ\circ22} = -6+0=-6 \end{aligned}\right\} \Rightarrow \; [b^{\circ\circ}_{\alpha\beta}] = \begin{bmatrix} 12 & -6 \\ 12 & -6 \end{bmatrix}.$$

The contractions with a *tensor nature* of tensor $\vec{t}$ are

$$-\mathcal{C}\binom{\alpha}{\gamma} - \mathcal{C}\binom{\alpha}{\delta} - \mathcal{C}\binom{\beta}{\gamma} - \mathcal{C}\binom{\beta}{\delta} - .$$

We execute them using two different procedures:

1. Direct procedure, according to the contraction definition:

$$[c^{\beta\circ}_{\circ\delta}] = \mathcal{C}\binom{\alpha}{\gamma}[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{1\beta\circ\circ}_{\circ\circ 1\delta}] + [t^{2\beta\circ\circ}_{\circ\circ 2\delta}]$$

$$\left.\begin{aligned}
c^{1\circ}_{\circ 1} &= t^{11\circ\circ}_{\circ\circ 11} + t^{21\circ\circ}_{\circ\circ 21} = 12 + 6 = 18\\
c^{1\circ}_{\circ 2} &= t^{11\circ\circ}_{\circ\circ 12} + t^{21\circ\circ}_{\circ\circ 22} = 0 + 0 = 0\\
c^{2\circ}_{\circ 1} &= t^{12\circ\circ}_{\circ\circ 11} + t^{22\circ\circ}_{\circ\circ 21} = -6 - 3 = -9\\
c^{2\circ}_{\circ 2} &= t^{12\circ\circ}_{\circ\circ 12} + t^{22\circ\circ}_{\circ\circ 22} = 0 + 0 = 0
\end{aligned}\right\} \Rightarrow \; [c^{\beta\circ}_{\circ\delta}] = \begin{bmatrix} 18 & 0 \\ -9 & 0 \end{bmatrix};$$

$$[d^{\beta\circ}_{\circ\gamma}] = \mathcal{C}\binom{\alpha}{\delta}[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{1\beta\circ\circ}_{\circ\circ\gamma 1}] + [t^{2\beta\circ\circ}_{\circ\circ\gamma 2}]$$

$$\left.\begin{aligned}
d^{1\circ}_{\circ 1} &= t^{11\circ\circ}_{\circ\circ 11} + t^{21\circ\circ}_{\circ\circ 12} = 12 + 0 = 12\\
d^{1\circ}_{\circ 2} &= t^{11\circ\circ}_{\circ\circ 21} + t^{21\circ\circ}_{\circ\circ 22} = 6 + 0 = 6\\
d^{2\circ}_{\circ 1} &= t^{12\circ\circ}_{\circ\circ 11} + t^{22\circ\circ}_{\circ\circ 12} = -6 + 0 = -6\\
d^{2\circ}_{\circ 2} &= t^{12\circ\circ}_{\circ\circ 21} + t^{22\circ\circ}_{\circ\circ 22} = -3 + 0 = -3
\end{aligned}\right\} \Rightarrow \; [d^{\beta\circ}_{\circ\gamma}] = \begin{bmatrix} 12 & 6 \\ -6 & -3 \end{bmatrix};$$

$$[f^{\alpha\circ}_{\circ\delta}] = \mathcal{C}\binom{\beta}{\gamma}[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{\alpha 1\circ\circ}_{\circ\circ 1\delta}] + [t^{\alpha 2\circ\circ}_{\circ\circ 2\delta}]$$

$$\left.\begin{aligned}
f^{1\circ}_{\circ 1} &= t^{11\circ\circ}_{\circ\circ 11} + t^{12\circ\circ}_{\circ\circ 21} = 12 - 3 = 9\\
f^{1\circ}_{\circ 2} &= t^{11\circ\circ}_{\circ\circ 12} + t^{12\circ\circ}_{\circ\circ 22} = 0 + 0 = 0\\
f^{2\circ}_{\circ 1} &= t^{21\circ\circ}_{\circ\circ 11} + t^{22\circ\circ}_{\circ\circ 21} = 12 - 3 = 9\\
f^{2\circ}_{\circ 2} &= t^{21\circ\circ}_{\circ\circ 12} + t^{22\circ\circ}_{\circ\circ 22} = 0 + 0 = 0
\end{aligned}\right\} \Rightarrow \; [f^{\alpha\circ}_{\circ\delta}] = \begin{bmatrix} 9 & 0 \\ 9 & 0 \end{bmatrix};$$

$$[g^{\alpha\circ}_{\circ\gamma}] = \mathcal{C}\binom{\beta}{\delta}[t^{\alpha\beta\circ\circ}_{\circ\circ\gamma\delta}] = [t^{\alpha 1\circ\circ}_{\circ\circ\gamma 1}] + [t^{\alpha 2\circ\circ}_{\circ\circ\gamma 2}]$$

$$\left.\begin{array}{l} g^{1\circ}_{\circ 1} = t^{11\circ\circ}_{\circ\circ 11} + t^{12\circ\circ}_{\circ\circ 12} = 12 + 0 = 12 \\ \\ g^{1\circ}_{\circ 2} = t^{11\circ\circ}_{\circ\circ 211} + t^{12\circ\circ}_{\circ\circ 22} = 6 + 0 = 6 \\ \\ g^{2\circ}_{\circ 1} = t^{21\circ\circ}_{\circ\circ 11} + t^{22\circ\circ}_{\circ\circ 12} = 12 + 0 = 12 \\ \\ g^{2\circ}_{\circ 2} = t^{21\circ\circ}_{\circ\circ 21} + t^{22\circ\circ}_{\circ\circ 22} = 6 + 0 = 6 \end{array}\right\} \Rightarrow [g^{\alpha\circ}_{\circ\gamma}] = \begin{bmatrix} 12 & 6 \\ 12 & 6 \end{bmatrix}.$$

2. Procedure based on the use of the simple contraction homomorphisms and of order $r = 4$. $\mathcal{C}\binom{\alpha}{\gamma} \to$ Model (2), Formula (5.70):

$$T'_4 = H_{4,16}(\alpha,\gamma) \bullet T_{16,1} = \big([I_2 \otimes E^t_1 | I_2 \otimes E^t_2] \otimes I_2\big) \bullet T_{16,1}$$
$$= \left[\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [1 \quad 0]\right] \middle| \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [0 \quad 1]\right] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \bullet T_{16,1}$$
$$= \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \bullet T_{16,1}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \\ 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 12+6 \\ 0+0 \\ -6-3 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ -9 \\ 0 \end{bmatrix},$$

and after condensation the result is

$$[c^{\beta\circ}_{\circ\delta}] = \begin{bmatrix} 18 & 0 \\ -9 & 0 \end{bmatrix}.$$

$\mathcal{C}\binom{\alpha}{\delta} \to$ Model (3), Formula (5.71):

$$T''_4 = H_{4,16}(\alpha,\delta) \bullet T_{16,1} = [I_2 \otimes I_2 \otimes E^t_1 | I_2 \otimes I_2 \otimes E^t_2] \bullet T_{16,1}$$
$$= \left[\left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes [1 \quad 0]\right] \middle| \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes [0 \quad 1]\right]\right] \bullet T_{16,1}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \\ 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 12+0 \\ 8+0 \\ -6+0 \\ -3+0 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ -6 \\ -3 \end{bmatrix},
$$

and after condensation we get

$$
[d^{\beta\,\circ}_{\circ\,\gamma}] = \begin{bmatrix} 12 & 6 \\ -6 & -3 \end{bmatrix}.
$$

$\mathcal{C}\binom{\beta}{\gamma} \rightarrow$ Model (4), Formula (5.72):

$$
T_4''' = H_{4,16}(\beta,\gamma) \bullet T_{16,1} = [I_2 \otimes \left[E_1^t | E_2^t\right] \otimes I_2] \bullet T_{16,1}
$$

$$
= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [1 \quad 0 \quad 0 \quad 1] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \bullet T_{16,1}
$$

$$
= \left(\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \bullet T_{16,1}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \\ 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 12-3 \\ 0+0 \\ 12-3 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 9 \\ 0 \end{bmatrix},
$$

and after condensation we get

$$
[f^{\alpha\,\circ}_{\circ\,\delta}] = \begin{bmatrix} 9 & 0 \\ 9 & 0 \end{bmatrix}.
$$

$\mathcal{C}\binom{\beta}{\gamma} \rightarrow$ Model (5), Formula (5.73):

$$
\begin{aligned}
T_4^{IV} &= H_{4,16}(\beta,\delta) \bullet T_{16,1} = \left(I_2 \otimes [I_2 \otimes E_1^t | I_2 \otimes E_2^t]\right) \bullet T_{16,1} \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [1 \quad 0] \middle| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [0 \quad 1] \right] \right) \bullet T_{16,1} \\
&= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \bullet T_{16,1} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \\ 12 \\ 0 \\ 6 \\ 0 \\ -6 \\ 0 \\ -3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 12+0 \\ 6+0 \\ 12+0 \\ 6+0 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 12 \\ 6 \end{bmatrix},
\end{aligned}
$$

and condensing yields

$$[g^{\alpha\circ}_{\circ\gamma}] = \begin{bmatrix} 12 & 6 \\ 12 & 6 \end{bmatrix}.$$

3. We will express each of the tensors previously obtained in a developed analytical form, and later we will try to factorize each of them, as a function of the factors $\vec{v}_1, \vec{v}_2, \vec{f}^3, \vec{f}^4$. Then

$$\vec{c} = c^{\beta\circ}_{\circ\delta}\vec{e}_\beta \otimes \vec{e}^{*\delta} = 18\vec{e}_1 \otimes \vec{e}^{*1} - 9\vec{e}_2 \otimes \vec{e}^{*1} = (2\vec{e}_1 - \vec{e}_2) \otimes (9\vec{e}^{*1})$$

and according to the statement data:

$$
\begin{aligned}
\vec{c} &= \vec{v}_2 \otimes 3\vec{f}^4 = 3(\vec{v}_2 \otimes \vec{f}^4) \\
\vec{d} &= d^{\beta\circ}_{\circ\gamma}\vec{e}_\beta \otimes \vec{e}^{*\gamma} = 12\vec{e}_1 \otimes \vec{e}^{*1} + 6\vec{e}_1 \otimes \vec{e}^{*2} - 6\vec{e}_2 \otimes \vec{e}^{*1} - 3\vec{e}_2 \otimes \vec{e}^{*2} \\
&= (12\vec{e}_1 - 6\vec{e}_2) \otimes \vec{e}^{*1} + (6\vec{e}_1 - 3\vec{e}_2) \otimes \vec{e}^{*2} \\
&= 2(2\vec{e}_1 - \vec{e}_2) \otimes 3\vec{e}^{*1} + (2\vec{e}_1 - \vec{e}_2) \otimes 3\vec{e}^{*2} \\
&= (2\vec{e}_1 - \vec{e}_2) \otimes 3(2\vec{e}^{*1} + \vec{e}^{*2}) \\
&= 3(2\vec{e}_1 - \vec{e}_2) \otimes (2\vec{e}^{*1}1 + \vec{e}^{*2}) \\
&= 3(\vec{v}_2 \otimes \vec{f}^3) \\
\vec{f} &= f^{\alpha\circ}_{\circ\delta}\vec{e}_\alpha \otimes \vec{e}^{*\delta}
\end{aligned}
$$

$$= 9\vec{e}_1 \otimes \vec{e}^{*1} + 9\vec{e}_2 \otimes \vec{e}^{*1}$$
$$= 3(\vec{e}_1 + \vec{e}_2) \otimes 3\vec{e}^{*1} = 3(\vec{v}_1 \otimes \vec{f}^4)$$
$$\vec{g} = g^{\alpha\circ}_{\circ\gamma}\vec{e}_\alpha \otimes \vec{e}^{*\gamma} = 12\vec{e}_1 \otimes \vec{e}^{*1} + 6\vec{e}_1 \otimes \vec{e}^{*2} + 12\vec{e}_2 \otimes \vec{e}^{*1} + 6\vec{e}_2 \otimes \vec{e}^{*2}$$
$$= 12(\vec{e}_1 + \vec{e}_2) \otimes \vec{e}^{*1} + 6(\vec{e}_1 + \vec{e}_2) \otimes \vec{e}^{*2}$$
$$= 6(\vec{e}_1 + \vec{e}_2) \otimes (2\vec{e}^{*1} + \vec{e}^{*2}) = 6(\vec{v}_1 \otimes \vec{f}^3).$$

□

Example 5.8 (Contracted tensor product). Consider the two tensors $\vec{a}$ and $\vec{b}$ given by their components with respect to the canonical basis of the linear space $\mathbb{R}^3$:

$$[a^{\alpha\beta}_{\circ\circ}] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix}; \quad [b^{\gamma\circ\epsilon}_{\circ\delta\circ}] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \\ - & - & - \\ 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & -1 \\ - & - & - \\ 0 & 3 & -2 \\ 5 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix}.$$

1. Obtain *all* possible contracted tensor products with both tensors.
2. Determine the type of homomorphism that directly relates two of the contracted products with the other two.

