Introduction to Part One

Linear series have long stood at the center of algebraic geometry. Systems of divisors were employed classically to study and define invariants of projective varieties, and it was recognized that varieties share many properties with their hyperplane sections. The classical picture was greatly clarified by the revolutionary new ideas that entered the field starting in the 1950s. To begin with, Serre's great paper [530], along with the work of Kodaira (e.g. [353]), brought into focus the importance of amplitude for line bundles. By the mid 1960s a very beautiful theory was in place, showing that one could recognize positivity geometrically, cohomologically, or numerically. During the same years, Zariski and others began to investigate the more complicated behavior of linear series defined by line bundles that may not be ample. This led to particularly profound insights in the case of surfaces [623]. In yet another direction, the classical theorems of Lefschetz comparing the topology of a variety with that of a hyperplane section were understood from new points of view, and developed in surprising ways in [258] and [30].

The present Part One is devoted to this body of work and its developments. Our aim is to give a systematic presentation, from a contemporary viewpoint, of the circle of ideas surrounding linear series and ample divisors on a projective variety.

We start in Chapter 1 with the basic theory of positivity for line bundles. In keeping with the current outlook, **Q**- and **R**-divisors and the notion of nefness play a central role, and the concrete geometry of nef and ample cones is given some emphasis. The chapter concludes with a section on Castelnuovo–Mumford regularity, a topic that we consider to merit inclusion in the canon of positivity.

Chapter 2 deals with linear series, our focus being the asymptotic geometry of linear systems determined by divisors that may not be ample. We study in particular the behavior of big divisors, whose role in birational geometry is similar to that of ample divisors in the biregular theory. The chapter also contains several concrete examples of the sort of interesting and challenging behavior that such linear series can display.

In Chapter 3 we turn to the theorems of Lefschetz and Bertini and their subsequent developments by Barth, Fulton–Hansen, and others. Here the surprising geometric properties of projective subvarieties of small codimension come into relief. The Lefschetz hyperplane theorem is applied in Chapter 4 to prove the classical vanishing theorems of Kodaira and Nakano. Chapter 4 also contains the vanishing theorem for big and nef divisors discovered by Kawamata and Viehweg, as well as one of the generic vanishing theorems from [242].

Finally, Chapter 5 takes up the theory of local positivity. This is a topic that has emerged only recently, starting with ideas of Demailly for quantifying how much of the positivity of a line bundle can be localized at a given point of a variety. Although some of the results are not yet definitive, the picture is surprisingly rich and structured.

Writing about linear series seems to lead unavoidably to conflict between the additive notation of divisors and the multiplicative language of line bundles. Our policy is to avoid explicitly mixing the two. However, on many occasions we adopt the compromise of speaking about divisors while using notation suggestive of bundles. We discuss this convention — as well as some of the secondary issues it raises — at the end of Section 1.1.A.

1

Ample and Nef Line Bundles

This chapter contains the basic theory of positivity for line bundles and divisors on a projective algebraic variety.

After some preliminaries in Section 1.1 on divisors and linear series, we present in Section 1.2 the classical theory of ample line bundles. The basic conclusion is that positivity can be recognized geometrically, cohomologically, or numerically. Section 1.3 develops the formalism of **Q**- and **R**-divisors, which is applied in Section 1.4 to study limits of ample bundles. These so-called nef divisors are central to the modern view of the subject, and Section 1.4 contains the core of the general theory. Most of the remaining material is more concrete in flavor. Section 1.5 is devoted to examples of ample cones and to further information about their structure, while Section 1.6 focuses on inequalities of Hodge type. After a brief review of the definitions and basic facts surrounding amplitude for a mapping, we conclude in Section 1.8 with an introduction to Castelnuovo–Mumford regularity.

We recall that according to our conventions we deal unless otherwise stated with complex algebraic varieties and schemes, and with closed points on them. However, as we go along we will point out that much of this material remains valid for varieties defined over algebraically closed fields of arbitrary characteristic.

1.1 Preliminaries: Divisors, Line Bundles, and Linear Series

In this section we collect some facts and notation that will be used frequently in the sequel. We start in Section 1.1.A by recalling some constructions involving divisors and line bundles, and turn in the second subsection to linear series. Section 1.1.C deals with intersection numbers and numerical equivalence, and we conclude in 1.1.D by discussing asymptotic formulations of the Riemann–Roch theorem. As a practical matter we assume that much of this material is familiar to the reader.[1] However, we felt it would be useful to include a brief summary in order to fix ideas.

1.1.A Divisors and Line Bundles

We start with a quick review of the definitions and facts concerning Cartier divisors, following [280, p. 140ff], [445, Chapters 9 and 10], and [344]. We take up first the very familiar case of reduced and irreducible varieties, and then pass to more general schemes.

Consider then an irreducible complex variety X, and denote by $\mathcal{M}_X = \mathbf{C}(X)$ the (constant) sheaf of rational functions on X. It contains the structure sheaf $\mathcal{O}_X$ as a subsheaf, and so there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{M}_X^*$ of sheaves of multiplicative abelian groups.

Definition 1.1.1. (Cartier divisors). A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. We denote by $\mathrm{Div}(X)$ the group of all such, so that

$$\mathrm{Div}(X) \ = \ \varGamma\big(X, \mathcal{M}_X^*/\mathcal{O}_X^*\big). \quad \square$$

Concretely, then, a divisor $D \in \mathrm{Div}(X)$ is represented by data $\big\{(U_i, f_i)\big\}$ consisting of an open covering $\{U_i\}$ of X together with elements $f_i \in \varGamma\big(U_i, \mathcal{M}_X^*\big)$, having the property that on $U_{ij} = U_i \cap U_j$ one can write

$$f_i \ = \ g_{ij} f_j \quad \text{for some} \quad g_{ij} \ \in \ \varGamma\big(U_{ij}, \mathcal{O}_X^*\big). \tag{1.1}$$

The function f_i is called a *local equation* for D at any point $x \in U_i$. Two such collections determine the same Cartier divisor if there is a common refinement $\{V_k\}$ of the open coverings on which they are defined so that they are given by data $\big\{(V_k, f_k)\big\}$ and $\big\{(V_k, f_k')\big\}$ with

$$f_k \ = \ h_k f_k' \quad \text{on} \quad V_k \quad \text{for some} \quad h_k \ \in \ \varGamma\big(V_k, \mathcal{O}_X^*\big).$$

The group operation on $\mathrm{Div}(X)$ is always written additively: if $D, D' \in \mathrm{Div}(X)$ are represented respectively by data $\big\{(U_i, f_i)\big\}$ and $\big\{(U_i, f_i')\big\}$, then $D + D'$ is given by $\big\{(U_i, f_i f_i')\big\}$. The *support* of a divisor $D = \big\{(U_i, f_i)\big\}$ is the set of points $x \in X$ at which a local equation of D at x is not a unit in $\mathcal{O}_x X$. D is *effective* if $f_i \in \varGamma\big(U_i, \mathcal{O}_X\big)$ is regular on U_i: this is written $D \succcurlyeq 0$. The notation $D \succcurlyeq D'$ indicates that $D - D'$ is effective.

Suppose now that X is a possibly non-reduced algebraic scheme. Then the same definition works except that one has to be more careful about what one means by $\mathcal{M}_X$, which now becomes the sheaf of total quotient rings of $\mathcal{O}_X$.[2] As

[1] For the novice, we give some suggestions and pointers to the literature.

[2] Grothendieck speaks of the sheaf of "meromorphic functions" on X (cf. [257, 20.1]). However since we are working with complex varieties this seems potentially confusing, and we prefer to follow the terminology of [280].

explained in [445, Chapter 9] there is a unique sheaf $\mathcal{M}_X$ on X characterized by the property that if $U = \operatorname{Spec}(A)$ is an affine open subset of X, then

$$\Gamma\big(U, \mathcal{M}_X\big) \;=\; \Gamma\big(U, \mathcal{O}_X\big)_{\mathrm{tot}} \;=\; A_{\mathrm{tot}}$$

is the total ring of fractions of A, i.e. the localization of A at the set of non zero-divisors.[3] Similarly, on the stalk level there is an isomorphism $\mathcal{M}_{X,x} = \big(\mathcal{O}_{X,x}\big)_{\mathrm{tot}}$. As before one has an inclusion $\mathcal{O}_X^* \subseteq \mathcal{M}_X^*$ of multiplicative groups of units, and Definition 1.1.1 — as well as the discussion following it — remains valid without change.

Convention 1.1.2. (Divisors). In Parts One and Two of this work we adopt the convention that when we speak of a "divisor" we always mean a Cartier divisor. (In Part Three it will be preferable to think instead of Weil divisors.) □

One should view Cartier divisors as "cohomological" objects, but one can also define "homological" analogues:

Definition 1.1.3. (Cycles and Weil divisors). Let X be a variety or scheme of pure dimension n. A *k-cycle* on X is a $\mathbf{Z}$-linear combination of irreducible subvarieties of dimension k. The group of all such is written $Z_k(X)$. A *Weil divisor* on X is an $(n-1)$-cycle, i.e. a formal sum of codimension one subvarieties with integer coefficients. We often use $\operatorname{WDiv}(X)$ in place of $Z_{n-1}(X)$ to denote the group of Weil divisors. □

Remark 1.1.4. (Cycle map for Cartier divisors). There is a *cycle map*

$$\operatorname{Div}(X) \longrightarrow \operatorname{WDiv}(X) \quad , \quad D \mapsto [D] \;=\; \sum \operatorname{ord}_V(D) \cdot [V]$$

where $\operatorname{ord}_V(D)$ is the order of D along a codimension-one subvariety. In general this homomorphism is neither injective nor surjective, although it is one-to-one when X is a normal variety and an isomorphism when X is non-singular. (See [208, Chapter 2.1] for details and further information.) □

A global section $f \in \Gamma\big(X, \mathcal{M}_X^*\big)$ determines in the evident manner a divisor

$$D \;=\; \operatorname{div}(f) \;\in\; \operatorname{Div}(X).$$

As usual, a divisor of this form is called *principal* and the subgroup of all such is $\operatorname{Princ}(X) \subseteq \operatorname{Div}(X)$. Two divisors D_1, D_2 are *linearly equivalent*, written $D_1 \equiv_{\mathrm{lin}} D_2$, if $D_1 - D_2$ is principal.

Let D be a divisor on X. Given a morphism $f : Y \longrightarrow X$, one would like to define a divisor f^*D on Y by pulling back the local equations for D. The following condition is sufficient to guarantee that this is meaningful:

[3] See [344] for a discussion of how one should define $\mathcal{M}_X$ on arbitrary open subsets.

Let $V \subseteq Y$ be any *associated subvariety* of Y, i.e. the subvariety defined by an associated prime of $\mathcal{O}_Y$ in the sense of primary decomposition. Then $f(V)$ should not be contained in the support of D.

If Y is reduced, the requirement is just that no component of Y map into the support of D.

A similar condition allows one to define the divisor of a section of a line bundle L on X. Specifically, let $s \in \Gamma\big(X, L\big)$ be a global section of L. Assume that s does not vanish on any associated subvariety of X — for example, if X is reduced, this just means that s shouldn't vanish identically on any component of X. Then a local equation of s determines in the natural way a divisor $\mathrm{div}(s) \in \mathrm{Div}(X)$. We leave it to the reader to formulate the analogous condition under which a "rational section" $s \in \Gamma\big(X, L \otimes_{\mathcal{O}_X} \mathcal{M}_X\big)$ gives rise to a divisor.

A Cartier divisor $D \in \mathrm{Div}(X)$ determines a line bundle $\mathcal{O}_X(D)$ on X, leading to a canonical homomorphism

$$\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \;\;,\;\; D \mapsto \mathcal{O}_X(D) \tag{1.2}$$

of abelian groups, where $\mathrm{Pic}(X)$ denotes as usual the Picard group of isomorphism classes of line bundles on X. Concretely, if D is given by data $\big\{(U_i, f_i)\big\}$ as above, then one can build $\mathcal{O}_X(D)$ by using the g_{ij} in (1.1) as transition functions. More abstractly, one can view (the isomorphism class of) $\mathcal{O}_X(D)$ as the image of D under the connecting homomorphism

$$\mathrm{Div}(X) \;=\; \Gamma\big(X, \mathcal{M}_X^*/\mathcal{O}_X^*\big) \longrightarrow H^1\big(X, \mathcal{O}_X^*\big) \;=\; \mathrm{Pic}(X)$$

determined by the exact sequence $0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^*/\mathcal{O}_X^* \longrightarrow 0$ of sheaves on X. Evidently,

$$\mathcal{O}_X(D_1) \;\cong\; \mathcal{O}_X(D_2) \quad \Longleftrightarrow \quad D_1 \equiv_{\mathrm{lin}} D_2.$$

If D is effective then $\mathcal{O}_X(D)$ carries a global section $s = s_D \in \Gamma\big(X, \mathcal{O}_X(D)\big)$ with $\mathrm{div}(s) = D$. In general $\mathcal{O}_X(D)$ has a rational section with the analogous property.

The question of whether every line bundle comes from a divisor is more delicate. On the positive side, there are two sufficient conditions:

Example 1.1.5. (Line bundles from divisors). There are a couple of natural hypotheses to guarantee that every line bundle arises from a divisor.

(i). If X is reduced and irreducible, or merely reduced, then the homomorphism in (1.2) is surjective.

(ii). If X is projective then the same statement holds even if it is non-reduced.

(If X is reduced and irreducible then any line bundle L has a rational section s, and one can take $D = \mathrm{div}(s)$. For the second statement — which is due to

Nakai [466] — one can use the theorem of Cartan–Serre–Grothendieck (Theorem 1.2.6) to reduce to the case in which L is globally generated (Definition 1.1.10). But then one can find a section $s \in \Gamma\big(X, L\big)$ that does not vanish on any of the associated subvarieties of X, in which case $D = \mathrm{div}(s)$ gives the required divisor.) □

On the other hand, there is also

Example 1.1.6. (Kleiman's example). Following [346] and [528] we construct a non-projective non-reduced scheme X on which the mapping (1.2) is not surjective.[4] Start by taking Y to be Hironaka's example of a smooth non-projective threefold containing two disjoint smooth rational curves A and B with $A + B \equiv_{\mathrm{num}} 0$ as described in [280, Appendix B, Example 3.4.1]. Now fix points $a \in A$, $b \in B$ and introduce nilpotents at a and b to produce a non-reduced scheme X containing Y. Note that X has depth zero at a and b, so these points must be disjoint from the support of every Cartier divisor on X. Observe also that $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is an isomorphism thanks to the fact that $\ker\big(\mathcal{O}_X^* \longrightarrow \mathcal{O}_Y^*\big)$ is supported on a finite set, and so has vanishing H^1 and H^2.

We claim that there exists a line bundle L on X with the property that

$$\int_A c_1(L) \ > \ 0. \tag{*}$$

In fact, it follows from Hironaka's construction that one can find a line bundle on Y satisfying the analogous inequality, and by what have said above this bundle extends to X. Suppose now that $L = \mathcal{O}_X(D)$ for some divisor D on X. Decompose the corresponding Weil divisor as a sum $[D] = \sum m_i[D_i]$ of prime divisors with $m_i \neq 0$. None of the D_i can pass through a or b, so each D_i is Cartier and

$$D \ = \ \sum m_i D_i$$

as Cartier divisors on X. Now $\big(D_i \cdot A\big) \geq 0$ and $\big(D_i \cdot B\big) \geq 0$ since each of the D_i — avoiding as they do the points a and b — meet A and B properly. On the other hand, it follows from (*) that there is at least one index i such that $\big(m_i D_i \cdot A\big) > 0$: in particular $m_i > 0$ and $\big(D_i \cdot A\big) > 0$. But $\big(D_i \cdot B\big) = -\big(D_i \cdot A\big)$ since $B \equiv_{\mathrm{num}} -A$ and therefore $\big(D_i \cdot B\big) < 0$, a contradiction. (As Schröer observes, the analogous but slightly simpler example appearing in [276, I.1.3] is erroneous.) □

Later on, the canonical bundle of a smooth variety will play a particularly important role:

Notation 1.1.7. (Canonical bundle and divisor). Let X be a nonsingular complete variety of dimension n. We denote by $\omega_X = \Omega_X^n$ the canonical line bundle on X, and by K_X any canonical divisor on X. Thus $\mathcal{O}_X(K_X) = \omega_X$. □

[4] We will use here the basic facts of intersection theory recalled later in this section.

Finally, a word about terminology. There is inevitably a certain amount of tension between the additive language of divisors and the multiplicative formalism of line bundles. Our convention is always to work additively with divisors and multiplicatively with line bundles. However, on many occasions it is natural or customary to stay in additive mode when nonetheless one has line bundles in mind. In these circumstances we will speak of divisors but use notation suggestive of bundles, as for instance in the following statement of the Kodaira vanishing theorem:

> Let L be an ample divisor on a smooth projective variety X of dimension n. Then $H^i\big(X, \mathcal{O}_X(K_X + L)\big) = 0$ for every $i > 0$.

(The mathematics here appears in Chapter 4.) While for the most part this convention seems to work well, it occasionally leads us to make extraneous projectivity or integrality hypotheses in order to be able to invoke Example 1.1.5. Specifically, we will repeatedly work with the Néron–Severi group $N^1(X)$ of X and the corresponding real vector space $N^1(X)_{\mathbf{R}} = N^1(X) \otimes \mathbf{R}$. Here additive notation seems essential, so we are led to view $N^1(X)$ as the group of divisors modulo numerical equivalence (Definition 1.1.15). On the other hand, functorial properties are most easily established by passing to $\mathrm{Pic}(X)$. For this to work smoothly one wants to know that every line bundle comes from a divisor, and this is typically guaranteed by simply assuming that X is either a variety or a projective scheme. We try to flag this artifice when it occurs.

1.1.B Linear Series

We next review some basic facts and definitions concerning linear series. For further information the reader can consult [276, Chapter I.2], [280, Chapter II, Sections 6, 7], and [248, Chapter 1.4].

Let X be a variety (or scheme), L a line bundle on X, and $V \subseteq H^0\big(X, L\big)$ a non-zero subspace of finite dimension. We denote by $|V| = \mathbf{P}_{\mathrm{sub}}(V)$ the projective space of one-dimensional subspaces of V. When X is a complete variety, $|V|$ is identified with the linear series of divisors of sections of V in the sense of [280, Chapter II, §7], and in general we refer to $|V|$ as a linear series.[5] Taking $V = H^0\big(X, L\big)$ — assuming that this space is finite-dimensional, as will be the case for instance if X is complete — yields the *complete linear series* $|L|$. Given a divisor D, we also write $|D|$ for the complete linear series associated to $\mathcal{O}_X(D)$.

Evaluation of sections in V gives rise to a morphism

[5] Observe that on a non-integral scheme it may happen that not every element of $|V|$ determines a divisor, since there may exist $s \in V$ for which $\mathrm{div}(s)$ is not defined. In this case calling $|V|$ a linear series is slightly unconventional. However we trust that no confusion will result.

$$\mathrm{eval}_V : V \otimes_{\mathbf{C}} \mathcal{O}_X \longrightarrow L$$

of vector bundles on X.

Definition 1.1.8. (Base locus and base ideal). The *base ideal* of $|V|$, written

$$\mathfrak{b}\big(|V|\big) \ = \ \mathfrak{b}\big(X, |V|\big) \ \subseteq \ \mathcal{O}_X,$$

is the image of the map $V \otimes_{\mathbf{C}} L^* \longrightarrow \mathcal{O}_X$ determined by eval_V. The *base locus*

$$\mathrm{Bs}\big(|V|\big) \ \subseteq \ X$$

of $|V|$ is the closed subset of X cut out by the base ideal $\mathfrak{b}\big(|V|\big)$. When we wish to emphasize the scheme structure on $\mathrm{Bs}\big(|V|\big)$ determined by $\mathfrak{b}\big(|V|\big)$ we will refer to $\mathrm{Bs}\big(|V|\big)$ as the *base scheme* of $|V|$. When $V = H^0\big(X, L\big)$ or $V = H^0\big(X, \mathcal{O}_X(D)\big)$ are finite-dimensional, we write respectively $\mathfrak{b}\big(|L|\big)$ and $\mathfrak{b}\big(|D|\big)$ for the base ideals of the indicated complete linear series. □

Very concretely, then, $\mathrm{Bs}\big(|V|\big)$ is the set of points at which all the sections in V vanish, and $\mathfrak{b}\big(|V|\big)$ is the ideal sheaf spanned by these sections.

Example 1.1.9. (Inclusions). Assuming for the moment that X is projective (or complete), fix a Cartier divisor D on X. Then for any integers $m, \ell \geq 1$, one has an inclusion

$$\mathfrak{b}\big(|\ell D|\big) \cdot \mathfrak{b}\big(|mD|\big) \ \subseteq \ \mathfrak{b}\big(|(\ell + m)D|\big).$$

(Use the natural homomorphism

$$H^0\big(X, \mathcal{O}_X(\ell D)\big) \otimes H^0\big(X, \mathcal{O}_X(mD)\big) \longrightarrow H^0\big(X, \mathcal{O}_X((\ell+m)D)\big)$$

determined by multiplication of sections.) □

The easiest linear series to deal with are those for which the base locus is empty.

Definition 1.1.10. (Free linear series). One says that $|V|$ is *free*, or *basepoint-free*, if its base locus is empty, i.e. if $\mathfrak{b}\big(|V|\big) = \mathcal{O}_X$. A divisor D or line bundle L is *free* if the corresponding complete linear series is so. In the case of line bundles one says synonymously that L is *generated by its global sections* or *globally generated*. □

In other words, $|V|$ is free if and only if for each point $x \in X$ one can find a section $s = s_x \in V$ such that $s(x) \neq 0$.

Assume now (in order to avoid trivialities) that $\dim V \geq 2$, and set $B = \mathrm{Bs}(|V|)$. Then $|V|$ determines a morphism

$$\phi \ = \ \phi_{|V|} \ : \ X - B \ \longrightarrow \ \mathbf{P}\big(V\big)$$

from the complement of the base locus in X to the projective space of one-dimensional quotients of V. Given $x \in X$, $\phi(x)$ is the hyperplane in V consisting of those sections vanishing at x. If one chooses a basis $s_0, \ldots, s_r \in V$, this amounts to saying that ϕ is given in homogeneous coordinates by the (somewhat abusive!) expression

$$\phi(x) \;=\; [s_0(x), \ldots, s_r(x)] \;\in\; \mathbf{P}^r.$$

When X is an irreducible variety it is sometimes useful to ignore the base locus, and view $\phi_{|V|}$ as a rational mapping $\phi : X \dashrightarrow \mathbf{P}(V)$. If $|V|$ is free then $\phi_{|V|} : X \longrightarrow \mathbf{P}(V)$ is a globally defined morphism.

At least when $B = \varnothing$ these constructions can be reversed, so that a morphism to projective space gives rise to a linear series. Specifically, suppose given a morphism

$$\phi \,:\, X \;\longrightarrow\; \mathbf{P} = \mathbf{P}(V)$$

from X to the projective space of one-dimensional quotients of a vector space V, and assume that $\phi(X)$ does not lie on any hyperplanes. Then pullback of sections via ϕ realizes $V = H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)\big)$ as a subspace of $H^0\big(X, \phi^*\mathcal{O}_{\mathbf{P}}(1)\big)$, and $|V|$ is a free linear series on X. Moreover, ϕ is identified with the corresponding morphism $\phi_{|V|}$.

Example 1.1.11. If X is a non-singular variety and $B \subseteq X$ has codimension ≥ 2, then a similar construction works starting with a morphism

$$\phi \;:\; X - B \;\longrightarrow\; \mathbf{P} = \mathbf{P}(V).$$

(In fact, $\phi^*\mathcal{O}_{\mathbf{P}}(1)$ extends uniquely to a line bundle L on X — corresponding to the divisor obtained by taking the closure of the pullback of a hyperplane — and ϕ^* realizes V as a subspace of $H^0\big(X, L\big)$, with $\mathrm{Bs}\big(|V|\big) \subseteq B$.) □

Example 1.1.12. (Projection). Suppose that $W \subseteq V$ is a subspace (say of dimension ≥ 2). Then $\mathrm{Bs}\big(|V|\big) \subseteq \mathrm{Bs}\big(|W|\big)$, so that $\phi_{|V|}$ and $\phi_{|W|}$ are both defined on $X - \mathrm{Bs}(|W|)$. Viewed as morphisms on this set one has the relation $\phi_{|W|} \;=\; \pi \circ \phi_{|V|}$, where

$$\pi \;:\; \mathbf{P}(V) - \mathbf{P}(V/W) \;\longrightarrow\; \mathbf{P}(W)$$

is linear projection centered along the subspace $\mathbf{P}(V/W) \subseteq \mathbf{P}(V)$. Note that if $|W|$ — and hence also $|V|$ — is free, and if X is complete, then $\pi|X$ is finite (since it is affine and proper). So in this case, the two morphisms

$$\phi_{|V|} : X \longrightarrow \mathbf{P}(V) \;\;,\;\; \phi_{|W|} : X \longrightarrow \mathbf{P}(W)$$

differ by a finite projection of $\phi_{|V|}(X)$. □

1.1.C Intersection Numbers and Numerical Equivalence

This subsection reviews briefly some definitions and facts from intersection theory.

Intersection numbers. Let X be a complete irreducible complex variety. Given Cartier divisors $D_1, \ldots, D_k \in \mathrm{Div}(X)$ together with an irreducible subvariety $V \subseteq X$ of dimension k, the intersection number

$$\big(D_1 \cdot \ldots \cdot D_k \cdot V\big) \ \in \ \mathbf{Z} \tag{1.3}$$

can be defined in various ways. To begin with, of course, the quantity in question arises as a special case of the theory in [208]. However intersection products of divisors against subvarieties do not require the full strength of that technology: a relatively elementary direct approach based on numerical polynomials was developed in the sixties by Snapper [546] and Kleiman [341]. Extensions and modern presentations of the Snapper–Kleiman theory appear in [363, VI.2], [114, Chapter 1.2], and [22, Chapter 1]. We prefer to minimize foundational discussions by working topologically, referring to [208] for additional properties as needed.[6] Some suggestions for the novice appear in Remark 1.1.13.

Specifically, in the above situation each of the line bundles $\mathcal{O}_X(D_i)$ has a Chern class

$$c_1\big(\mathcal{O}_X(D_i)\big) \ \in \ H^2\big(X;\mathbf{Z}\big),$$

the cohomology group in question being ordinary singular cohomology of X with its classical topology. The cup product of these classes is then an element

$$c_1\big(\mathcal{O}_X(D_1)\big) \cdot \ldots \cdot c_1\big(\mathcal{O}_X(D_k)\big) \ \in \ H^{2k}\big(X;\mathbf{Z}\big) :$$

here and elsewhere we write $\alpha \cdot \beta$ or simply $\alpha\beta$ for the cup product of elements $\alpha, \beta \in H^*\big(X;\mathbf{Z}\big)$. Denoting by $[V] \in H_{2k}\big(X;\mathbf{Z}\big)$ the fundamental class of V, cap product leads finally to an integer

$$\Big(c_1\big(\mathcal{O}_X(D_1)\big) \cdot \ldots \cdot c_1\big(\mathcal{O}_X(D_k)\big)\Big) \cap [V] \ \in \ H_0\big(X;\mathbf{Z}\big) \ = \ \mathbf{Z}, \tag{1.4}$$

which of course is nothing but the quantity appearing in (1.3). We generally use one of the notations

$$\big(D_1 \cdot \ldots \cdot D_k \cdot V\big) \quad , \qquad \int_V D_1 \cdot \ldots \cdot D_k$$

[6] The essential foundational savings materialize when we deal with higher-codimension intersection theory — e.g. Chern classes of vector bundles — in Part Two of this book. Working topologically allows one to bypass complications involved in specifying groups to receive the classes in question. It then seemed natural to use topologically based intersection theory throughout.

(or a small variant thereof) for the intersection product in question. By linearity one can replace V by an arbitrary k-cycle, and evidently this product depends only on the linear equivalence class of the D_i. If $D_1 = \ldots = D_k = D$ we write $\big(D^k \cdot V\big)$, and when $V = X$ is irreducible of dimension n we often use the abbreviation $\big(D_1 \cdot \ldots \cdot D_n\big) \in \mathbf{Z}$. Intersection numbers involving line bundles in place of divisors are of course defined analogously.

Similar constructions work when X or V are possibly non-reduced complete complex schemes provided only that V has pure dimension k. The homology and cohomology groups of X are those of the underlying Hausdorff space: in other words, $H^*\big(X;\mathbf{Z}\big)$ and $H_*\big(X;\mathbf{Z}\big)$ do not see the scheme structure of X. However, one introduces the cycle $[V]$ of V, viz. the algebraic k-cycle

$$[V] \;=\; \sum_{V_i} \big(\mathrm{length}_{\mathcal{O}_{V_i}} \mathcal{O}_V\big) \cdot [V_i]$$

on X, where $\{V_i\}$ are the irreducible components of V (with their reduced scheme structures), and $\mathcal{O}_{V_i}$ is the local ring of V along V_i. By linearity we get a corresponding class $[V] = \sum \big(\mathrm{length}_{\mathcal{O}_{V_i}} \mathcal{O}_V\big) \cdot [V_i] \in H_{2k}\big(X;\mathbf{Z}\big)$. Then the cap product appearing in (1.4) defines the intersection number

$$\big(D_1 \cdot \ldots \cdot D_k \cdot [V]\big) \;=\; \int_{[V]} D_1 \cdot \ldots \cdot D_k \;\in\; \mathbf{Z}.$$

Somewhat abusively we often continue to write simply $\big(D_1 \cdot \ldots \cdot D_k \cdot V\big) \in \mathbf{Z}$, it being understood that one has to take into account any multiple components of V. If V has pure dimension d and $k \leq d$ then we define

$$\begin{aligned}\big(D_1 \cdot \ldots \cdot D_k \cdot [V]\big) \;=_{\mathrm{def}}\; &\Big(c_1\big(\mathcal{O}_X(D_1)\big) \cdot \ldots \cdot c_1\big(\mathcal{O}_X(D_k)\big)\Big) \cap [V] \\ &\qquad \in\; H_{2d-2k}\big(X;\mathbf{Z}\big). \end{aligned} \tag{1.5}$$

These intersection classes are compatible with the constructions in [208] and they satisfy the usual formal properties, as in [208, Chapter 2].[7] For instance if X has pure dimension n and D is an effective Cartier divisor on X, then we may view D as a subscheme of X and

$$c_1\big(\mathcal{O}_X(D)\big) \cap [X] \;=\; [D] \;\in\; H_{2(n-1)}\big(X;\mathbf{Z}\big) \tag{1.6}$$

([208, Chapter 2.5]). The same formula holds even if D is not effective provided that one interprets the right-hand side as the homology class of the Weil

[7] More precisely, the constructions of [208, Chapter 2] yield a class

$$\Big(c_1\big(\mathcal{O}_X(D_1)\big) \cdot \ldots \cdot c_1\big(\mathcal{O}_X(D_k)\big)\Big) \cap [V] \;\in\; A_{d-k}(X)$$

in the Chow group that maps to the class in (1.5) under the cycle map $A_{d-k}(X) \longrightarrow H_{2(d-k)}\big(X;\mathbf{Z}\big)$.

divisor determined by D (Example 1.1.3). Similarly, if V is an irreducible variety, then

$$c_1\big(\mathcal{O}_X(D)\big) \cap [V] \;=\; c_1\big(\,\mathcal{O}_X(D)\,|\,V\,\big) \cap [V] \;=\; [\overline{D}],$$

where $\overline{D} \in \mathrm{Div}(V)$ is a divisor on V with $\mathcal{O}_V(\overline{D}) = \mathcal{O}_X(D)\,|\,V$. This inductively leads to the important fact that the intersection class $\big(D_1 \cdot \ldots \cdot D_k \cdot [V]\big)$ in (1.5) is represented by an algebraic $(d-k)$-cycle on X. In fact it is even represented by a $(d-k)$-cycle on $\mathrm{Supp}(D_1) \cap \ldots \cap \mathrm{Supp}(D_k) \cap V$.

Remark 1.1.13. (Advice for the novice). The use of topological definitions as our "official" foundation for intersection theory might not be the most accessible approach for a novice. So we say here a few words about what we actually require, and where one can learn it. In the present volume, all one needs for the most part is to be able to define the intersection number

$$\big(D_1 \cdot \ldots \cdot D_n\big) \;=\; \int_X D_1 \cdot \ldots \cdot D_n \;\in\; \mathbf{Z}$$

of n Cartier divisors $D_1, \ldots, D_n$ on an n-dimensional irreducible projective (or complete) variety X. The most important features of this product (which in fact characterize it in the projective case) are:

(i). The integer $\big(D_1 \cdot \ldots \cdot D_n\big)$ is symmetric and multilinear as a function of its arguments;

(ii). $\big(D_1 \cdot \ldots \cdot D_n\big)$ depends only on the linear equivalence classes of the D_i;

(iii). If $D_1, \ldots, D_n$ are effective divisors that meet transversely at smooth points of X, then

$$\big(D_1 \cdot \ldots \cdot D_n\big) \;=\; \#\,\{\, D_1 \cap \ldots \cap D_n \,\}.$$

Given an irreducible subvariety $V \subseteq X$ of dimension k, the intersection number

$$\big(D_1 \cdot \ldots \cdot D_k \cdot V\big) \;\in\; \mathbf{Z} \tag{*}$$

is then defined by replacing each divisor D_i with a linearly equivalent divisor D_i' whose support does not contain V, and intersecting the restrictions of the D_i' on V.[8] It is also important to know that if D_n is reduced, irreducible and effective, then one can compute $(D_1 \cdot \ldots \cdot D_n)$ by taking $V = D_n$ in (*). The intersection product satisfies the projection formula: if $f : Y \longrightarrow X$ is a generically finite surjective proper map, then

$$\int_Y f^*D_1 \cdot \ldots \cdot f^*D_n \;=\; \big(\deg f\,\big) \cdot \int_X D_1 \cdot \ldots \cdot D_n.$$

By linearity, one can replace V in (*) by an arbitrary k-cycle, and the analogous constructions when X or V carries a possibly non-reduced scheme structure are handled as above by passing to cycles.

[8] This is independent of the choice of D_i' thanks to (ii).

The case $\dim X = 2$ is treated very clearly in Chapter 5, Section 1, of [280], and this is certainly the place for a beginner to start. The extension to higher dimensions might to some extent be taken on faith. Alternatively, as noted above the theory is developed in detail via the method of Snapper and Kleiman in [363, Chapter 6.2], [114, Chapter 1.2] or [22, Chapter 1]. In this approach the crucial Theorem 1.1.24 is established along the way to defining intersection products. A more elementary presentation appears in [532, Chapter 4] provided that one is willing to grant 1.1.24. □

Numerical equivalence. We continue to assume that X is a complete algebraic scheme over $\mathbf{C}$. Of the various natural equivalence relations defined on $\mathrm{Div}(X)$, we will generally deal with the weakest:

Definition 1.1.14. (Numerical equivalence). Two Cartier divisors

$$D_1\,,\, D_2 \ \in \ \mathrm{Div}(X)$$

are *numerically equivalent*, written $D_1 \equiv_{\mathrm{num}} D_2$, if

$$\big(D_1 \cdot C\big) \ = \ \big(D_2 \cdot C\big) \ \text{ for every irreducible curve } C \subseteq X,$$

or equivalently if $(D_1 \cdot \gamma) = (D_2 \cdot \gamma)$ for all one-cycles γ on X. Numerical equivalence of line bundles is defined in the analogous manner. A divisor or line bundle is *numerically trivial* if it is numerically equivalent to zero, and $\mathrm{Num}(X) \subseteq \mathrm{Div}(X)$ is the subgroup consisting of all numerically trivial divisors. □

Definition 1.1.15. (Néron–Severi group). The *Néron–Severi group* of X is the group

$$N^1(X) = \mathrm{Div}(X)/\mathrm{Num}(X)$$

of numerical equivalence classes of divisors on X. □

The first basic fact is that this group is finitely generated:

Proposition 1.1.16. (Theorem of the base). *The Néron–Severi group $N^1(X)$ is a free abelian group of finite rank.*

Definition 1.1.17. (Picard number). The rank of $N^1(X)$ is called the *Picard number* of X, written $\rho(X)$. □

Proof of Proposition 1.1.16. A divisor D on X determines a cohomology class

$$[D]_{\mathrm{hom}} \ = \ c_1(\mathcal{O}_X(D)) \ \in \ H^2\big(X;\mathbf{Z}\big),$$

and if $[D]_{\mathrm{hom}} = 0$ then evidently D is numerically trivial. Therefore the group $\mathrm{Hom}(X)$ of cohomologically trivial Cartier divisors is a subgroup of $\mathrm{Num}(X)$. It follows that $N^1(X)$ is a quotient of a subgroup of $H^2\big(X;\mathbf{Z}\big)$, and in particular is finitely generated. It is torsion-free by construction. □

The next point is that intersection numbers respect numerical equivalence:

Lemma 1.1.18. *Let X be a complete variety or scheme, and let*

$$D_1, \dots, D_k, D'_1, \dots, D'_k \in \mathrm{Div}(X)$$

be Cartier divisors on X. If $D_i \equiv_{\mathrm{num}} D'_i$ for each i, then

$$\big(D_1 \cdot D_2 \cdot \ldots \cdot D_k \cdot [V]\big) \;=\; \big(D'_1 \cdot D'_2 \cdot \ldots \cdot D'_k \cdot [V]\big)$$

for every subscheme $V \subseteq X$ of pure dimension k.

The lemma allows one to discuss intersection numbers among numerical equivalence classes:

Definition 1.1.19. (Intersection of numerical equivalence classes). Given classes $\delta_1, \dots, \delta_k \in N^1(X)$, we denote by

$$\int_{[V]} \delta_1 \cdot \ldots \cdot \delta_k \qquad \text{or} \qquad \big(\delta_1 \cdot \ldots \cdot \delta_k \cdot [V]\big)$$

the intersection number of any representatives of the classes in question. □

Proof of Lemma 1.1.18. We assert first that if $E \equiv_{\mathrm{num}} 0$ is a numerically trivial Cartier divisor on X, then $\big(E \cdot D_2 \cdot \ldots \cdot D_k \cdot [V]\big) \;=\; 0$ for any Cartier divisors $D_2, \dots, D_k$. In fact, $c_1\big(\mathcal{O}_X(D_2)\big) \cdot \ldots \cdot c_1\big(\mathcal{O}_X(D_k)\big) \cap [V]$ is represented by a one-cycle γ on X, and so $\big(E \cdot D_2 \cdot \ldots \cdot D_k \cdot [V]\big) \;=\; (E \cdot \gamma) \;=\; 0$ by definition of numerical equivalence. This shows that $\big(D_1 \cdot D_2 \cdot \ldots \cdot D_k \cdot [V]\big) = \big(D'_1 \cdot D_2 \cdot \ldots \cdot D_k \cdot [V]\big)$ provided that $D_1 \equiv_{\mathrm{num}} D'_1$, and the lemma follows by induction on k. □

Remark 1.1.20. (Characterization of numerically trivial line bundles). A useful characterization of numerically trivial line bundles is established in Kleiman's exposé [52, XIII, Theorem 4.6] in SGA6. Specifically, consider a line bundle L on a complete scheme X. Then L is numerically trivial if and only if there is an integer $m \neq 0$ such that $L^{\otimes m} \in \mathrm{Pic}^0(X)$, i.e. such that $L^{\otimes m}$ is a deformation of the trivial line bundle. We sketch a proof in Section 1.4.D in the projective case based on a vanishing theorem of Fujita and Grothendieck's Quot schemes. (We also give there a fuller explanation of the statement.) □

Remark 1.1.21. (Lefschetz $(1\,,1)$-theorem). When X is a non-singular projective variety, Hodge theory gives an alternative description of $N^1(X)$. Set

$$H^2\big(X;\mathbf{Z}\big)_{\text{t.f.}} \;=\; H^2\big(X;\mathbf{Z}\big)/\text{ (torsion) }.$$

It follows from the result quoted in the previous remark that if D is a numerically trivial divisor then $[D]_{\mathrm{hom}} \in H^2\big(X;\mathbf{Z}\big)$ is a torsion class. Therefore

$N^1(X)$ embeds into $H^2\big(X;\mathbf{Z}\big)_{\text{t.f.}}$. On the other hand, the Lefschetz $(1,1)$-theorem asserts that a class $\alpha \in H^2\big(X;\mathbf{Z}\big)$ is algebraic if and only if α has type $(1,1)$ under the Hodge decomposition of $H^2\big(X;\mathbf{C}\big)$ (cf. [248, Chapter 1, §2]). Therefore

$$N^1(X) \;=\; H^2\big(X;\mathbf{Z}\big)_{\text{t.f.}} \cap H^{1,1}\big(X;\mathbf{C}\big). \quad \square$$

Finally we say a word about functoriality. Let $f : Y \longrightarrow X$ be a morphism of complete varieties or projective schemes. If $\alpha \in \mathrm{Pic}(X)$ is a class mapping to zero in $N^1(X)$, then it follows from the projection formula that $f^*(\alpha)$ is numerically trivial on Y. Therefore the pullback mapping on Picard groups determines thanks to Example 1.1.5 a functorial induced homomorphism $f^* : N^1(X) \longrightarrow N^1(Y)$.

Remark 1.1.22. (Non-projective schemes). As indicated at the end of Section 1.1.A, the integrality and projectivity hypotheses in the previous paragraph arise only in order to use the functorial properties of line bundles to discuss divisors. To have a theory that runs smoothly for possibly non-projective schemes, it would be better — as in [341] — to take $N^1(X)$ to be the additive group of numerical equivalence classes of line bundles: we leave this modification to the interested reader. As explained above we prefer to stick with the classical language of divisors. $\square$

1.1.D Riemann–Roch

We will often have occasion to draw on asymptotic forms of the Riemann–Roch theorem, and we give a first formulation here. More detailed treatments appear in [363, VI.2], [114, Chapter 1.2], and [22, Chapter 1], to which we will refer for proofs.

We start with a definition:

Definition 1.1.23. (Rank and cycle of a coherent sheaf). Let X be an irreducible variety (or scheme) of dimension n, and $\mathcal{F}$ a coherent sheaf on X. The *rank* $\mathrm{rank}(\mathcal{F})$ of $\mathcal{F}$ is the length of the stalk of $\mathcal{F}$ at the generic point of X. If X is reduced, then

$$\mathrm{rank}(\mathcal{F}) \;=\; \dim_{\mathbf{C}(X)} \mathcal{F} \otimes \mathbf{C}(X).$$

If X is reducible (but still of dimension n), then one defines similarly the rank of $\mathcal{F}$ along any n-dimensional irreducible component V of X: $\mathrm{rank}_V(\mathcal{F}) = \mathrm{length}_{\mathcal{O}_v} \mathcal{F}_v$, where $\mathcal{F}_v$ is the stalk of $\mathcal{F}$ at the generic point v of V. The *cycle* of $\mathcal{F}$ is the n-cycle

$$Z_n(\mathcal{F}) \;=\; \sum_V \mathrm{rank}_V(\mathcal{F}) \cdot [V],$$

the sum being taken over all n-dimensional components of X. $\square$

One then has

Theorem 1.1.24. (Asymptotic Riemann–Roch, I). *Let X be an irreducible projective variety of dimension n, and let D be a divisor on X. Then the Euler characteristic $\chi\big(X, \mathcal{O}_X(mD)\big)$ is a polynomial of degree $\leq n$ in m, with*

$$\chi\big(X, \mathcal{O}_X(mD)\big) \;=\; \frac{\big(D^n\big)}{n!}\, m^n \;+\; O(m^{n-1}). \tag{1.7}$$

More generally, for any coherent sheaf $\mathcal{F}$ on X,

$$\chi\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) \;=\; \mathrm{rank}(\mathcal{F}) \cdot \frac{\big(D^n\big)}{n!} \cdot m^n \;+\; O(m^{n-1}). \tag{1.8}$$

Corollary 1.1.25. *In the setting of the theorem, if $H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) = 0$ for $i > 0$ and $m \gg 0$ then*

$$h^0\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) \;=\; \mathrm{rank}(\mathcal{F}) \cdot \frac{\big(D^n\big)}{n!}\, m^n \;+\; O(m^{n-1}) \tag{1.9}$$

for large m. More generally, (1.9) *holds provided that*

$$h^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) \;=\; O(m^{n-1})$$

for $i > 0$. $\square$

Remark 1.1.26. (Reducible schemes). The formula (1.7) remains valid if X is a possibly reducible complete scheme of pure dimension n provided as usual that we interpret $\big(D^n\big)$ as the intersection number $\int_{[X]} D^n$. The same is true of (1.8) provided that the first term on the right is replaced by

$$\Big(\int_{Z_n(\mathcal{F})} D^n\Big) \cdot \frac{m^n}{n!}.$$

With the analogous modifications, the corollary likewise extends to possibly reducible complete schemes. $\square$

We do not prove Theorem 1.1.24 here. The result is established in a relatively elementary fashion via the approach of Snapper–Kleiman in [363, Corollary VI.2.14] (see also [22, Chapter 1]). Debarre [114, Theorem 1.5] gives a very accessible account of the main case $\mathcal{F} = \mathcal{O}_X$. However, one can quickly obtain 1.1.24 and 1.1.26 as special cases of powerful general results: see the next example.

Example 1.1.27. (Theorem 1.1.24 via Hirzebruch–Riemann–Roch). Theorem 1.1.24 and the extension in 1.1.26 yield easily to heavier machinery. In fact, if X is a non-singular variety then the Euler characteristic in question is computed by the Hirzebruch–Riemann–Roch theorem:

$$\chi\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) \;=\; \int_X \mathrm{ch}(\mathcal{F} \otimes \mathcal{O}_X(mD)) \cdot \mathrm{Td}(X).$$

Viewing $\mathrm{ch}(\mathcal{F} \otimes \mathcal{O}_X(mD))$ as a polyomial in m, one has

$$\begin{aligned} \mathrm{ch}(\mathcal{F} \otimes \mathcal{O}_X(mD)) &= \mathrm{ch}(\mathcal{F}) \cdot \mathrm{ch}\big(\mathcal{O}_X(mD)\big) \\ &= \ \mathrm{rank}(\mathcal{F}) \cdot \frac{c_1\big(\mathcal{O}_X(D)\big)^n}{n!}\, m^n \ + \ \text{lower-degree terms}, \end{aligned}$$

which gives (1.8) in this case. On an arbitrary complete scheme one can invoke similarly the general Riemann–Roch theorem for singular varieties [208, Corollary 18.3.11 and Example 18.3.6].[9] □

Finally we record two additional results for later reference. The first asserts that Euler characteristics are multiplicative under étale covers:

Proposition 1.1.28. (Étale multiplicativity of Euler characteristics). *Let $f : Y \longrightarrow X$ be a finite étale covering of complete schemes, and let $\mathcal{F}$ be any coherent sheaf on X. Then*

$$\chi\big(Y\,,\, f^*\mathcal{F}\big) \ = \ \deg\,(Y \longrightarrow X) \cdot \chi\big(X\,,\, \mathcal{F}\big).$$

This follows for example from the Riemann–Roch theorem and [208, Example 18.3.9]. An elementary direct approach — communicated by Kleiman — is outlined in Example 1.1.30.

Example 1.1.29. The Riemann–Hurwitz formula for branched coverings of curves shows that Proposition 1.1.28 fails in general if f is not étale. □

Example 1.1.30. (Kleiman's proof of Proposition 1.1.28). We sketch a proof of 1.1.28 when X is projective. Set $d = \deg(f)$. Arguing as in Example 1.4.42 one reduces to the case in which X is an integral variety and $\mathcal{F} = \mathcal{O}_X$, and by induction on dimension one can assume that the result is known for all sheaves supported on a proper subset of X. Since f is finite one has $\chi(Y, \mathcal{O}_Y) = \chi(X, f_*\mathcal{O}_Y)$, so the issue is to show that

$$\chi(X, f_*\mathcal{O}_Y) \ = \ d \cdot \chi(X, \mathcal{O}_X). \tag{*}$$

For this, choose an ample divisor H on X. Then for $p \gg 0$ one can construct exact sequences

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_X(-pH)^d \longrightarrow f_*\mathcal{O}_Y \longrightarrow \mathcal{G}_1 \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_X(-pH)^d \longrightarrow \mathcal{O}_X^d \longrightarrow \mathcal{G}_2 \longrightarrow 0, \end{gathered} \tag{**}$$

where $\mathcal{G}_1$, $\mathcal{G}_2$ are supported on proper subsets of X. Now suppose one knew that 1.1.28 held for $\mathcal{F} = f_*\mathcal{O}_Y$, i.e. suppose that one knows

$$\chi(Y, f^*f_*\mathcal{O}_Y) \ = \ d \cdot \chi(X, f_*\mathcal{O}_Y). \tag{***}$$

[9] Note however that the term $\tau_{X,n}(\mathcal{F})$ is missing from the last displayed formula in [208, 18.3.6].

Since we can assume that 1.1.28 holds for $\mathcal{G}_1$ and $\mathcal{G}_2$, the exact sequences (**) will then yield (*). So it remains to prove (***).

To this end, consider the fibre square

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where $W = Y \times_X Y$. Since f is étale, W splits as the disjoint union of a copy of Y and another scheme W' étale of degree $d-1$ over Y. So by induction on d, we can assume that $\chi(W, \mathcal{O}_W) = \chi(Y, g_*\mathcal{O}_W) = d \cdot \chi(Y, \mathcal{O}_Y)$. On the other hand, $f^*f_*\mathcal{O}_Y = g_*g^*\mathcal{O}_Y = g_*\mathcal{O}_W$ since f is flat ([280, III.9.3]), and then (***) follows. ☐

The second result, allowing one to produce very singular divisors, will be useful in Chapters 5 and 10.

Proposition 1.1.31. (Constructing singular divisors). *Let X be an irreducible projective (or complete) variety of dimension n, and let D be a divisor on X with the property that $h^i\big(X, \mathcal{O}_X(mD)\big) = O(m^{n-1})$ for $i > 0$. Fix a positive rational number α with*

$$0 \ < \ \alpha^n \ < \ \big(D^n\big).$$

Then when $m \gg 0$ there exists for any smooth point $x \in X$ a divisor $E = E_x \in |mD|$ with

$$\operatorname{mult}_x(E) \ \geq \ m \cdot \alpha. \tag{1.10}$$

Here $\operatorname{mult}_x(E)$ denotes as usual the multiplicity of the divisor E at x, i.e. the order of vanishing at x of a local equation for E. The proof will show that there is one large value of m that works simultaneously at all smooth points $x \in X$.

Proof. Producing a divisor with prescribed multiplicity at a given point involves solving the system of linear equations determined by the vanishing of an appropriate number of partial derivatives of a defining equation. To prove the Proposition we simply observe that under the stated assumptions there are more variables than equations. Specifically, the number of sections of $\mathcal{O}_X(mD)$ is estimated by 1.1.25:

$$h^0\big(X, \mathcal{O}_X(mD)\big) \ = \ \frac{(D^n)}{n!}m^n \ + \ O(m^{n-1}).$$

On the other hand, it is at most

$$\binom{n+c-1}{n} \ = \ \frac{c^n}{n!} \ + \ O(c^{n-1})$$

conditions for a section of $\mathcal{O}_X(mD)$ to vanish to order $\geq c$ at a smooth point $x \in X$. Taking $m \gg 0$ and suitable $m \cdot \alpha < c < m \cdot \big(D^n\big)^{1/n}$, we get the required divisor. □

Remark 1.1.32. (Other ground fields). The discussion in this section goes through with only minor changes if X is an algebraic variety or scheme defined over an algebraically closed field of any characteristic. (In Section 1.1.C one would use the algebraic definition of intersection numbers, and a different argument is required to prove that $N^1(X)$ has finite rank: see [341, Chapter IV]). □

1.2 The Classical Theory

Given a divisor D on a projective variety X, what should it mean for D to be positive? The most appealing idea from an intuitive point of view is to ask that D be a hyperplane section under some projective embedding of X — one says then that D is *very ample*. However this turns out to be rather difficult to work with technically: already on curves it can be quite subtle to decide whether or not a given divisor is very ample. It is found to be much more convenient to focus instead on the condition that some positive multiple of D be very ample; in this case D is *ample*. This definition leads to a very satisfying theory, which was largely worked out in the fifties and early sixties. The fundamental conclusion is that on a projective variety, amplitude can be characterized geometrically (which we take as the definition), cohomologically (theorem of Cartan–Serre–Grothendieck) or numerically (Nakai–Moishezon–Kleiman criterion).

This section is devoted to an overview of the classical theory of ample line bundles. One of our purposes is to set down the basic facts in a form convenient for later reference. The cohomological material in particular is covered (in greater generality and detail) in Hartshorne's text [280], to which we will refer where convenient. Chapter I of Hartshorne's earlier book [276] contains a nice exposition of the theory, and in several places we have drawn on his discussion quite closely.

We begin with the basic definition.

Definition 1.2.1. (Ample and very ample line bundles and divisors on a complete scheme). Let X be a complete scheme, and L a line bundle on X.

(i). L is *very ample* if there exists a closed embedding $X \subseteq \mathbf{P}$ of X into some projective space $\mathbf{P} = \mathbf{P}^N$ such that

$$L \;=\; \mathcal{O}_X(1) \;=_{\text{def}}\; \mathcal{O}_{\mathbf{P}^N}(1) \mid X.$$

(ii). L is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

A Cartier divisor D on X is ample or very ample if the corresponding line bundle $\mathcal{O}_X(D)$ is so. □

Remark 1.2.2. (Amplitude). We will interchangeably use "ampleness" and "amplitude" to describe the property of being ample. We feel that the euphonious quality of the latter term compensates for the fact that it may not be completely standard in the present context. (Confusion with other meanings of amplitude seems very unlikely.) □

Example 1.2.3. (Ample line bundle on curves). If X is an irreducible curve, and L is a line bundle on X, then L is ample if and only if $\deg(L) > 0$. □

Example 1.2.4. (Varieties with Pic = Z). If X is a projective variety with $\mathrm{Pic}(X) = \mathbf{Z}$, then any non-zero effective divisor on X is ample. (Use 1.2.6 (iv).) This applies, for instance, when X is a projective space or a Grassmannian. □

Example 1.2.5. (Intersection products). If $D_1, \ldots, D_n$ are ample divisors on an n-dimensional projective variety X, then $\big(D_1 \cdot \ldots \cdot D_n\big) > 0$. (One can assume that each D_i is very ample, and then the inequality reduces to Example 1.2.3.) □

1.2.A Cohomological Properties

The first basic fact is that amplitude can be detected cohomologically:

Theorem 1.2.6. (Cartan–Serre–Grothendieck theorem). *Let L be a line bundle on a complete scheme X. The following are equivalent:*

(i). *L is ample.*

(ii). *Given any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_1 = m_1(\mathcal{F})$ having the property that*

$$H^i\big(X, \mathcal{F} \otimes L^{\otimes m}\big) = 0 \quad \textit{for all} \;\; i > 0 \;,\; m \geq m_1(\mathcal{F}).$$

(iii). *Given any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_2 = m_2(\mathcal{F})$ such that $\mathcal{F} \otimes L^{\otimes m}$ is generated by its global sections for all $m \geq m_2(\mathcal{F})$.*

(iv). *There is a positive integer $m_3 > 0$ such that $L^{\otimes m}$ is very ample for every $m \geq m_3$.*

Remark 1.2.7. (Serre vanishing). The conclusion in (ii) is often referred to as Serre's vanishing theorem. □

Outline of Proof of Theorem 1.2.6. (i) ⇒ (ii). We assume to begin with that L is very ample, defining an embedding of X into some projective space $\mathbf{P}$.

In this case, extending $\mathcal{F}$ by zero to a coherent sheaf on $\mathbf{P}$, we are reduced to the vanishing of $H^i\big(\mathbf{P}, \mathcal{F}(m)\big)$ for $m \gg 0$, which is the content of [280, Theorem III.5.2]. In general, when L is merely ample, fix m_0 such that $L^{\otimes m_0}$ is very ample. Then apply the case already treated to each of the sheaves $\mathcal{F}, \mathcal{F} \otimes L, \ldots, \mathcal{F} \otimes L^{\otimes m_0 - 1}$.

(ii) $\Rightarrow$ (iii). Fix a point $x \in X$, and denote by $\mathfrak{m}_x \subset \mathcal{O}_X$ the maximal ideal sheaf of x. By (ii) there is an integer $m_2(\mathcal{F}, x)$ such that

$$H^1\big(X, \mathfrak{m}_x \cdot \mathcal{F} \otimes L^{\otimes m}\big) = 0 \quad \text{for} \quad m \geq m_2(\mathcal{F}, x).$$

It then follows from the exact sequence

$$0 \longrightarrow \mathfrak{m}_x \cdot \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \;/\; \mathfrak{m}_x \cdot \mathcal{F} \longrightarrow 0$$

upon twisting by $L^{\otimes m}$ and taking cohomology that $\mathcal{F} \otimes L^{\otimes m}$ is globally generated in a neighborhood of x for every $m \geq m_2(\mathcal{F}, x)$. By quasi-compactness we can then choose a single natural number $m_2(\mathcal{F})$ that works for all $x \in X$.