Solution:

1. A tensor product tensor is

$$t^{\alpha\beta\gamma\circ\epsilon}_{\circ\circ\circ\delta\circ} = a^{\alpha\beta}_{\circ\circ} \otimes b^{\gamma\circ\epsilon}_{\circ\delta\circ}.$$

There are two possible tensor contractions: $\mathcal{C}\binom{\alpha}{\delta} - \mathcal{C}\binom{\beta}{\delta}$, because contractions $\mathcal{C}\binom{\gamma}{\delta}$ and $\mathcal{C}\binom{\epsilon}{\delta}$ correspond to indices of the same factor.

$$[u^{\beta\gamma\epsilon}_{\circ\circ\circ}] = \mathcal{C}\begin{pmatrix}\alpha \\ \delta\end{pmatrix}[t^{\alpha\beta\gamma\circ\epsilon}_{\circ\circ\circ\delta\circ}] = [t^{\theta\beta\gamma\circ\epsilon}_{\circ\circ\circ\theta\circ}] = [a^{\theta\beta}_{\circ\circ} \cdot b^{\gamma\circ\epsilon}_{\circ\theta\circ}].$$

If $[a^{\theta\beta}_{\circ\circ}]^t = [a^{\beta\theta}_{\circ\circ}]$, for $\gamma = 1$, we get

$$[u^{\beta 1\epsilon}_{\circ\circ\circ}] = [a^{\beta\theta}_{\circ\circ}] \cdot [b^{1\circ\epsilon}_{\circ\theta\circ}] = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 11 \\ 3 & -11 & -7 \\ -2 & 1 & 5 \end{bmatrix},$$

for $\gamma = 2$:

$$[u^{\beta 2\epsilon}_{\circ\circ\circ}] = [a^{\beta\theta}_{\circ\circ}] \cdot [b^{2\circ\epsilon}_{\circ\theta\circ}] = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -1 \\ -3 & -2 & 5 \\ 3 & -1 & -2 \end{bmatrix},$$

and for $\gamma = 3$:

$$[u^{\beta 3 \epsilon}_{\circ\circ\circ}] = [a^{\beta\theta}_{\circ\circ}] \cdot [b^{3\circ\epsilon}_{\circ\theta\circ}] = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & -2 \\ 5 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 2 \\ -6 & 11 & -10 \\ -2 & -2 & 2 \end{bmatrix}.$$

So that letting $\beta = 1$, and assigning to ϵ the values $1, 2, 3$ in the above three matrices we arrive at

$$[u^{1\gamma\epsilon}_{\circ\circ\circ}] = \begin{bmatrix} 1 & 7 & 11 \\ 7 & 2 & -1 \\ 6 & 1 & 2 \end{bmatrix}.$$

Similarly, for $\beta = 2$ and ϵ taking values $1, 2, 3$, we get

$$[u^{2\gamma\epsilon}_{\circ\circ\circ}] = \begin{bmatrix} 3 & -11 & -7 \\ -3 & -2 & 5 \\ -6 & 11 & -10 \end{bmatrix}.$$

Finally, for $\beta = 3$ and values $1, 2, 3$ we obtain

$$[u^{3\gamma\epsilon}_{\circ\circ\circ}] = \begin{bmatrix} -2 & 1 & 5 \\ 3 & -1 & -2 \\ -2 & -2 & 2 \end{bmatrix}.$$

Then, the first contracted product is

$$[u^{\beta\gamma\epsilon}_{\circ\circ\circ}] = \begin{bmatrix} 1 & 7 & 11 \\ 7 & 2 & -1 \\ 6 & 1 & 2 \\ - & - & - \\ 3 & -11 & -7 \\ -3 & -2 & 5 \\ -6 & 11 & -10 \\ - & - & - \\ -2 & 1 & 5 \\ 3 & -1 & -2 \\ -2 & -2 & 2 \end{bmatrix},$$

and the second is

$$[v^{\alpha\gamma\epsilon}_{\circ\circ\circ}] = \mathcal{C}\binom{\beta}{\delta}[t^{\alpha\beta\gamma\circ\epsilon}_{\circ\circ\circ\delta\circ}] = [t^{\alpha\theta\gamma\circ\epsilon}_{\circ\circ\circ\theta\circ}] = [a^{\alpha\theta}_{\circ\circ} \cdot b^{\gamma\circ\epsilon}_{\circ\theta\circ}].$$

For $\gamma = 1$, we get

$$[v^{\alpha 1 \epsilon}_{\circ\circ\circ}] = [a^{\alpha\theta}_{\circ\circ} \cdot b^{1\circ\epsilon}_{\circ\theta\circ}] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 1 \\ 0 & -4 & -5 \\ -2 & -4 & 7 \end{bmatrix},$$

for $\gamma = 2$:

$$[v^{\alpha 2 \epsilon}_{\circ\circ\circ}] = [a^{\alpha\theta}_{\circ\circ} \cdot b^{2\circ\epsilon}_{\circ\theta\circ}] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ -3 & -1 & 1 \\ 5 & -3 & -1 \end{bmatrix},$$

and for $\gamma = 3$:

$$[v^{\alpha 3 \epsilon}_{\circ\circ\circ}] = [a^{\alpha\theta}_{\circ\circ} \cdot b^{3\circ\epsilon}_{\circ\theta\circ}] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 3 & -2 \\ 5 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 6 & -2 \\ -3 & 1 & -2 \\ -7 & 3 & -2 \end{bmatrix}.$$

Doing exactly the same as in the previous case, $\beta = 1, 2, 3$, and ϵ successively equal to $1, 2, 3$ in each jump of β, we obtain the second contracted product

$$[v^{\alpha\gamma\epsilon}_{\circ\circ\circ}] = \begin{bmatrix} 7 & 8 & 1 \\ 1 & 6 & 4 \\ 15 & 6 & -2 \\ - & - & - \\ 0 & -4 & -5 \\ -3 & -1 & 1 \\ -3 & 1 & -2 \\ - & - & - \\ -2 & -4 & 7 \\ 5 & -3 & -1 \\ -7 & 3 & -2 \end{bmatrix}.$$

Another tensor product is $p^{\gamma\circ\epsilon\alpha\beta}_{\circ\delta\circ\circ\circ} = b^{\gamma\circ\epsilon}_{\circ\delta\circ} \otimes a^{\alpha\beta}_{\circ\circ}$.

There are two possible tensor contractions (of contracted product): $\mathcal{C}\binom{\alpha}{\delta} - \mathcal{C}\binom{\beta}{\delta}$. Thus, we have

$$[w^{\gamma\epsilon\beta}_{\circ\circ\circ}] = \mathcal{C}\begin{pmatrix}\alpha \\ \delta\end{pmatrix}[p^{\gamma\circ\epsilon\alpha\beta}_{\circ\delta\circ\circ\circ}] = [p^{\gamma\circ\epsilon\theta\beta}_{\circ\theta\circ\circ\circ}] = [b^{\gamma\circ\epsilon}_{\circ\theta\circ} \cdot a^{\theta\beta}_{\circ\circ}].$$

For $\gamma = 1$, taking into account that $[b^{1\epsilon\circ}_{\circ\circ\theta}] = [b^{1\circ\epsilon}_{\circ\theta\circ}]^t$, the result is

$$[w^{1\epsilon\beta}_{\circ\circ\circ}] = [b^{1\circ\epsilon}_{\circ\theta\circ} \cdot a^{\theta\beta}_{\circ\circ}] = [b^{1\epsilon\circ}_{\circ\circ\theta}] \cdot [a^{\theta\beta}_{\circ\circ}] = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix} \bullet \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & -2 \\ 7 & -11 & 1 \\ 11 & -7 & 5 \end{bmatrix}.$$

For $\gamma = 2$, we get

$$[w^{2\epsilon\beta}_{\circ\circ\circ}] = [b^{2\circ\epsilon}_{\circ\theta\circ} \cdot a^{\theta\beta}_{\circ\circ}] = [b^{2\epsilon\circ}_{\circ\circ\theta}] \cdot [a^{\theta\beta}_{\circ\circ}] = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & -3 & 3 \\ 2 & -2 & -1 \\ -1 & 5 & -2 \end{bmatrix},$$

and for $\gamma = 3$, is

$$[w^{3\epsilon\beta}_{\circ\circ\circ}] = [b^{3\circ\epsilon}_{\circ\theta\circ} \cdot a^{\theta\beta}_{\circ\circ}] = [b^{3\epsilon\circ}_{\circ\circ\theta}] \cdot [a^{\theta\beta}_{\circ\circ}] = \begin{bmatrix} 0 & 5 & 3 \\ 3 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix} \bullet \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -6 & -2 \\ 1 & 11 & -2 \\ 2 & -10 & 2 \end{bmatrix}.$$

So that the present contraction becomes

$$[w^{\gamma\epsilon\beta}_{\circ\circ\circ}] = \left[\begin{array}{rrr} 1 & 3 & -2 \\ 7 & -11 & 1 \\ 11 & -7 & 5 \\ \hline 7 & -3 & 3 \\ 2 & -2 & -1 \\ -1 & 5 & -2 \\ \hline 6 & -6 & -2 \\ 1 & 11 & -2 \\ 2 & -10 & 2 \end{array}\right].$$

Finally, the following contraction remains to be calculated:

$$[s^{\gamma\epsilon\alpha}_{\circ\circ\circ}] = \mathcal{C}\binom{\beta}{\delta}[p^{\gamma\circ\epsilon\alpha\beta}_{\circ\delta\circ\circ\circ}] = [p^{\gamma\circ\epsilon\alpha\theta}_{\circ\theta\circ\circ\circ}] = [b^{\gamma\circ\epsilon}_{\circ\theta\circ} \cdot a^{\alpha\theta}_{\circ\circ}].$$

This time we will transpose the matrices associated with both factors, in order to be able to execute them in matrix form.
For $\gamma = 1$, we obtain

$$[s^{1\epsilon\alpha}_{\circ\circ\circ}] = [b^{1\circ\epsilon}_{\circ\theta\circ} \cdot a^{\alpha\theta}_{\circ\circ}] = [b^{1\epsilon\circ}_{\circ\circ\theta}] \cdot [a^{\theta\alpha}_{\circ\circ}] = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 0 & -2 \\ 8 & -4 & -4 \\ 1 & -5 & 7 \end{bmatrix}.$$

For $\gamma = 2$:

$$[s^{2\epsilon\alpha}_{\circ\circ\circ}] = [b^{2\circ\epsilon}_{\circ\theta\circ} \cdot a^{\alpha\theta}_{\circ\circ}] = [b^{2\epsilon\circ}_{\circ\circ\theta}] \cdot [a^{\theta\alpha}_{\circ\circ}] = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -3 & 5 \\ 6 & -1 & -3 \\ 4 & 1 & -1 \end{bmatrix},$$

and for $\gamma = 3$:

$$[s^{3\epsilon\alpha}_{\circ\circ\circ}] = [b^{3\circ\epsilon}_{\circ\theta\circ} \cdot a^{\alpha\theta}_{\circ\circ}] = [b^{3\epsilon\circ}_{\circ\circ\theta}] \cdot [a^{\theta\alpha}_{\circ\circ}] = \begin{bmatrix} 0 & 5 & 3 \\ 3 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 15 & -3 & -7 \\ 6 & 1 & 3 \\ -2 & -2 & -2 \end{bmatrix},$$

which yields the contracted tensor

$$[s^{\gamma\,\epsilon\,\alpha}_{\circ\,\circ\,\circ}] = \begin{bmatrix} 7 & 0 & -2 \\ 8 & -4 & -4 \\ 1 & -5 & 7 \\ - & - & - \\ 1 & -3 & 5 \\ 6 & -1 & -3 \\ 4 & 1 & -1 \\ - & - & - \\ 15 & -3 & -7 \\ 6 & 1 & 3 \\ -2 & -2 & -2 \end{bmatrix}.$$

2. A careful examination of the tensor $[w^{\gamma\,\epsilon\,\beta}_{\circ\,\circ\,\circ}]$ reveals that it is a certain permutation of $[u^{\beta\,\gamma\,\epsilon}_{\circ\,\circ\,\circ}]$ and, since all dummy indices change position, it is a rotation. Compared with the Example 5.5 of rotation tensors, we finally establish that $[w^{\gamma\,\epsilon\,\beta}_{\circ\,\circ\,\circ}] = [u^{\beta\,\gamma\,\epsilon}_{\circ\,\circ\,\circ}]^{R(1)}$.

   Similarly, we establish that $[s^{\gamma\,\epsilon\,\alpha}_{\circ\,\circ\,\circ}] = [v^{\alpha\,\gamma\,\epsilon}_{\circ\,\circ\,\circ}]^{R(1)}$, an interesting relation, which enables us to avoid half of the operations in the previous question.

□

5.10 Eigentensors

Given an arbitrary tensor, T, we examine what possible tensors exist of a given order, r, that in a *contracted tensor product* with the given tensor, become a tensor that is λ times ($\lambda \in K$) the initial tensor, that is, the following tensor equation is satisfied, with T and $r = 3$:

$$\mathcal{C}\left(\begin{array}{c|c} \alpha & \phi \\ \theta & \beta \end{array}\right)(T^{\alpha\,\circ\,\circ\,\delta}_{\circ\,\beta\,\gamma\,\circ} \otimes X^{\circ\,\phi\,\circ}_{\theta\,\circ\,w}) = \lambda X^{\circ\,\phi\,\circ}_{\theta\,\circ\,w}. \tag{5.81}$$

First case:

Data tensor: $A = [a^{\alpha\,\circ}_{\circ\,\beta}]$, of second order, over $n = dim V^2(K) = 2$.

Test tensor $r = 1$: vector $X = [x^\theta] \equiv \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$

According to (5.81), we must have

$$\mathcal{C}\begin{pmatrix} \theta \\ \beta \end{pmatrix}[A \otimes X] = [a^{\alpha\,\circ}_{\circ\,\beta} \cdot \delta^{\beta\,\circ}_{\circ\,\theta} \cdot x^{\theta}_{\circ}] = \left[a^{\alpha\,\circ}_{\circ\,\theta} \cdot x^{\theta}_{\circ}\right] = A \bullet X = \lambda X \tag{5.82}$$

and the relation (5.82) leads to the classic relation

$$[A - \lambda I] \bullet X = \Omega, \tag{5.83}$$

which is solved in algebras with the eigenvalues and eigenvectors associated with matrix A, for the eigenvalues λ_1 and λ_2 of the characteristic polynomial.

We do not insist on this, since we assume that it is well known by the reader. Let A_1 and A_2 be the matrices of eigenvectors associated with the eigenvalues λ_1 and λ_2 (assuming they coexist in K); we assume from now on that they are *known*.

The solutions in this first case are

$$\begin{aligned} X_1 &= A_1 \text{ arbitrary eigenvector of the matrix } A, \text{ associated with } \lambda_1. \\ X_2 &= A_2 \text{ arbitrary eigenvector of the matrix } A, \text{ associated with } \lambda_2. \end{aligned} \tag{5.84}$$

Second case:

Data tensor: $A = [a^{\alpha\circ}_{\circ\beta}]$, of second order, over $n = dim\, V^2(K) = 2$.

Test tensor $r = 2$: matrix $X = [x^{\gamma\circ}_{\circ\delta}] \equiv \begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

According to (5.81), the first term must be

Let $P = [p^{\alpha\circ\gamma\circ}_{\circ\beta\circ\delta}] = A \otimes X$; There are several possible contractions:

First possible contraction

$$[q^{\alpha\circ}_{\circ\delta}] = \mathcal{C}\begin{pmatrix} \gamma \\ \beta \end{pmatrix} P \equiv \mathcal{C}\begin{pmatrix} \gamma \\ \beta \end{pmatrix} \left[p^{\alpha\circ\gamma\circ}_{\circ\beta\circ\delta}\right] \tag{5.85}$$

Equation (5.85) is stated by “extension”:

$$q_{\sigma'} = H_{n^2,n^4}(\beta,\gamma) P_{\sigma,1} \tag{5.86}$$

with the help of the homomorphism (5.72).

The details are

$$A = \begin{bmatrix} a^{1\circ}_{\circ 1} & a^{1\circ}_{\circ 2} \\ a^{2\circ}_{\circ 1} & a^{2\circ}_{\circ 2} \end{bmatrix}; \quad X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}; \quad P = A \otimes X$$

$$P = \left[\begin{array}{cc|cc} a^{1\circ}_{\circ 1}x & a^{1\circ}_{\circ 1}y & a^{1\circ}_{\circ 2}x & a^{1\circ}_{\circ 2}y \\ a^{1\circ}_{\circ 1}z & a^{1\circ}_{\circ 1}t & a^{1\circ}_{\circ 2}z & a^{1\circ}_{\circ 2}t \\ \hline a^{2\circ}_{\circ 1}x & a^{2\circ}_{\circ 1}y & a^{2\circ}_{\circ 2}x & a^{2\circ}_{\circ 2}y \\ a^{2\circ}_{\circ 1}z & a^{2\circ}_{\circ 1}t & a^{2\circ}_{\circ 2}z & a^{2\circ}_{\circ 2}t \end{array}\right],$$

which in our case is $n = 2; \sigma = n^4 = 2^4 = 16; \sigma' = n^2 = 2^2 = 4$, and then, (5.72) leads to

$$\begin{aligned} H_{4,16}(\beta,\gamma) &= I_2 \otimes [E_1^t | E_2^t] \otimes I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [1 \quad 0 \quad 0 \quad 1] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and (5.86) gives

$$q_4 = H_{4,16}(\beta,\gamma) \bullet P_{16} = \begin{bmatrix} a^{1\circ}_{\circ 1}x + a^{1\circ}_{\circ 2}z \\ a^{1\circ}_{\circ 1}y + a^{1\circ}_{\circ 2}t \\ a^{2\circ}_{\circ 1}x + a^{2\circ}_{\circ 2}z \\ a^{2\circ}_{\circ 1}y + a^{2\circ}_{\circ 2}t \end{bmatrix}$$

and once condensed, we identify with the right-hand of (5.81):

$$\begin{bmatrix} a^{1\circ}_{\circ 1}x + a^{1\circ}_{\circ 2}z & a^{1\circ}_{\circ 1}y + a^{1\circ}_{\circ 2}t \\ a^{2\circ}_{\circ 1}x + a^{2\circ}_{\circ 2}z & a^{2\circ}_{\circ 1}y + a^{2\circ}_{\circ 2}t \end{bmatrix} = \lambda \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

and passing all terms to the left-hand side leads to the matrix system:

$$\begin{cases} [A - \lambda I] \bullet \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [A - \lambda I] \bullet \begin{bmatrix} y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases},$$

the solutions of which are the eigenvalues and eigenvectors of the classic, which has been solved in the first case.

Thus, the solution *matrices*, built by blocks are the following:

$$\begin{matrix} X_1 = [A_1|\mu A_1] \text{ automatrix associated with } \lambda_1 \\ X_2 = [A_2|\nu A_2] \text{ automatrix associated with } \lambda_2 \end{matrix} ; \quad \forall \mu, \nu \in K. \tag{5.87}$$

Second possible contraction

$$[q^{\circ\gamma}_{\beta\circ}] = \mathcal{C}\begin{pmatrix} \alpha \\ \delta \end{pmatrix} [p^{\alpha\circ\gamma\circ}_{\circ\beta\circ\delta}], \tag{5.88}$$

which once stretched leads to the new $q_{\sigma'}$:

$$q_{\sigma'} = H_{4,16}(\alpha, \delta) \cdot P_\sigma. \tag{5.89}$$

With the help of the homomorphism (5.71) we obtain

$$H_{4,16}(\alpha,\delta) = [I_4 \otimes E_1^t | I_4 \otimes E_2^t]$$
$$= \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes [1 \quad 0] \; \middle| \; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes [0 \quad 1] \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and (5.89) gives

$$q_4 = H_{4,16}(\alpha, \delta) P_{16} = \begin{bmatrix} a^{1\circ}_{\circ 1} x + a^{2\circ}_{\circ 1} y \\ a^{1\circ}_{\circ 1} z + a^{2\circ}_{\circ 1} t \\ a^{1\circ}_{\circ 2} x + a^{2\circ}_{\circ 2} y \\ a^{1\circ}_{\circ 2} z + a^{2\circ}_{\circ 2} t \end{bmatrix},$$

which once condensed and according to (5.88) leads to $[q^{\circ\gamma}_{\beta\circ}]$:

$$[q^{\circ\gamma}_{\beta\circ}] = \begin{bmatrix} a^{1\circ}_{\circ 1} x + a^{2\circ}_{\circ 1} y & a^{1\circ}_{\circ 1} z + a^{2\circ}_{\circ 1} t \\ a^{1\circ}_{\circ 2} x + a^{2\circ}_{\circ 2} y & a^{1\circ}_{\circ 2} z + a^{2\circ}_{\circ 2} t \end{bmatrix}.$$

According to (5.81) matrix $[q^{\circ\gamma}_{\beta\circ}]$ must be equal to $\lambda X \equiv \lambda[x^{\gamma\circ}_{\circ\delta}]$, which requires transposing one of them, then

$$[q^{\circ\gamma}_{\beta\circ}]^t = \lambda X; \quad \begin{bmatrix} a^{1\circ}_{\circ 1} x + a^{2\circ}_{\circ 1} y & a^{1\circ}_{\circ 2} x + a^{2\circ}_{\circ 2} y \\ a^{1\circ}_{\circ 1} z + a^{2\circ}_{\circ 1} t & a^{1\circ}_{\circ 2} z + a^{2\circ}_{\circ 2} t \end{bmatrix} = \lambda \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

and passing all terms to the left-hand side, and adequately sorting the equations, yields the matrix system

$$\begin{cases} [A^t - \lambda I] \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ [A^t - \lambda I] \bullet \begin{bmatrix} z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases},$$

the solutions of which are the same eigenvalues λ_1 and λ_2 as in possibility (a), but the eigenvectors A'_1 and A'_2 are those corresponding to matrix A^t. So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = A'_1 \to [\, x \quad y \,] = A'^t_1 \text{ eigenvector of } \lambda_1$$

$$\begin{bmatrix} z \\ t \end{bmatrix} = \mu A'_1 \to [\, z \quad t \,] = \mu A'^t_1 \text{ eigenvector of } \lambda_1$$

and similarly A'_2 and $\nu A'_2$ for $\lambda = \lambda_2$.

Finally, we give the following matrices, built by blocks as left solutions:

$$\left.\begin{array}{l} X_1 = \begin{bmatrix} A_1'^t \\ ----- \\ \mu A_1'^t \end{bmatrix} \text{ automatrix associated with } \lambda_1 \\ \\ X_2 = \begin{bmatrix} A_2'^t \\ ----- \\ \nu A_2'^t \end{bmatrix} \text{ automatrix associated with } \lambda_2 \end{array}\right\} \forall \mu, \nu \in K, \tag{5.90}$$

which satisfy

$$X_1 \bullet A = \lambda_1 X_1 \text{ and } X_2 \bullet A = \lambda_2 X_2.$$

Third case:

Finally, we will study the autotensor of order $r = 3$.

Data tensor: $A = [a^{\alpha\circ}_{\circ\beta}]$, of second order, over n.

Test tensor $r = 3$: (tensor of order 3). Among several possible choices, we select the tensor $X = [x^{\gamma\circ\epsilon}_{\circ\delta\circ}]$.

Let $P = [p^{\alpha\circ\gamma\circ\epsilon}_{\circ\beta\circ\delta\circ}] = A \otimes X$.

X is a contra–cova–contravariant tensor. The possible contraction tensor products are:

$$M = \mathcal{C}\begin{pmatrix}\alpha\\ \delta\end{pmatrix} P = \mathcal{C}\begin{pmatrix}\alpha\\ \delta\end{pmatrix}[p^{\alpha\circ\gamma\circ\epsilon}_{\circ\beta\circ\delta\circ}] = [m^{\circ\gamma\epsilon}_{\beta\circ\circ}], \text{ cova-contra-contravariant}$$

$$N = \mathcal{C}\begin{pmatrix}\epsilon\\ \beta\end{pmatrix} P = \mathcal{C}\begin{pmatrix}\epsilon\\ \beta\end{pmatrix}[p^{\alpha\circ\gamma\circ\epsilon}_{\circ\beta\circ\delta\circ}] = [n^{\alpha\gamma\circ}_{\circ\circ\delta}], \text{ contra-contra-covavariant}$$

$$Q = \mathcal{C}\begin{pmatrix}\gamma\\ \beta\end{pmatrix} P = \mathcal{C}\begin{pmatrix}\gamma\\ \beta\end{pmatrix}[p^{\alpha\circ\gamma\circ\epsilon}_{\circ\beta\circ\delta\circ}] = [q^{\alpha\circ\epsilon}_{\circ\delta\circ}], \text{ contra-cova-contravariant}$$

So, the only valid option is the third one. Since the dimensions of the tensors to be contracted and contracted are, respectively, for $n = 2 : \sigma = 2^3 \times 2^2 = 32$ and $\sigma' = \sigma/2^2 = 32/4 = 8$, the following tensor equations must be satisfied

$$Q = \mathcal{C}\begin{pmatrix}\gamma\\ \beta\end{pmatrix} P = \mathcal{C}\begin{pmatrix}\gamma\\ \beta\end{pmatrix}[A \otimes X] = \lambda X. \tag{5.91}$$

We start from

$$X = [x^{\gamma\circ\epsilon}_{\circ\delta\circ}] = \begin{bmatrix} a & b \\ c & d \\ - & - \\ e & f \\ g & h \end{bmatrix}; \quad A = [a^{\alpha\circ}_{\circ\beta}] = \begin{bmatrix} a^{1\circ}_{\circ 1} & a^{1\circ}_{\circ 2} \\ a^{2\circ}_{\circ 1} & a^{2\circ}_{\circ 2} \end{bmatrix}$$

Having performed the contraction, the fundamental relation (5.91) can be stated as

$$Q = [q^{\alpha \circ \epsilon}_{\circ \delta \circ}] = \begin{bmatrix} q^{1 \circ 1}_{\circ 1 \circ} & q^{1 \circ 2}_{\circ 1 \circ} \\ q^{1 \circ 1}_{\circ 2 \circ} & q^{1 \circ 2}_{\circ 2 \circ} \\ -- & -- \\ q^{2 \circ 1}_{\circ 1 \circ} & q^{2 \circ 2}_{\circ 1 \circ} \\ q^{2 \circ 1}_{\circ 2 \circ} & q^{2 \circ 2}_{\circ 2 \circ} \end{bmatrix} = \begin{bmatrix} a^{1\circ}_{\circ 1} a + a^{1\circ}_{\circ 2} e & a^{1\circ}_{\circ 1} b + a^{1\circ}_{\circ 2} f \\ a^{1\circ}_{\circ 1} c + a^{1\circ}_{\circ 2} g & a^{1\circ}_{\circ 1} d + a^{1\circ}_{\circ 2} h \\ ------ & ------ \\ a^{2\circ}_{\circ 1} a + a^{2\circ}_{\circ 2} e & a^{2\circ}_{\circ 1} b + a^{2\circ}_{\circ 2} f \\ a^{2\circ}_{\circ 1} c + a^{2\circ}_{\circ 2} g & a^{2\circ}_{\circ 1} d + a^{2\circ}_{\circ 2} h \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \\ -- & -- \\ e & f \\ g & h \end{bmatrix} \tag{5.92}$$

passing all terms to the left-hand side, and grouping adequately the equations, we obtain the systems

$$\begin{cases} [A - \lambda I] \bullet \begin{bmatrix} a \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad [A - \lambda I] \bullet \begin{bmatrix} b \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \\ [A - \lambda I] \bullet \begin{bmatrix} c \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad [A - \lambda I] \bullet \begin{bmatrix} d \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}.$$

that can be summarized as

$$[A - \lambda I] \bullet \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} = \Omega_{2,4}.$$

Their interpretation is evident: the matrix solution appears as a permutation of X, and the columns of such a matrix, must be eigenvectors of the eigenvalue λ_1 for X_1, or, for the solution X_2, eigenvectors of the eigenvalue λ_2.