(iii) $\Rightarrow$ (iv). It follows first of all from (iii) that there exists a positive integer p_1 such that $L^{\otimes m}$ is globally generated for all $m \geq p_1$. Denote by

$$\phi_m : X \longrightarrow \mathbf{P}H^0\big(X, L^{\otimes m}\big)$$

the corresponding map to projective space. We need to show that we can arrange for ϕ_m to be an embedding by taking $m \gg 0$, for which it is sufficient to prove that ϕ_m is one-to-one and unramified ([280, II.7.3]). To this end, consider the set

$$U_m = \big\{\, y \in X \;\big|\; L^{\otimes m} \otimes \mathfrak{m}_y \text{ is globally generated} \,\big\}.$$

This is an open set (Example 1.2.9), and $U_m \subset U_{m+p}$ for $p \geq p_1$ thanks to the fact that $L^{\otimes p}$ is generated by its global sections. Given any point $x \in X$ we can find by (iii) an integer $m_2(x)$ such that $x \in U_m$ for all $m \geq m_2(x)$, and therefore $X = \cup U_m$. By quasi-compactness there is a single integer $m_3 \geq p_1$ such that $L^{\otimes m} \otimes \mathfrak{m}_x$ is generated by its global sections for every $x \in X$ whenever $m \geq m_3$. But the global generation of $L^{\otimes m} \otimes \mathfrak{m}_x$ implies that $\phi_m(x) \neq \phi_m(x')$ for all $x' \neq x$, and that ϕ_m is unramified at x. Thus ϕ_m is an embedding for all $m \geq m_3$, as required.

(iv) $\Rightarrow$ (i): Definitional. □

Remark 1.2.8. (Amplitude on non-complete schemes). One can also discuss ample line bundles on possibly non-complete schemes. In this more general setting, property (iii) from 1.2.6 is taken as the definition of amplitude. □

Example 1.2.9. Let B be a globally generated line bundle on a complete variety or scheme X. Then

$$U =_{\text{def}} \left\{ y \in X \mid B \otimes \mathfrak{m}_y \text{ is globally generated} \right\}$$

is an open subset of X. (Since B is globally generated, it suffices by Nakayama's lemma to prove the openness of the set

$$V =_{\text{def}} \left\{ y \in X \mid H^0\big(X, B\big) \longrightarrow B \otimes \mathcal{O}_X/\mathfrak{m}_y^2 \text{ is surjective } \right\}.$$

But this follows from the existence of a coherent sheaf $\mathcal{P}$ on X, whose fibre at y is canonically $\mathcal{P}(y) = B \otimes \mathcal{O}_X/\mathfrak{m}_y^2$, together with a map $u : H^0\big(X, B\big) \otimes \mathcal{O}_X \longrightarrow \mathcal{P}$ that fibre by fibre is given by evaluation of sections. In fact if $u(y)$ is surjective at one point y then it is surjective in a neighborhood of y by the coherence of $\operatorname{coker}(u)$. As for $\mathcal{P}$, it is the sheaf $\mathcal{P} = P^2_X(B)$ of second-order principal parts of B: starting with the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal on $X \times X$ one takes

$$P^2_X(B) \;=\; \mathrm{pr}_{2,*}\big(\mathrm{pr}_1^* B \otimes (\mathcal{O}_{X\times X}/\mathcal{I}_\Delta^2)\big).$$

See [257, Chapter 16] for details on these sheaves.) □

Example 1.2.10. (Sums of divisors). Let D and E be (Cartier) divisors on a projective scheme X. If D is ample, then so too is $mD+E$ for all $m \gg 0$. In fact, $mD+E$ is very ample if $m \gg 0$. (For the second assertion choose positive integers m_1, m_2 such that mD is very ample for $m \geq m_1$ and $mD+E$ is free when $m \geq m_2$. Then $mD+E$ is very ample once $m \geq m_1 + m_2$.) □

Example 1.2.11. (Ample line bundles on a product). If L and M are ample line bundles on projective schemes X and Y respectively, then $\mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* M$ is ample on $X \times Y$. □

Remark 1.2.12. (Matsusaka's theorem). According to Theorem 1.2.6, if L is an ample line bundle on a projective variety X then there is an integer $m(L)$ such that $L^{\otimes m}$ is very ample for $m \geq m(L)$. However, the proof of the theorem fails to give any concrete information about the value of this integer. So it is interesting to ask what geometric information $m(L)$ depends on, and whether one can give effective estimates. A theorem of Matsusaka [420] and Kollár–Matsusaka [366] states that if X is a smooth projective variety of dimension n, then one can find $m(L)$ depending only on the intersection numbers $\int c_1(L)^n$ and $\int c_1(L)^{n-1} c_1(X)$. Siu [537] used the theory of multiplier ideals to give an effective statement, which was subsequently improved and clarified by Demailly [126]. A proof of the theorem of Kollár–Matsusaka via the approach of Siu–Demailly appears in Section 10.2. An example due to Kollár, showing that in general one cannot take $m(L)$ independent of L, is presented in Example 1.5.7. □

Proposition 1.2.13. (Finite pullbacks, I). *Let $f : Y \longrightarrow X$ be a finite mapping of complete schemes, and L an ample line bundle on X. Then f^*L is an ample line bundle on Y. In particular, if $Y \subseteq X$ is a subscheme of X, then the restriction $L \mid Y$ of L to Y is ample.*

Remark 1.2.14. See Corollary 1.2.28 for a partial converse. □

Proof of Proposition 1.2.13. Let $\mathcal{F}$ be a coherent sheaf on Y. Then $f_*\big(\mathcal{F} \otimes f^*L^{\otimes m}\big) = f_*\mathcal{F} \otimes L^{\otimes m}$ by the projection formula, and $R^j f_*\big(\mathcal{F} \otimes f^*L^{\otimes m}\big) = 0$ for $j > 0$ thanks to the finiteness of f. Therefore

$$H^i\big(Y, \mathcal{F} \otimes f^*L^{\otimes m}\big) = H^i\big(X, f_*\mathcal{F} \otimes L^{\otimes m}\big)$$

for all i, and the statement then follows from the characterization (ii) of amplitude in Theorem 1.2.6. □

Corollary 1.2.15. (Globally generated line bundles). *Suppose that L is globally generated, and let*

$$\phi = \phi_{|L|} : X \longrightarrow \mathbf{P} = \mathbf{P}H^0\big(X, L\big)$$

be the resulting map to projective space defined by the complete linear system $|L|$. Then L is ample if and only if ϕ is a finite mapping, or equivalently if and only if

$$\int_C c_1(L) > 0$$

for every irreducible curve $C \subseteq X$.

Proof. The preceding proposition shows that if ϕ is finite, then L is ample. In this case evidently $\int_C c_1(L) > 0$ for every irreducible curve $C \subseteq X$. Conversely, if ϕ is not finite then there is a subvariety $Z \subseteq X$ of positive dimension that is contracted by ϕ to a point. Since $L = \phi^*\mathcal{O}_{\mathbf{P}}(1)$, we see that L restricts to a trivial line bundle on Z. In particular, $L \mid Z$ is not ample, and so thanks again to the previous proposition, neither is L. Moreover, if $C \subseteq Z$ is any irreducible curve, then $\int_C c_1(L) = 0$. □

The next result allows one in practice to restrict attention to reduced and irreducible varieties.

Proposition 1.2.16. *Let X be a complete scheme, and L a line bundle on X.*

(i). *L is ample on X if and only if L_{red} is ample on X_{red}.*

(ii). *L is ample on X if and only if the restriction of L to each irreducible component of X is ample.*

Proof. In each case the "only if" statement is a consequence of the previous proposition. So for (i) we need to show that if L_{red} is ample on X_{red}, then L itself is already ample. To this end we again use characterization (ii) of Theorem 1.2.6. Fix a coherent sheaf $\mathcal{F}$ on X, and let $\mathcal{N}$ be the nilradical of $\mathcal{O}_X$, so that $\mathcal{N}^r = 0$ for some r. Consider the filtration

$$\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \supset \cdots \supset \mathcal{N}^r \cdot \mathcal{F} = 0.$$

The quotients $\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}$ are coherent $\mathcal{O}_{X_{\mathrm{red}}}$-modules, and therefore

$$H^j\big(X, (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes L^{\otimes m}\big) \ = \ 0 \quad \text{for } \ j > 0 \text{ and } m \gg 0$$

thanks to the amplitude of $L \mid X_{\mathrm{red}}$. Twisting the exact sequences

$$0 \longrightarrow \mathcal{N}^{i+1}\mathcal{F} \longrightarrow \mathcal{N}^i\mathcal{F} \longrightarrow \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \longrightarrow 0$$

by $L^{\otimes m}$ and taking cohomology, we then find by decreasing induction on i that

$$H^j\big(X, \mathcal{N}^i\mathcal{F} \otimes L^{\otimes m}\big) = 0 \quad \text{for } \ j > 0 \text{ and } m \gg 0.$$

When $i = 0$ this gives the vanishings required for 1.2.6 (ii). The proof of (ii) is similar. Specifically, supposing as we may that X is reduced, let $X = X_1 \cup \cdots \cup X_r$ be its decomposition into irreducible components, and assume that $L \mid X_i$ is ample for every i. Fix a coherent sheaf $\mathcal{F}$ on X, let $\mathcal{I}$ be the ideal sheaf of X_1 in X, and consider the exact sequence

$$0 \longrightarrow \mathcal{I} \cdot \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \ / \ \mathcal{I} \cdot \mathcal{F} \longrightarrow 0. \tag{*}$$

The outer terms of (*) are supported on $X_2 \cup \cdots \cup X_r$ and X_1 respectively. So by induction on the number of irreducible components, we may assume that

$$H^j\big(X, \mathcal{I}\mathcal{F} \otimes L^{\otimes m}\big) \ = \ H^j\big(X, (\mathcal{F}/\mathcal{I}\mathcal{F}) \otimes L^{\otimes m}\big) \ = \ 0$$

for $j > 0$ and $m \gg 0$. It then follows from (*) that $H^j\big(X, \mathcal{F} \otimes L^{\otimes m}\big) = 0$ when $j > 0$ and $m \gg 0$, as required. □

A theorem of Grothendieck [256, III.4.7.1] shows that — in an extremely strong sense — amplitude is an open condition in families.

Theorem 1.2.17. (Amplitude in families). *Let $f : X \longrightarrow T$ be a proper morphism of schemes, and L a line bundle on X. Given $t \in T$, write*

$$X_t = f^{-1}(t) \quad , \quad L_t = L \mid X_t.$$

Assume that L_0 is ample on X_0 for some point $0 \in T$. Then there is an open neighborhood U of 0 in T such that L_t is ample on X_t for all $t \in U$.

Observe that we do not assume that f is flat.

Proof of Theorem 1.2.17. We follow a proof given by Kollár and Mori [368, Proposition 1.41]. The statement being local on T, we suppose that $T = \mathrm{Spec}(A)$ is affine.

We assert to begin with that for any coherent sheaf $\mathcal{F}$ on X, there is a positive integer $m(\mathcal{F}, L)$ such that

$$R^i f_*\big(\mathcal{F} \otimes L^{\otimes m}\big) = 0 \ \text{ in a neighborhood } U_m \subseteq T \text{ of } 0 \tag{*}$$

for all $i \geq 1$ and $m \geq m(\mathcal{F}, L)$. In fact, this is certainly true for large i — e.g. $i > \dim X_0$ — and we proceed by decreasing induction on i. Assuming then that (*) is known for given $i \geq 2$ and all $\mathcal{F}$, we need to show that it holds also for $i-1$.

To this end, consider the maximal ideal $\mathfrak{m}_0 \subset A$ of 0 in T, and choose generators $u_1, \ldots, u_p \in \mathfrak{m}_0$. This gives rise to a presentation

$$A^{\oplus p} \xrightarrow{\ u\ } A \longrightarrow A/\mathfrak{m}_0 \longrightarrow 0$$

of $A/\mathfrak{m}_0$, where $u(a_1, \ldots, a_p) = \sum a_i u_i$. Pulling back by f and tensoring by $\mathcal{F}$ we arrive at an exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f^*u \otimes 1) \arrow[r] & \mathcal{O}_X^p \otimes \mathcal{F} \arrow[rr, "f^*u\otimes 1"] \arrow[dr] & & \mathcal{F} \arrow[r] & \mathcal{F} \otimes \mathcal{O}_{X_0} \arrow[r] & 0 \\
& & & \operatorname{im}(f^*u \otimes 1) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0\ . & &
\end{tikzcd}$$

By the induction hypothesis applied to the kernel sheaf,

$$R^i f_*\big(\ker(f^*u \otimes 1) \otimes L^{\otimes m}\big) \ = \ 0 \ \text{ near } 0$$

for $m \gg 0$. Furthermore, since L_0 is ample, the higher direct images of $\mathcal{F} \otimes \mathcal{O}_{X_0} \otimes L^{\otimes m}$ — which are just the cohomology groups on X_0 of the sheaves in question — vanish when m is large. It then follows upon tensoring by $L^{\otimes m}$ and chasing through the above diagram that the map

$$R^{i-1} f_*(\mathcal{F} \otimes L^{\otimes m}) \otimes_{\mathcal{O}_T} \mathcal{O}_T^p \xrightarrow{1\otimes u} R^{i-1} f_*(\mathcal{F} \otimes L^{\otimes m})$$

on direct images is surjective in a neighborhood U'_m of 0 for $m \gg 0$. On the other hand, by construction $1 \otimes u$ factors through the inclusion $\mathfrak{m}_0 \cdot R^{i-1} f_*(\mathcal{F} \otimes L^{\otimes m}) \subset R^{i-1} f_*(\mathcal{F} \otimes L^{\otimes m})$. In other words, if m is sufficiently large, then

$$R^{i-1} f_*(\mathcal{F} \otimes L^{\otimes m}) \ = \ \mathfrak{m}_0 \cdot R^{i-1} f_*(\mathcal{F} \otimes L^{\otimes m})$$

in a neighborhood of 0. But by Nakayama's lemma, this implies that

$$R^{i-1} f_*(\mathcal{F} \otimes L^{\otimes m}) \ = \ 0$$

near 0, as required. Thus we have verified (*).

We assert next that the canonical mapping

$$\rho_m : f^* f_* L^{\otimes m} \longrightarrow L^{\otimes m}$$

is surjective along X_t for all t in a neighborhood U''_m of 0 provided that m is sufficiently large. To see this, apply (*) to the ideal sheaf $\mathcal{I}_{X_0/X}$ of X_0 in X. One finds that

$$f_*(L^{\otimes m}) \longrightarrow f_*(L^{\otimes m} \otimes \mathcal{O}_{X_0}) \;=\; H^0\big(X_0, L_0^{\otimes m}\big) \qquad (**)$$

is surjective when $m \gg 0$. But all sufficiently large powers of the ample line bundle L_0 are globally generated. Composing (**) with the evaluation $H^0\big(X_0, L_0^{\otimes m}\big) \otimes \mathcal{O}_{X_0} \longrightarrow L_0^{\otimes m}$, then shows that ρ_m is surjective along X_0 for $m \gg 0$. By the coherence of coker ρ_m, it follows that ρ_m is also surjective along X_t for t near 0, as claimed.

Shrinking T we can suppose that ρ_m is globally surjective for some fixed large integer m. Now $f_*(L^{\otimes m})$ is itself globally generated since T is affine. Choosing finitely many sections generating $f_*(L^{\otimes m})$ and pulling back to X, we arrive at a surjective homomorphism $f^*\mathcal{O}_T^{r+1} \twoheadrightarrow L^{\otimes m}$ of sheaves on X. This defines a mapping

$$\phi : X \longrightarrow \mathbf{P}(\mathcal{O}_T^{r+1}) = \mathbf{P}^r \times T$$

over T. The amplitude of L_0 implies that ϕ is finite on X_0, and hence $\phi_t = \phi | X_t : X_t \longrightarrow \mathbf{P}^r$ is likewise finite for t in a neighborhood of 0. Thus $L_t^{\otimes m} = \phi_t^* \mathcal{O}_{\mathbf{P}^r}(1)$ is indeed ample. □

Remark 1.2.18. Observe for later reference that the final step of the proof just completed shows that there is a neighborhood $U = U_m$ of 0 in T such that the mapping

$$\phi : X_U =_{\text{def}} f^{-1}(U) \longrightarrow \mathbf{P}^r \times U$$

over U determined by $L^{\otimes m}$ is finite. □

We close this discussion by presenting some useful applications of Serre vanishing.

Example 1.2.19. (Asymptotic Riemann–Roch, II). Let D be an ample Cartier divisor on an irreducible projective variety X of dimension n. Then

$$h^0\big(X, \mathcal{O}_X(mD)\big) \;=\; \frac{(D^n)}{n!} \cdot m^n + O(m^{n-1}).$$

More generally, if $\mathcal{F}$ is any coherent sheaf on X then

$$h^0\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) \;=\; \operatorname{rank}(\mathcal{F}) \frac{(D^n)}{n!} \cdot m^n + O(m^{n-1}).$$

(This follows immediately from Theorem 1.1.24 by virtue of the vanishing

$$H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) = 0$$

for $i > 0$ and $m \gg 0$. In fact, in the case at hand the dimensions in question are given for $m \gg 0$ by polynomials with the indicated leading terms.) This extends to reducible or non-reduced schemes X as in Remark 1.1.26; we leave the statement to the reader. □

Example 1.2.20. (Upper bounds on $\mathbf{h^0}$). If E is any divisor on an irreducible projective variety X of dimension n, then there is a constant $C > 0$ such that

$$h^0\big(X, \mathcal{O}_X(mE)\big) \ \le \ Cm^n \quad \text{for all} \ \ m.$$

(Fix an ample divisor D on X. Then $aD - E$ is effective for some $a \gg 0$, and consequently $h^0\big(X, \mathcal{O}_X(mE)\big) \le h^0\big(X, \mathcal{O}_X(maD)\big)$. The assertion then follows from Example 1.2.19.) See Example 1.2.33 for a generalization. □

Example 1.2.21. (Resolutions of a sheaf). Let X be a projective variety, and D an ample divisor on X. Then any coherent sheaf $\mathcal{F}$ on X admits a (possibly non-terminating) resolution of the form

$$\ldots \longrightarrow \oplus\, \mathcal{O}_X(-p_1D) \longrightarrow \oplus\, \mathcal{O}_X(-p_0D) \longrightarrow \mathcal{F} \longrightarrow 0$$

for suitable integers $0 \ll p_0 \ll p_1 \ll \ldots$. (Choose $p_0 \gg 0$ such that $\mathcal{F} \otimes \mathcal{O}_X(p_0D)$ is globally generated. Fixing a collection of generating sections determines a surjective map $\oplus\, \mathcal{O}_X \longrightarrow \mathcal{F} \otimes \mathcal{O}_X(p_0D)$. Twisting by $\mathcal{O}_X(-p_0D)$ then gives rise to a surjection $\oplus\, \mathcal{O}_X(-p_0D) \longrightarrow \mathcal{F}$, and one continues by applying the same argument to the kernel of this map.) Even though they may be infinite, one can sometimes use such resolutions to reduce cohomological questions about coherent sheaves to the case of line bundles. The next example provides an illustration. □

Example 1.2.22. (Surjectivity of multiplication maps). Let X be a projective variety or scheme, and let D and E be ample Cartier divisors on X. Then there is a positive integer $m_0 = m_0(D, E)$ such that the natural maps

$$H^0\big(X, \mathcal{O}_X(aD)\big) \otimes H^0\big(X, \mathcal{O}_X(bE)\big) \longrightarrow H^0\big(X, \mathcal{O}_X(aD + bE)\big)$$

are surjective whenever $a, b \ge m_0$. More generally, for any coherent sheaves $\mathcal{F}, \mathcal{G}$ on X, there is an integer $m_1 = m_1(D, E, \mathcal{F}, \mathcal{G})$ such that

$$H^0\Big(\,X\,,\,\mathcal{F} \otimes \mathcal{O}_X(aD)\Big) \otimes H^0\Big(\,X\,,\,\mathcal{G} \otimes \mathcal{O}_X(bE)\Big) \longrightarrow \\ H^0\Big(\,X\,,\,\mathcal{F} \otimes \mathcal{G} \otimes \mathcal{O}_X(aD + bE)\Big)$$

is surjective for $a, b \ge m_1$. (For the first statement consider on $X \times X$ the exact sequence

$$0 \longrightarrow \mathcal{I}_\Delta \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0, \tag{*}$$

$\Delta \subset X \times X$ being the diagonal. Writing (aD, bE) for the divisor $\mathrm{pr}_1^*(aD) + \mathrm{pr}_2^*(bE)$ on $X \times X$, the displayed sequence (*) shows that it suffices to verify that

$$H^1\big(X \times X, \mathcal{I}_\Delta\big((aD, bE)\big)\big) = 0 \tag{**}$$

for $a, b \ge m_0$. To this end, apply 1.2.21 to the ample divisor (D, E) to construct a resolution

$$\ldots \to \oplus \mathcal{O}_{X\times X}\big((-p_1D, -p_1E)\big) \longrightarrow \oplus \mathcal{O}_{X\times X}\big((-p_0D, -p_0E)\big) \longrightarrow \mathcal{I}_\Delta \to 0.$$

By Proposition B.1.2 from Appendix B, it is enough for (**) to produce an integer m_0 such that

$$H^i\Big(X \times X \, , \, \mathcal{O}_{X\times X}\big((a-p_{i-1})D, (b-p_{i-1})E\big) \Big) = 0$$

whenever $i > 0$ and $a, b \geq m_0$. But this is non-trivial only when $i \leq \dim X \times X$, and the cohomology group in question is computed by the Künneth formula. So the existence of the required integer m_0 follows immediately from Serre vanishing. The second statement is similar, except that one works with $\mathcal{I}_\Delta \otimes \mathrm{pr}_1^*(\mathcal{F}) \otimes \mathrm{pr}_2^*(\mathcal{G})$ in place of $\mathcal{I}_\Delta$, observing that (*) remains exact after tensoring through by $\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathrm{pr}_2^*(\mathcal{G})$ thanks to flatness. Alternatively, one could use Fujita's vanishing theorem (Theorem 1.4.35) to bypass 1.2.21.) □

1.2.B Numerical Properties

A second very fundamental fact is that amplitude is characterized numerically:

Theorem 1.2.23. (Nakai–Moishezon–Kleiman criterion). *Let L be a line bundle on a projective scheme X. Then L is ample if and only if*

$$\int_V c_1(L)^{\dim(V)} \ > \ 0 \tag{1.11}$$

for every positive-dimensional irreducible subvariety $V \subseteq X$ (including the irreducible components of X).

Kleiman's paper [340] contains an illuminating discussion of the history of this basic result. In brief, it was originally established by Nakai [464] for smooth surfaces. Moishezon [433] proved 1.2.23 for non-singular varieties of higher dimension, and suggested in [434] a definition of intersection numbers that led to its validity on singular varieties as well. Nakai [465] subsequently extended the statement to arbitrary projective algebraic schemes, and finally Kleiman [341, Chapter III] treated the case of arbitrary complete schemes. The (now standard) proof we will give is due to Kleiman. In spite of the collaborative nature of Theorem 1.2.23, we will generally refer to it in the interests of brevity simply as Nakai's criterion.

Before giving the proof we mention two important consequences. First, it follows from the theorem that the amplitude of a divisor depends only on its numerical equivalence class:

Corollary 1.2.24. (Numerical nature of amplitude). *If $D_1, D_2 \in \mathrm{Div}(X)$ are numerically equivalent Cartier divisors on a projective variety or scheme X, then D_1 is ample if and only if D_2 is.* □

In particular it makes sense to discuss the amplitude of a class $\delta \in N^1(X)$:

Definition 1.2.25. (Ample classes). A numerical equivalence class $\delta \in N^1(X)$ is *ample* if it is the class of an ample divisor. Ample (algebraic) classes in $H^2\big(X, \mathbf{Z}\big)$ or $H^2\big(X, \mathbf{Q}\big)$ are defined in the same way. □

Example 1.2.26. (Varieties with Picard number 1). If X is a projective variety having Picard number $\rho(X) = 1$, then any non-zero effective divisor on X is ample. This extends Example 1.2.4, and applies for example to a very general abelian variety having a polarization of fixed type. □

Remark 1.2.27. The structure of the cone of all ample classes on a fixed projective variety is discussed further in Section 1.4. Several examples are worked out in Section 1.5. □

The second corollary shows that amplitude can be tested after pulling back by a finite surjective morphism:

Corollary 1.2.28. (Finite pullbacks, II). *Let $f : Y \longrightarrow X$ be a finite and surjective mapping of projective schemes, and let L be a line bundle on X. If f^*L is ample on Y, then L is ample on X.*

Proof. Let $V \subseteq X$ be an irreducible variety. Since f is surjective, there is an irreducible variety $W \subseteq Y$ mapping (finitely) onto V: starting with $f^{-1}(V)$, one constructs W by taking irreducible components and cutting down by general hyperplanes. Then by the projection formula

$$\int_W c_1(f^*L)^{\dim W} \;=\; \deg(W \longrightarrow V) \cdot \int_V c_1(L)^{\dim V},$$

so the assertion follows from the theorem. □

We now turn to the very interesting proof of the Nakai–Moishezon–Kleiman criterion. The argument will lead to several other results as well.

Proof of Theorem 1.2.23. Suppose first that L is ample. Then $L^{\otimes m}$ is very ample for some $m \gg 0$, and

$$m^{\dim V} \cdot \int_V \big(c_1(L)\big)^{\dim V} \;=\; \int_V \big(\, c_1(L^{\otimes m})\,\big)^{\dim V}$$

is the degree of V in the corresponding projective embedding of X. Consequently, this integral is strictly positive. (Alternatively, one could invoke Example 1.2.5.)

Conversely, assuming the positivity of the intersection numbers appearing in the theorem, we prove that L is ample. By Proposition 1.2.16 we are free to suppose that X is reduced and irreducible. The result being clear if $\dim X = 1$, we put $n = \dim X$ and assume inductively that the theorem is known for all schemes of dimension $\leq n - 1$. It is convenient at this point to switch to additive notation, so write $L = \mathcal{O}_X(D)$ for some divisor D on X.

We assert first that

$$H^0\big(X, \mathcal{O}_X(mD)\big) \neq 0 \quad \text{for} \quad m \gg 0.$$

In fact, asymptotic Riemann–Roch (Theorem 1.1.24) gives to begin with that

$$\chi\big(X, \mathcal{O}_X(mD)\big) \;=\; m^n \,\frac{(D^n)}{n!} \;+\; O(m^{n-1}), \tag{*}$$

and $(D^n) = \int_X c_1(L)^n > 0$ by assumption. Now write $D \equiv_{\text{lin}} A - B$ as a difference of very ample effective divisors A and B (using e.g. Example 1.2.10). We have two exact sequences:

$$0 \longrightarrow \mathcal{O}_X(mD - B) \xrightarrow{\cdot A} \mathcal{O}_X((m+1)D) \longrightarrow \mathcal{O}_A((m+1)D) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_X(mD - B) \xrightarrow{\cdot B} \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_B(mD) \longrightarrow 0.$$

By induction, $\mathcal{O}_A(D)$ and $\mathcal{O}_B(D)$ are ample. Consequently the higher cohomology of each of the two sheaves on the right vanishes when $m \gg 0$. So we find that if $m \gg 0$, then

$$H^i\big(X, \mathcal{O}_X(mD)\big) \;=\; H^i\big(X, \mathcal{O}_X(mD-B)\big) \;=\; H^i\big(X, \mathcal{O}_X((m+1)D)\big)$$

for $i \geq 2$. In other words, if $i \geq 2$ then the dimensions $h^i\big(X, \mathcal{O}_X(mD)\big)$ are eventually constant. Therefore

$$\chi\big(X, \mathcal{O}_X(mD)\big) \;=\; h^0\big(X, \mathcal{O}_X(mD)\big) - h^1\big(X, \mathcal{O}_X(mD)\big) \;+\; C$$

for some constant C and $m \gg 0$. So it follows from (*) that $H^0\big(X, \mathcal{O}_X(mD)\big)$ is non-vanishing when m is sufficiently large, as asserted. Since D is ample if and only if mD is, there is no loss in generality in replacing D by mD. Therefore we henceforth suppose that D is effective.