Built by blocks they are

$$X_1 = \begin{bmatrix} \begin{bmatrix} 1 & \mu \\ \nu & \rho \end{bmatrix} \otimes [1 \quad 0] A_1 \\ ------------ \\ \begin{bmatrix} 1 & \mu \\ \nu & \rho \end{bmatrix} \otimes [0 \quad 1] A_1 \end{bmatrix}_{4 \times 2}$$

$$X_2 = \begin{bmatrix} \begin{bmatrix} 1 & \mu' \\ \nu' & \rho' \end{bmatrix} \otimes [1 \quad 0] A_2 \\ ------------ \\ \begin{bmatrix} 1 & \mu \\ \nu & \rho \end{bmatrix} \otimes [0 \quad 1] A_2 \end{bmatrix}_{4 \times 2} ; \forall \mu, \nu, \ldots \rho, \mu', \nu', \ldots \rho' \in K. \tag{5.93}$$

The reader has now enough tools and experience to solve again the problem using the direct homomorphism model 5 in Section 5.8.4, on P_σ. that is, the tensor components of $A \otimes X$ in a column matrix. Then, it can be checked that the resulting matrix $Q_{\sigma'} = H_{\sigma',\sigma} \bullet P_\sigma$ is the stretched expression of the matrix Q in (5.92). Then, the solution, that must be (5.93), can be obtained.

5.11 Generalized multilinear mappings

We analyze here the mapping of a linear space absolute direct product $\begin{pmatrix} r \\ \times V_i^{n_i} \\ 1 \end{pmatrix}(K)$ into an arbitrary linear space $W^m(K)$.

As is well known, we call this an "absolute total" linear space or "total product" linear space, which is denoted by

$$V_1^{n_1} \times V_2^{n_2} \times V_r^{n_r}(K) \text{ or } \begin{pmatrix} r \\ \times V_i^{n_i} \\ 1 \end{pmatrix}(K) \tag{5.94}$$

to a linear space, the vectors of which are r-tuples of vectors chosen one per each factor linear space and in order:

$$(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r) \in \begin{pmatrix} r \\ \times V_i^{n_i} \\ 1 \end{pmatrix}(K); \quad \vec{v}_i \in V_i^{n_i}(K) \tag{5.95}$$

and its dimension $n = n_1 + n_2 + \cdots + n_r$.

Next, we establish two formal axioms that must be satisfied by the generalized multilineal mappings:

1. F is a mapping that associates with each r-tuple of vectors in $\begin{pmatrix} r \\ \times V_i^{n_i} \\ 1 \end{pmatrix}(K)$, a vector $\vec{w} \in W^m(K)$:

$$F : \begin{pmatrix} r \\ \times V_i^{n_i} \\ 1 \end{pmatrix}(K) \to W^m(K) \tag{5.96}$$

for all r-tuple it is

$$F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r) = \vec{w} \in W^m(K). \tag{5.97}$$

2. This mapping is multilinear:

$$\begin{aligned} F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_h' + \vec{v}_h'', \ldots, \vec{v}_r) = F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_h', \ldots, \vec{v}_r) \\ + F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_h'', \ldots, \vec{v}_r) \end{aligned} \tag{5.98}$$

$$F(\vec{v}_1, \vec{v}_2, \ldots, \lambda\vec{v}_h, \ldots, \vec{v}_r) = \lambda F(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_h, \ldots, \vec{v}_r); \quad 1 \le h \le r. \tag{5.99}$$

Based on these axioms, we will establish how data are presented and what the operative formulas are for practical use. First, we select bases for the intervening linear spaces, and thus, to the vectors of components:

$$\begin{array}{lllll} \vec{v}_1 = \vec{e}_{\beta_1} x^{\beta_1 \circ}_{\circ\, 1} & \text{with} & \vec{v}_1 \in V_1^{n_1}(K) & \text{and} & 1 \leq \beta_1 \leq n_1 \\ \vec{v}_2 = \vec{e}_{\beta_2} x^{\beta_2 \circ}_{\circ\, 2} & \text{with} & \vec{v}_2 \in V_2^{n_2}(K) & \text{and} & 1 \leq \beta_2 \leq n_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vec{v}_i = \vec{e}_{\beta_i} x^{\beta_i \circ}_{\circ\, i} & \text{with} & \vec{v}_i \in V_i^{n_i}(K) & \text{and} & 1 \leq \beta_i \leq n_i \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vec{v}_r = \vec{e}_{\beta_r} x^{\beta_r \circ}_{\circ\, r} & \text{with} & \vec{v}_r \in V_r^{n_r}(K) & \text{and} & 1 \leq \beta_r \leq n_r, \end{array} \tag{5.100}$$

where $\forall x^{\beta_i \circ}_{\circ\, i}$ is *data*.

When introducing these data in (5.97), on account of (5.98) and (5.99), we obtain

$$\vec{w} = F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r) = x^{\beta_1 \circ}_{\circ\, 1} \cdot x^{\beta_2 \circ}_{\circ\, 2} \cdot \cdots \cdot x^{\beta_r \circ}_{\circ\, r} F(\vec{e}_{\beta_1}, \vec{e}_{\beta_2}, \ldots, \vec{e}_{\beta_r}). \tag{5.101}$$

This expression with contracted dummy indices has a total of $\sigma = n_1 \cdot n_2 \cdot \cdots \cdot n_r$ summands, which correspond with the possibilities of the r-tuples $(\vec{e}_{\beta_1}, \vec{e}_{\beta_2}, \ldots, \vec{e}_{\beta_r})$.

Assume now that the σ basic mappings:

$$F(\vec{e}_{\beta_1}, \vec{e}_{\beta_2}, \ldots, \vec{e}_{\beta_r}) = \vec{w}(\beta_1, \beta_2, \ldots, \beta_r); \quad \vec{w}(\beta_1, \beta_2, \ldots, \beta_r) \in W^m(K) \tag{5.102}$$

are given (again data).

We also assume that vectors $\vec{w}(\beta_1, \beta_2, \cdots, \beta_r)$ are data of the following form.

If the basis of the linear space $W^m(K)$ is $\{\vec{\epsilon}_k\}_1^m$, expressing the vector $\vec{w}(\beta_1, \beta_2, \ldots, \beta_r)$ as a *vector covariant tensor*:

$$\begin{aligned} \vec{w}(\beta_1, \beta_2, \ldots, \beta_r) = {} & w^{1 \circ\ \circ \cdots \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r} \vec{\epsilon}_1 + w^{2 \circ\ \circ \cdots \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r} \vec{\epsilon}_2 + \cdots \\ & + w^{k \circ\ \circ \cdots \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r} \vec{\epsilon}_k + \cdots + w^{m \circ\ \circ \cdots \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r} \vec{\epsilon}_m, \end{aligned} \tag{5.103}$$

where the vector coefficients are mounted with the corresponding covariant tensors, the m covariant tensors are *the data that characterize* the mapping $F(\vec{e}_{\beta_1}, \vec{e}_{\beta_2}, \ldots, \vec{e}_{\beta_r}) = \vec{w}(\beta_1, \beta_2, \ldots, \beta_r)$. (In reality $\vec{w}(\beta_1, \beta_2, \ldots, \beta_r)$ is a vector covariant tensor built with vectors of $W^m(K)$, instead of scalars of K; the reader can see this by executing the sum indicated in (5.103) by separate summands, and then grouping them *into a single entity*.

Assuming that F is delivered as indicated, in (5.103), and entering it in (5.101) we obtain the image of the stated multilinear mapping, by means of the final calculation formula:

$$\vec{w} = x^{\beta_1 \circ}_{\circ\, 1} \cdot x^{\beta_2 \circ}_{\circ\, 2} \cdot \cdots \cdot x^{\beta_r \circ}_{\circ\, r} \big(w^{1 \circ\ \circ \cdots \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r} \vec{\epsilon}_1 + w^{2 \circ\ \circ \cdots \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r} \vec{\epsilon}_2 + \cdots + w^{m \circ\ \circ \cdots \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r} \vec{\epsilon}_m \big), \tag{5.104}$$

which is built with m contracted products of the contravariant components of the data vectors by the covariant components of the multilinear mapping F.

One perfectly detects that in Formula (5.104) the notation used has a free index in the *interior* of the coefficients (the index h of $w^{h\,\circ\,\circ\,\cdots\,\circ}_{\circ\,\beta_1\beta_2\cdots\beta_r}$) but it is useful for the calculation; it is the "vector" index of the basis $\{\vec{\epsilon}_h\}$ of $W^m(K)$.

If in Formula (5.104) we take as fixed, for example, the vectors $(\vec{v}_2)_0, (\vec{v}_3)_0, \ldots, (\vec{v}_r)_0$, leaving as dummy the $\vec{v}_1$, since they are constant during all the multilinear mappings F all $(x^{\beta_h\,\circ}_{\circ\,h})_0; 2 \leq h \leq r$ the multilinear mapping degenerates into a homomorphism H_1 that applies $H_1 : V_1^{n_1}(K) \to W^m(K)$; similarly, if we fix as constant other vectors $\vec{v}_h$ with the exception of a given vector. This is the way most authors *define* multilinear mappings, which in the authors present opinion is correct, but not useful from a practical point of view, because none of them arrives at a concrete expression, like the one in (5.104).

5.11.1 Theorems of similitude with tensor mappings

Theorem 5.6 (Similitude). *There exists a univocal correspondence between the σ r-tuples $(\vec{e}_{\beta_1}, \vec{e}_{\beta_2}, \ldots, \vec{e}_{\beta_r})$; $1 \leq \beta_i \leq n_i; i \in I_r$ that appear in Formula (5.101) and the σ basic tensor products, of the basis $\mathcal{B}' = \{\vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \ldots \otimes \vec{e}_{\beta_r}\}$ of the tensor space $V_1^{n_1} \otimes V_2^{n_2} \otimes \cdots \otimes V_r^{n_r}(K) \equiv \left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}\right)(K)$*

$$(\vec{e}_{\beta_1}, \vec{e}_{\beta_2}, \cdots, \vec{e}_{\beta_r}) \overset{\longrightarrow}{\longleftarrow} \vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r}. \tag{5.105}$$

□

It should be surprising for any reader the evidence of the above theorem's final expression. Next, we give a second theorem that is based on the one above.

Theorem 5.7 (Similitude). *There exists a unique multilinear mapping:*

$$F' : \left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}\right)(K) \to W^m(K),$$

such that

$$F'(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r) = F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r) = \vec{w}; \quad \vec{w} \in W^m(K); \quad \forall \vec{v}_i \in V_i^{n_i}(K). \tag{5.106}$$

□

So that the problem of solving images by means of the multilinear mapping $F : \left(\overset{r}{\underset{1}{\times}} V_i^{n_i}\right)(K) \to W^m(K)$ can be solved indistinctly, with the tensor multilinear morphism $F' : \left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}\right)(K) \to W^m(K)$, by simply changing the notation with the help of Formula (5.105).

Finally, we consider two tensor spaces: the tensor space

$$A \equiv [V_1^{n_1} \otimes V_2^{n_2} \otimes \cdots \otimes V_r^{n_r}(K)] \otimes [V_1^{n_1} \otimes V_2^{n_2} \otimes \cdots \otimes V_r^{n_r}(K)]^*$$

and the tensor space B, the set of all (tensor) multilinear *endomorphisms* that operate inside the tensor space $V_1^{n_1} \otimes V_2^{n_2} \otimes \cdots \otimes V_r^{n_r}(K)$:

$$B = \mathcal{ML}\left[V_1^{n_1} \otimes V_2^{n_2} \otimes \cdots \otimes V_r^{n_r}(K), V_1^{n_1} \otimes V_2^{n_2} \otimes \cdots \otimes V_r^{n_r}(K)\right].$$

Theorem 5.8 (Similitude). *There exists a unique isomorphism Φ*

$$\Phi : \left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}(K)\right) \otimes \left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}(K)\right)^* \overset{\rightarrow}{\leftarrow} \mathcal{ML}\left[\left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}\right)(K), \left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}\right)(K)\right] \tag{5.107}$$

such that with each tensor $(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r) \otimes (\vec{u}_1 \otimes \vec{u}_2 \otimes \cdots \otimes \vec{u}_r)^* \in A$ it associates a tensor multilinear endomorphism

$$T_{(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r) \otimes (\vec{u}_1 \otimes \vec{u}_2 \otimes \cdots \otimes \vec{u}_r)^*} \in B,$$

that transforms the multivectors $\vec{w} = \vec{w}_1 \otimes \vec{w}_2 \otimes \cdots \otimes \vec{w}_r \in \left(\overset{r}{\underset{1}{\otimes}} V_i^{n_i}(K)\right)$ into the following form:

$$T(\vec{w}) = [(\vec{w}_1 \otimes \vec{w}_2 \otimes \cdots \otimes \vec{w}_r) \bullet (\vec{u}_1 \otimes \vec{u}_2 \otimes \cdots \otimes \vec{u}_r)^*]\,(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r). \tag{5.108}$$

□

Theorems 5.7 and 5.8 will be proved by means of concrete models in the proposed examples, so that the interested reader will be able to obtain the general proofs.

5.11.2 Tensor mapping types

If we reconstruct Formula (5.101) adapted for generalized tensor mapping or as a mapping of the correspondence (5.105):

$$\vec{w} = F(\vec{t}) = F(t^{\beta_1 \beta_2 \cdots \beta_r}_{\circ\ \circ\ \cdots\ \circ}\vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r}) = t^{\beta_1 \beta_2 \cdots \beta_r}_{\circ\ \circ\ \cdots\ \circ} F(\vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r}) \tag{5.109}$$

and we do the same with (5.102) and (5.103):

$$F(\vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r}) = \vec{w}(\beta_1, \beta_2, \cdots, \beta_r), \tag{5.110}$$

the development of the tensor mapping is performed using the same expression (5.104) but with these changes.

It is obvious that in (5.104) the tensor coefficients $w_{\circ\,\beta_1\,\beta_2\cdots\beta_r}^{h\,\circ\;\circ\;\cdots\;\circ}$ with $1 \leq h \leq m$, can be in some cases *symmetric*, or *anti-symmetric* for the covariant subindices leading to the existence of *tensor mappings F -symmetric and F-anti-symmetric.*

The tensor F-anti-symmetric mappings will be studied in later chapters.

It must be clarified, however, that the tensor mapping type F is completely *independent* of the tensor type over which it is applied, in other words, for example *it is not necessary* to transform symmetric tensors with symmetric mappings.

5.11.3 Direct n-dimensional tensor endomorphisms

We study here the particular case of tensor mappings. Consider the tensor space $\overset{r}{\underset{1}{\otimes}} V_i^n(K) \equiv V_1^n \otimes V_2^n \otimes \cdots \otimes V_r^n(K)$ tensor product of r n-dimensional linear spaces of dimension $\sigma = n^r$, over the same field K. We assume that in each of the linear spaces $V_i^n(K)$ acts an endomorphism of associated square matrix H_i of order n, which transforms the vectors $\vec{v}_i \in V_i^n(K)$ in $H_i(\vec{v}_i) = \vec{w}_i \in V_i^n(K)$.

We look for the heterogeneous tensor endomorphism H_σ, which applies the prototype multivector $\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r \in \overset{r}{\underset{1}{\otimes}} V_i^n(K)$ on the image multivector $\vec{w}_1 \otimes \vec{w}_2 \otimes \cdots \otimes \vec{w}_r \in \overset{r}{\underset{1}{\otimes}} V_i^n(K)$, that is,

$$H_\sigma(\vec{v}) = \vec{w} \Leftrightarrow H_\sigma(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r) = \vec{w}_1 \otimes \vec{w}_2 \otimes \cdots \otimes \vec{w}_r. \tag{5.111}$$

We solve the problem in a direct form until we find H_σ. Later, the result will be related with the formulas in Section 5.11.

If we notate in tensor form the individual endomorphisms, if $\vec{v}_i = x_{(i)}{}^{\alpha_i}_{\circ}\vec{e}_{\alpha_i}$ and $\vec{w}_i = y_{(i)}{}^{\beta_j}_{\circ}\vec{e}_{\beta_j}$ with $\alpha_i, \beta_j \in I_n;\ i, j \in I_r$, the result is

$$y_{(i)}{}^{\beta_j}_{\circ} = h_{(i)}{}^{\beta_j\ \circ}_{\circ\ \alpha_j} x_{(i)}{}^{\alpha_j}_{\circ}. \tag{5.112}$$

Replacing in $\vec{w} = \vec{w}_1 \otimes \vec{w}_2 \otimes \cdots \otimes \vec{w}_r$ the expression of each vector, we arrive at

$$\begin{aligned} \vec{w} &= (y_{(1)}{}^{\beta_1}_{\circ}\vec{e}_{\beta_1}) \otimes (y_{(2)}{}^{\beta_2}_{\circ}\vec{e}_{\beta_2}) \otimes \cdots \otimes (y_{(r)}{}^{\beta_r}_{\circ}\vec{e}_{\beta_r}) \\ &= \left(y_{(1)}{}^{\beta_1}_{\circ} y_{(2)}{}^{\beta_2}_{\circ} \cdots y_{(r)}{}^{\beta_r}_{\circ}\right) \vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r} \end{aligned} \tag{5.113}$$

and replacing (5.112) we get

$$\begin{aligned} \vec{w} = &\left[\left(h_{(1)}{}^{\beta_1\ \circ}_{\circ\ \alpha_1} x_{(1)}{}^{\alpha_1}_{\circ}\right)\left(h_{(2)}{}^{\beta_2\ \circ}_{\circ\ \alpha_2} x_{(2)}{}^{\alpha_2}_{\circ}\right) \cdots \left(h_{(r)}{}^{\beta_r\ \circ}_{\circ\ \alpha_r} x_{(r)}{}^{\alpha_r}_{\circ}\right)\right] \\ &\vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r}, \end{aligned} \tag{5.114}$$

which after operating and grouping yields

$$\begin{aligned}\vec{w} &= H_\sigma(\vec{v}) = H_\sigma(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r)\\ &= \left(x_{(1)}{}^{\alpha_1}_{\circ} x_{(2)}{}^{\alpha_2}_{\circ} \cdots x_{(r)}{}^{\alpha_r}_{\circ}\right)\left(h_{(1)}{}^{\beta_1\ \circ}_{\circ\ \alpha_1} h_{(2)}{}^{\beta_2\ \circ}_{\circ\ \alpha_2} \cdots h_{(r)}{}^{\beta_r\ \circ}_{\circ\ \alpha_r}\right)\\ &\quad \vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r}. \end{aligned} \tag{5.115}$$

If we write

$$h^{\beta_1\ \circ\ \beta_2\ \circ\ \cdots \beta_r\ \circ}_{\circ\ \alpha_1\ \circ\ \alpha_2 \cdots\ \circ\ \alpha_r} = h_{(1)}{}^{\beta_1\ \circ}_{\circ\ \alpha_1} h_{(2)}{}^{\beta_2\ \circ}_{\circ\ \alpha_2} \cdots h_{(r)}{}^{\beta_r\ \circ}_{\circ\ \alpha_r}, \tag{5.116}$$

Expression (5.115) becomes

$$\begin{aligned}\vec{w} &= H_\sigma(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r)\\ &= \left(x_{(1)}{}^{\alpha_1}_{\circ} x_{(2)}{}^{\alpha_2}_{\circ} \cdots x_{(r)}{}^{\alpha_r}_{\circ}\right)\left(h^{\beta_1\ \circ\ \beta_2\ \circ\ \cdots \beta_r\ \circ}_{\circ\ \alpha_1\ \circ\ \alpha_2 \cdots\ \circ\ \alpha_r}\right)\\ &\quad \vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r}. \end{aligned} \tag{5.117}$$

Since $x_{(j)}{}^{\alpha_j}_{\circ}$ are the vector data $\vec{v}_j$ and $h_{(j)}{}^{\beta_j\ \circ}_{\circ\ \alpha_j}$ are the endomorphism data inside each $V_j^n(K)$, Formula (5.117) solves the problem stated in this section.

In matrix form, expression (5.116) is solved in the matrix

$$H_\sigma = H_1 \otimes H_2 \otimes \cdots \otimes H_r. \tag{5.118}$$

If the column matrix $V_{\sigma,1}$ is an extension of the components of $\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r$, and the column matrix $W_{\sigma,1}$ is an extension of the components of $\vec{w}_1 \otimes \vec{w}_2 \otimes \cdots \otimes \vec{w}_r$, then, expression (5.117) leads to the endomorphism (in matrix form)

$$W_{\sigma,1} = H_\sigma \bullet V_{\sigma,1}. \tag{5.119}$$

If we consider

$$m \equiv n^r; \vec{\epsilon}_k \equiv \vec{e}_{\beta_1} \otimes \vec{e}_{\beta_2} \otimes \cdots \otimes \vec{e}_{\beta_r},$$

with $1 \le k \le m$ and finally $h^{\beta_1\ \circ\ \beta_2\ \circ\ \cdots \beta_r\ \circ}_{\circ\ \alpha_1\ \circ\ \alpha_2 \ldots\ \circ\ \alpha_r} \equiv w^{k\ \circ\ \circ\ \cdots\ \circ}_{\circ \beta_1 \beta_2 \cdots \beta_r}$, the tensor equation (5.117) represents a variant of Formula (5.104).

One can easily conclude that Formulas (5.104) and (5.119) can be applied to tensors in $\overset{r}{\underset{1}{\otimes}} V_i^n(K)$ not coming from tensor products, as it was indicated in Formula (5.109) and will be in the following formulas.

Example 5.9 (Proof of Theorem 5.7). In this example we prove the tensor similitude Theorem 5.7 for the homogeneous case with the help of tensor and matrix tools.

Consider the homogeneous linear space "total product" (initial space):

$$\left(\begin{matrix} r \\ \times (V^n)(K) \\ 1 \end{matrix}\right) \equiv V^n \times V^n \times V^n \cdots \times V^n(K)$$

of dimension $(r \cdot n)$. Let the r-tuple $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r)$ be one of its vectors, where $\forall \vec{v}_i \in V^n(K)$, and consider the final linear space $W^m(K)$. The vectors $\vec{v}_i = x^{\alpha_i \circ}_{\circ\ i} \vec{e}_{\alpha_i}$ are given by its components $(x^{\alpha_i \circ}_{\circ\ i})$.

Consider a multilinear mapping F that applies the initial space on the final space by means of the following report, the coefficients f of which are data tensors:

$$F : \begin{pmatrix} r \\ \times (V^n)(K) \\ 1 \end{pmatrix} \to W^m(K)$$

$$F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r) = f^{\beta\ \circ\ \ \circ\ \cdots\ \circ}_{\circ\ \alpha_1 \alpha_2 \cdots \alpha_r} x^{\alpha_1 \circ}_{\circ\ 1} x^{\alpha_2 \circ}_{\circ\ 2} \cdots x^{\alpha_r \circ}_{\circ\ r} \vec{\epsilon}_\beta, \tag{5.120}$$

where $\{\vec{\epsilon}_\beta\}$ is the basis of the linear space $W^m(K)$, with $\alpha_i \in I_n; 1 \leq \beta \leq m$.

Developing the sum associated with index β in (5.120) to obtain its matrix expression, we get

$$\begin{aligned}
&F(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r) = f^{1\ \circ\ \ \circ\ \cdots\ \circ}_{\circ\ \alpha_1 \alpha_2 \cdots \alpha_r} x^{\alpha_1 \circ}_{\circ\ 1} x^{\alpha_2 \circ}_{\circ\ 2} \cdots x^{\alpha_r \circ}_{\circ\ r} \vec{\epsilon}_1 \\
&+ f^{2\ \circ\ \ \circ\ \cdots\ \circ}_{\circ\ \alpha_1 \alpha_2 \cdots \alpha_r} x^{\alpha_1 \circ}_{\circ\ 1} x^{\alpha_2 \circ}_{\circ\ 2} \cdots x^{\alpha_r \circ}_{\circ\ r} \vec{\epsilon}_2 + \cdots + f^{m\ \circ\ \ \circ\ \cdots\ \circ}_{\circ\ \alpha_1 \alpha_2 \cdots \alpha_r} x^{\alpha_1 \circ}_{\circ\ 1} x^{\alpha_2 \circ}_{\circ\ 2} \cdots x^{\alpha_r \circ}_{\circ\ r} \vec{\epsilon}_m \\
&= [\vec{\epsilon}_1 \vec{\epsilon}_2 \cdots \vec{\epsilon}_m] \bullet \begin{bmatrix}
f^{1 \circ \circ \cdots \circ}_{\circ 1 1 \cdots 1} & f^{1 \circ \circ \cdots \circ}_{\circ 1 1 \cdots 2} & \cdots & f^{1\ \circ\ \ \circ\ \cdots\ \circ}_{\circ\ \alpha_1 \alpha_2 \cdots \alpha_r} & \cdots & f^{1 \circ \circ \cdots \circ}_{\circ n n \cdots n} \\
f^{2 \circ \circ \cdots \circ}_{\circ 1 1 \cdots 1} & f^{2 \circ \circ \cdots \circ}_{\circ 1 1 \cdots 2} & \cdots & f^{2\ \circ\ \ \circ\ \cdots\ \circ}_{\circ\ \alpha_1 \alpha_2 \cdots \alpha_r} & \cdots & f^{2 \circ \circ \cdots \circ}_{\circ n n \cdots n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
f^{m \circ \circ \cdots \circ}_{\circ 1 1 \cdots 1} & f^{m \circ \circ \cdots \circ}_{\circ 1 1 \cdots 2} & \cdots & f^{m\ \circ\ \ \circ\ \cdots\ \circ}_{\circ\ \alpha_1 \alpha_2 \cdots \alpha_r} & \cdots & f^{m \circ \circ \cdots \circ}_{\circ n n \cdots n}
\end{bmatrix} \\
&\bullet \begin{bmatrix}
x^{1 \circ}_{\circ 1} x^{1 \circ}_{\circ 2} \cdots x^{1 \circ}_{\circ r} \\
x^{1 \circ}_{\circ 1} x^{1 \circ}_{\circ 2} \cdots x^{2 \circ}_{\circ r} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
x^{n \circ}_{\circ 1} x^{n \circ}_{\circ 2} \cdots x^{n \circ}_{\circ r}
\end{bmatrix},
\end{aligned} \tag{5.121}$$

the symbolic matrix expression of which, with declaration of the sizes of the matrices appearing (with $\sigma = n^r$) is

$$F(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r) = [\vec{\epsilon}_1 \vec{\epsilon}_2 \cdots \vec{\epsilon}_m] H_{m,\sigma} \left[x^{\alpha_1 \circ}_{\circ\ 1} x^{\alpha_2 \circ}_{\circ\ 2} \cdots x^{\alpha_r \circ}_{\circ\ r} \right]_{\sigma,1}, \tag{5.122}$$

which is the matrix expression of the multilinear mapping F.