We next show that $\mathcal{O}_X(mD)$ is generated by its global sections if $m \gg 0$. Since D is assumed to be effective this is evidently true away from $\mathrm{Supp}(D)$, so the issue is to show that no point of D is a base point of the linear series $|\mathcal{O}_X(mD)|$. Consider to this end the exact sequence

$$0 \longrightarrow \mathcal{O}_X\big((m-1)D\big) \xrightarrow{\cdot D} \mathcal{O}_X\big(mD\big) \longrightarrow \mathcal{O}_D\big(mD\big) \longrightarrow 0. \tag{*}$$

As before, $\mathcal{O}_D(D)$ is ample by induction. Consequently $\mathcal{O}_D(mD)$ is globally generated and $H^1\big(X, \mathcal{O}_D(mD)\big) = 0$ for $m \gg 0$. It then follows first of all from (*) that the natural homomorphism

$$H^1\big(X, \mathcal{O}_X\big((m-1)D\big)\big) \longrightarrow H^1\big(X, \mathcal{O}_X\big(mD\big)\big) \tag{**}$$

is surjective for every $m \gg 0$. The spaces in question being finite-dimensional, the maps in (**) must actually be isomorphisms for sufficiently large m. Therefore the restriction mappings

$$H^0\big(X, \mathcal{O}_X(mD)\big) \longrightarrow H^0\big(X, \mathcal{O}_D(mD)\big)$$

are surjective for $m \gg 0$. But since $\mathcal{O}_D(mD)$ is globally generated, it follows that no point of $\mathrm{Supp}(D)$ is a basepoint of $|mD|$, as required.

Finally, the amplitude of $\mathcal{O}_X(mD)$ — and hence also of $\mathcal{O}_X(D)$ — now follows from Corollary 1.2.15 since by assumption $\big(mD \cdot C\big) > 0$ for every irreducible curve $C \subseteq X$. □

Remark 1.2.29. (Nakai's criterion on proper schemes). The statement of Theorem 1.2.23 remains true for any complete scheme X, without assuming at the outset that X is projective. The projectivity hypothesis was used in the previous proof to write the given divisor D as a difference of two very ample divisors, but with a little more care one can modify this step to work on any complete X. See [341, Chapter III] or [276, p. 31] for details. □

We conclude this subsection with several other applications of the line of reasoning that led to the Nakai criterion.

Example 1.2.30. (Divisors with ample normal bundle). We outline a result due to Hartshorne [276, III.4.2] concerning divisors having ample normal bundles. Let X be a projective variety, and let $D \subset X$ be an effective Cartier divisor on X whose normal bundle $\mathcal{O}_D(D)$ is ample. Then:

(i). For $m \gg 0$, $\mathcal{O}_X(mD)$ is globally generated.

(ii). For $m \gg 0$, the restriction

$$H^0\big(X, \mathcal{O}_X(mD)\big) \longrightarrow H^0\big(D, \mathcal{O}_D(mD)\big)$$

is surjective.

(iii). There is a proper birational morphism

$$f : X \longrightarrow \overline{X}$$

from X to a projective variety $\overline{X}$ such that f is an isomorphism in a neighborhood of D, and $\overline{D} =_{\mathrm{def}} f(D)$ is an ample effective divisor on $\overline{X}$.

(For (i) and (ii), argue as in the proof of Theorem 1.2.23. For (iii) assume that m is sufficiently large so that (i) and (ii) hold, and in addition $\mathcal{O}_D(mD)$ is very ample. Then take $\overline{X}$ to be the image of the Stein factorization of the mapping

$$\phi : X \longrightarrow \mathbf{P} = \mathbf{P}H^0\big(X, \mathcal{O}_X(mD)\big),$$

defined by $|\mathcal{O}_X(mD)|$, with $f : X \longrightarrow \overline{X}$ the evident morphism. By (ii) and the assumption on $\mathcal{O}_D(mD)$, f is finite over a neighborhood of $f(D)$. Then since $f_*\mathcal{O}_X = \mathcal{O}_{\overline{X}}$ it follows that f maps a neighborhood of D isomorphically to its image in the variety $\overline{X}$. In particular, f is birational.) □

Example 1.2.31. (Irreducible curves of positive self-intersection on a surface). Let X be a smooth projective surface, and let $C \subseteq X$ be an irreducible curve with $\big(C^2\big) > 0$. Then $\mathcal{O}_X(mC)$ is free for $m \gg 0$. $\square$

Example 1.2.32. (Further characterizations of amplitude). Kleiman [341, Chapter 3, §1] gives some additional characterizations of amplitude. Let D be a Cartier divisor on a projective algebraic scheme X. Then D is ample if and only if it satisfies either of the following properties:

(i). For every irreducible subvariety $V \subseteq X$ of positive dimension, there is a positive integer $m = m(V)$, together with a non-zero section $0 \neq s = s_V \in H^0\big(V, \mathcal{O}_V(mD)\big)$, such that s vanishes at some point of V.

(ii). For every irreducible subvariety $V \subseteq X$ of positive dimension,

$$\chi\big(V, \mathcal{O}_V(mD)\big) \to \infty \quad \text{as } m \to \infty.$$

(It is enough to prove this when X is reduced and irreducible. Suppose that (i) holds. Taking first $V = X$ and replacing D by a multiple, we can assume that D is effective. By induction on dimension $\mathcal{O}_D(D)$ is ample, and so by 1.2.30 (i) $\mathcal{O}_X(mD)$ is free for $m \gg 0$. But the hypothesis implies that $\mathcal{O}_X(D)$ is non-trivial on every curve, and hence Corollary 1.2.15 applies. Supposing (ii) holds, one can again assume inductively that $\mathcal{O}_E(D)$ is ample for every effective divisor E on X. Then the proof of Theorem 1.2.23 goes through with little change, except that one uses (ii) rather than Riemann–Roch to control $\chi(X, \mathcal{O}_X(mD))$, while in order to apply 1.2.15 one notes that (ii) implies that $\mathcal{O}_X(D)$ restricts to an ample bundle on every irreducible curve in X.) $\square$

The next example shows that on a scheme of dimension n, dimensions of cohomology groups can grow at most like a polynomial of degree n:

Example 1.2.33. (Growth of cohomology). Let X be a projective scheme of dimension n and D a divisor on X. If $\mathcal{F}$ is any coherent sheaf on X then

$$h^i\big(X, \mathcal{F}(mD)\big) \;=\; O(m^n)$$

for every i. (Write $D = A - B$ as the difference of very ample divisors having the property that neither A nor B contains any of the subvarieties of X defined by the associated primes of $\mathcal{F}$. Then the two sequences

$$0 \longrightarrow \mathcal{F}\big(mD - B\big) \xrightarrow{\cdot A} \mathcal{F}\big((m+1)D\big) \longrightarrow \mathcal{F} \otimes \mathcal{O}_A\big((m+1)D\big) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{F}\big(mD - B\big) \xrightarrow{\cdot B} \mathcal{F}\big(mD\big) \longrightarrow \mathcal{F} \otimes \mathcal{O}_B\big(mD\big) \longrightarrow 0$$

are exact. By induction on dimension one finds that

$$\big|\, h^i\big(X, \mathcal{F}((m+1)D)\big) - h^i\big(X, \mathcal{F}(mD)\big) \,\big| \;=\; O(m^{n-1}),$$

and the assertion follows.) $\square$

Example 1.2.34. In the setting of 1.2.33, it can easily happen that the higher cohomology groups have maximal growth. For instance, if X is smooth and $-D$ is ample, then $h^n\big(X, \mathcal{O}_X(mD)\big) = h^0\big(X, \mathcal{O}_X(K_X - mD)\big)$ by Serre duality, and the latter group grows like m^n. For a more interesting example, let X be the blowing-up $\mathrm{Bl}_p(\mathbf{P}^2)$ of $\mathbf{P}^2$ at a point, with exceptional divisor E. Then $h^1\big(X, \mathcal{O}_X(mE)\big) = \binom{m}{2}$ has quadratic growth. □

Example 1.2.35. (Growth of cohomology of pullbacks). Let

$$\mu : X' \longrightarrow X$$

be a surjective and generically finite mapping of projective varieties or schemes of dimension n. Fix a divisor D on X and put $D' = \mu^* D$. Then for every $i \geq 0$,

$$h^i\big(X', \mathcal{O}_{X'}(mD')\big) \;=\; h^i\big(X, (\mu_* \mathcal{O}_{X'}) \otimes \mathcal{O}_X(mD)\big) + O(m^{n-1}).$$

(This follows from the Leray spectral sequence and Example 1.2.33 in view of the fact that the higher direct images $R^j \mu_* \mathcal{O}_{X'}$ $(j > 0)$ are supported on proper subschemes of X.) □

Example 1.2.36. (Higher cohomology of nef divisors, I). Kollár [363, V.2.15] shows that one can adapt the proof of Nakai's criterion to establish the following

> **Theorem.** *Let X be a projective scheme of dimension n, and D a divisor on X having the property that*
>
> $$\big(D^{\dim V} \cdot V\big) \;\geq\; 0 \quad \textit{for all irreducible subvarieties } V \subseteq X.$$
>
> *Then*
> $$h^i\big(X, \mathcal{O}_X(mD)\big) \;=\; O(m^{n-1}) \quad \text{for } i \geq 1. \tag{1.12}$$

We outline Kollár's argument here. As we will see in Section 1.4, the hypothesis on D is equivalent to the assumption that it be nef. A stronger statement is established in Theorem 1.4.40 using a vanishing theorem of Fujita.

(i). Arguing as in the proof of 1.2.23, one shows by induction on dimension that

$$\big|\, h^i\big(X, \mathcal{O}_X\big((m+1)D\big)\big) \,-\, h^i\big(X, \mathcal{O}_X\big(mD\big)\big) \,\big| \;=\; O(m^{n-2})$$

provided that $i \geq 2$. This yields (1.12) for $i \geq 2$.

(ii). Combining (i) with asymptotic Riemann–Roch (Theorem 1.1.24), it follows that

$$h^0\big(X, \mathcal{O}_X\big(mD\big)\big) \,-\, h^1\big(X, \mathcal{O}_X\big(mD\big)\big) \;=\; \frac{(D^n)}{n!} \cdot m^n \,+\, O(m^{n-1}).$$

If $h^0\big(X, \mathcal{O}_X(mD)\big) = 0$ for all $m > 0$, then the left-hand side is non-positive. But since by assumption $\big(D^n\big) \geq 0$, this forces $\big(D^n\big) = 0$ and we get the required estimate on $h^1\big(X, \mathcal{O}_X(mD)\big)$.

(iii). In view of (ii), we can assume that $H^0\big(X, \mathcal{O}_X(m_0D)\big) \neq 0$ for some $m_0 > 0$. Fix $E \in |m_0D|$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X\big((m-m_0)D\big) \longrightarrow \mathcal{O}_X\big(mD\big) \longrightarrow \mathcal{O}_E\big(mD\big) \longrightarrow 0.$$

Applying the induction hypothesis to E we find that

$$\begin{aligned} h^1\big(X, \mathcal{O}_X\big(mD\big)\big) - h^1\big(X, \mathcal{O}_X\big((m-m_0)D\big)\big) \ &\leq \ h^1\big(E, \mathcal{O}_E(mD)\big) \\ &= \ O(m^{n-2}), \end{aligned}$$

and the case $i = 1$ of (1.12) follows. $\square$

Remark 1.2.37. (Complete schemes). With a little more care, one can show that 1.2.33 and 1.2.36 remain valid for arbitrary complete schemes. See [114, Proposition 1.31]. $\square$

Remark 1.2.38. (Other ground fields). Except for Matsusaka's theorem (Example 1.2.12) all of the results and arguments appearing so far in this section remain valid without change for varieties defined over an algebraically closed field of arbitrary characteristic. $\square$

1.2.C Metric Characterizations of Amplitude

The final basic result we recall here is that when X is smooth — and so may be considered as a complex manifold — amplitude can be detected analytically. The discussion will be rather brief, and we refer for instance to [604] or [248, Chapter 1, Sections 1, 2, and 4], for background and details.

We start with some remarks on positivity of differential forms. Let X be a complex manifold. For $x \in X$, write $T_xX_{\mathbf{R}}$ for the tangent space to the underlying $\mathcal{C}^\infty$ real manifold, and let $J : T_xX_{\mathbf{R}} \longrightarrow T_xX_{\mathbf{R}}$ be the endomorphism determined by the complex structure on X, so that $J^2 = -\mathrm{Id}$. Given a $\mathcal{C}^\infty$ 2-form ω on X, we denote by $\omega(v,w) = \omega_x(v,w) \in \mathbf{R}$ the result of evaluating ω on a pair of real tangent vectors $v, w \in T_xX_{\mathbf{R}}$.

Definition 1.2.39. (Positive, (1 , 1) and Kähler forms). The 2-form ω has *type* $(1,1)$ if

$$\omega_x(Jv, Jw) \ = \ \omega_x(v,w)$$

for every $v, w \in T_xX_{\mathbf{R}}$ and every $x \in X$. A $(1,1)$-form is *positive* if

$$\omega_x(v, Jv) > 0$$

for every $x \in X$ and all non-zero tangent vectors $0 \neq v \in T_xX_{\mathbf{R}}$. A *Kähler form* is a closed positive $(1,1)$-form, i.e. a positive form ω of type $(1,1)$ such that $d\omega = 0$. $\square$

Example 1.2.40. (Local description). Let $z_1, \ldots, z_n$ be local holomorphic coordinates on the complex manifold X. A two-form ω is of type $(1,1)$ if and only if it can be written locally as

$$\omega \;=\; \frac{i}{2} \sum h_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta,$$

where $\big(h_{\alpha\beta}\big)$ is a Hermitian matrix of $\mathbf{C}$-valued $\mathcal{C}^\infty$ functions on X. It is positive if and only if $\big(h_{\alpha\beta}\big)$ is positive definite at each point $x \in X$. $\square$

Example 1.2.41. (Hermitian metrics). Let H be a Hermitian metric on X, given by Hermitian forms $H_x(\ ,\ )$ on $T_x X_{\mathbf{R}}$ varying smoothly with $x \in X$.[10] Then the negative imaginary part

$$\omega \;=\; -\mathrm{Im}\, H$$

of H is a $(1,1)$-form on X, and if H is positive definite then ω is positive. Conversely a positive $(1,1)$-form ω determines a positive definite Hermitian metric by the rule

$$H_x(v,w) \;=\; \omega_x(v, Jw) - i\omega_x(v,w). \quad \square$$

Fix a Kähler form ω on X. Let $\varDelta$ be the unit disk, with complex coordinate $z = x + iy$, and suppose that $\mu : \varDelta \longrightarrow X$ is a holomorphic mapping. Then $\mu^*\omega$ is likewise positive of type $(1,1)$, and hence

$$\mu^*\omega \;=\; \phi(x,y)\,\cdot dx \wedge dy,$$

where $\phi(x,y)$ is a positive $\mathcal{C}^\infty$ function on $\varDelta$. Therefore if $C \subseteq X$ is a one-dimensional complex submanifold, then $\int_C \omega > 0$ (provided that the integral is finite). Similarly, for any complex submanifold $V \subseteq X$, the integrand computing $\int_V \omega^{\dim V}$ is everywhere positive. So it is suggestive to think of a positive $(1,1)$-form as one satisfying "pointwise" inequalities of Nakai–Moishezon–Kleiman type.

Example 1.2.42. (Fubini–Study form on $\mathbf{P}^n$, I). Complex projective space $\mathbf{P}^n$ carries a very beautiful $\mathrm{SU}(n+1)$-invariant Kähler form ω_{FS}. Following [17, Appendix 3], we construct it by first building an $\mathrm{SU}(n{+}1)$-invariant Hermitian metric H_{FS} on $\mathbf{P}^n$. The Fubini–Study form ω_{FS} will then arise as the negative imaginary part $\omega_{\mathrm{FS}} = -\mathrm{Im}\, H_{\mathrm{FS}}$ of this Fubini–Study metric.

Consider the standard Hermitian inner product $\langle v, w\rangle = {}^t v \cdot \overline{w}$ on $V = \mathbf{C}^{n+1}$. Set $V^0 = V - \{0\}$ and denote by

$$\rho \,:\, V^0 \longrightarrow \mathbf{P}^n \;=\; \mathbf{P}_{\mathrm{sub}}(V)$$

the canonical map. Define to begin with a Hermitian metric H' on V^0 by associating to $x \in V^0$ the Hermitian inner product

[10] Our convention is that H_x should be complex linear in the first argument and conjugate linear in the second.

$$H'_x(v,w) \;=\; \left\langle \frac{v}{|x|}\,,\,\frac{w}{|x|}\right\rangle \qquad \text{for} \;\; v\,,\,w \;\in\; T_xV^0 \;=\; V$$

and $|x| = \sqrt{\langle x, x\rangle}$. The metric H' is constructed so as to be invariant under the natural $\mathbf{C}^*$-action on V^0. Now $\rho^*T\mathbf{P}^n$ is canonically a quotient of TV^0, and so H' induces in the usual manner a $\mathbf{C}^*$-invariant metric on $\rho^*T\mathbf{P}^n$, which then descends to a Hermitian metric H_{FS} on $T\mathbf{P}^n$.

More explicitly, write $W_x \subseteq V$ for the H'_x-orthogonal complement to $\mathbf{C}\cdot x \subseteq V$, and let $\pi_x : V \longrightarrow W_x$ be orthogonal projection:

$$\pi_x(v) \;=\; v \,-\, \frac{\langle v, x\rangle}{\langle x, x\rangle}\cdot x.$$

Then W_x is identified with $T_{\rho(x)}\mathbf{P}^n$ and π_x with $d\rho_x$, and

$$\begin{aligned} H_{\rho(x)}\big(d\rho_x v, d\rho_x w\big)_{\mathrm{FS}} \;&=\; H'_x(\pi_x v, \pi_x w) \\ &=\; \frac{\langle v,w\rangle\langle x,x\rangle \,-\, \langle v,x\rangle\langle x,w\rangle}{\langle x,x\rangle^2}. \end{aligned}$$

If we take the usual affine local coordinates $z_1, \ldots, z_n$ on $\mathbf{P}^n$ — corresponding to $x = (1, z_1, \ldots, z_n) \in V^0$ — then one finds[11] that

$$\begin{aligned} \omega_{\mathrm{FS}} \;&=_{\mathrm{def}}\; -\mathrm{Im}\, H_{\mathrm{FS}} \\ &=_{\mathrm{locally}}\; \frac{i}{2}\cdot\left(\frac{\sum dz_\alpha \wedge d\bar{z}_\alpha}{1+\sum |z_\alpha|^2} \;-\; \frac{\big(\sum\; \bar{z}_\alpha dz_\alpha\big)\wedge\big(\sum z_\alpha d\bar{z}_\alpha\big)}{\big(1+\sum|z_\alpha|^2\big)^2}\right). \end{aligned}$$

By construction H_{FS} is invariant under the natural $\mathrm{SU}(n+1)$-action on $\mathbf{P}^n$, and hence so too is ω_{FS}.

We next verify that ω_{FS} is indeed a Kähler form. The positivity of ω_{FS} follows using Example 1.2.41 from the fact that H_{FS} is positive definite. Alternatively, since ω_{FS} is $\mathrm{SU}(n+1)$-invariant it is enough to prove positivity at any one point $p \in \mathbf{P}^n$, and when $p = [1, 0, \ldots, 0]$ this is clear from the local description. Following [453, Lemma 5.20], $\mathrm{SU}(n+1)$-invariance also leads to a quick proof that ω_{FS} is closed. In fact, given $p \in \mathbf{P}^n$ choose an element $\gamma \in \mathrm{SU}(n+1)$ such that $\gamma(p) = p$ while $d\gamma_p = -\mathrm{Id}$. Then for any three tangent vectors $u, v, w \in T_p\mathbf{P}^n$ one has

[11] It may be useful to consider here an n-dimensional vector space $W = \mathbf{C}^n$, with its standard Hermitian product $\langle u, v\rangle \;=\; {}^t u \cdot \overline{v}$. Then as w varies over W the expressions

$$\eta_w(u,v) \;=\; -\mathrm{Im}\big(\,\langle u,v\rangle\,\big) \;\;,\;\; \eta'_w(u,v) \;=\; -\mathrm{Im}\big(\,\langle u,w\rangle\langle w,v\rangle\,\big)$$

define $(1,1)$-forms η and η' on W, which in terms of standard linear coordinates $w_1, \ldots, w_n$ on W are given by $\eta = \frac{i}{2}\cdot\sum dw_\alpha \wedge d\overline{w}_\alpha$ and $\eta' = \frac{i}{2}\cdot\big(\sum \overline{w}_\alpha dw_\alpha\big)\wedge\big(\sum w_\alpha d\overline{w}_\alpha\big)$.

$$d\omega_{\mathrm{FS}}(u,v,w) \;=\; \gamma^*(d\omega_{\mathrm{FS}})(u,v,w) \;=\; d\omega_{\mathrm{FS}}(-u,-v-w),$$

and hence $d\omega_{\mathrm{FS}} = 0$. □

Example 1.2.43. (Fubini–Study form on $\mathbf{P}^n$, II). Another approach to the Fubini–Study form involves the Hopf map. Keeping the notation of the previous example, consider the unit sphere

$$\mathbf{C}^{n+1} \;\supseteq\; S^{2n+1} \;=\; S$$

with respect to the standard inner product $\langle\ ,\ \rangle$, with

$$p \,:\, S \longrightarrow \mathbf{P}^n$$

the Hopf mapping. Denote by ω_{std} the standard symplectic form on $\mathbf{C}^{n+1}$, i.e.

$$\omega_{\mathrm{std}} \;=\; \sum dx_\alpha \wedge dy_\alpha,$$

where $z_\alpha = x_\alpha + iy_\alpha$ are the usual complex coordinates on $\mathbf{C}^{n+1}$. Then ω_{FS} is characterized as the unique symplectic form on $\mathbf{P}^n$ having the property that

$$p^*\omega_{\mathrm{FS}} \;=\; \omega_{\mathrm{std}} \mid S.$$

(This follows from the construction in the previous example.) □

Suppose now given a holomorphic line bundle L on X on which a Hermitian metric h has been fixed. We write $|\ \ |_h$ for the corresponding length function on the fibres of L. The Hermitian line bundle (L,h) determines a *curvature form*

$$\Theta(L,h) \;\in\; C^\infty(X, \Lambda^{1,1}T^*X_{\mathbf{R}}) :$$

this is a closed $(1,1)$-form on X having the property that $\frac{i}{2\pi}\Theta(L,h)$ represents $c_1(L)$. If $s \in \Gamma(U,L)$ is a local holomorphic section of L that doesn't vanish at any point of an open set U, then for instance one can define $\Theta(L,h)$ locally by the formula

$$\Theta \;=\; -\partial\bar{\partial}\log|s|_h^2\ ,$$

this being independent of the choice of s.

The analytic approach to positivity is to ask that the form $\frac{i}{2\pi}\Theta(L,h)$ representing $c_1(L)$ be positive:

Definition 1.2.44. (Positive line bundles). The line bundle L is *positive* (in the sense of Kodaira) if it carries a Hermitian metric h such that $\frac{i}{2\pi}\Theta(L,h)$ is a Kähler form. □

Example 1.2.45. (Fubini–Study metric on the hyperplane bundle). The hyperplane bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ on $\mathbf{P}^n$ carries a Hermitian metric h whose curvature form is a multiple of the Fubini–Study form ω_{FS} (Example 1.2.42). In fact, the standard Hermitian product $\langle v,w\rangle = {}^t v \cdot \overline{w}$ on $V = \mathbf{C}^{n+1}$ gives rise to a Hermitian metric on the trivial bundle $V_{\mathbf{P}^n}$ on $\mathbf{P}^n = \mathbf{P}_{\text{sub}}(V)$. Then $\mathcal{O}_{\mathbf{P}^n}(-1)$ inherits a metric as a sub-bundle of $V_{\mathbf{P}^n}$, which in turn determines a metric h on $\mathcal{O}_{\mathbf{P}^n}(1)$. Very explicitly, write $[x] \in \mathbf{P}^n$ for the point corresponding to a vector $x \in V - \{0\}$ and consider a section $s \in V^* = H^0\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)\big)$. Then h is determined by the rule

$$\big|s\big(\,[x]\,\big)\big|_h^2 \;=\; \frac{|s(x)|^2}{\langle x, x\rangle},$$

where the numerator on the right is the squared modulus of the result of evaluating the linear functional s on the vector x.

If we work with the usual affine coordinates $z = (z_1, \ldots, z_n)$ on $\mathbf{P}^n$ corresponding to the point $x = (1, z_1, \ldots, z_n) \in V$ and take $s \in V^*$ to be the functional given by projection onto the zeroth coordinate, then

$$\big|s\big(\,[x]\,\big)\big|_h^2 \;=\; \frac{1}{1 + \sum |z_\alpha|^2}.$$

An explicit calculation [248, p. 30] shows that

$$\begin{aligned}
\frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbf{P}^n}(1), h) \;&=_{\text{locally}}\; -\frac{i}{2\pi}\cdot \partial\bar{\partial}\log\left(\frac{1}{1+\sum|z_\alpha|^2}\right) \\
&= \;\frac{i}{2\pi}\cdot\left(\frac{\sum dz_\alpha \wedge d\bar{z}_\alpha}{1+\sum|z_\alpha|^2} \;-\; \frac{\big(\sum \bar{z}_\alpha dz_\alpha\big)\wedge\big(\sum z_\alpha d\bar{z}_\alpha\big)}{\big(1+\sum|z_\alpha|^2\big)^2}\right) \\
&= \;\frac{1}{\pi}\cdot\omega_{\text{FS}}.
\end{aligned}$$

In particular, $\mathcal{O}_{\mathbf{P}^n}(1)$ is positive in the sense of Kodaira. □

The beautiful fact is that if L is a positive line bundle on a compact Kähler manifold X, then X is algebraic and L is ample:

Theorem. (Kodaira embedding theorem). *Let X be a compact Kähler manifold, and L a holomorphic line bundle on X. Then L is positive if and only if there is a holomorphic embedding*

$$\phi : X \hookrightarrow \mathbf{P}$$

of X into some projective space such that $\phi^*\mathcal{O}_{\mathbf{P}}(1) = L^{\otimes m}$ for some $m > 0$.

One direction is elementary: if such an embedding exists, then the pullback of the standard Fubini–Study metric on $\mathcal{O}_{\mathbf{P}}(1)$ determines a positive metric on $L^{\otimes m}$ and hence also on L. Conversely, if one assumes that X is already a projective variety — in which case it follows by the GAGA theorems that L is an algebraic line bundle — then the amplitude of a positive line bundle is a consequence of the Nakai criterion. Indeed, $\frac{i}{2\pi}\Theta(L, h)$ represents $c_1(L)$, and as we have noted, the positivity of this form implies the positivity of the intersection numbers appearing in 1.2.23.[12]

The deeper assertion is that as soon as L is a positive line bundle on a compact Kähler manifold, some power of L has enough sections to define a projective embedding of X. This is traditionally proved by establishing for positive line bundles an analogue of the sort of vanishings appearing in Theorem 1.2.6. We refer to [248, Chapter 2, §4] for details.

1.3 Q-Divisors and R-Divisors

For questions of positivity, it is very useful to be able to discuss small perturbations of a given divisor class. The natural way to do so is through the formalism of **Q**- and **R**-divisors, which we develop in this section. As an application, we establish that amplitude is an open condition on numerical equivalence classes.

1.3.A Definitions for Q-Divisors

As one would expect, a **Q**-divisor is simply a **Q**-linear combination of integral Cartier divisors:

Definition 1.3.1. (Q-divisors). Let X be an algebraic variety or scheme. A **Q**-*divisor* on X is an element of the **Q**-vector space

$$\mathrm{Div}_{\mathbf{Q}}(X) \ =_{\mathrm{def}} \ \mathrm{Div}(X) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

We represent a **Q**-divisor $D \in \mathrm{Div}_{\mathbf{Q}}(X)$ as a finite sum

$$D \ = \ \sum \ c_i \cdot A_i, \tag{1.13}$$

where $c_i \in \mathbf{Q}$ and $A_i \in \mathrm{Div}(X)$. By clearing denominators we can also write $D = cA$ for a single rational number c and integral divisor A, and if $c \neq 0$ then $cA = 0$ if and only if A is a torsion element of $\mathrm{Div}(X)$. A **Q**-divisor D is *integral* if it lies in the image of the natural map $\mathrm{Div}(X) \longrightarrow \mathrm{Div}_{\mathbf{Q}}(X)$. The **Q**-divisor D is *effective* if it is of the form $D = \sum c_i A_i$ with $c_i \geq 0$ and A_i effective. □

Definition 1.3.2. (Supports). Let $D \in \mathrm{Div}_{\mathbf{Q}}(X)$ be a **Q**-divisor. A codimension one subset $E \subseteq X$ *supports* D, or *is a support* of D, if D admits a representation (1.13) in which the union of the supports of the A_i is contained in E. □

[12] Recall ([248, p. 32]) that if V is singular, one computes $\int_V c_1(L)^{\dim V}$ by integrating the appropriate power of $\frac{i}{2\pi}\Theta(L,h)$ over the smooth locus of V.

Since the expression (1.13) may not be unique, E is not canonically determined. But this does not cause any problems.

All the usual operations and properties of Cartier divisors extend naturally to this setting simply by tensoring with $\mathbf{Q}$:

Definition 1.3.3. (Equivalences and operations on Q-divisors). Assume henceforth that X is complete.

(i). Given a subvariety or subscheme $V \subseteq X$ of pure dimension k, a $\mathbf{Q}$-valued intersection product

$$\mathrm{Div}_{\mathbf{Q}}(X) \times \ldots \times \mathrm{Div}_{\mathbf{Q}}(X) \longrightarrow \mathbf{Q},$$
$$\big(D_1, \ldots, D_k\big) \mapsto \int_{[V]} D_1 \cdot \ldots \cdot D_k \;=\; \big(D_1 \cdot \ldots \cdot D_k \cdot [V]\big)$$

is defined via extension of scalars from the analogous product on $\mathrm{Div}(X)$.

(ii). Two $\mathbf{Q}$-divisors $D_1, D_2 \in \mathrm{Div}_{\mathbf{Q}}(X)$ are *numerically equivalent*, written

$$D_1 \;\equiv_{\mathrm{num}}\; D_2,$$

(or $D_1 \;\equiv_{\mathrm{num},\mathbf{Q}}\; D_2$ when confusion seems possible) if

$$\big(D_1 \cdot C\big) \;=\; \big(D_2 \cdot C\big)$$

for every curve $C \subseteq X$. We denote by $N^1(X)_{\mathbf{Q}}$ the resulting finite-dimensional $\mathbf{Q}$-vector space of numerical equivalence classes of $\mathbf{Q}$-divisors.

(iii). Two $\mathbf{Q}$-divisors $D_1, D_2 \in \mathrm{Div}_{\mathbf{Q}}(X)$ are *linearly equivalent*, written

$$D_1 \equiv_{\mathrm{lin},\mathbf{Q}} D_2 \;\; \text{(or simply } D_1 \equiv_{\mathrm{lin}} D_2\text{)}$$

if there is an integer r such that rD_1 and rD_2 are integral and linearly equivalent in the usual sense, i.e. if $r\big(D_1 - D_2\big)$ is the image of a principal divisor in $\mathrm{Div}(X)$.

(iv). If $f : Y \longrightarrow X$ is a morphism such that the image of every associated subvariety of Y meets a support of $D \in \mathrm{Div}_{\mathbf{Q}}(X)$ properly, then $f^*D \in \mathrm{Div}_{\mathbf{Q}}(Y)$ is defined by extension of scalars from the corresponding pullback on integral divisors (this being independent of the representation of D in (1.13)).

(v). If $f : Y \longrightarrow X$ is an arbitrary morphism of complete varieties or projective schemes, extension of scalars gives rise to a functorial induced homomorphism $f^* : N^1(X)_{\mathbf{Q}} \longrightarrow N^1(Y)_{\mathbf{Q}}$ compatible with the divisor-level pullback defined in (iv). □

Remark 1.3.4. More concretely, these operations and equivalences are determined from those on integral divisors by writing $D = \sum c_i A_i$ — or, after clearing denominators, $D = cA$ — and expanding by linearity. So for instance

$D \equiv_{\text{num},\mathbf{Q}} 0$ if and only if $\sum c_i(A_i \cdot C) = 0$ for every curve $C \subseteq X$. Note also that there is an isomorphism

$$N^1(X)_{\mathbf{Q}} \;=\; N^1(X) \otimes_{\mathbf{Z}} \mathbf{Q}. \quad \square$$

Remark 1.3.5. It can happen that two integral divisors in distinct linear equivalence classes become linearly equivalent in the sense of (iii) when considered as $\mathbf{Q}$-divisors. For this reason one has to be careful when dealing with $\mathbf{Q}$-linear equivalence. For the most part we will work with numerical equivalence, where this problem does not arise. $\square$

Continue to assume that X is complete. The definition of ampleness for $\mathbf{Q}$-divisors likewise presents no problems:

Definition 1.3.6. (Amplitude for Q-divisors). A $\mathbf{Q}$-divisor

$$D \;\in\; \mathrm{Div}_{\mathbf{Q}}(X)$$

is *ample* if any one of the following three equivalent conditions is satisfied:

(i). D is of the form $D = \sum c_i A_i$ where $c_i > 0$ is a positive rational number and A_i is an ample Cartier divisor.

(ii). There is a positive integer $r > 0$ such that $r \cdot D$ is integral and ample.

(iii). D satisfies the statement of Nakai's criterion, i.e.

$$\left(D^{\dim V} \cdot V\right) \;>\; 0$$

for every irreducible subvariety $V \subseteq X$ of positive dimension. $\square$

(The equivalence of (i)–(iii) is immediate.) As before, amplitude is preserved by numerical equivalence, and we speak of ample classes in $N^1(X)_{\mathbf{Q}}$.

As an illustration, we prove that amplitude is an open condition under small perturbations of a divisor:

Proposition 1.3.7. *Let X be a projective variety, H an ample $\mathbf{Q}$-divisor on X, and E an arbitrary $\mathbf{Q}$-divisor. Then $H + \varepsilon E$ is ample for all sufficiently small rational numbers $0 \leq |\varepsilon| \ll 1$. More generally, given finitely many $\mathbf{Q}$-divisors $E_1, \ldots, E_r$ on X,*

$$H + \varepsilon_1 E_1 + \ldots + \varepsilon_r E_r$$

is ample for all sufficiently small rational numbers $0 \leq |\varepsilon_i| \ll 1$.

Proof. Clearing denominators, we may assume that H and each E_i are integral. By taking $m \gg 0$ we can arrange for each of the $2r$ divisors $mH \pm E_1, \ldots, mH \pm E_r$ to be ample (Example 1.2.10). Now provided that $|\varepsilon_i| \ll 1$ we can write any divisor of the form

$$H + \varepsilon_1 E_2 + \ldots + \varepsilon_r E_r$$

as a positive **Q**-linear combination of H and some of the **Q**-divisors $H \pm \frac{1}{m} E_i$. But a positive linear combination of ample **Q**-divisors is ample. □

Remark 1.3.8. (Weil Q-divisors). As in Example 1.1.3, write $\mathrm{WDiv}(X)$ for the additive group of Weil divisors on an irreducible variety X. The group of Weil **Q**-divisors is defined to be

$$\mathrm{WDiv}_{\mathbf{Q}}(X) \ =_{\mathrm{def}} \ \mathrm{WDiv}(X) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

So a Weil **Q**-divisor is just a **Q**-linear combination of codimension-one subvarieties. As before, there is a cycle class map $[\ \ ] : \mathrm{Div}_{\mathbf{Q}}(X) \longrightarrow \mathrm{WDiv}_{\mathbf{Q}}(X)$.

Now assume that X is normal. Then the cycle mapping is injective, and in this case one can identify $\mathrm{Div}_{\mathbf{Q}}(X)$ with a subgroup of $\mathrm{WDiv}_{\mathbf{Q}}(X)$. This provides a convenient and concrete way of manipulating Cartier **Q**-divisors on a normal variety. A Weil **Q**-divisor $E \in \mathrm{WDiv}_{\mathbf{Q}}(X)$ is said to be **Q**-Cartier if it lies in $\mathrm{Div}_{\mathbf{Q}}(X)$. Thus all the operations and equivalence relations defined in Definition 1.3.3 make sense for **Q**-Cartier Weil **Q**-divisors provided that X is normal. (However we do not attempt to pass to Weil divisors when X fails to be normal.) □

Example 1.3.9. To illustrate the preceding remark, consider the quadric cone $X \subset \mathbf{P}^3$ with vertex O, and let $E \subset X$ be a ruling of X, i.e. a line through O (Figure 1.1). Viewed as a Weil divisor, E is not Cartier. But if A

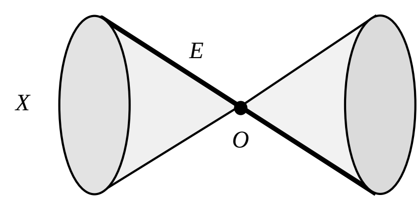


Figure 1.1. Ruling of quadric cone

is the Cartier divisor obtained by intersecting X with the hyperplane in $\mathbf{P}^3$ tangent to X along E, then $A = 2 \cdot E$. Thus E is **Q**-Cartier, and in particular we can compute its self-intersection:

$$\begin{aligned} \big(E \cdot E\big) \ &= \big(\tfrac{1}{2}A \cdot \tfrac{1}{2}A\big) \\ &= \ \tfrac{1}{4}\big(A \cdot A\big) \\ &= \ \tfrac{1}{4} \cdot 2 \\ &= \ \tfrac{1}{2}. \quad \square \end{aligned}$$

1.3.B R-Divisors and Their Amplitude

The definition of **R**-divisors proceeds in an exactly analogous fashion. Thus one defines the real vector space

$$\mathrm{Div}_{\mathbf{R}}(X) = \mathrm{Div}(X) \otimes \mathbf{R}$$

of **R**-divisors on X. Supposing X is complete, there is an associated **R**-valued intersection theory, giving rise in particular to the notion of numerical equivalence. Very concretely, an **R**-divisor D is represented by a finite sum $D = \sum c_i A_i$ where $c_i \in \mathbf{R}$ and $A_i \in \mathrm{Div}(X)$. It is numerically trivial if and only if $\sum c_i\big(A_i \cdot C\big) = 0$ for every curve $C \subseteq X$. The resulting vector space of equivalence classes is denoted by $N^1(X)_{\mathbf{R}}$. We say that D is *effective* if $D = \sum c_i A_i$ with $c_i \geq 0$ and A_i effective. Pullbacks and supports of **R**-divisors are likewise defined as before.

Example 1.3.10. One has an isomorphism

$$N^1(X)_{\mathbf{R}} \ = \ N^1(X) \otimes_{\mathbf{Z}} \mathbf{R}.$$

(Use the fact — to be established shortly in the proof of Proposition 1.3.13 — that a numerically trivial **R**-divisor is an **R**-linear combination of numerically trivial integral divisors.) □

For ampleness of **R**-divisors, however, the situation is slightly more subtle (Remark 1.3.12). We take as our definition the analogue of (i) in 1.3.6:

Definition 1.3.11. (Amplitude for R-divisors). Assume that X is complete. An **R**-divisor D on X is *ample* if it can be expressed as a finite sum

$$D = \sum c_i A_i$$

where $c_i > 0$ is a positive real number and A_i is an ample Cartier divisor. □

Observe that a finite positive **R**-linear combination of ample **R**-divisors is therefore ample.

Remark 1.3.12. (Nakai inequalities for R-divisors). If D is an ample **R**-divisor then certainly

$$\big(D^{\dim V} \cdot V\big) \ > \ 0 \qquad (*)$$

for every irreducible $V \subseteq X$ of positive dimension. However, now it is no longer clear that these inequalities characterize amplitude. For instance, if $D = \sum c_i A_i$ with $c_i > 0$ and A_i integral and ample, then

$$\big(D^{\dim V} \cdot V\big) \ \geq \ \big(\sum c_i\big)^{\dim V}$$

thanks to the fact that the intersection product on integral Cartier divisors is **Z**-valued. In particular, for fixed ample D the intersection numbers in (*) are bounded away from zero. On the other hand, one could imagine the existence of an **R**-divisor D satisfying (*) but for which the intersection numbers in question cluster toward 0 as V varies over all subvarieties of a given dimension. Surprisingly enough, however, these difficulties don't actually occur: a theorem of Campana and Peternell [78] states that the Nakai inequalities (*) do in fact imply that an **R**-divisor D on a projective variety is ample. This appears as Theorem 2.3.18 below. However we prefer to develop the general theory without appealing to this result. □

As before, amplitude depends only on numerical equivalence classes:

Proposition 1.3.13. (Ample classes for R-divisors). *The amplitude of an* **R***-divisor depends only upon its numerical equivalence class.*

Proof. It is sufficient to show that if D and B are **R**-divisors, with D ample and $B \equiv_{\text{num}} 0$, then $D + B$ is again ample. To this end, observe first that B is an **R**-linear combination of numerically trivial integral divisors. Indeed, the condition that an **R**-divisor

$$B = \sum r_i B_i \quad (\, r_i \in \mathbf{R}\ ,\ B_i \in \operatorname{Div}(X)\,)$$

be numerically trivial is given by finitely many integer linear equations on the r_i, determined by integrating over a set of generators of the subgroup of $H_2\big(X, \mathbf{Z}\big)$ spanned by algebraic 1-cycles on X. The assertion then follows from the fact that any real solution to these equations is an **R**-linear combination of integral ones.

We are now reduced to showing that if A and B are integral divisors, with A ample and $B \equiv_{\text{num}} 0$, then $A + rB$ is ample for any $r \in \mathbf{R}$. If r is rational we already know this. In general, we can fix rational numbers $r_1 < r < r_2$, together with a real number $t \in [0, 1]$, such that $r = tr_1 + (1 - t)r_2$. Then

$$A + rB \ = \ t\big(A + r_1 B\big) \ + \ (1 - t)\big(A + r_2 B\big),$$

exhibiting $A + rB$ as a positive **R**-linear combination of ample **Q**-divisors. □

Example 1.3.14. (Openness of amplitude for R-divisors). The statement of Proposition 1.3.7 remains valid for **R**-divisors. In other words:

> Let X be a projective variety and H an ample **R**-divisor on X. Given finitely many **R**-divisors $E_1, \ldots, E_r$, the **R**-divisor
>
> $$H + \varepsilon_1 E_1 + \ldots + \varepsilon_r E_r$$
>
> is ample for all sufficiently small real numbers $0 \leq |\varepsilon_i| \ll 1$.

(When H and each E_i are rational this follows from the proof of Proposition 1.3.7, and one reduces the general case to this one. To begin with, since each

E_j is a finite $\mathbf{R}$-linear combination of integral divisors, there is no loss of generality in assuming at the outset that all of the E_j are integral. Now write $H = \sum c_i A_i$ with $c_i > 0$ and A_i ample and integral, and fix a rational number $0 < c < c_1$. Then

$$H \ + \ \sum \varepsilon_j E_j \ = \ \big(cA_1 + \sum \varepsilon_j E_j\big) \ + \ \big(c_1 - c\big)A_1 \ + \ \sum_{i \geq 2} c_i A_i.$$

The first term on the right is governed by the case already treated, and the remaining summands are ample.) □

Example 1.3.15. If X is projective, then the finite-dimensional vector space $N^1(X)_{\mathbf{R}}$ is spanned by the classes of ample divisors on X. (Use 1.3.14.) □

Remark 1.3.16. (More general ground fields). The discussion in this section again goes through with only minor changes if X is a projective scheme over an arbitrary algebraically closed ground field. (In the proof of 1.3.13 one would replace $H_2\big(X; \mathbf{Z}\big)$ with the group $N_1(X)$ of numerical equivalence classes of curves (Definition 1.4.25).) □

1.4 Nef Line Bundles and Divisors

We have seen that if X is a projective variety, then a class $\delta \in N^1(X)_{\mathbf{Q}}$ is ample if and only if it satisfies the Nakai inequalities:

$$\int_V \delta^{\dim V} \ > \ 0 \ \text{ for all irreducible } V \subseteq X \ \text{ with } \dim V \ > \ 0.$$

This suggests that limits of ample classes should be characterized by the corresponding weak inequalities

$$\int_V \delta^{\dim V} \ \geq \ 0 \quad \text{for all } \ V \subseteq X. \tag{*}$$

It is a basic and remarkable fact (Kleiman's theorem) that it suffices to test (*) when V is a curve. For this reason, it turns out to be very profitable to work systematically with such limits of ample classes. From the contemporary viewpoint these so-called nef divisors lie at the heart of the theory of positivity for line bundles.

We start in Section 1.4.A with the definition and basic properties. The most important material appears in Section 1.4.B, which contains Kleiman's theorem and its consequences. It is reinterpreted in the following subsection, where we introduce the ample and nef cones. Finally we discuss in Section 1.4.D an extremely useful vanishing theorem due to Fujita.

1.4.A Definitions and Formal Properties

We begin with the definition.

Definition 1.4.1. (Nef line bundles and divisors). Let X be a complete variety or scheme. A line bundle L on X is *numerically effective*, or *nef*, if

$$\int_C c_1(L) \ \geq \ 0$$

for every irreducible curve $C \subseteq X$. Similarly, a Cartier divisor D on X (with $\mathbf{Z}$, $\mathbf{Q}$, or $\mathbf{R}$ coefficients) is *nef* if

$$\big(D \cdot C\big) \ \geq \ 0$$

for all irreducible curves $C \subset X$. □

The definition evidently depends only on the numerical equivalence class of L or D, and so one has a notion of nef classes in $N^1(X), N^1(X)_{\mathbf{Q}}$, and $N^1(X)_{\mathbf{R}}$. Note that any ample class is nef, as is the sum of two nef classes.

Remark 1.4.2. The terminology "nef," although now standard, did not come into use until the mid 1980s: it was introduced by Reid.[13] The concept previously appeared in the literature under various different names. For example, in his paper [623] Zariski speaks of "arithmetically effective" divisors. Kleiman used "numerically effective" in [341]. In that paper, a divisor satisfying the conclusion of Theorem 1.4.9 was called "pseudoample." □

Remark 1.4.3. (Chow's lemma). In dealing with nefness, one can often use Chow's Lemma to reduce statements about complete varieties or schemes to the projective case. Specifically, suppose that X is a complete variety (or scheme). Then there exists a projective variety (or scheme) X', together with a surjective morphism $f : X' \longrightarrow X$ that is an isomorphism over a dense open subset of X. In the relative setting, an analogous statement holds starting from a proper morphism $X \longrightarrow T$. See [280, Exercise II.4.10] for details. □

The formal properties of nefness are fairly immediate:

Example 1.4.4. (Formal properties of nefness). Let X be a complete variety or scheme, and L a line bundle on X.

(i). Let $f : Y \longrightarrow X$ be a proper mapping. If L is nef, then $f^*(L)$ is a nef line bundle on Y. In particular, restrictions of nef bundles to subschemes remain nef.

(ii). In the situation of (i), if f is surjective and $f^*(L)$ is nef on Y, then L itself is nef.

[13] Reid was motivated by the fact that nef is also an abbreviation for "numerically eventually free."

(iii). L is nef if and only if L_{red} is nef on X_{red}.

(iv). L is nef if and only if its restriction to each irreducible component of X is nef.

(For (ii) one needs to check that if $f : Y \longrightarrow X$ is a surjective morphism of (possibly non-projective) complete varieties, and if $C \subset X$ is an irreducible curve, then there is a curve $C' \subset Y$ mapping onto C. To this end, one can use Chow's lemma to reduce to the case in which Y is projective, where the assertion is clear. See [341, Chapter I, Section 4, Lemma 1].) □

Example 1.4.5. Let X be a complete variety (or scheme) and L a globally generated line bundle on X. Then L is nef. □

Example 1.4.6. (Divisors with nef normal bundle). Let X be a complete variety, and $D \subseteq X$ an effective divisor on X. If the normal bundle $N_{D/X} = \mathcal{O}_D(D)$ to D in X is nef, then D itself is a nef divisor. In particular, if X is a surface and $C \subseteq X$ is an irreducible curve with $(C^2) \geq 0$, then C is nef. (This generalizes the previous example.) □

Example 1.4.7. (Nefness on homogeneous varieties). Let X be a complete variety, and suppose that a connected algebraic group G acts transitively on X. Then any effective divisor D on X is nef. This applies for instance to arbitrary flag manifolds and abelian varieties. (Fix an irreducible curve $C \subset X$. Then the translate gD of D by a general element $g \in G$ meets C properly. Moreover, $gD \equiv_{\mathrm{num}} D$ since G is connected. Therefore

$$\big(D \cdot C\big) \ = \ \big(gD \cdot C\big) \ \geq \ 0,$$

and hence D is nef.) □

Remark 1.4.8. (Metric characterizations of nefness). Let X be a complex projective manifold, and L a line bundle on X. In the spirit of Section 1.2.C, it is natural to ask whether one can recognize the nefness of L metrically. If L carries a Hermitian metric h such that the corresponding Chern form $c_1(L) = \frac{i}{2\pi}\Theta(L,h)$ is non-negative, then certainly L is nef. However, there are examples [133, Example 1.7] of nef bundles that do not admit such metrics.

On the other hand, one can in effect use Corollary 1.4.10 below to reduce to the case of positive bundles. Specifically, fix a Kähler form ω on X. Then L is nef if and only if for every $\varepsilon > 0$ there exists a Hermitian metric h_ε on L such that

$$\frac{i}{2\pi}\ \Theta(L, h_\varepsilon) \ > \ -\varepsilon \cdot \omega$$

in the sense that $\frac{i}{2\pi}\Theta(L, h_\varepsilon) + \varepsilon\,\omega$ is a Kähler form. This also gives a way of defining nefness on arbitrary compact Kähler manifolds (which might not contain any curves). We refer to [133, §1.A] for details, and to [126, §6] for a survey. □

1.4.B Kleiman's Theorem

The fundamental result concerning nef divisors is due to Kleiman [341]:

Theorem 1.4.9. (Kleiman's theorem). *Let X be a complete variety (or scheme). If D is a nef $\mathbf{R}$-divisor on X, then*

$$\big(D^k \cdot V\big) \ \geq \ 0$$

for every irreducible subvariety $V \subseteq X$ of dimension k. Similarly,

$$\int_V c_1(L)^{\dim V} \ \geq \ 0$$

for every nef line bundle L on X.

Before giving the proof, we present several applications and corollaries.

The essential content of Kleiman's theorem is to characterize nef divisors as limits of ample ones. The next statement gives a first illustration of this principle; another formulation appears in Theorem 1.4.23.

Corollary 1.4.10. *Let X be a projective variety or scheme, and D a nef $\mathbf{R}$-divisor on X. If H is any ample $\mathbf{R}$-divisor on X, then*

$$D \ + \ \varepsilon \cdot H$$

is ample for every $\varepsilon > 0$. Conversely, if D and H are any two divisors such that $D + \varepsilon H$ is ample for all sufficiently small $\varepsilon > 0$, then D is nef.

Proof. If $D + \varepsilon H$ is ample for $\varepsilon > 0$, then

$$\big(D \cdot C\big) + \varepsilon\big(H \cdot C\big) \ = \ \big((D + \varepsilon H) \cdot C\big) \ > \ 0$$

for every irreducible curve C. Letting $\varepsilon \to 0$ it follows that $(D \cdot C) \geq 0$, and hence that D is nef.

Assume conversely that D is nef and H is ample. Replacing εH by H, it suffices to show that $D + H$ is ample. To this end, the main point is to verify that $D + H$ satisfies the Nakai-type inequalities appearing in Definition 1.3.6 (iii). Provided that $D+H$ is (numerically equivalent to) a rational divisor, this will establish that it is ample; the general case will follow by an approximation argument.

So fix an irreducible subvariety $V \subseteq X$ of dimension $k > 0$. Then

$$\Big((D + H)^k \cdot V\Big) \ = \ \sum_{s=0}^{k} \binom{k}{s} \Big(H^s \cdot D^{k-s} \cdot V\Big). \tag{*}$$

Since H is a positive $\mathbf{R}$-linear combination of integral ample divisors, the intersection $(H^s \cdot V)$ is represented by an effective real $(k-s)$-cycle. Applying

Kleiman's theorem to each of the components of this cycle, it follows that $(H^s \cdot D^{k-s} \cdot V) \geq 0$. Thus each of the terms in (*) is non-negative, and the last intersection number $(H^k \cdot V)$ is strictly positive. Therefore $\big((D+H)^k \cdot V\big) > 0$ for every V, and in particular if $D + H$ is rational then it is ample.

It remains to prove that $D+H$ is ample even when it is irrational. To this end, choose ample divisors $H_1, \ldots, H_r$ whose classes span $N_1(X)_{\mathbf{R}}$. By the open nature of amplitude (Example 1.3.14), the $\mathbf{R}$-divisor

$$H(\varepsilon_1, \ldots, \varepsilon_r) = H - \varepsilon_1 H_1 - \ldots - \varepsilon_r H_r$$

remains ample for all $0 \leq \varepsilon_i \ll 1$. But the classes of these divisors fill up an open[14] subset of $N^1(X)_{\mathbf{R}}$, and consequently there exist $0 < \varepsilon_i \ll 1$ such that $D' = D + H(\varepsilon_1, \ldots, \varepsilon_r)$ represents a rational class in $N^1(X)_{\mathbf{R}}$. The case of the corollary already treated shows that D' is ample. Consequently so too is

$$D + H \ = \ D' + \varepsilon_1 H_1 + \ldots + \varepsilon_r H_r,$$

as required. □

The corollary in turn gives rise to a test for amplitude involving only intersections with curves:

Corollary 1.4.11. *Let X be a projective variety or scheme, and H an ample $\mathbf{R}$-divisor on X. Fix an $\mathbf{R}$-divisor D on X. Then D is ample if and only if there exists a positive number $\varepsilon > 0$ such that*

$$\frac{\big(D \cdot C\big)}{\big(H \cdot C\big)} \ \geq \ \varepsilon \tag{1.14}$$

for every irreducible curve $C \subset X$.

In other words, the amplitude of a divisor D is characterized by the requirement that the degree of any curve C with respect to D be uniformly bounded below in terms of the degree C with respect to a known ample divisor. (See Example 1.5.3 for a concrete illustration of how this can fail.)

Proof of Corollary 1.4.11. The inequality (1.14) is equivalent to the condition that $D - \varepsilon H$ be nef. So assuming (1.14) holds, it follows from the previous Corollary 1.4.10 that

$$D \ = \ \big(D - \varepsilon H\big) + \varepsilon H$$

is ample. Conversely, if D is ample then $D - \varepsilon H$ is even ample for $0 \leq \varepsilon \ll 1$ (Example 1.3.14). □

---

[14] Openness refers here to the usual topology on the finite-dimensional real vector space $N^1(X)_{\mathbf{R}}$.

Example 1.4.12. If H_1 and H_2 are ample divisors on a projective variety X, then there are positive rational numbers $M, m > 0$ such that

$$m \cdot \big(H_1 \cdot C\big) \;\leq\; \big(H_2 \cdot C\big) \;\leq\; M \cdot \big(H_1 \cdot C\big)$$

for every irreducible curve $C \subset X$. (Choose M and m such that $M \cdot H_1 - H_2$ and $H_2 - m \cdot H_1$ are both ample.) □

Seshadri's criterion for amplitude is another application. Aside from its intrinsic interest, this result forms the basis for our discussion of local positivity in Chapter 5. As a matter of notation, given an irreducible curve C, $\mathrm{mult}_x C$ denotes the multiplicity of C at a point $x \in C$.

Theorem 1.4.13. (Seshadri's criterion). *Let X be a projective variety and D a divisor on X. Then D is ample if and only if there exists a positive number $\varepsilon > 0$ such that*

$$\frac{\big(D \cdot C\big)}{\mathrm{mult}_x C} \;\geq\; \varepsilon \tag{1.15}$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x.

In other words, we ask that the degree of every curve be uniformly bounded below in terms of its singularities.

Proof of Theorem 1.4.13. We first show that (1.15) holds when D is ample. To this end note that if E_x is an effective divisor which passes through x and meets an irreducible curve C properly, then the local intersection number $i(E_x, C; x)$ of E_x and C at x is bounded below by $\mathrm{mult}_x C$. In particular,

$$\big(E_x \cdot C\big) \;\geq\; \mathrm{mult}_x C.$$

But if D is ample, so that mD is very ample for some $m \gg 0$, then for every x and C one can find an effective divisor $E_x \equiv_{\mathrm{lin}} mD$ with the stated properties. Therefore $(D \cdot C) \geq \frac{1}{m}\mathrm{mult}_x C$ for all x and C.

Conversely, suppose that (1.15) holds for some $\varepsilon > 0$. Arguing by induction on $n = \dim X$, we can assume that $\mathcal{O}_V(D)$ is ample for every irreducible proper subvariety $V \subset X$. In particular, $\big(D^{\dim V} \cdot V\big) > 0$ for every proper $V \subset X$ of positive dimension. By Nakai's criterion, it therefore suffices to show that $(D^n) > 0$.