Next, we will discover a multilinear tensor morphism F'. Remembering that

$$\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r = x^{\alpha_1 \circ}_{\circ\ 1} x^{\alpha_2 \circ}_{\circ\ 2} \cdots x^{\alpha_r \circ}_{\circ\ r} \vec{e}_{\alpha_1} \otimes \vec{e}_{\alpha_2} \otimes \cdots \otimes \vec{e}_{\alpha_r},$$

and applying Theorems 5.6 and 5.7 we choose the following equality:

$$F'(\vec{e}_{\alpha_1} \otimes \vec{e}_{\alpha_2} \otimes \cdots \otimes \vec{e}_{\alpha_r}) \equiv F(\vec{e}_{\alpha_1}, \vec{e}_{\alpha_2}, \cdots, \vec{e}_{\alpha_r})$$

and then

$$\begin{aligned} F'(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r) &= F'(x^{\alpha_1}_{\circ\,1}{}^{\circ} x^{\alpha_2}_{\circ\,2}{}^{\circ} \cdots x^{\alpha_r}_{\circ\,r}{}^{\circ} \vec{e}_{\alpha_1} \otimes \vec{e}_{\alpha_2} \otimes \cdots \otimes \vec{e}_{\alpha_r}) \\ &= F'(\vec{e}_{\alpha_1} \otimes \vec{e}_{\alpha_2} \otimes \cdots \otimes \vec{e}_{\alpha_r})(x^{\alpha_1}_{\circ\,1}{}^{\circ} x^{\alpha_2}_{\circ\,2}{}^{\circ} \cdots x^{\alpha_r}_{\circ}) \\ &= F(\vec{e}_{\alpha_1}, \vec{e}_{\alpha_2}, \cdots, \vec{e}_{\alpha_r})(x^{\alpha_1}_{\circ\,1}{}^{\circ} x^{\alpha_2}_{\circ\,2}{}^{\circ} \cdots x^{\alpha_r}_{\circ}). \end{aligned} \tag{5.123}$$

If now we apply (5.122) to the vectors $(\vec{e}_{\alpha_1}, \vec{e}_{\alpha_2}, \ldots, \vec{e}_{\alpha_r})$ in matrix form we get

$$F(\vec{e}_{\alpha_1}, \vec{e}_{\alpha_2}, \ldots, \vec{e}_{\alpha_r}) = [\vec{\epsilon}_1, \vec{\epsilon}_2, \ldots, \vec{\epsilon}_m] H_{m,\sigma} \bullet [E_{\alpha_1} \otimes E_{\alpha_2} \otimes \cdots \otimes E_{\alpha_r}]_{\sigma,1}\,, \tag{5.124}$$

where $\{E_{\alpha_i}\}$ is the matrix canonical basis of $V^n(K)$.

The matrix

$$\begin{aligned} &H'_{m,\sigma} = H_{m,\sigma} \\ &\bullet [E_1 \otimes E_1 \otimes \cdots \otimes E_1 | \cdots | E_{\alpha_1} \otimes E_{\alpha_2} \otimes \cdots \otimes E_{\alpha_r} | \ldots | E_n \otimes E_n \otimes \cdots \otimes E_n]_{\sigma,\sigma} \end{aligned} \tag{5.125}$$

represents the operator F', and then the final expression for Formula (5.123) is

$$F'(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_r) = [\vec{\epsilon}_1, \vec{\epsilon}_2, \ldots, \vec{\epsilon}_n] H'_{m,\sigma} \left[x^{\alpha_1}_{\circ\,1}{}^{\circ} x^{\alpha_2}_{\circ\,2}{}^{\circ} \cdots x^{\alpha_r}_{\circ} \right]_{\sigma,1}. \tag{5.126}$$

Developing Equation (5.125) one gets

$$H'_{m,\sigma} = H_{m,\sigma} \bullet I_n \equiv H_{m,\sigma},$$

which proves our theorem.

$\square$

Example 5.10 (Confirmation of Theorem 5.7). We wish to prove the similitude Theorem 5.7 by means of the following model. Consider two linear spaces $U^m(K)$ and $V^n(K)$ referred to their bases $\{\vec{e}_{\alpha_1}\}_1^m$ and $\{\vec{e}_{\alpha_2}\}_1^n$, respectively, and the two vectors

$$\vec{u}(x^1, x^2, \ldots, x^m) \in U^m(K) \text{ and } \vec{v}(y^1, y^2, \ldots, y^n) \in V^n(K).$$

Consider also another linear space $W^{m\times n}(K)$ referred to a basis $\{\vec{\epsilon}_k\}_1^{m\times n}$, and a bilinear mapping:

$$F : U^m \times V^n(K) \to W^{m\times n}(K),$$

which transforms the vector duples of the "direct product" space $U^m \times V^n(K)$, into vectors of $W^{m\times n}(K)$ by means of

$$\vec{w} \equiv F(\vec{u}, \vec{v}) = [x^1 x^2 \cdots x^m] \begin{bmatrix} \vec{w}_{11} & \vec{w}_{12} & \cdots & \vec{w}_{1n} \\ \vec{w}_{21} & \vec{w}_{22} & \cdots & \vec{w}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \vec{w}_{m1} & \vec{w}_{m2} & \cdots & \vec{w}_{mn} \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix}, \tag{5.127}$$

where each of the vectors $\vec{w}_{\alpha\beta}$ in this matrix comes from $W^{m\times n}(K)$ and can be written as

$$\vec{w}_{\alpha\beta} = w^{1\circ\circ}_{\circ\alpha\beta}\vec{\epsilon}_1 + w^{2\circ\circ}_{\circ\alpha\beta}\vec{\epsilon}_2 + \cdots + w^{m\times n\circ\circ}_{\circ\ \ \alpha\beta}\vec{\epsilon}_{m\times n};\ \ 1 \le \alpha \le m;\ \ 1 \le \beta \le n. \tag{5.128}$$

1. Give a matrix expression of the image vector $\vec{w}$.
2. Prove the existence of the mapping $F'(\vec{u} \otimes \vec{v})$ in Theorem 5.7.
3. Answer questions 1 and 2 for the particular case

$$m = 2;\ \ n = 3;\ \ \vec{u}(2, -1);\ \ \vec{v}(3, 2, 1);$$
$$\vec{w}_{11} = 2\vec{\epsilon}_2 - 3\vec{\epsilon}_3;\ \ \vec{w}_{12} = \vec{0};\ \ \vec{w}_{13} = 5\vec{\epsilon}_1 + 2\vec{\epsilon}_2 - \vec{\epsilon}_4 + \vec{\epsilon}_6;$$
$$\vec{w}_{21} = \vec{\epsilon}_1 + \vec{\epsilon}_6;\ \ \vec{w}_{22} = \vec{\epsilon}_2 - \vec{\epsilon}_5;\ \ \vec{w}_{23} = \vec{\epsilon}_1 + \vec{\epsilon}_2 - \vec{\epsilon}_3 - \vec{\epsilon}_4 + \vec{\epsilon}_6.$$

Solution:

1. Expression (5.127) can be written in tensor form as

$$\vec{w} = F(\vec{u}, \vec{v}) = x^{\alpha}_{\circ} y^{\beta}_{\circ} \vec{w}_{\alpha\beta}, \tag{5.129}$$

and developing the sums associated with the dummy indices α and β, and writing them as a matrix product and, as required, representing the *vector* matrices as row matrices, we finally get

$$\vec{w} = F(\vec{u}, \vec{v}) = [\vec{w}_{11}\vec{w}_{12} \cdots \vec{w}_{1n}\vec{w}_{21}\vec{w}_{22} \cdots \vec{w}_{2n} \cdots \vec{w}_{m1}\vec{w}_{m2} \cdots \vec{w}_{mn}] \begin{bmatrix} x^1_\circ y^1_\circ \\ x^1_\circ y^2_\circ \\ \vdots \\ x^1_\circ y^n_\circ \\ x^2_\circ y^1_\circ \\ x^2_\circ y^2_\circ \\ \vdots \\ x^2_\circ y^n_\circ \\ \vdots \\ x^m_\circ y^1_\circ \\ x^m_\circ y^2_\circ \\ \vdots \\ x^m_\circ y^n_\circ \end{bmatrix},$$

which is the answer to the first question.

2. Substituting vectors $\vec{w}_{\alpha\beta}$ in Formula (5.128) into the last expression and grouping in matrix form yields

$$\vec{w} = F(\vec{u},\vec{v})$$

$$= [\vec{\epsilon}_1\vec{\epsilon}_2\cdots\vec{\epsilon}_{m\times n}] \begin{bmatrix} w^{1\circ\circ}_{\circ 11} & w^{1\circ\circ}_{\circ 12} & \cdots & w^{1\circ\circ}_{\circ\alpha\beta} & \cdots & w^{1\circ\circ}_{\circ mn} \\ w^{2\circ\circ}_{\circ 11} & w^{2\circ\circ}_{\circ 12} & \cdots & w^{2\circ\circ}_{\circ\alpha\beta} & \cdots & w^{2\circ\circ}_{\circ mn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w^{m\times n\circ\circ}_{\circ\ 11} & w^{m\times n\circ\circ}_{\circ\ 12} & \cdots & w^{m\times n\circ\circ}_{\circ\ \alpha\beta} & \cdots & w^{m\times n\circ\circ}_{\circ\ mn} \end{bmatrix} \begin{bmatrix} x^1_\circ y^1_\circ \\ x^1_\circ y^2_\circ \\ \vdots \\ x^1_\circ y^n_\circ \\ x^2_\circ y^1_\circ \\ x^2_\circ y^2_\circ \\ \vdots \\ x^2_\circ y^n_\circ \\ \vdots \\ x^m_\circ y^1_\circ \\ x^m_\circ y^2_\circ \\ \vdots \\ x^m_\circ y^n_\circ \end{bmatrix}. \tag{5.130}$$

Note that Expression (5.130) is the matrix expression of a multilinear mapping $F'(\vec{u}\otimes\vec{v})$ by means of the *central data matrix*, which "stacks" tensor F'.
Consequently $F(\vec{u},\vec{v}) = F'(\vec{u}\otimes\vec{v}) = \vec{w}$, which is Theorem 5.7, answering the second question.

3. Next, we illustrate this numerically.

$$m\cdot n = 2\times 3 = 6;\quad \vec{u} = [\,\vec{e}_1 \quad \vec{e}_2\,] \begin{bmatrix} 2 \\ -1 \end{bmatrix};\quad \vec{v} = [\,\vec{e}\,'_1 \quad \vec{e}\,'_2 \quad \vec{e}\,'_3\,] \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$

where $\vec{u}\in U^2(\mathbb{R})$ and $\vec{v}\in V^3(\mathbb{R})$.
Let $\vec{z} = \vec{u}\otimes\vec{v} = \left([\,\vec{e}_1 \quad \vec{e}_2\,]\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)\otimes\left([\,\vec{e}\,'_1 \quad \vec{e}\,'_2 \quad \vec{e}\,'_3\,]\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right)$; by extension we get

$$\vec{z} = ([\,\vec{e}_1 \quad \vec{e}_2\,]\otimes[\,\vec{e}\,'_1 \quad \vec{e}\,'_2 \quad \vec{e}\,'_3\,]) \bullet \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}^t \otimes \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}^t\right)^t$$

$$= [\,\vec{e}_1 \otimes \vec{e}\,'_1 \;\; \vec{e}_1 \otimes \vec{e}\,'_2 \;\; \vec{e}_1 \otimes \vec{e}\,'_3 \;\; \vec{e}_2 \otimes \vec{e}\,'_1 \;\; \vec{e}_2 \otimes \vec{e}\,'_2 \;\; \vec{e}_2 \otimes \vec{e}\,'_3\,] \bullet \begin{bmatrix} 6 \\ 4 \\ 2 \\ -3 \\ -2 \\ -1 \end{bmatrix}.$$

The vector $\vec{z} = \vec{u} \otimes \vec{v} \in U^2 \otimes V^3(\mathbb{R})$ will be useful later.
Using Formula (5.127), we obtain

$$\vec{w} = F(\vec{u}, \vec{v}) = [\,2 \quad -1\,] \begin{bmatrix} 2\vec{\epsilon}_2 - 3\vec{\epsilon}_3 & \vec{0} & 5\vec{\epsilon}_1 + 2\vec{\epsilon}_2 - \vec{\epsilon}_4 + \vec{\epsilon}_6 \\ \vec{\epsilon}_1 + \vec{\epsilon}_6 & \vec{\epsilon}_2 - \vec{\epsilon}_5 & \vec{\epsilon}_1 + \vec{\epsilon}_2 - \vec{\epsilon}_3 - \vec{\epsilon}_4 + \vec{\epsilon}_6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= [\,2 \quad -1\,] \begin{bmatrix} 0 & 0 & 5 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \vec{\epsilon}_1 + [\,2 \quad -1\,] \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \vec{\epsilon}_2$$

$$+[\,2 \quad -1\,] \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \vec{\epsilon}_3 + [\,2 \quad -1\,] \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \vec{\epsilon}_4$$

$$+[\,2 \quad -1\,] \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \vec{\epsilon}_5 + [\,2 \quad -1\,] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \vec{\epsilon}_6$$

$$= 6\vec{\epsilon}_1 + 13\vec{\epsilon}_2 - 17\vec{\epsilon}_3 - \vec{\epsilon}_4 + 2\vec{\epsilon}_5 - 2\vec{\epsilon}_6;$$

$$\vec{w} = [\,\vec{\epsilon}_1 \quad \vec{\epsilon}_2 \quad \cdots \quad \vec{\epsilon}_6\,] \begin{bmatrix} 6 \\ 13 \\ -17 \\ -1 \\ 2 \\ -2 \end{bmatrix},$$

which answers the first question. Next, we build the central matrix of the multilinear mapping F', the structure of which has been given in Formula (5.130), Thus, we arrange the data vector components $\vec{w}_{\alpha\beta}$ as columns, and then, we apply the mentioned formula

$$\vec{w} = F'(\vec{z}) = F'(\vec{u} \otimes \vec{v})$$

$$= [\,\vec{\epsilon}_1 \quad \vec{\epsilon}_2 \quad \cdots \quad \vec{\epsilon}_6\,] \begin{bmatrix} 0 & 0 & 5 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 & 1 & 1 \\ -3 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ 4 \\ 2 \\ -3 \\ -2 \\ -1 \end{bmatrix}$$

$$= [\,\vec{e}_1 \otimes \vec{e}_1' \;\; \vec{e}_1 \otimes \vec{e}_2' \;\; \vec{e}_1 \otimes \vec{e}_3' \;\; \vec{e}_2 \otimes \vec{e}_1' \;\; \vec{e}_2 \otimes \vec{e}_2' \;\; \vec{e}_2 \otimes \vec{e}_3'\,] \begin{bmatrix} 6 \\ 13 \\ -17 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \quad (5.131)$$

which gives the answer to the second question.
As one can see the result is the same as the one obtained in the first question, which is in agreement with the tensor similitude Theorem 5.7.