To this end, fix any smooth point $x \in X$, and consider the blowing-up

$$\mu : X' \;=\; \mathrm{Bl}_x(X) \longrightarrow X$$

of X at x, with exceptional divisor $E = \mu^{-1}(x)$. We claim that the $\mathbf{R}$-divisor

$$\mu^* D - \varepsilon \cdot E$$

is nef on X'. Granting this, Theorem 1.4.9 implies:

$$\begin{aligned}\big(D^n\big)_X - \varepsilon^n \ &= \ \Big(\big(\mu^* D - \varepsilon \cdot E\big)^n\Big)_{X'} \\ &\geq 0\end{aligned}$$

(where we indicate with a subscript the variety on which intersection numbers are being computed). Therefore $(D^n) > 0$, as required. For the nefness of $\big(\mu^* D - \varepsilon \cdot E\big)$, fix an irreducible curve $C' \subset X'$ not contained in E and set $C = \mu(C')$, so that C' is the proper transform of C. Then

$$\big(C' \cdot E\big) \ = \ \mathrm{mult}_x C$$

thanks to [208, p. 79] (see Lemma 5.1.10). On the other hand,

$$\big(C' \cdot \mu^* D\big)_{X'} \ = \ \big(C \cdot D\big)_X$$

by the projection formula. So the hypothesis (1.15) implies that $\big((\mu^* D - \varepsilon \cdot E) \cdot C'\big) \geq 0$. Since $\mathcal{O}_E(E)$ is a negative line bundle on the projective space E the same inequality certainly holds if $C' \subset E$. Therefore $\big(\mu^* D - \varepsilon E\big)$ is nef and the proof is complete. □

As a final application, we prove a result about the variation of nefness in families:

Proposition 1.4.14. (Nefness in families). *Let $f : X \longrightarrow T$ be a surjective proper morphism of varieties, and let L be a line bundle on X. For $t \in T$ put*

$$X_t = f^{-1}(t) \quad , \quad L_t = L \mid X_t.$$

If L_0 is nef for some given $0 \in T$, then there is a countable union $B \subset T$ of proper subvarieties of T, not containing 0, such that L_t is nef for all $t \in T - B$.

Proof. First, we can assume by Chow's lemma that f is projective. Next, after possibly shrinking T, we can write $L = \mathcal{O}_X(D)$ where D is a Cartier divisor on X whose support does not contain any of the fibres X_t. Fix also a Cartier divisor A on X such that $A_t = A|X_t$ is ample for all t. According to Corollary 1.4.10, D_t is nef if and only if $D_t + \frac{1}{m} A_t$ is ample for every positive integer $m > 0$. By assumption this holds when $t = 0$, and it follows from Theorem 1.2.17 that the locus on T where $D_t + \frac{1}{m} A_t$ fails to be ample is contained in a proper algebraic subset $B_m \subset T$ not containing 0. Then take $B = \cup B_m$. □

Remark 1.4.15. It seems to be unknown whether one actually needs a countable union of subvarieties in 1.4.14. □

We now turn to the proof of Kleiman's theorem.

Proof of Theorem 1.4.9. One can assume that X is irreducible and reduced, and by Chow's Lemma one can assume in addition that X is projective. We proceed by induction on $n = \dim X$, the assertion being evident if X is a curve. We therefore suppose that

$$(D^k \cdot V) \geq 0 \quad \text{for all irreducible } V \subset X \text{ of dimension} \leq n-1, \qquad (*)$$

and the issue is to show that $(D^n) \geq 0$. Until further notice we suppose that D is a $\mathbf{Q}$-divisor: the argument reducing the general case to this one appears at the end of the proof.

Fix an ample divisor H on X, and consider for $t \in \mathbf{R}$ the self-intersection number

$$P(t) =_{\text{def}} \big(D + tH\big)^n \in \mathbf{R}.$$

Expanding out the right-hand side, we can view $P(t)$ as a polynomial in t, and we are required to verify that $P(0) \geq 0$. Aiming for a contradiction, we assume to the contrary that $P(0) < 0$.

Note first that if $k < n$, then

$$\big(D^k \cdot H^{n-k}\big) \geq 0. \qquad (**)$$

In fact, H being ample, H^{n-k} is represented by an effective rational k-cycle. So (**) follows by applying the induction hypothesis (*) to the components of this cycle. In particular, for $k < n$ the coefficient of t^{n-k} in $P(t)$ is non-negative. Since by assumption $P(0) < 0$, it follows that $P(t)$ has a single real root $t_0 > 0$.

We claim next that for any rational number $t > t_0$, the $\mathbf{Q}$-divisor $D + tH$ is ample. To verify this, it is equivalent to check that

$$\Big(\big(D + tH\big)^k \cdot V\Big) > 0$$

for every irreducible $V \subseteq X$ of dimension k. When $V = X$ this follows from the fact that $P(t) > P(t_0) = 0$. If $V \subsetneq X$ one expands out the intersection number in question as a polynomial in t. As in (*) and (**) all the coefficients are non-negative, while the leading coefficient $\big(H^k \cdot V\big)$ is strictly positive. The claim is established.

Now write $P(t) = Q(t) + R(t)$, where

$$\begin{aligned} Q(t) &= \Big(D \cdot \big(D + tH\big)^{n-1}\Big), \\ R(t) &= \Big(tH \cdot \big(D + tH\big)^{n-1}\Big). \end{aligned}$$

If $t > t_0$ then $(D+tH)$ is ample, and hence $\big(D \cdot (D+tH)^{n-1}\big)$ is the intersection of a nef divisor with an effective 1-cycle. Therefore $Q(t) \geq 0$ for all rational $t > t_0$, and consequently $Q(t_0) \geq 0$ by continuity. On the other hand, thanks to (**) all the coefficients of $R(t)$ are non-negative, and the highest one (H^n) is strictly positive. Therefore $R(t_0) > 0$. But then $P(t_0) > 0$, a contradiction. Thus we have proven the theorem in the case that D is rational.

It remains only to check that the theorem holds when D is an arbitrary nef $\mathbf{R}$-divisor. To this end, choose ample divisors $H_1, \ldots, H_r$ whose classes span $N^1(X)_{\mathbf{R}}$. Then $\varepsilon_1 H_1 + \ldots + \varepsilon_r H_r$ is ample for all $\varepsilon_i > 0$. In particular,

$$D(\varepsilon_1, \ldots, \varepsilon_r) \;=\; D + \varepsilon_1 H_1 + \ldots + \varepsilon_r H_r,$$

being the sum of a nef and an ample $\mathbf{R}$-divisor, is (evidently) nef. But the classes of these divisors fill up an open subset in $N^1(X)_{\mathbf{R}}$, and therefore we can find arbitrarily small $0 < \varepsilon_i \ll 1$ such that $D(\varepsilon_1, \ldots, \varepsilon_r)$ is (numerically equivalent to) a rational divisor. For such divisors, the case of the Theorem already treated shows that

$$\Big(D(\varepsilon_1, \ldots, \varepsilon_r)^k \cdot V \Big) \;\geq\; 0$$

for all irreducible V of dimension k. Letting the $\varepsilon_i \to 0$, it follows that $(D^k \cdot V) \geq 0$. □

Example 1.4.16. (Intersection products of nef classes). Let

$$\delta_1, \ldots, \delta_n \;\in\; N^1(X)_{\mathbf{R}}$$

be nef classes on a complete variety or scheme X. Then

$$\int_X \delta_1 \cdot \ldots \cdot \delta_n \;\geq\; 0.$$

(By Chow's lemma, one can assume that X is projective. Then using Corollary 1.4.10 one can perturb the δ_i slightly so that they become ample classes. But in this case the assertion is clear.) □

Remark 1.4.17. Let X be a projective variety and L a line bundle on X. By analogy with Kleiman's theorem, one might be tempted to wonder whether the strict positivity of the degrees $\int_C c_1(L)$ of L on every curve $C \subseteq X$ implies that $\int_V c_1(L)^{\dim V} > 0$ for every $V \subseteq X$. However, this is not the case. A counter-example due to Mumford is outlined in Example 1.5.2 below. □

Example 1.4.18. (Minimal surfaces). Let X be a smooth projective surface of non-negative Kodaira dimension, i.e. with the property that $|mK_X| \neq \varnothing$ for some $m > 0$. Then X is minimal — in other words, it contains no smooth rational curves having self-intersection (-1) — if and only if the canonical divisor K_X is nef. (Fix $D \in |mK_X|$, and write $D = \sum a_i C_i$ with $a_i > 0$ and C_i irreducible. If $C \subseteq X$ is an irreducible curve with $\big(K_X \cdot C\big) < 0$, then evidently C must appear as one of the C_i, say $C = C_1$. Then $\big(a_1 C \cdot C\big) \leq \big(D \cdot C\big) < 0$, and it follows from the adjunction formula that C is a (-1)-curve.) □

Remark 1.4.19. (Higher-dimensional minimal varieties). The previous example points to a notion of minimality for varieties of higher dimension.

A non-singular projective variety is *minimal* if its canonical divisor K_X is nef. More generally, a *minimal variety* is a normal projective variety, having only canonical singularities, with K_X nef.[15] Kawamata, Shokurov, Reid, and others have shown that minimal varieties share many of the excellent properties of minimal surfaces (cf. [326] or [368] for an overview). By analogy with the case of surfaces, it is natural to ask whether every smooth projective variety — say of general type, to fix ideas — is birationally equivalent to a minimal variety. Mori [440] proved that this is so in dimension three, but in arbitrary dimensions the question remains open as of this writing. However the *minimal model program* of trying to construct such models has led to many important developments. We again refer to [368] or [114] for a survey. Section 1.5.F describes some related work. □

1.4.C Cones

The meaning of Theorem 1.4.9 is clarified by introducing some natural and important cones in the Néron–Severi space $N^1(X)_{\mathbf{R}}$ and its dual. This viewpoint was pioneered by Kleiman in [341, Chapter 4].

Let X be a complete complex variety or scheme. We start by defining the nef and ample cones. As a matter of terminology, if V is a finite-dimensional real vector space, by a *cone* in V we understand a set $K \subseteq V$ stable under multiplication by positive scalars.[16]

Definition 1.4.20. (Ample and nef cones). The *ample cone*

$$\mathrm{Amp}(X) \ \subset \ N^1(X)_{\mathbf{R}}$$

of X is the convex cone of all ample $\mathbf{R}$-divisor classes on X. The *nef cone*

$$\mathrm{Nef}(X) \ \subset \ N^1(X)_{\mathbf{R}}$$

is the convex cone of all nef $\mathbf{R}$-divisor classes. □

It follows from the definitions that one could equivalently define $\mathrm{Amp}(X)$ to be the convex cone in $N^1(X)_{\mathbf{R}}$ spanned by the classes of all ample integral (or rational) divisors, i.e. the convex hull of all positive real multiples of such classes.

Remark 1.4.21. As soon as $\rho(X) = \dim N^1(X)_{\mathbf{R}} \geq 3$ the structure of these cones can become quite complicated. For example, they may or may not be polyhedral. Several concrete examples are worked out in the next section. □

[15] See [516], [364], [368], or [114] for the relevant definitions. One of the requirements is that K_X exist as a $\mathbf{Q}$-Cartier divisor, so that the intersection products $\big(K_X \cdot C\big)$ defining nefness make sense.

[16] In order that the ample classes form a cone in $N^1(X)_{\mathbf{R}}$, we do not require that cones contain the origin.

Remark 1.4.22. (Visualization). It is sometimes convenient to represent a cone by drawing its intersection with a hyperplane not passing through the origin. For example, the pentagonal cone shown in Figure 1.2 would be drawn as a pentagon in the plane. □

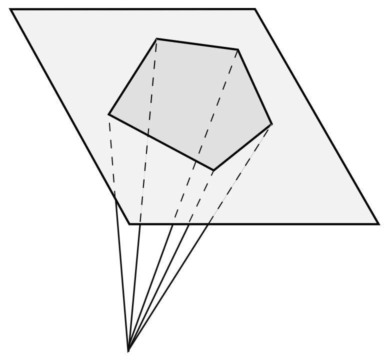


Figure 1.2. Representing cones

We view $N^1(X)_{\mathbf{R}}$ as a finite-dimensional vector space with its standard Euclidean topology. This allows one in particular to discuss closures and interiors of sets of numerical equivalence classes of **R**-divisors.

At least in the projective case, Kleiman's theorem is equivalent to the fact that the nef cone is the closure of the ample cone.

Theorem 1.4.23. (Kleiman, [341]). *Let X be any projective variety or scheme.*

(i). *The nef cone is the closure of the ample cone:*

$$\mathrm{Nef}(X) \;=\; \overline{\mathrm{Amp}(X)}.$$

(ii). *The ample cone is the interior of the nef cone:*

$$\mathrm{Amp}(X) \;=\; \mathrm{int}\Big(\mathrm{Nef}(X)\Big).$$

Proof. It is evident that the nef cone is closed, and it follows from Example 1.3.14 that Amp(X) is open. This gives the inclusions

$$\overline{\mathrm{Amp}(X)} \;\subseteq\; \mathrm{Nef}(X) \qquad \text{and} \qquad \mathrm{Amp}(X) \;\subseteq\; \mathrm{int}\big(\mathrm{Nef}(X)\big).$$

The remaining two inclusions

$$\mathrm{Nef}(X) \;\subseteq\; \overline{\mathrm{Amp}(X)} \qquad \text{and} \qquad \mathrm{int}\big(\mathrm{Nef}(X)\big) \;\subseteq\; \mathrm{Amp}(X) \tag{*}$$

are consequences of Corollary 1.4.10. In fact let H be an ample divisor on X. If D is any nef $\mathbf{R}$-divisor then 1.4.10 shows that $D+\varepsilon H$ is ample for all $\varepsilon > 0$. Therefore D is a limit of ample divisors, establishing the first inclusion in (*). For the second, observe that if the class of D lies in the interior of $\mathrm{Nef}(X)$, then $D - \varepsilon H$ remains nef for $0 < \varepsilon \ll 1$. Consequently

$$D \;=\; \big(D - \varepsilon H\big) + \varepsilon H$$

is ample thanks again to Corollary 1.4.10. □

Remark 1.4.24. (Non-projective complete varieties). Kleiman [341] shows that 1.4.23 (ii) holds on a possibly non-projective complete variety X assuming only that the Zariski topology on X is generated by the complements of effective Cartier divisors: this is automatic, for instance, if X is smooth or $\mathbf{Q}$-factorial. In this setting, $\mathrm{Nef}(X)$ has non-empty interior if and only if X is actually projective. □

Another perspective is provided by introducing the vector space of curves dual to $N^1(X)_{\mathbf{R}}$:

Definition 1.4.25. (Numerical equivalence classes of curves). Let X be a complete variety. We denote by $Z_1(X)_{\mathbf{R}}$ the $\mathbf{R}$-vector space of *real one-cycles* on X, consisting of all finite $\mathbf{R}$-linear combinations of irreducible curves on X. An element $\gamma \in Z_1(X)_{\mathbf{R}}$ is thus a formal sum

$$\gamma \;=\; \sum a_i \cdot C_i$$

where $a_i \in \mathbf{R}$ and $C_i \subset X$ is an irreducible curve. Two one-cycles $\gamma_1, \gamma_2 \in Z_1(X)_{\mathbf{R}}$ are *numerically equivalent* if

$$\big(D \cdot \gamma_1\big) \;=\; \big(D \cdot \gamma_2\big)$$

for every $D \in \mathrm{Div}_{\mathbf{R}}(X)$. The corresponding vector space of *numerical equivalence classes* of one-cycles is written $N_1(X)_{\mathbf{R}}$. Thus by construction one has a perfect pairing

$$N^1(X)_{\mathbf{R}} \times N_1(X)_{\mathbf{R}} \longrightarrow \mathbf{R} \quad , \quad (\delta, \gamma) \mapsto \big(\delta \cdot \gamma\big) \in \mathbf{R}.$$

In particular, $N_1(X)_{\mathbf{R}}$ is a finite dimensional real vector space on which we put the standard Euclidean topology. (Of course one defines numerical equivalence of integral and rational one-cycles similarly.) □

The relevant cones in $N_1(X)_{\mathbf{R}}$ are those spanned by effective curves:

Definition 1.4.26. (Cone of curves). Let X be a complete variety. The *cone of curves*

$$\mathrm{NE}(X) \;\subseteq\; N_1(X)_{\mathbf{R}}$$

is the cone spanned by the classes of all effective one-cycles on X. Concretely,

$$\mathrm{NE}(X) \;=\; \Big\{ \sum a_i[C_i] \;\Big|\; C_i \subset X \text{ an irreducible curve, } a_i \geq 0 \Big\}.$$

Its closure

$$\overline{\mathrm{NE}}(X) \;\subseteq\; N_1(X)_{\mathbf{R}}$$

is the *closed cone of curves* on X. □

The notation $\mathrm{NE}(X)$ seems to have been introduced by Mori in his fundamental paper [438]. The abbreviation is suggested by the observation — to be established momentarily — that $\overline{\mathrm{NE}}(X)$ is dual to the cone of numerically effective divisors. □

Remark 1.4.27. An example in which $\mathrm{NE}(X)$ is not itself closed is given in 1.5.1. □

A basic fact is that $\overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X)$ are dual:

Proposition 1.4.28. *In the situation of Definition 1.4.26,* $\overline{\mathrm{NE}}(X)$ *is the closed cone dual to* $\mathrm{Nef}(X)$*, i.e.*

$$\overline{\mathrm{NE}}(X) \;=\; \big\{ \;\gamma \in N_1(X)_{\mathbf{R}} \;\mid\; \big(\delta \cdot \gamma\big) \geq 0 \quad \textit{for all} \;\; \delta \in \mathrm{Nef}(X) \;\big\}.$$

Proof. This is a consequence of the theory of duality for cones. Specifically, suppose that $K \subseteq V$ is a closed convex cone in a finite-dimensional real vector space. Recall that the dual of K is defined to be the cone in V^* given by

$$K^* \;=\; \big\{ \,\phi \in V^* \mid \phi(x) \geq 0 \;\forall\; x \in K \,\big\}.$$

The duality theorem for cones (cf. [35, p. 162]) states that under the natural identification of V^{**} with V, one has $K^{**} = K$. In the situation at hand, take

$$V \;=\; N_1(X)_{\mathbf{R}} \quad , \quad K = \overline{\mathrm{NE}}(X).$$

Then $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)^*$ by definition. Consequently

$$\overline{\mathrm{NE}}(X) \;=\; \mathrm{Nef}(X)^*,$$

which is the assertion of the proposition. □

Continue to assume that X is complete, and fix a divisor $D \in \mathrm{Div}_{\mathbf{R}}(X)$, not numerically trivial. We denote by

$$\phi_D : N_1(X)_{\mathbf{R}} \longrightarrow \mathbf{R}$$

the linear functional determined by intersection with D, and we set

$$\begin{aligned} D^{\perp} \;&=\; \{\, \gamma \in N_1(X)_{\mathbf{R}} \mid \big(D \cdot \gamma\,\big) = 0 \,\}, \\ D_{>0} \;&=\; \{\, \gamma \in N_1(X)_{\mathbf{R}} \;\mid\; \big(\, D \cdot \gamma \,\big) > 0 \,\}. \end{aligned}$$

Thus $D^{\perp} = \ker \phi_D$ is a hyperplane and $D_{>0}$ an open half-space in $N_1(X)_{\mathbf{R}}$. One defines $D_{\geq 0}\;,\; D_{\leq 0}\;,\; D_{<0} \;\subset\; N_1(X)_{\mathbf{R}}$ similarly.

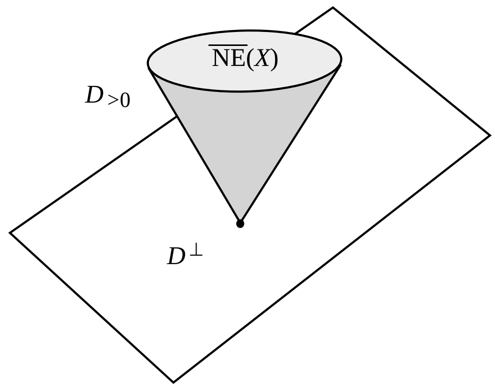


Figure 1.3. Test for amplitude via the cone of curves

Theorem 1.4.29. (Amplitude via cones). *Let X be a projective variety (or scheme), and let D be an $\mathbf{R}$-divisor on X. Then D is ample if and only if*

$$\overline{\mathrm{NE}}(X) - \{0\} \quad \subseteq \quad D_{>0}.$$

Equivalently, choose any norm $\| \ \|$ on $N_1(X)_{\mathbf{R}}$, and denote by

$$S \ = \ \{ \ \gamma \in N_1(X)_{\mathbf{R}} \ \mid \ \|\gamma\| = 1 \ \}$$

the "unit sphere" of classes in $\mathrm{N}_1(X)_{\mathbf{R}}$ of length 1. Then D is ample if and only if

$$\Big(\overline{\mathrm{NE}}(X) \ \cap \ S\Big) \ \subseteq \ \Big(D_{>0} \ \cap S \ \Big). \tag{1.16}$$

The theorem is illustrated in Figure 1.3: D is ample if and only if the closed cone $\overline{\mathrm{NE}}(X)$ (except the origin) lies entirely in the positive halfspace determined by D. The result is sometimes known as Kleiman's criterion for amplitude.

Proof of Theorem 1.4.29. We assume that (1.16) holds, and show that D is ample. To this end, consider the linear functional $\phi_D : N_1(X)_{\mathbf{R}} \longrightarrow \mathbf{R}$ determined by intersection with D. Then $\phi_D(\gamma) > 0$ for all $\gamma \in \big(\overline{\mathrm{NE}}(X) \cap S\big)$. But $\overline{\mathrm{NE}}(X) \ \cap \ S$ is compact, and therefore ϕ_D is bounded away from zero on this set. In other words, there exists a positive real number $\varepsilon > 0$ such that

$$\phi_D(\gamma) \geq \varepsilon \quad \text{for all} \ \ \gamma \in \overline{\mathrm{NE}}(X) \cap S.$$

Thus

$$\big(D \cdot C\big) \ \geq \ \varepsilon \cdot \|C\| \tag{*}$$

for every irreducible curve $C \subseteq X$. On the other hand, choose ample divisors $H_1, \ldots, H_r$ on X whose classes form a basis of $N^1(X)_{\mathbf{R}}$. Then $\| \ \|$ is equivalent to the "taxicab" norm

$$\|\gamma\|_{\text{taxi}} \;=\; \sum \big|\big(H_i\cdot\gamma\big)\big|.$$

Setting $H = \sum H_i$ it therefore follows from (*) that for suitable $\varepsilon' > 0$,

$$\big(D\cdot C\big) \;\geq\; \varepsilon'\cdot\big(H\cdot C\big)$$

for every irreducible curve $C \subseteq X$. But then the amplitude of D is a consequence of Corollary 1.4.11. We leave the converse to the reader (cf. [363, Proposition II.4.8]). □

Example 1.4.30. Let X be a projective variety. Then the closed cone of curves $\overline{\mathrm{NE}}(X) \subset N_1(X)_{\mathbf{R}}$ does not contain any infinite straight lines. In other words, if $\gamma \in N_1(X)_{\mathbf{R}}$ is a class such that both $\gamma, -\gamma \in \overline{\mathrm{NE}}(X)$, then $\gamma = 0$. □

Example 1.4.31. (Finiteness of integral classes of bounded degree). Let X be a projective variety, and H an ample divisor on X. Denote by $N_1(X) = N_1(X)_{\mathbf{Z}}$ the group of numerical equivalence classes of integral one-cycles, and put

$$\overline{\mathrm{NE}}(X)_{\mathbf{Z}} \;=\; \overline{\mathrm{NE}}(X)\cap N_1(X)_{\mathbf{Z}}.$$

Then for any positive number $M > 0$, the set

$$\Big\{\gamma \;\in\; \overline{\mathrm{NE}}(X)_{\mathbf{Z}} \;\Big|\; \big(H\cdot\gamma\big) \leq M\;\Big\}$$

is finite. (One can choose ample $\mathbf{R}$-divisors $H_1, \ldots, H_r$ forming a basis of $N^1(X)_{\mathbf{R}}$ such that $H = \sum H_i$. If $\gamma \in \overline{\mathrm{NE}}(X)_{\mathbf{Z}}$, then

$$\big(H\cdot\gamma\big) \;=\; \sum\big(H_i\cdot\gamma\big) \;=\; \sum\big|\big(H_i\cdot\gamma\big)\big|$$

is the norm $\|\gamma\|_{\text{taxi}}$ of γ in the "taxi-cab" norm determined by the H_i. So the set in question is contained in the closed ball of radius M with respect to this norm. Being compact, this ball contains only finitely many integer points.) □

Definition 1.4.32. (Extremal rays). Let $K \subseteq V$ be a closed convex cone in a finite-dimensional real vector space. An *extremal ray* $\mathbf{r} \subseteq K$ is a one-dimensional subcone having the property that if $v + w \in \mathbf{r}$ for some vectors $v, w \in K$, then necessarily $v, w \in \mathbf{r}$. □

An extremal ray is contained in the boundary of K.

Example 1.4.33. (Curves on a surface). If X is a smooth projective surface, then a one-cycle is the same thing as a divisor. Hence

$$N^1(X)_{\mathbf{R}} \;=\; N_1(X)_{\mathbf{R}},$$

and in particular the various cones we have defined all live in the same finite-dimensional vector space.

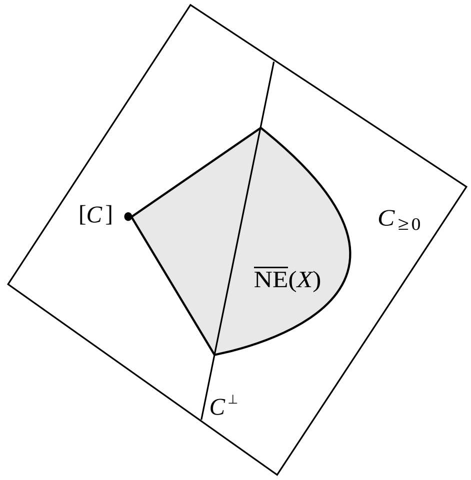


Figure 1.4. Curve of negative self-intersection on a surface

(i). One has the inclusion

$$\mathrm{Nef}(X) \ \subseteq \ \overline{\mathrm{NE}}(X),$$

with equality if and only if $\left(C^2\right) \geq 0$ for every irreducible curve $C \subset X$.

(ii). If $C \subset X$ is an irreducible curve with $\left(C^2\right) \leq 0$, then $\overline{\mathrm{NE}}(X)$ is spanned by $[C]$ and the subcone

$$\overline{\mathrm{NE}}(X)_{C \geq 0} \ =_{\mathrm{def}} \ C_{\geq 0} \cap \overline{\mathrm{NE}}(X).$$

(iii). In the situation of (ii), $[C]$ lies on the boundary of $\overline{\mathrm{NE}}(X)$. If in addition $\left(C^2\right) < 0$ then $[C]$ spans an extremal ray in that cone.

The conclusion of (iii) when $\left(C^2\right) < 0$ is illustrated in Figure 1.4 (drawn according to the convention of Remark 1.4.22). (It is evident that $\mathrm{Amp}(X) \subseteq \mathrm{NE}(X)$, and the inclusion in (i) follows by passing to closures. For (ii), observe that if $C' \subset X$ is any effective curve not containing C as a component, then $\left(C \cdot C'\right) \geq 0$. If $\left(C^2\right) < 0$, then $[C]$ does not lie in $\overline{\mathrm{NE}}(X)_{C \geq 0}$, and the second assertion of (iii) follows.) Kollár analyzes the cone of curves on an algebraic surface in more detail in [363, II.4]. $\square$

Remark 1.4.34. (Positive characteristics). The material so far in this section goes through for varieties defined over an algebraically closed field of arbitrary characteristic. $\square$

1.4.D Fujita's Vanishing Theorem

We now discuss a theorem of Fujita [195] showing that Serre-type vanishings can be made to operate uniformly with respect to twists by nef divisors.

Fujita's result is very useful in applications. The proof will call on vanishing theorems for big and nef line bundles to be established in Section 4.3, so from a strictly logical point of view Fujita's statement is somewhat out of sequence here.[17] We felt however that an early presentation is justified by the insight it provides.