□

Example 5.11 (Proof of Theorem 5.8). Consider the linear spaces $V^n(K)$ and its dual $V_*^n(K)$ referred to the reciprocal bases $\{\vec{e}_\beta\}$ and $\{\vec{e}^{*\alpha}\}$.

Consider also the linear space of all linear operators T that transform vectors inside $V^n(K)$, that is,

$$T : V^n(K) \to V^n(K); \;\; T \in \mathcal{L}[V^n(K), V^n(K)],$$

where $\mathcal{L}$ refers to *linear* operators and $V^n(K), V^n(K)$ to the endomorphism *initial* and *final* linear spaces, respectively.

Let $\{\vec{e}_{\alpha\beta}\}$ be the canonical basis of $\mathcal{L}[V^n(K), V^n(K)]$,

$$\vec{e}_{\alpha\beta} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{n\times n},$$

with a *one* in the position associated with row α and column β and zero otherwise; there exist n^2 basic vectors.

Consider two data vectors

$$\vec{u} = ||\vec{e}^{*\alpha}|| \begin{bmatrix} \overset{\circ}{u}_1 \\ \overset{\circ}{u}_2 \\ \vdots \\ \overset{\circ}{u}_n \end{bmatrix}, \vec{u}^* \in V_*^n(K) \text{ and } \vec{v} = ||\vec{e}_\beta|| \begin{bmatrix} \underset{\circ}{v}^1 \\ \underset{\circ}{v}^2 \\ \vdots \\ \underset{\circ}{v}^n \end{bmatrix}, \vec{v} \in V^n(K).$$

If we build in matrix form the vector $\vec{v} \otimes \vec{u}^* \in V^n \otimes V_*^n(K)$ and make the matrix of order $n \times n$ of the product equal to $\Phi(\vec{v} \otimes \vec{u}^*)$, we apply the tensor space $V^n \otimes V_*^n(K)$ in the space $\mathcal{L}[V^n(K), V^n(K)]$. Show that the endomorphism transforms the vectors as stated in the tensor similitude Theorem 5.8.

Solution: We calculate the vector $\vec{v} \otimes \vec{u}^*$:

$$\vec{v} \otimes \vec{u}^* = \left([\vec{e}_1 \vec{e}_2 \cdots \vec{e}_n] \begin{bmatrix} v^1_\circ \\ v^2_\circ \\ \vdots \\ v^n_\circ \end{bmatrix} \right) \otimes \left([\vec{e}^{*1} \vec{e}^{*2} \cdots \vec{e}^{*n}] \begin{bmatrix} u^\circ_1 \\ u^\circ_2 \\ \vdots \\ u^\circ_n \end{bmatrix} \right)^t$$

$$= [\vec{e}_1 \vec{e}_2 \cdots \vec{e}_n] \left(\begin{bmatrix} v^1_\circ \\ v^2_\circ \\ \vdots \\ v^n_\circ \end{bmatrix} \bullet \begin{bmatrix} u^\circ_1 & u^\circ_2 & \cdots & u^\circ_n \end{bmatrix} \right)_\otimes \begin{bmatrix} \vec{e}^{*1} \\ \vec{e}^{*2} \\ \vdots \\ \vec{e}^{*n} \end{bmatrix}$$

$$= [\vec{e}_1 \vec{e}_2 \cdots \vec{e}_n] \left(\begin{bmatrix} v^1_\circ u^\circ_1 & v^1_\circ u^\circ_2 & \cdots & v^1_\circ u^\circ_n \\ v^2_\circ u^\circ_1 & v^2_\circ u^\circ_2 & \cdots & v^2_\circ u^\circ_n \\ \ldots & \ldots & \ldots & \ldots \\ v^n_\circ u^\circ_1 & v^n_\circ u^\circ_2 & \cdots & v^n_\circ u^\circ_n \end{bmatrix} \right)_\otimes \begin{bmatrix} \vec{e}^{*1} \\ \vec{e}^{*2} \\ \ldots \\ \vec{e}^{*n} \end{bmatrix},$$

where the $\otimes$ operator appears as a subindex to refer to a quadratic form of tensor products.

Following the stated conditions, we have

$$\Phi(\vec{v} \otimes \vec{u}^*) = \begin{bmatrix} v^1_\circ u^\circ_1 & v^1_\circ u^\circ_2 & \cdots & v^1_\circ u^\circ_n \\ v^2_\circ u^\circ_1 & v^2_\circ u^\circ_2 & \cdots & v^2_\circ u^\circ_n \\ \ldots & \ldots & \ldots & \ldots \\ v^n_\circ u^\circ_1 & v^n_\circ u^\circ_2 & \cdots & v^n_\circ u^\circ_n \end{bmatrix}, \tag{5.132}$$

which gives the endomorphism matrix. Next, following Theorem 5.8 we examine how the vectors $\vec{w} \in V^n(K)$ (since in this example there exists only one space as primary and dual factors) are transformed.

We call the matrix in (5.132) $T_{\vec{v}\otimes\vec{u}^*}$, and transforming a vector $\vec{w} \in V^n(K)$ with the operator T we get

$$T_{\vec{v}\otimes\vec{u}^*}(\vec{w}) = \begin{bmatrix} v^1_\circ u^\circ_1 & v^1_\circ u^\circ_2 & \cdots & v^1_\circ u^\circ_n \\ v^2_\circ u^\circ_1 & v^2_\circ u^\circ_2 & \cdots & v^2_\circ u^\circ_n \\ \ldots & \ldots & \ldots & \ldots \\ v^n_\circ u^\circ_1 & v^n_\circ u^\circ_2 & \cdots & v^n_\circ u^\circ_n \end{bmatrix} \begin{bmatrix} w^1_\circ \\ w^2_\circ \\ \vdots \\ w^n_\circ \end{bmatrix}$$

$$= \begin{bmatrix} (u^\circ_1 w^1_\circ + u^\circ_2 w^2_\circ + \cdots + u^\circ_n w^n_\circ) v^1_\circ \\ (u^\circ_1 w^1_\circ + u^\circ_2 w^2_\circ + \cdots + u^\circ_n w^n_\circ) v^2_\circ \\ \cdots \\ (u^\circ_1 w^1_\circ + u^\circ_2 w^2_\circ + \cdots + u^\circ_n w^n_\circ) v^n_\circ \end{bmatrix}$$

$$= (\overset{\circ}{u}_1 w^1_\circ + \overset{\circ}{u}_2 w^2_\circ + \cdots + \overset{\circ}{u}_n w^n_\circ) \begin{bmatrix} v^1_\circ \\ v^2_\circ \\ \vdots \\ v^n_\circ \end{bmatrix}$$

$$= (\vec{w} \bullet \vec{u}^*) \begin{bmatrix} v^1_\circ \\ v^2_\circ \\ \vdots \\ v^n_\circ \end{bmatrix}. \tag{5.133}$$

The tensor conclusion of (5.133) is that

$$T_{\vec{v}\otimes\vec{u}^*}(\vec{w}) = (\vec{w} \bullet \vec{u}^*)\vec{v}. \tag{5.134}$$

From (5.132) to (5.133) we conclude that the equality

$$\Phi(\vec{v} \otimes \vec{u}^*) = T_{\vec{v}\otimes\vec{u}^*}$$

has the property (5.134) and then, Theorem 5.8 has been proved with the present model.

The isomorphism character is detected if we apply (5.132) to the vectors $\vec{e}_i$ and $\vec{e}^{*j}$ of matrices $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, with the 1 in row i, and $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, with the 1 in the row j, respectively:

$$\Phi(\vec{e}_i \otimes \vec{e}^{*j}) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \equiv \vec{\epsilon}_{ij}.$$

Thus, it is shown that this multilinear endomorphism associates the basis of $V^n \otimes V^n_*(K)$ with the basis of $\mathcal{L}[V^n(K), V^n(K)]$, and then, in this particular case it is an isomorphism.

□

Example 5.12 (Total and tensor products). Consider the *total product* homogeneous linear space

$$\begin{pmatrix} 3 \\ \times V_k^{n_0} \\ 1 \end{pmatrix} (\mathbb{R}) \equiv V_1^{n_0} \times V_2^{n_0} \times V_3^{n_0}(\mathbb{R})$$

of dimension $n = 3n_0 = 9$, and let the tuple $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be one of its vectors, where $\vec{v}_k \in V_k^3(\mathbb{R})$. The bases for each factor linear space will be denoted by $\{\vec{e}_i(k)\}; 1 \leq i, k \leq 3$, and thus, we have

$$\vec{v}_k = ||\vec{e}(k)||X_k; \quad X_k = \begin{bmatrix} x^1(k) \\ x^2(k) \\ x^3(k) \end{bmatrix}, \ \forall x^i(k) \in \mathbb{R}.$$

The basis of the total product linear space $\begin{pmatrix} 3 \\ \times V_k^3 \\ 1 \end{pmatrix} (\mathbb{R})$ will be notated

$$B = \{\vec{e}_1(1) \ \ \vec{e}_2(1) \ \ \vec{e}_3(1) \ , \vec{e}_1(2) \ \ \vec{e}_2(2) \ \ \vec{e}_3(2) \ , \vec{e}_1(3) \ \ \vec{e}_2(3) \ \ \vec{e}_3(3)\}$$

and therefore, the matrix representation of the 3-tuple $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ results

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = ||B||X,$$

where X is the block column matrix

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

Three morphisms "$f(k)$" apply each space factor $V_k^3(\mathbb{R})$ into a linear space $W^m(\mathbb{R})$ of dimension $m = 4$ and basis $\{\varepsilon_\ell\}_1^4$.

The matrix representation of a vector $\vec{w} \in W^4(\mathbb{R})$ is $\vec{w} = ||\vec{\varepsilon}_\ell||Y$. Assuming that the associated matrix representation, relative to such bases, of morphisms $f(k)$ are the data matrix $H_{4,3}(k)$, the morphism matrix representations become

$$Y_k = H_{4,3}(k) \bullet X_k; \quad 1 \leq k \leq 3.$$

Finally, let us build an homomorphism f that applies the initial total product linear space into the final space $W^m(\mathbb{R})$:

$$f : \begin{pmatrix} 3 \\ \times V_k^3 \\ 1 \end{pmatrix} (\mathbb{R}) \to W^m(\mathbb{R}); \quad f(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \vec{w}$$

which matrix representation is:

$$Y = H_{m,n} \bullet X;$$

where $H_{m,n} = [H(1) \ H(2) \ H(3)]$ is built with $H_{4,3}(k)$ matrices as blocks.

Assuming now that the data matrices are:

$$H(1) = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 5 & 3 \\ 2 & 4 & 3 \\ 4 & 2 & 3 \end{bmatrix}; \quad H(2) = \begin{bmatrix} 8 & -7 & 4 \\ 4 & 5 & -2 \\ -4 & 2 & 0 \\ 2 & 3 & 6 \end{bmatrix}; \quad H(3) = \begin{bmatrix} 9 & 0 & 6 \\ -6 & -3 & 9 \\ 7 & 2 & -4 \\ 10 & -1 & 11 \end{bmatrix};$$

1. Give the representation of morphism f specifying the components of each matrix.
2. If
$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = (2\vec{e}_1 + 3\vec{e}_2 - \vec{e}_3, 5\vec{e}_2 - 2\vec{e}_3, \vec{e}_1 - \vec{e}_2 + 3\vec{e}_3) \tag{5.135}$$
find the image vector $\vec{w} = f(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.
3. In $\left(\begin{matrix} 3 \\ \times V_k^3 \\ 1 \end{matrix}\right)(\mathbb{R})$, consider the multivector relation
$$(2\vec{e}_1, 2\vec{e}_2 - 5\vec{e}_3, 2\vec{e}_3) = (\vec{e}_1, 2\vec{e}_2, \vec{e}_3) + (\vec{e}_1, -5\vec{e}_3, \vec{e}_3).$$
Using this relation, examine if f is a *multilinear transformation* for the addition of the total product linear space.
4. Find a basis of the null space relative to morphism f, verifying that the dimension of the resulting basis is coherent with the dimension of the range space.
5. Based on the knowledge we already have on f, build a multilinear mapping
$$F : \left(\begin{matrix} 3 \\ \times V_k^3 \\ 1 \end{matrix}\right)(\mathbb{R}) \to W^4(\mathbb{R}).$$
To get it, one must answer the following questions:
(a) Determine matrix $M_{n,\sigma}$ where $\sigma = n_0^r = 3^3 = 27$; the matrix columns X of $M_{n,\sigma}$ are the matrix representations of the σ 3-tuples $\big(\vec{e}_{\beta_1}(1), \vec{e}_{\beta_2}(2), \vec{e}_{\beta_3}(3)\big), \forall \beta_i, 1 \le \beta_i \le 3$, in the B basis.
(b) Set condition
$$F(\vec{e}_{\beta_1}, \vec{e}_{\beta_2}, \vec{e}_{\beta_3}) = f(\vec{e}_{\beta_1}(1), \vec{e}_{\beta_2}(2), \vec{e}_{\beta_3}(3)), \forall \beta_i, 1 \le \beta_i \le 3,$$
through the matrix relation:
$$H(F)_{m,\sigma} = H_{m,n} \bullet M_{n,\sigma}.$$
Give the matrix $H_{m,\sigma}$ associated with the multilinear application F.
(c) Determine matrix $X_{\sigma,1}$ as the representation of multivector $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ given in (5.135), but now with the appropriate components as shown in formula (5.122).
6. Determine the image vector $\vec{w}' = F(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, in accordance with matrix equation $Y'_{m,1} = H_{m,\sigma} \bullet X_{\sigma,1}$.
7. Determine if $\vec{w}' = 3\vec{w}$.

Solution:

1. The matrix representation of morphism f is:

$$\begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 4 & 8 & -7 & 4 & 9 & 0 & 6 \\ 1 & 5 & 3 & 4 & 5 & -2 & -6 & -3 & 9 \\ 2 & 4 & 3 & -4 & 2 & 0 & 7 & 2 & -4 \\ 4 & 2 & 3 & 2 & 3 & 6 & 10 & -1 & 11 \end{bmatrix} \begin{bmatrix} x^1(1) \\ x^2(1) \\ x^3(1) \\ x^1(2) \\ x^2(2) \\ x^3(2) \\ x^1(3) \\ x^2(3) \\ x^3(3) \end{bmatrix}$$

2. Since

$$\begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 4 & 8 & -7 & 4 & 9 & 0 & 6 \\ 1 & 5 & 3 & 4 & 5 & -2 & -6 & -3 & 9 \\ 2 & 4 & 3 & -4 & 2 & 0 & 7 & 2 & -4 \\ 4 & 2 & 3 & 2 & 3 & 6 & 10 & -1 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \\ 5 \\ -2 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 67 \\ 16 \\ 58 \end{bmatrix}$$

   we have

$$\vec{w} = f(\vec{v}_1, \vec{v}_2, \vec{v}_3) = -\vec{\varepsilon}_1 + 67\vec{\varepsilon}_2 + 16\vec{\varepsilon}_3 + 58\vec{\varepsilon}_4.$$

3. Since f is a morphism, we have

$$\begin{aligned} f(2\vec{e}_1, 2\vec{e}_2 - 5\vec{e}_3, 2\vec{e}_3) &= f\big((\vec{e}_1, 2\vec{e}_2, \vec{e}_3) + (\vec{e}_1, -5\vec{e}_3, \vec{e}_3)\big) \\ &= f(\vec{e}_1, 2\vec{e}_2, \vec{e}_3) + f(\vec{e}_1, -5\vec{e}_3, \vec{e}_3) \end{aligned}$$

   If f were a multilinear mapping, it should be

$$f(2\vec{e}_1, 2\vec{e}_2 - 5\vec{e}_3, 2\vec{e}_3) = f(2\vec{e}_1, 2\vec{e}_2, 2\vec{e}_3) + f(2\vec{e}_1, -5\vec{e}_3, 2\vec{e}_3)$$

   so, f is not a multilinear mapping.

4. A basis of the null space is

$$B_N = \begin{bmatrix} -1 & 0 & -176 & -766 & -768 \\ -1 & 0 & -82 & 196 & 442 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 405 & 207 & -1 \\ 0 & 0 & 302 & 46 & -118 \\ 0 & 0 & 0 & 0 & 424 \\ 0 & -2 & 0 & 212 & 0 \\ 0 & 13 & 848 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

As matrix $H(2)$ has rank 4, this is the rank of matrix $H_{4,9}$. Thus, we have

dim(null space)+dim(range space) = 5+4 = 9 = dim(total product space).

5. (a) The matrix representations of $(\vec{e}_1(1), \vec{e}_1(2), \vec{e}_1(3)), (\vec{e}_1(1), \vec{e}_1(2), \vec{e}_2(3))$ in the B basis are:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

so that following this we get matrix $M_{n,\sigma}$:

$$M_{9,27} = \begin{bmatrix}
1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \\
1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0 \\
0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0 \\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1 \\
1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0 \\
0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0 \\
0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1
\end{bmatrix}$$

(b) The relation $H(F)_{m,\sigma} = H_{m,n} \bullet M_{n,\sigma}$, that in this case is

$$H(F)_{4,27} = H_{4,9} \bullet M_{9,27}$$

becomes

$$\begin{bmatrix}
22&13&19&7&-2&4&18&9&15&20&11&17&5&-4&2&16&7&13&21&12&18&6&-3&3&17&8&14 \\
-1&2&14&0&3&15&-7&-4&8&3&6&18&4&7&19&-3&0&12&1&4&16&2&5&17&-5&-2&10 \\
5&0&-6&11&6&0&9&4&-2&7&2&-4&13&8&2&11&6&0&6&1&-5&12&7&1&10&5&-1 \\
16&5&17&17&6&18&20&9&21&14&3&15&15&4&16&18&7&19&15&4&16&16&5&17&19&8&20
\end{bmatrix}$$

(c) Applying formula (2.21) one gets

$$X_{27,1} = X_1 \otimes X_2 \otimes X_3 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 10 \\ -10 \\ 30 \\ -4 \\ 4 \\ -12 \\ 0 \\ 0 \\ 0 \\ 15 \\ -15 \\ 45 \\ -6 \\ 6 \\ -18 \\ 0 \\ 0 \\ 0 \\ -5 \\ 5 \\ -15 \\ 2 \\ -2 \\ 6 \end{pmatrix}$$

6. The matrix representation of F is:

$$Y'_{4,1} = H_{4,27} \bullet X_{27,1} = \begin{bmatrix} -57 \\ 762 \\ 153 \\ 663 \end{bmatrix}$$

and then

$$\vec{w}\,' = -57\vec{\varepsilon}_1 + 762\vec{\varepsilon}_2 + 153\vec{\varepsilon}_3 + 663\vec{\varepsilon}_4.$$

7. It is clear that

$$\begin{aligned} \vec{w}' &= 3(-19\vec{\varepsilon}_1 + 254\vec{\varepsilon}_2 + 51\vec{\varepsilon}_3 + 221\vec{\varepsilon}_4) \\ &\neq 3(-\vec{\varepsilon}_1 + 67\vec{\varepsilon}_2 + 16\vec{\varepsilon}_3 + 58\vec{\varepsilon}_4.) \\ &= 3\vec{w} \end{aligned}$$

□

5.12 Exercises

5.1. In the tensor space $\overset{4}{\underset{1}{\otimes}} R^2_*$, we consider the totally covariant homogeneous tensor T, given by its matrix representation:

$$[t_{\alpha\beta\gamma\delta}^{\circ\circ\circ\circ}] = \begin{bmatrix} a_1 & b_1 & | & a_2 & b_2 \\ c_1 & d_1 & | & c_2 & d_2 \\ - & - & + & - & - \\ a_3 & b_3 & | & a_4 & b_4 \\ c_3 & d_3 & | & c_4 & d_4 \end{bmatrix}.$$

Obtain the "permutation" matrices P_1, P_2 and P_3 associated with the three rotation isomers (1),(2),(3) mentioned in Example 5.5, point 3, that transform the tensor $T_{\sigma,1}$ into its "extended" isomers.

5.2. Consider the homogeneous tensors P, Q and D (the last is the Kronecker delta), all of them associated with the linear space $V^n(\mathbb{R})$. Determine if the tensors A, B, C, contracted products of the data tensors, are their isomers:

$$A: \ \delta_{\alpha\circ}^{\circ\beta} p_{\beta\gamma\circ}^{\circ\circ\lambda}; \quad B: \ \delta_{\alpha\circ}^{\circ\beta} q_{\gamma\beta\lambda}^{\circ\circ\circ} \delta_{\circ\mu}^{\lambda\circ}; \quad C: \ \delta_{\alpha\circ}^{\circ\beta} \delta_{\alpha\circ}^{\circ\beta} \delta_{\beta\circ}^{\circ\lambda} \delta_{\gamma\circ}^{\circ\lambda}.$$

5.3. Two tensors T of order $r_1 = 2$ and U of order $r_2 = 3$ are defined over a certain linear space $V^3(\mathbb{R})$ referred to a certain basis $\{\vec{e}_\alpha\}$. Their matrix representations are

$$[t_{\circ\circ}^{\alpha\beta}] = \begin{bmatrix} 2 & 3 & 0 \\ -2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad (\alpha \text{ row }, \beta \text{ column})$$

$$[u_{\circ\lambda\circ}^{\gamma\circ\mu}] = \begin{bmatrix} 0 & 0 & 1 & | & 2 & 1 & 0 & | & 0 & 1 & 3 \\ 1 & 3 & 5 & | & 1 & 0 & -1 & | & 1 & -1 & 4 \\ 0 & 2 & 0 & | & -1 & 1 & 4 & | & 6 & 1 & 0 \end{bmatrix},$$

where γ is the row block, λ is the column of each block and μ is the block column. (beware of the matrix block disposition of this tensor).

1. Determine, as contractions of the tensors $Q_1 = T \otimes U$ and $Q_2 = U \otimes T$ (of order $r = 5$), the contracted products that follow:

$$A: \ t_{\circ\circ}^{\theta\beta} u_{\circ\theta\circ}^{\gamma\circ\mu}; \quad B: \ t_{\circ\circ}^{\alpha\theta} u_{\circ\theta\circ}^{\gamma\circ\mu}; \quad F: \ u_{\circ\theta\circ}^{\gamma\circ\mu} t_{\circ\circ}^{\theta\beta}; \quad G: \ u_{\circ\theta\circ}^{\gamma\circ\mu} t_{\circ\circ}^{\alpha\theta}.$$

   Note: the matrix representations of tensors A, B, F, G must have the same ordering criterion as the one given in the statement for tensors of order $r = 3$.
2. Since the tensor U does not satisfy the correct axiomatic ordering in its matrix representation, give the matrix P of the permutation that transforms $U_{\sigma,1}$ in the isomer $U'_{\sigma,1}$ the condensation of which leads to tensor U' with the correct ordering.
3. Examine if P is an orthogonal matrix.
4. Give A', B', F', G', the correct contracted products, with the usual matrix representation.

5. Give the matrices $H_{A'}, H_{B'}, H_{F'}, H_{G'}$ corresponding to the contraction homomorphisms executed in the previous question, over the "extended" tensors.
6. If we recover the isomers from tensors A', B', F', G' by means of matrix P^{-1} (inverse permutation), do we get the results of question 1?. Check this result.
7. We perform a change-of-basis, in the linear space $V^3(\mathbb{R})$ of matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

Give the new tensors $\hat{T}$ and $\hat{U}$ that would present this statement.
8. Solve for $\hat{T}$ and $\hat{U}$, questions 1 to 6.

5.4. Consider the tensor $T \in V^3 \otimes V_*^3(\mathbb{R})$, with matrix representation

$$[t^{\alpha\circ}_{\circ\beta}] = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}.$$

Give two right-autotensors A and B, contracted $(T \otimes A) = \lambda A$ and contracted $(T \otimes B) = \mu B$, where A is of order $(r = 2)$ symmetric and B of order $(r = 3)$.

5.5. Consider the linear space $V^3(\mathbb{R})$ referred to the basis $\{\vec{e}_\alpha\}$. We take a particular vector $(\vec{V}_1, \vec{V}_2, \vec{V}_3) \in \overset{3}{\underset{1}{\times}} V^3(\mathbb{R})$ belonging to the *total* product linear space, the matrix of which associated with the basis $\{\vec{e}_\alpha\}$ is $[\,X_1 \quad X_2 \quad X_3\,] = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$.

A multilinear transformation $F : \overset{3}{\underset{1}{\times}} V^3(\mathbb{R}) \to W^4(\mathbb{R})$ that applies the total product linear space in $W^4(\mathbb{R})$, is given by (5.104):

$$F[(\vec{V}_1, \vec{V}_2, \vec{V}_3)] = \vec{W} \in W^4(K),$$

which results from the total contraction of the four covariant tensors of order $(r = 3)$ that appear as vector components of

$$F^{h\circ\circ\circ}_{\circ\alpha\beta\gamma}\vec{\epsilon}_h = (\alpha-1)\vec{\epsilon}_1 + (\alpha-\beta+2)\vec{\epsilon}_2 + (\beta-\gamma-3)\vec{\epsilon}_3 + (\gamma+4)\vec{\epsilon}_4; \quad 1 \le \alpha, \beta, \gamma \le 3,$$

with the vector $\vec{V}_1 \otimes \vec{V}_2 \otimes \vec{V}_3 \in \overset{3}{\underset{1}{\otimes}} V^3(\mathbb{R})$.

Give the image vector $\vec{W}$ of the multilinear mapping.

5.6. In the linear space $V^3(\mathbb{R})$ referred to a certain basis $\{\vec{e}_\alpha\}$, we consider three linear operators with associated matrices:

$$H_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix};\ H_2 = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix};\ H_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & -2 \end{bmatrix}.$$

1. Obtain the eigenvalues of H_1, H_2, H_3 in increasing order. They will be notated as $(\lambda_1, \lambda_2, \lambda_3)$, (μ_1, μ_2, μ_3) and (ν_1, ν_2, ν_3), for H_1, H_2 and H_2, respectively.
2. Obtain the eigenvectors $(X_1, X_2, X_3)_{H_i}$ associated with each operator, giving their components in columns.
3. If $\vec{W}_i = H_i(\vec{V}_i)$, $1 \leq i \leq 3$ are the images of the vectors:

$$\vec{V}_1 = ||\vec{e}_\alpha|| \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix};\ \vec{V}_2 = ||\vec{e}_\alpha|| \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix};\ \vec{V}_3 = ||\vec{e}_\alpha|| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

   determine the multivector $\vec{V}_1 \otimes \vec{V}_2 \otimes \vec{V}_3 \in \overset{3}{\underset{1}{\otimes}} V^3(\mathbb{R})$ and the multivector $\vec{W}_1 \otimes \vec{W}_2 \otimes \vec{W}_3 \in \overset{3}{\underset{1}{\otimes}} V^3(\mathbb{R})$.
4. Obtain the matrix H_σ associated with the direct endomorphism that transforms $H_\sigma(\vec{V}_1 \otimes \vec{V}_2 \otimes \vec{V}_3) = \vec{W}_1 \otimes \vec{W}_2 \otimes \vec{W}_3$.
5. Determine the eigenvalues of H_σ.
6. Determine the eigenvectors of H_σ (remember Section 1.3.4).
7. Solve questions 4, 5 and 6 using the computer and assuming that the solutions of 1, 2 and 3 are known.