Here is Fujita's theorem:

Theorem 1.4.35. (Fujita's vanishing theorem). *Let X be a complex projective scheme and let H be an ample (integral) divisor on X. Given any coherent sheaf $\mathcal{F}$ on X, there exists an integer $m(\mathcal{F}, H)$ such that*

$$H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)\big) \;=\; 0 \quad \textit{for all} \quad i > 0\ ,\ m \geq m(\mathcal{F}, H),$$

and any nef divisor D on X.

The essential point here is that the integer $m(\mathcal{F}, H)$ is independent of the nef divisor D.

Proof of Theorem 1.4.35. Arguing as in the proof of Proposition 1.2.16, it suffices to prove the theorem under the additional assumption that X is irreducible and reduced. Moreover, by induction on $\dim X$ one can assume that the theorem is known for all sheaves $\mathcal{F}$ supported on a proper subscheme of X. To streamline the discussion, we will henceforth say that the theorem holds for a given coherent sheaf $\mathcal{G}$ if the statement is true when $\mathcal{F} = \mathcal{G}$.

We claim next that it is enough to exhibit any one integer $a \in \mathbf{Z}$ such that the theorem holds for $\mathcal{O}_X(aH)$. In fact, according to Example 1.2.21 an arbitrary coherent sheaf $\mathcal{F}$ admits a (possibly infinite) resolution by bundles of the form $\oplus\, \mathcal{O}_X(-pH)$. Using Proposition B.1.2 and Remark B.1.4 from Appendix B this reduces one to proving the stated vanishing for finitely many such bundles. On the other hand, if the theorem holds for $\mathcal{O}_X(aH)$ for any one integer a, then it follows formally (by suitably adjusting $m(\mathcal{F}, H)$) that it holds for any finite collection of the line bundles $\mathcal{O}_X(bH)$.

Let $\mu : X' \longrightarrow X$ be a resolution of singularities, and consider the torsion-free sheaf

$$\mathcal{K}_X = \mu_* \mathcal{O}_{X'}(K_{X'})$$

where $K_{X'}$ is a canonical divisor on X'.[18] If $a \gg 0$ there is an injective homomorphism

$$u \;:\; \mathcal{K}_X \longrightarrow \mathcal{O}_X(aH)$$

of coherent sheaves on X, deduced from a non-zero section of $\mathcal{O}_{X'}\big(\mu^*(aH) - K_{X'}\big)$. The cokernel of u is supported on a proper subscheme of X, so one can

[17] Needless to say, the proof of vanishing in 4.3 does not draw on Fujita's result. The interested reader can go directly to Chapter 4 after skimming Sections 2.1.A, 2.2.A, and 3.1.

[18] This is the Grauert–Riemenschneider canonical sheaf on X: see Example 4.3.12.

assume that the theorem holds for $\operatorname{coker}(u)$. Therefore it is enough to show that the theorem holds for $\mathcal{F} = \mathcal{K}_X$, for this then implies the statement for $\mathcal{O}_X(aH)$.

Finally, take $\mathcal{F} = \mathcal{K}_X$. Here Example 4.3.12 applies, but we recall the argument. The vanishing theorem of Grauert–Riemenschneider (Theorem 4.3.9) guarantees that

$$R^j \mu_* \mathcal{O}_{X'}(K_{X'}) \ = \ 0 \quad \text{for} \quad j > 0.$$

Thus Proposition B.1.1 in Appendix B yields

$$H^i\Big(X \, , \mathcal{K}_X \otimes \mathcal{O}_X(aH + D) \Big) \ = \ H^i\Big(X' \, , \mathcal{O}_{X'}\big(K_{X'} + \mu^*(aH + D)\big) \Big) \quad (*)$$

for all i. On the other hand, if $a > 0$ then $\mu^*(aH + D)$ is a nef divisor on X' whose top self-intersection number is positive (i.e. it is "big" in the sense of 2.2.1: see 2.2.16). But then the group on the right in (*) vanishes thanks to Theorem 4.3.1. □

Remark 1.4.36. (Positive characteristics). Fujita uses an argument with the Frobenius to show that the theorem also holds over algebraically closed ground fields of positive characteristic. □

We next indicate some applications of Fujita's result. The first shows that the set of all numerically trivial line bundles on a projective variety forms a bounded family.

Proposition 1.4.37. (Boundedness of numerically trivial line bundles). *Let X be a projective variety or scheme. Then there is a scheme T (of finite type!) together with a line bundle $\mathcal{L}$ on $X \times T$ having the property that any numerically trivial line bundle L on X arises as the restriction*

$$\mathcal{L}_t \ = \ \mathcal{L} \,|\, X_t \quad \textit{for some} \quad t \ \in \ T,$$

where $X_t = X \times \{t\}$.

Proof. It is equivalent to prove the boundedness of the bundles $L \otimes B$ for any fixed line bundle B independent of L. With this in mind, choose a very ample line bundle $\mathcal{O}_X(1)$ on X. Since any numerically trivial line bundle is nef, 1.4.35 shows that there exists an integer $m_0 \gg 0$ such that

$$H^i\big(X, L \otimes \mathcal{O}_X(m_0 - i)\big) \ = \ 0$$

for $i > 0$ and every numerically trivial bundle L. By an elementary result of Mumford — appearing below as Theorem 1.8.3 — this implies first that

$$A_L \ =_{\text{def}} \ L \otimes \mathcal{O}_X(m_0)$$

is globally generated. Theorem 1.8.3 also gives $H^i\big(X, L \otimes \mathcal{O}_X(m)\big) = 0$ for $i > 0$ and $m \geq m_0$, so that

$$\begin{aligned} h^0\big(X, L \otimes \mathcal{O}_X(m)\big) &= \chi\big(X, L \otimes \mathcal{O}_X(m)\big) \\ &= \chi\big(X, \mathcal{O}_X(m)\big) \end{aligned}$$

for $m \geq m_0$ and every numerically trivial L: in the second equality we are using Riemann–Roch to know that twisting by L does not affect the Euler characteristic. In particular, all the bundles A_L have the same Hilbert polynomial, and they can each be written as a quotient of the trivial bundle $\mathcal{O}_X^N$ for $N = \chi\big(X, \mathcal{O}_X(m_0)\big)$. But Grothendieck's theory of Quot schemes implies that the set of all quotients of $\mathcal{O}_X^N$ having fixed Hilbert polynomial is parametrized by a scheme H of finite type, and there exists moreover a universal quotient sheaf $\mathcal{O}_{X \times H}^N \twoheadrightarrow \mathcal{F}$ flat over H. (See [300, Chapter 2.2] for a nice account of Grothendieck's theory.) We then obtain T as the open subscheme of H consisting of points $t \in H$ for which $\mathcal{F}_t$ is locally free on $X = X_t$ (cf. [300, Lemma 2.18]). □

Proposition 1.4.37 has as a consequence the characterization of numerically trivial bundles mentioned in Remark 1.1.20, at least for projective schemes:

Corollary 1.4.38. (Characterization of numerically trivial line bundles). *Let X be a projective variety or scheme, and L a line bundle on X. Then L is numerically trivial if and only if there is an integer $m > 0$ such that $L^{\otimes m}$ is a deformation of the trivial line bundle.*

Keeping notation as in 1.4.37, the conclusion means that there exists an irreducible scheme T, points $0, 1 \in T$, and a line bundle $\mathcal{L}$ on $X \times T$ such that

$$\mathcal{L}_1 \;=\; L^{\otimes m} \;\text{ and }\; \mathcal{L}_0 \;=\; \mathcal{O}_X.$$

The proof will show that one could even take T to be a smooth connected curve.

Proof of Corollary 1.4.38. If $L^{\otimes m}$ is a deformation of $\mathcal{O}_X$ then evidently L is numerically trivial. Conversely, according to the previous result all numerically trivial line bundles fall into finitely many irreducible families. Therefore there must be two distinct integers $p \neq q$ such that $L^{\otimes p}$ and $L^{\otimes q}$ lie in the same family. But then $L^{\otimes (p-q)}$ is a deformation of the trivial bundle. The statement immediately before the proof follows from the fact that any two points on an irreducible variety can be joined by a map from a smooth irreducible curve to the variety (Example 3.3.5). □

Remark 1.4.39. It is established in [52, XIII, Theorem 4.6] that the corollary continues to hold if X is complete but possibly non-projective. □

We conclude with a result that strengthens the statement of Example 1.2.36. It shows that the growth of the cohomology of a nef line bundle is bounded in terms of the degree of the cohomology:

Theorem 1.4.40. (Higher cohomology of nef divisors, II). *Let X be a projective variety or scheme of dimension n, and D a nef divisor on X. Then for any coherent sheaf $\mathcal{F}$ on X,*

$$h^i\big(X, \mathcal{F}(mD)\big) \ = \ O(m^{n-i}). \tag{1.17}$$

Proof. We may suppose by induction that the statement is known for all schemes of dimension $\leq n-1$. Thanks to Fujita's theorem, there exists a very ample divisor H having the property that $H^i\big(X, \mathcal{F}(mD+H)\big) = 0$ for $i > 0$ and every $m \geq 0$. Assuming as we may that H doesn't contain any of the subvarieties of X defined by the associated primes of $\mathcal{F}$, we have the exact sequence

$$0 \longrightarrow \mathcal{F}(mD) \xrightarrow{\cdot H} \mathcal{F}(mD+H) \longrightarrow \mathcal{F}(mD+H) \otimes \mathcal{O}_H \longrightarrow 0.$$

Therefore when $i \geq 1$,

$$h^i\big(X, \mathcal{F}(mD)\big) \ \leq \ h^{i-1}\big(H, \mathcal{F}(mD+H) \otimes \mathcal{O}_H\big) \ = \ O\big(m^{(n-1)-(i-1)}\big)$$

as required. □

Combining the Theorem with 1.1.25 one has:

Corollary 1.4.41. (Asymptotic Riemann–Roch, III). *Let X be an irreducible projective variety or scheme of dimension n, and let D be a nef divisor on X. Then*

$$h^0\big(X, \mathcal{O}_X(mD)\big) \ = \ \frac{(D^n)}{n!} \cdot m^n \ + \ O(m^{n-1}).$$

More generally,

$$h^0\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) \ = \ \operatorname{rank}(\mathcal{F}) \cdot \frac{(D^n)}{n!} \cdot m^n \ + \ O(m^{n-1}) \tag{1.18}$$

for any coherent sheaf $\mathcal{F}$ on X. □

Example 1.4.42. Kollár [363, VI.2.15] shows that in the situation of Theorem 1.4.40 one can prove a slightly weaker statement without using Fujita's vanishing theorem. Specifically, with X, $\mathcal{F}$, and D as in 1.4.40 he establishes the bound

$$h^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(mD)\big) \ = \ O(m^{n-1}) \quad \text{for} \quad i \geq 1. \tag{1.19}$$

In particular, one can avoid Fujita's result for Corollary 1.4.41. (When $\mathcal{F} = \mathcal{O}_X$, Kollár's argument was outlined in Example 1.2.36. In general, arguing as in Proposition 1.2.16, one first reduces to the case in which X is reduced and irreducible. If $\operatorname{rank}(\mathcal{F}) = 0$ then $\mathcal{F}$ is supported on a proper subscheme, so the statement follows by induction on dimension. Assuming $\operatorname{rank}(\mathcal{F}) = r > 0$ fix a very ample divisor H. Then $\mathcal{F}(pH)$ is globally generated for $p \gg 0$, so there is an injective homomorphism $u : \mathcal{O}_X^r(-pH) \longrightarrow \mathcal{F}$ whose cokernel is supported on a proper subscheme of X. Then (1.19) for the given sheaf $\mathcal{F}$ is implied by the analogous assertion for the line bundle $\mathcal{O}_X(-pH)$. This in turn follows from the previously treated case of $\mathcal{O}_X$ by choosing a general divisor $A \in |pH|$ and arguing by induction on dimension from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-A) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_A \longrightarrow 0.$$

We refer to [363, VI.2.15] for details.) □

1.5 Examples and Complements

This section gives some concrete examples of ample and nef cones, and presents some further information about their structure. We begin with ruled surfaces in Section 1.5.A. The product of a curve with itself is discussed in 1.5.B, where in particular we prove an interesting theorem of Kouvidakis. Abelian varieties are treated in Section 1.5.C, while 1.5.D contains telegraphic summaries of some other situations in which ample cones have been studied. Section 1.5.E is concerned with results of Campana and Peternell describing the local structure of the nef cone. We conclude in 1.5.F with a brief summary (without proofs) of Mori's cone theorem. We warn the reader that the present section assumes a somewhat broader background than has been required up to now.

1.5.A Ruled Surfaces

As a first example, we work out the nef and effective cones for ruled surfaces. At one point we draw on some facts concerning semistability that are discussed and established later in Section 6.4. The reader may consult [280, Chapter V, §2] or [114, Chapter 1.9] for a somewhat different perspective.

Let E be a smooth projective curve of genus g, let U be a vector bundle on E of rank two, and set $X = \mathbf{P}(U)$ with

$$\pi : X = \mathbf{P}(U) \longrightarrow E$$

the bundle projection. For ease of computation we assume that U has even degree. After twisting by a suitable divisor, we can then suppose without loss of generality that $\deg U = 0$.

Recall that $N^1(X)_{\mathbf{R}}$ is generated by the two classes

$$\xi \;=\; c_1\big(\mathcal{O}_{\mathbf{P}(U)}(1)\big) \quad \text{and} \quad f \;=\; [F],$$

where F is a fibre of π. The intersection form on X is determined by the relations

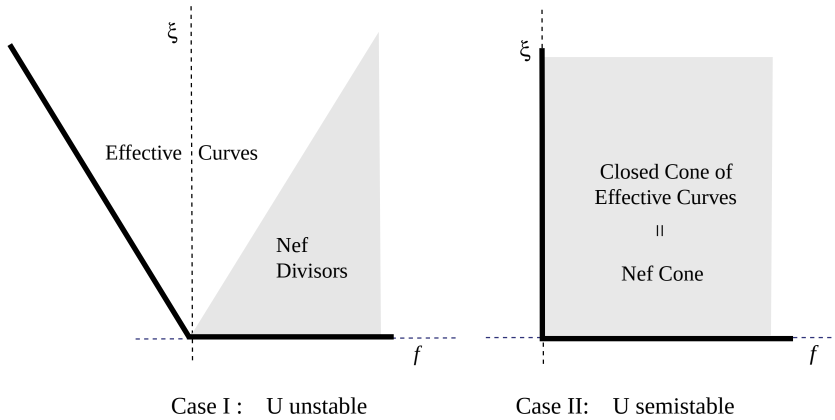


Figure 1.5. Néron–Severi group of a ruled surface

$$\left(\xi^2\right) = \deg U = 0 \quad , \quad \left(\xi \cdot f\right) = 1 \quad , \quad \left(f^2\right) = 0.$$

In particular, $\left((af + b\xi)^2\right) = 2ab$. If we represent the class $(af + b\xi)$ by the point (a, b) in the f-ξ plane, it follows that the nef cone $\mathrm{Nef}(X)$ must lie within the first quadrant $a, b \geq 0$. Moreover, the fibre F is evidently nef (e.g. by Example 1.4.6). Therefore the non-negative "f-axis" forms one of the two boundaries of $\mathrm{Nef}(X)$. Equivalently, f lies on the boundary of $\overline{\mathrm{NE}}(X)$.

The second ray bounding $\mathrm{Nef}(X)$ depends on the geometry of U. Specifically, there are two possibilities:

Case I: U is unstable. By definition, a rank-two bundle U of degree 0 is unstable if it has a line bundle quotient A of negative degree $a = \deg(A) < 0$. Assuming such a quotient exists,

$$C \ = \ \mathbf{P}(A) \ \subset \ \mathbf{P}(U) \ = \ X$$

is an effective curve in the class $af + \xi$. One has $\left(C^2\right) = 2a < 0$, and it follows from Example 1.4.33 that the ray spanned by $[C]$ bounds $\overline{\mathrm{NE}}(X)$. Therefore $\mathrm{Nef}(X)$ is bounded by the dual ray generated by $(-af + \xi)$. The situation is illustrated in Figure 1.5.

Case II: U is semistable. By definition, a bundle of degree 0 is semistable if it does not admit any quotients of negative degree. It is a basic fact that if U is semistable then so too are all the symmetric powers S^mU of U (Corollary 6.4.14). In the present situation this implies that if A is a line bundle of degree a such that $H^0\big(E, S^mU \otimes A\big) \neq 0$, then $a \geq 0$. Now suppose that $C \subset X$ is an effective curve. Then C arises as a section of $\mathcal{O}_{\mathbf{P}(U)}(m) \otimes \pi^*A$ for some integer $m \geq 0$ and some line bundle A on E. On the other hand,

$$H^0\big(\mathbf{P}(U), \mathcal{O}_{\mathbf{P}(U)}(m) \otimes \pi^*A\big) \ = \ H^0\big(E, S^mU \otimes A\big),$$

so by what we have just said $a = \deg A \geq 0$. In other words, the class $(af + m\xi)$ of C lies in the first quadrant. So in this case $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)$ and the cones in question fill up the first quadrant of the f-ξ plane.

Example 1.5.1. (Ruled surfaces where NE(X) is not closed). In the setting of Case II, it is interesting to ask whether the "positive ξ-axis" $\mathbf{R}_+ \cdot \xi$ actually lies in the cone $\mathrm{NE}(X)$ of effective curves, or merely in its closure. In other words, we ask whether there exists an irreducible curve $C \subset X$ with $[C] = m\xi$ for some $m \geq 1$. The presence of such a curve is equivalent to the existence of a line bundle A of degree 0 on E such that $H^0\big(E, S^m U \otimes A\big) \neq 0$, which implies that $S^m U$ is semistable but not strictly stable. Using a theorem of Narasimhan and Seshadri [474] describing stable bundles in terms of unitary representations of the fundamental group $\pi_1(E)$, Hartshorne checks in [276, I.10.5] that if E has genus $g(E) \geq 2$ then there exist bundles U of degree 0 on E having the property that

$$H^0\big(E, S^m U \otimes A\big) = 0 \quad \text{for all} \;\; m \geq 1 \tag{1.20}$$

whenever $\deg A \leq 0$: in fact this holds for a "sufficiently general" semistable bundle U. Thus there is no effective curve C on the resulting surface $X = \mathbf{P}(U)$ with class $[C] = m\xi$, and therefore the positive ξ-axis does not itself lie in the cone of effective curves. This example is due to Mumford. □

Example 1.5.2. (Non-ample bundle that is positive on all curves). Mumford observed that the phenomenon just described also yields an example of a surface X carrying a line bundle L such that $\int_C c_1(L) > 0$ for every irreducible curve $C \subseteq X$ on X, but where L fails to be ample. In fact, let U be a bundle satisfying the condition in (1.20) and take $X = \mathbf{P}(U)$ and $L = \mathcal{O}_{\mathbf{P}(U)}(1)$. This shows that it is not enough to check intersections with curves in Nakai's criterion. By the same token it gives an example in which the linear functional ϕ_ξ determined by intersection with ξ is positive on the cone of curves $\mathrm{NE}(X)$ for a non-ample bundle ξ, explaining why one passes to the closed cone $\overline{\mathrm{NE}}(X)$ in Theorem 1.4.29. □

Example 1.5.3. The line bundle $L = \mathcal{O}_{\mathbf{P}(U)}(1)$ constructed in Example 1.5.1 is nef but not ample. It is instructive to see explicitly how the condition in Corollary 1.4.11 fails for L (as of course it must). In fact, a theorem of Segre, Nagata, and Ghione (Examples 7.2.13, 7.2.14) implies that for every m there is a line bundle A_m on E of degree $\leq g$ such that $H^0\big(E, S^m U \otimes A_m\big) \neq 0$. As above, this gives rise to a curve $C_m \subset X$ with $\int_{C_m} c_1(L) = \deg A_m$ bounded. For the reference ample class H one can take for instance $H = \xi + f$. Then one sees that the intersection numbers $\big(C_m \cdot H\big)$ go to infinity with m, and so

$$\lim_{m \to \infty} \frac{\big(C_m \cdot c_1(L)\big)}{\big(C_m \cdot H\big)} \;=\; 0. \quad \square$$

1.5.B Products of Curves

Our next examples involve products of curves, and we start by establishing notation. Denote by E a smooth irreducible complex projective curve of genus $g = g(E)$. We set

$$X = E \times E,$$

with projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \longrightarrow E$. Fixing a point $P \in E$, consider in $N^1(X)_{\mathbf{R}}$ the three classes

$$f_1 = [\,\{P\} \times E\,] \ , \ f_2 = [\,E \times \{P\}\,] \ , \ \delta = [\,\Delta\,],$$

where $\Delta \subset E \times E$ is the diagonal (Figure 1.6). Provided that $g(E) \geq 1$ these

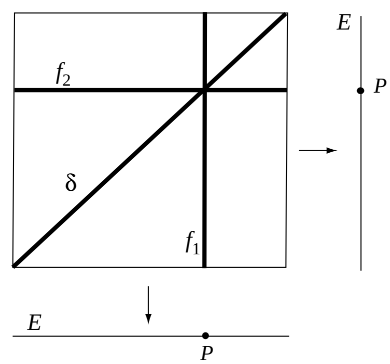


Figure 1.6. Cartesian product of curve with itself

classes are independent, and if E has general moduli then it is known that they span $N_1(X)_{\mathbf{R}}$. Intersections among them are governed by the formulae

$$\begin{aligned} \big(\delta \cdot f_1\big) \ = \ \big(\delta \cdot f_2\big) \ = \ \big(f_1 \cdot f_2\big) \ &= \ 1, \\ \big((f_1)^2\big) \ = \ \big((f_2)^2\big) \ &= \ 0, \\ \big(\delta^2\big) \ &= \ 2 - 2g. \end{aligned}$$

Elliptic curves. Assume that $g(E) = 1$. Then $X = E \times E$ is an abelian surface, and one has:

Lemma 1.5.4. *Any effective curve on X is nef, and consequently*

$$\overline{\mathrm{NE}}(X) \ = \ \mathrm{Nef}(X).$$

A class $\alpha \in N^1(X)_{\mathbf{R}}$ is nef if and only if

$$\big(\alpha^2\big) \ \geq \ 0 \ \ , \ \ \big(\alpha \cdot h\big) \ \geq \ 0,$$

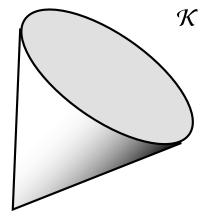
for some ample class h. In particular, if

$$\alpha \;=\; x \cdot f_1 + y \cdot f_2 + z \cdot \delta,$$

then α is nef if and only if

$$\begin{aligned} xy + xz + yz \;&\geq\; 0 \\ x + y + z \;&\geq\; 0. \end{aligned} \tag{*}$$

If we identify $x \cdot f_1 + y \cdot f_2 + z \cdot \delta$ in the natural way with the point $(x, y, z) \in \mathbf{R}^3$, then the equations (*) define a circular cone $\mathcal{K} \subset \mathbf{R}^3$. When $\rho(X) = 3$ — which as we have noted is the case for a sufficiently general elliptic curve E — it is precisely the nef cone, i.e. $\mathcal{K} = \mathrm{Nef}(X)$. In general, $\mathcal{K}$ is the intersection of $\mathrm{Nef}(X)$ with a linear subspace of $N^1(X)_{\mathbf{R}}$. In either event, the proposition shows that $\mathrm{Nef}(X)$ is not polyhedral. (See also Example 1.5.6 and Proposition 1.5.17.) Kollár analyzes this example in more detail in [363, Chapter II, Exercise 4.16].



Proof of Lemma 1.5.4. The first statement is a special case of Example 1.4.7. A standard and elementary argument with Riemann–Roch (cf. [280, V.1.8]) shows that if D is an integral divisor on X such that $\big(D^2\big) > 0$ and $\big(D \cdot H\big) > 0$ for some ample H, then for $m \gg 0$, mD is linearly equivalent to an effective divisor. The second statement follows, and one deduces (*) by taking $h = f_1 + f_2 + \delta$. □

Remark 1.5.5. (Irrational polyhedral cones). One can use this example to construct a projective variety V with $\rho(V) = 2$ for which $\mathrm{Nef}(V) \subseteq \mathbf{R}^2$ is an irrational polyhedron: see Example 5.4.17. □

Remark 1.5.6. (Arbitrary abelian surfaces). An analogous statement holds on an arbitrary abelian surface X. Specifically, $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)$ and $\alpha \in N^1(X)_{\mathbf{R}}$ is nef if and only if $\big(\alpha^2\big) \geq 0$ and $\big(\alpha \cdot h\big) \geq 0$ for some ample class h. Moreover, if $\rho(X) = r$ then in suitable linear coordinates $x_1, \ldots, x_r$ on $N^1(X)_{\mathbf{R}}$, $\mathrm{Nef}(X)$ is the cone given by

$$x_1^2 - x_2^2 - \ldots - x_r^2 \geq 0 \;\;,\;\; x_1 \geq 0.$$

(Note that in any event $r \leq 4$.) Abelian varieties of arbitrary dimension are discussed below. □

Example 1.5.7. (An example of Kollár). If D is an ample divisor on a variety Y, then by definition there is a positive integer $m(D)$ such that $\mathcal{O}_Y(mD)$ is very ample when $m \geq m(D)$. We reproduce from [152, Example 3.7] Kollár's example of a surface Y on which the integer $m(D)$ cannot be bounded independently of D. (By contrast, if Y is a smooth curve of genus g then one can take $m(D) = 2g + 1$.) Keeping notation as above, start with the product $X = E \times E$ of an elliptic curve with itself, and for each integer $n \geq 2$, form the divisor

$$A_n = n \cdot F_1 + (n^2 - n + 1) \cdot F_2 - (n-1) \cdot \Delta,$$

F_1, F_2 being fibres of the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \longrightarrow E$. One has $\big(A_n \cdot A_n\big) = 2$ and $\big(A_n \cdot (F_1 + F_2)\big) = n^2 - 2n + 3 > 0$. It follows from 1.5.4 that A_n is ample.

Now set $R = F_1 + F_2$, let $B \in |2R|$ be a smooth divisor, and take for Y the double cover $f : Y \longrightarrow X$ of X branched along B (see Proposition 4.1.6 for the construction of such coverings). Let $D_n = f^* A_n$. Then D_n is ample, and we claim that the natural inclusion

$$H^0\big(X, \mathcal{O}_X(nA_n)\big) \longrightarrow H^0\big(Y, \mathcal{O}_Y(nD_n)\big) \tag{*}$$

is an isomorphism. It follows that $n \cdot D_n$ cannot be very ample, and hence that $m\big(D_n\big) > n$. For the claim, observe that $f_* \mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-R)$, and therefore

$$f_*\big(\mathcal{O}_Y(nD_n)\big) \;=\; \mathcal{O}_X(nA_n) \oplus \mathcal{O}_X(nA_n - R).$$

So to verify that (*) is bijective, it suffices to prove that

$$H^0\big(X, \mathcal{O}_X(nA_n - R)\big) \;=\; 0.$$

But this follows from the computation that $\big(\, (n \cdot A_n - R)^2 \,\big) < 0$. □

Curves of higher genus. Suppose now that $g = g(E) \geq 2$. In this case the ample cone of $X = E \times E$ is already quite subtle, and not fully understood in general. Here we will give a few computations emphasizing the interplay between amplitude on X and the classical geometry of E. Some related results appear in Section 5.3.A.

Following [602], it is convenient to make a change of variables and replace δ by the class

$$\delta' = \delta - (f_1 + f_2).$$

This brings the intersection product into the simpler form

$$\big(\delta' \cdot f_1\big) = \big(\delta' \cdot f_2\big) = 0 \quad , \quad \big(\delta' \cdot \delta'\big) = -2g.$$

Given a positive real number $t > 0$, we focus particularly on understanding when the class

$$e_t \;=\; t(f_1 + f_2) - \delta'$$

is nef.[19] Since $(f_1 + f_2)$ is ample, e_t becomes ample for $t \gg 0$. For the purpose of the present discussion set

[19] This amounts to determining part of the ample cone of the symmetric product $S^2 E$: see Example 1.5.14.

$$t(E) \ = \ \inf\big\{ t>0 \mid e_t \text{ is nef } \big\}. \tag{1.21}$$

Since $\big(e_t \cdot e_t\big) = 2t^2 - 2g$, we find that in any event $t(E) \geq \sqrt{g}$.

An interesting result of Kouvidakis [370] shows that for many classes of curves, the invariant $t(E)$ reflects in a quite precise way the existence of special divisors on E. As a matter of terminology, we say that a branched covering

$$\pi : E \longrightarrow \mathbf{P}^1$$

is *simple* if π has only simple ramification (locally given by $z \mapsto z^2$) and if no two ramification points in E lie over the same point of $\mathbf{P}^1$. Writing $B \subset \mathbf{P}^1$ for the branch locus of π, this condition guarantees that $\pi_1(\mathbf{P}^1 - B)$ acts via monodromy as the full symmetric group on a general fibre $\pi^{-1}(y)$ (cf. [54, Lemma 1.3]).

Theorem 1.5.8. (Theorem of Kouvidakis). *Assume that E admits a simple branched covering*

$$\pi : E \longrightarrow \mathbf{P}^1$$

of degree $d \leq [\sqrt{g}\ ] + 1$. Then $t(E) = \frac{g}{d-1}$.

Corollary 1.5.9. *If E is a very general curve of genus g, then*

$$\sqrt{g} \ \leq \ t(E) \ \leq \ \frac{g}{[\sqrt{g}\ ]}.$$

The assumption on E means that the conclusion holds for all curves parameterized by the complement of a countable union of proper subvarieties of the moduli space M_g.

Remark 1.5.10. It is natural to conjecture that $t(E) = \sqrt{g}$ for very general E of sufficiently large genus. When g is a perfect square, this follows from the corollary. Ciliberto and Kouvidakis [92] have shown that the statement is implied by a conjecture of Nagata (Remark 5.1.14) provided that $g \geq 10$. When E is a general curve of genus 3 it follows from computations of Kouvidakis [370] and Bauer and Szemberg [41] that $t(E) = \frac{9}{5}$. □

Remark 1.5.11. If E is a very general curve of large genus, Kollár asks whether the diagonal $\Delta \subseteq E \times E$ is the only irreducible curve of negative self-intersection. □

Proof of Corollary 1.5.9. By the Riemann existence theorem, there exists a curve E_0 of genus g admitting a simple covering $\pi : E_0 \longrightarrow \mathbf{P}^1$ of degree $d = [\sqrt{g}] + 1$. Theorem 1.5.8 shows that

$$t(E_0) = \frac{g}{[\sqrt{g}\ ]}.$$

The corollary then follows by letting E_0 vary in a complete family of curves of genus g, and applying Proposition 1.4.14 to the corresponding family of products. □

For the theorem the essential observation is the following

Lemma 1.5.12. *Assume that there exists a reduced irreducible curve*

$$C_0 \subset E \times E$$

with $[C_0] = e_s = s(f_1 + f_2) - \delta'$ *for some* $s \le \sqrt{g}$*. Then* e_t *is nef if and only if* $t \ge \frac{g}{s}$*, and consequently* $t(E) = \frac{g}{s}$*.*

Proof. This could be deduced from Example 1.4.33 (ii), but it is simplest to argue directly. Suppose then that $t \ge \frac{g}{s}$. One has $\big(e_t \cdot C_0\big) = 2st - 2g \ge 0$, so it remains to show that if $C_1 \subset E \times E$ is any irreducible curve distinct from C_0, then $\big(e_t \cdot C_1\big) \ge 0$. The intersection product on $E \times E$ being non-degenerate, we can write

$$[C_1] \ = \ x_1 f_1 + x_2 f_2 - y\delta' + \alpha$$

where $\alpha \in N^1(X)_{\mathbf{R}}$ is a class orthogonal to f_1, f_2, and δ'. By intersecting with f_2 and f_1, we find that $x_1, x_2 \ge 0$. Moreover $\big(C_1 \cdot C_0\big) \ge 0$ since C_0 and C_1 meet properly, which yields

$$s(x_1 + x_2) - 2gy \ge 0. \tag{*}$$

But $\big(C_1 \cdot e_t\big) = t(x_1 + x_2) - 2gy$, and the two inequalities in the hypothesis of the Lemma imply that $t \ge s$. Therefore (*) shows that $\big(C_1 \cdot e_t\big) \ge 0$, as required. □

Proof of Theorem 1.5.8. Thanks to the lemma, it suffices to produce an irreducible reduced curve $C = C_f \subset E \times E$ having class $(d-1)(f_1 + f_2) - \delta'$, and these exist very naturally. Specifically, consider in $E \times E$ the fibre product

$$E \times E \ \supset \ E \times_{\mathbf{P}^1} E \ = \ \big\{(x, y) \mid \pi(x) = \pi(y)\big\}.$$

This contains the diagonal Δ_E as a component, and we take C to be the residual divisor: set theoretically, C is the closure of the set of all pairs $(x, y) \in E \times E$ with $x \ne y$ such that $\pi(x) = \pi(y)$. Thus

$$\begin{aligned} [C] \ &= \ d(f_1 + f_2) - \delta \\ &= \ (d-1)(f_1 + f_2) - \delta'. \end{aligned}$$

Moreover C is clearly reduced (being generically so), and it is irreducible thanks to the fact that the monodromy of f is the full symmetric group. □

Example 1.5.13. (Characterization of hyperelliptic curves). It follows from Theorem 1.5.8 that if E is hyperelliptic, then $t(E) = g$. In fact, this characterizes hyperelliptic curves: if E is non-hyperelliptic, then

$$t(E) \ \le \ g - 1.$$

This strengthens and optimizes a result of Taraffa [564], who showed that

$$t(E) \ \le \ 2\sqrt{g^2-g} \ \approx \ 2g$$

for any curve E of genus g. (The main point is to check that any non-hyperelliptic curve E carries a simple covering $\pi : E \longrightarrow \mathbf{P}^1$ of degree g (cf. [54, Lemma 1.4]). Then, as in the proof of (1.5.8), let $C = C_\pi$ be the curve residual to Δ_E in $E \times_{\mathbf{P}^1} E \subset E \times E$. The class of C is given by $[C] = (g-1)(f_1+f_2)-\delta'$, and C is irreducible since f is simple. But $\big(C^2\big) \ge 0$, and consequently C is nef.) $\square$

Example 1.5.14. (Symmetric products). Following [370] let $Y = S^2E$ be the second symmetric product of E, so that there is a natural double cover $p : X = E \times E \longrightarrow S^2E = Y$. This gives rise to an inclusion $p^* : N^1(Y)_{\mathbf{R}} \longrightarrow N^1(X)_{\mathbf{R}}$ realizing the Néron–Severi space of Y as a subspace in that of X. The classes

$$f \ = \ f_1 + f_2 \quad \text{and} \quad \delta'$$

lie in $N^1(Y)_{\mathbf{R}}$, and when E has general moduli they span it. The invariant $t(E)$ determines one of the rays bounding the intersection

$$\mathrm{Nef}(Y) \cap \big(\mathbf{R}\cdot\delta' + \mathbf{R}\cdot f\big)$$

of $\mathrm{Nef}(Y)$ with the subspace spanned by δ' and f. The other bounding ray is generated by $gf + \delta'$. In other words, $(sf+\delta') \in N^1(Y)_{\mathbf{R}}$ is nef iff $s \ge g$. (The class $gf+\delta'$ in question is the pullback of the theta divisor Θ_E of the Jacobian of E under the Abel–Jacobi map $u : S^2E \longrightarrow \mathrm{Jac}(E)$. Hence $gf+\delta'$ is nef, and since f is ample so too is $(sf+\delta')$ when $s \ge g$. On the other hand, $\delta = [\Delta]$ is effective and

$$\begin{aligned}\big((sf+\delta')\cdot(\delta)\big) \ &= \ \big((sf+\delta')\cdot(\delta'+f)\big)\\ &= \ 2s-2g.\end{aligned}$$

Consequently $(sf+\delta')$ is not nef when $s < g$.) $\square$

Remark 1.5.15. (Higher-dimensional symmetric product). Let E be a general curve of even genus $g = 2k$. Using deep results of Voisin from [598], Pacienza [490] works out the nef cone of the symmetric product $Y = S^kE$. Since $\rho(Y) = 2$, the cone in question is determined as above by two slopes: in the case at hand, Pacienza shows that they are rational. $\square$

Example 1.5.16. (Vojta's divisor). Fix real numbers $r, s > 0$ and let

$$a_1 \ = \ a_1(r) \ = \ \sqrt{(g+s)r} \ \ , \ \ a_2 \ = \ a_2(r) \ = \ \sqrt{\frac{(g+s)}{r}}.$$

Put $v_r = a_1f_1 + a_2f_2 + \delta'$, so that $\big(v_r \cdot v_r\big) = 2s$. If

$$r > \frac{(g+s)(g-1)}{s},$$

then v_r is nef. (See [602, Proposition 1.5].) This divisor — or more precisely the height function it determines — plays an important role in Vojta's proof of the Mordell conjecture (i.e. Faltings' theorem). See [382] for a nice overview. □

1.5.C Abelian Varieties

In this subsection, following [78], we describe the ample and nef cones of an abelian variety of arbitrary dimension.

Let X be an abelian variety of dimension n, and let H be a fixed ample divisor on X. The essential point is the following, which generalizes Lemma 1.5.4.

Proposition 1.5.17. *An arbitrary* $\mathbf{R}$*-divisor* D *on* X *is ample if and only if*

$$\left(D^k \cdot H^{n-k}\right) > 0 \tag{1.22}$$

for all $0 \le k \le n$*, and* D *is nef if and only if* $\left(D^k \cdot H^{n-k}\right) \ge 0$ *for all* k*.*

Sketch of Proof. If D is ample or nef, then the stated inequalities follow from Examples 1.2.5 and 1.4.16. Conversely, we show that if D is an integral divisor satisfying (1.22), then D is ample: since there are only finitely many inequalities involved, the corresponding statement for $\mathbf{R}$-divisors follows. To this end, consider the polynomial $P(t) = P_D(t) \in \mathbf{Z}[t]$ defined in the usual way by the expression

$$P_D(t) = \Big(\big(D + t \cdot H\big)^n \Big).$$

The given inequalities imply that $P(t) > 0$ for every real number $t \ge 0$. So it is enough to show that if D is *not* ample, then $P_D(t)$ has a non-negative real root $t_0 \ge 0$. But this follows from the theory of the index of a line bundle on an abelian variety ([447, pp. 154–155]). In brief, write $X = V/\Lambda$ as the quotient of an n-dimensional complex vector space modulo a lattice, and let h_D, h_H be the Hermitian forms on V determined by D and H. To say that D fails to be ample means that h_D is not positive definite. On the other hand, $h_D + t \cdot h_H$ is positive definite for $t \gg 0$. Hence there exists $t_0 \ge 0$ such that $h_D + t_0 \cdot h_H$ is indefinite. But then $P_D(t_0) = 0$. (See [383, p. 79] for a more detailed account.) □

Corollary 1.5.18. *If* $\delta \in N^1(X)_{\mathbf{R}}$ *is a nef class that is not ample, then* $\left(\delta^n\right) = 0$*.*

Proof. If δ is nef but not ample, then the proposition implies that

$$\left(\delta^k \cdot h^{n-k}\right) = 0 \text{ for some } k \in [1, n],$$

h being a fixed ample class. But the Hodge-type inequalities in Corollary 1.6.3 show that

$$\Big(\delta^k \cdot h^{n-k}\Big)^n \;\geq\; \Big(\delta^n\Big)^k \cdot \Big(h^n\Big)^{n-k}.$$

Since both terms on the right are non-negative, the assertion follows. □

Remark 1.5.19. (Examples of hypersurfaces of large degree bounding the nef cone). Corollary 1.5.18 gives rise to explicit examples in which the nef cone is locally bounded by polynomial hypersurfaces of large degree: see [78, Example 2.6]. General results of Campana–Peternell about this nef boundary appear in Section 1.5.E. □

Remark 1.5.20. Bauer [37] shows that the nef cone $\mathrm{Nef}(X)$ of an abelian variety is rational polyhedral if and only if X is isogeneous to a product of abelian varieties of mutually distinct isogeny types, each having Picard number one. □

1.5.D Other Varieties

We survey here a few other examples of varieties whose ample cones have been studied in the literature. Our synopses are very brief: relevant definitions and details can be found in the cited references.

Blow-ups of $\mathbf{P}^2$. Let X be the blowing up of the projective plane at ten or more very general points. Denote by $e_i \in N^1(X)$ the classes of the exceptional divisors, and let ℓ be the pullback to X of the hyperplane class on $\mathbf{P}^2$. We may fix $0 < \varepsilon \ll 1$ such that $h =_{\mathrm{def}} \ell - \varepsilon \cdot \sum e_i$ is an ample class.

It is known that one can find (-1)-curves of arbitrarily high degree on X (see [280, Exercise V.4.15]). In other words, there exists a sequence $C_i \subseteq X$ of smooth rational curves with

$$\big(C_i \cdot C_i\big) = -1 \;\text{ and }\; \big(C_i \cdot h\big) \to \infty \;\text{ with }\; i.$$

By 1.4.32, each $[C_i]$ generates an extremal ray in $\overline{\mathrm{NE}}(X)$. On the other hand, let K_X denote as usual a canonical divisor on X. Then $\big(C_i \cdot K_X\big) = -1$ does not grow with i. This means that the rays $\mathbf{R}_+ \cdot [C_i]$ generated by the C_i cluster in $N_1(X)_{\mathbf{R}}$ towards the plane $K_X^{\perp}$ defined by the vanishing of K_X. The situation — which is an illustrative instance of Mori's cone theorem (Theorem 1.5.33) — is illustrated schematically in Figure 1.7. It is conjectured that $\overline{\mathrm{NE}}(X)$ is circular on the region $(K_X)_{>0}$ — i.e. that $\overline{\mathrm{NE}}(X) \cap (K_X)_{>0}$ consists of classes of non-negative self-intersection — but this is not known. (The cone of curves of X is governed by a conjecture of Hirschowitz [289]. See [426] for an account of some recent work on this question using classical methods, and Remarks 5.1.14 and 5.1.23 for a related conjecture of Nagata and an analogue in symplectic geometry.)

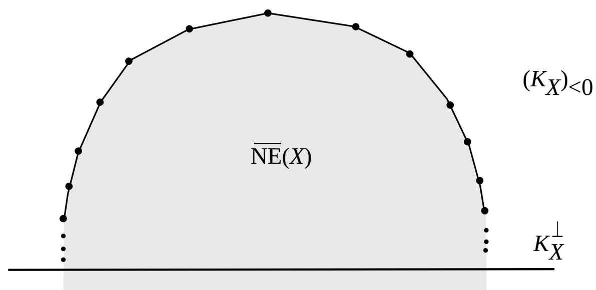


Figure 1.7. Cone of curves on blow-up of $\mathbf{P}^2$.

K3 surfaces. Kovács [371] has obtained very precise information about the cone of curves of a $K3$ surface X. He shows for example that if $\rho(X) \geq 3$, then either X does not contain any curves of negative self-intersection, or else $\overline{\mathrm{NE}}(X)$ is spanned by the classes of smooth rational curves on X. He deduces as a corollary that either $\overline{\mathrm{NE}}(X)$ is circular, or else has no circular part at all.

Holomorphic symplectic fourfolds. Hassett and Tschinkel [283] have studied the ample cone on holomorphic symplectic varieties of dimension four. They formulate a conjecture giving a Hodge-theoretic description of this cone (at least in important cases), and they present some evidence for this conjecture. Huybrechts [298] [299] establishes some related results on the Kähler cone of hyper-Kähler manifolds. Divisors on hyper-Kähler manifolds are also studied by Boucksom in [67].

$\overline{\mathbf{M}}_{\mathbf{g},\mathbf{n}}$. A very basic question, going back to Mumford, is to describe the ample cone of the Deligne–Mumford compactification $\overline{M}_{g,n}$ of the moduli space parameterizing n-pointed curves of genus g. The case of $g = 0$ is already very interesting and subtle: the variety $\overline{M}_{0,n}$ has a rich combinatorial structure, and has been the focus of considerable attention (cf. [328], [310]). Fulton conjectured that the closed cone of curves $\overline{\mathrm{NE}}(\overline{M}_{0,n})$ is generated by certain natural one-dimensional boundary strata in $\overline{M}_{0,n}$. This has been verified for $n \leq 6$ by Farkas and Gibney [183] following earlier work of Farber and Keel. Gibney, Keel, and Morrison [222] show that the truth of Fulton's conjecture for all n would in fact lead to a description of the ample cone of $\overline{M}_{g,n}$ in all genera. Fulton conjectured analogously that the closed cone $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ of pseudoeffective divisors (Definition 2.2.25) should be generated by boundary divisors, but counterexamples were given by Keel and by Vermeire [587]. This pseudoeffective cone has been studied for small n by Hassett and Tschinkel [284]. $\square$

1.5.E Local Structure of the Nef Cone

We present here an interesting theorem of Campana and Peternell [78] describing the local structure of the nef cone at a general point. It depends on their result — which we prove later as Theorem 2.3.18 — that the Nakai–Moishezon inequalities characterize the amplitude of real divisor classes.

We start by introducing some notation and terminology intended to streamline the discussion. Let X be an irreducible projective variety or scheme of dimension n and $V \subseteq X$ a subvariety. Then intersection with V determines a real homogeneous polynomial on $N^1(X)_{\mathbf{R}}$:

$$\varphi_V : N^1(X)_{\mathbf{R}} \longrightarrow \mathbf{R} \quad , \quad \varphi_V(\xi) \ = \ \int_V \xi^{\dim V}.$$

Thus $\deg \varphi_V = \dim V$, and φ_V is given by an integer polynomial with respect to the natural integral structure on $N^1(X)_{\mathbf{R}}$. By Kleiman's theorem (Theorem 1.4.9), all the φ_V are non-negative on $\mathrm{Nef}(X)$.

Definition 1.5.21. (Null cone). The *null cone* $\mathcal{N}_V \subseteq N^1(X)_{\mathbf{R}}$ determined by V is the zero-locus of φ_V:

$$\mathcal{N}_V \ = \ \big\{ \xi \in N^1(X)_{\mathbf{R}} \mid \varphi_V(\xi) = 0 \big\}. \quad \square$$

Note that φ_V and $\mathcal{N}_V$ depend only on the homology class of V. Consequently as V varies over all subvarieties of X, only countably many distinct functions and cones occur.

Example 1.5.22. When X is a smooth surface, $\mathcal{N}_X$ is the familiar quadratic cone of classes of self-intersection zero. □

Example 1.5.23. If X is the blow-up of $\mathbf{P}^n$ at k points, then in suitable coordinates $\mathcal{N}_X$ is the Fermat-type hypersurface defined by the equation

$$x^n \ = \ y_1^n + \ldots + y_k^n. \quad \square$$

Example 1.5.24. (Singularities of the null cone). Let $f : X \longrightarrow Y$ be a surjective morphism of irreducible projective varieties or schemes with $\dim X = n$ and $\dim Y = d$. Then φ_X vanishes with multiplicity $n - d$ along the image of $f^* : N^1(Y)_{\mathbf{R}} \longrightarrow N^1(X)_{\mathbf{R}}$. (Given $\xi = f^*\eta$, consider $\varphi_X(\xi + h)$ for any ample $h \in \mathbf{N}^1(X)_{\mathbf{R}}$.) This result is due to Wiśniewski [610]: see that paper or [363, Exercise III.1.9] for some applications. □

Remark 1.5.25. (Calabi–Yau threefolds). Wilson [608], [609] uses a deep and careful analysis of the null cone $\mathcal{N}_X$ — especially its arithmetic properties — to prove some interesting results about the geometry of a Calabi–Yau threefold X. For example, he proves that if $\rho(X) > 19$, then X is a resolution of singularities of a "Calabi–Yau model" (i.e. a Calabi–Yau threefold that may have mild singularities) of smaller Picard number. Wilson also shows that the ample cones of Calabi–Yau threefolds are invariant under deformations if and only if none of the manifolds in question contains a smooth elliptic ruled surface. □

Definition 1.5.26. (Nef boundary). The *nef boundary* $\mathcal{B}_X \subseteq N^1(X)_{\mathbf{R}}$ of X is the boundary of the nef cone:

$$\mathcal{B}_X \ = \ \partial\,\mathrm{Nef}(X).$$

It is topologized as a subset of the Euclidean space $N^1(X)_{\mathbf{R}}$. □

Example 1.5.27. We may restate Corollary 1.5.18 as asserting that if X is an abelian variety then $\mathcal{B}_X \ \subseteq \ \mathcal{N}_X$, i.e. its nef boundary lies on the null cone of X. □

The results of Campana and Peternell are summarized in the next two statements:

Theorem 1.5.28. *Given any point $\xi \in \mathcal{B}_X$ on the nef boundary, there is a subvariety $V \subseteq X$ such that $\varphi_V(\xi) \ = \ 0$.*

In other words, any point in the nef boundary actually lies on one of the null cones $\mathcal{N}_V \subseteq N^1(X)_{\mathbf{R}}$. The possibility that $V = X$ is of course not excluded.

Because there are only countably many such cones, at most points the nef boundary $\mathcal{B}_X$ must then look locally like one of them:

Theorem 1.5.29. *There is an open dense set $\mathrm{CP}(X) \ \subseteq \ \mathcal{B}_X$ with the property that for every point $\xi \in \mathrm{CP}(X)$, there is an open neighborhood $U = U(\xi)$ of ξ in $N^1(X)_{\mathbf{R}}$, together with a subvariety $V \subseteq X$ of X (depending on ξ), such that*

$$\mathcal{B}_X \cap U(\xi) \ = \ \mathcal{N}_V \cap U(\xi).$$

In other words, $\mathcal{B}_X$ is cut out in $U(\xi)$ by the polynomial φ_V.

The conclusion of the theorem is illustrated schematically in Figure 1.8, which shows (with the convention of Remark 1.4.22) null cones bounding Nef(X). Note however that in general there may be infinitely many cones $\mathcal{N}_V$ required to fill out the nef boundary in Theorem 1.5.28, leading to clustering phenomena not shown in the picture.

Remark 1.5.30. The proof of 1.5.29 will show that we can take

$$d\varphi_V(\xi) \neq 0 \ \text{ for } \ \xi \in \mathrm{CP}(X).$$

It follows that $\mathcal{N}_V$ is smooth at ξ, and moreover that we can choose $U(\xi)$ so that $\varphi_V < 0$ on $U(\xi) - \mathrm{Nef}(X)$. In other words, $U(\xi) \cap \mathrm{Nef}(X)$ is defined on $U(\xi)$ by the inequality $\{\varphi_V \geq 0\}$. □

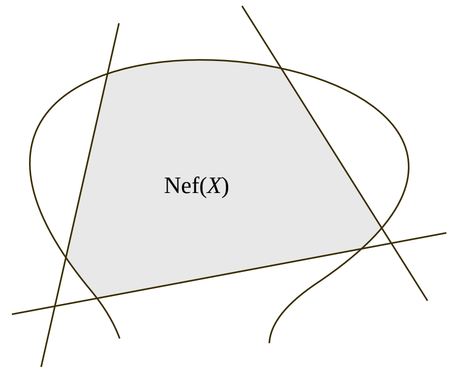


Figure 1.8. Null cones bounding Nef(X)

Example 1.5.31. If $\xi \in \mathrm{CP}(X)$, and if $\mathcal{N}_V$ is the corresponding null cone as in 1.5.30, then the dual hypersurface

$$\mathcal{N}_V{}^* \subseteq N^1(X)^*_{\mathbf{R}} = N_1(X)_{\mathbf{R}}$$

of hyperplanes tangent to $\mathcal{N}_V$ bounds $\overline{\mathrm{NE}}(X)$ in a neighborhood of the tangent plane to $\mathcal{N}_V$ at ξ. □

Remark 1.5.32. (Numerical characterization of the Kähler cone). Demailly and Paun [132] have recently established the striking result that an analogue of Theorem 1.5.28 remains true on any compact Kähler manifold. Given such a manifold X one considers the cone Kahler(X) of all Kähler classes in $H^{1,1}\big(X, \mathbf{R}\big)$ (Definition 1.2.39). The main result of [132] is that Kahler(X) is a connected component of the set of all classes $\alpha \in H^{1,1}\big(X, \mathbf{R}\big)$ such that

$$\int_V \alpha^{\dim V} > 0 \qquad (*)$$

for every irreducible analytic subvariety $V \subseteq X$ of positive dimension. It follows for instance that if X contains no proper subvarieties, then the Kähler cone of X is a connected component of the set of classes with positive self-intersection. When X is projective, Demailly and Paun prove moreover that Kahler(X) actually coincides with the set of classes satisfying (*). These results of course imply 1.5.28 (at least on smooth varieties), but in fact they are stronger since in general $N^1(X)_{\mathbf{R}}$ might span a proper subspace of $H^{1,1}\big(X, \mathbf{R}\big)$. □

Proof of Theorem 1.5.28. This is a restatement of the Nakai criterion for **R**-divisors (Theorem 2.3.18). In fact, if $\xi \in \mathcal{B}_X$ then certainly $\varphi_V(\xi) \geq 0$ for all $V \subseteq X$. But if $\varphi_V(\xi) > 0$ for every V then the result in question implies that $\xi \in \mathrm{Amp}(X)$. □

Proof of Theorem 1.5.29. Given a subvariety $V \subseteq X$, let

$$O_V \;=\; \Big\{ \xi_1 \in \mathcal{B}_X \;\Big|\; \text{some neighborhood of } \xi_1 \text{ in } \mathcal{B}_X \text{ lies in } \mathcal{N}_V \Big\},$$

and put

$$O \;=\; \bigcup_{V \subseteq X} O_V.$$

Clearly O is open, and we claim that it is dense in $\mathcal{B}_X$. For in the contrary case we would find a point $z \in \mathcal{B}_X$ having a compact neighborhood K in $\mathcal{B}_X$ such that $\big(K - \mathcal{N}_V\big)$ is dense in K for every V. Since there are only countably many distinct $\mathcal{N}_V$ as V varies over all subvarieties of X, Baire's theorem implies that

$$\bigcap_V \big(K - \mathcal{N}_V\big) \;=\; K - \bigcup_V \mathcal{N}_V$$

is dense, hence non-empty. But by Theorem 1.5.28, $K \subseteq \bigcup_V \mathcal{N}_V$, a contradiction.

Now let $P \subseteq \mathcal{B}_X$ denote the set of all points satisfying the conclusion of the theorem. Again P is open by construction, so the issue is to show that it is dense. To this end, fix any point $\xi_1 \in O$ and a subvariety $V \subseteq X$ of minimal dimension such that $\xi_1 \in O_V$. Choose also a very ample divisor H on X meeting V properly, and let $h \in N^1(X)_{\mathbf{R}}$ denote its numerical equivalence class. Then $\xi_1 \notin O_{(V \cap H)}$ thanks to the minimality of V. Therefore ξ_1 is a limit of points $\xi \in O_V$ such that

$$\int_V \big(\xi^{\dim V - 1} \cdot h\big) \;>\; 0. \tag{*}$$

We will show that if $\xi \in O_V$ is any point satisfying (*), then $\xi \in P$. This implies the required density of P, and will complete the proof.

So fix such a point ξ. We claim first that $d\varphi_V(\xi) \neq 0$, and hence that $\mathcal{N}_V$ is non-singular near ξ. In fact,

$$\big(\dim V\big) \cdot \int_V \big(\xi^{\dim V - 1} \cdot h\big) \;=\; \lim_{t \to 0} \frac{1}{t} \cdot \int_V \big(\xi + t \cdot h\big)^{\dim V}$$

is the directional derivative of φ_V at ξ in the direction h, which is non-vanishing by (*). We now argue that if $U(\xi)$ is a small convex neighborhood of ξ in $N^1(X)_{\mathbf{R}}$, then

$$\mathcal{B}_X \cap U(\xi) \;=\; \mathcal{N}_V \cap U(\xi). \tag{**}$$

This will show that $\xi \in P$, as required. For (**), the point roughly speaking is that $\mathcal{B}_X \cap U(\xi)$ — being a piece of the boundary of a closed convex set with non-empty interior — is a topological manifold, and since $\mathcal{B}_X \cap U(\xi) \subseteq \mathcal{N}_V \cap U(\xi)$ for sufficiently small $U(\xi)$, the sets in question must coincide. In more detail, let $L \subseteq N^1(X)_{\mathbf{R}}$ be the embedded affine tangent space to $\mathcal{N}_V$ at ξ, and let

$$\pi \,:\, N^1(X)_{\mathbf{R}} \longrightarrow L$$

be an affine linear projection. Thus π restricts to an isomorphism $\mathcal{N}_V \longrightarrow L$ in a neighborhood of ξ. On the other hand, since $\mathcal{B}_X$ is the boundary of a closed convex set, the image under π of a convex neighborhood of ξ in $\mathcal{B}_X$ is a convex neighborhood of $\pi(\xi)$ in L. But $\mathcal{B}_X \cap U(\xi) \subseteq \mathcal{N}_V$ by construction, so this implies that $\mathcal{B}_X \cap U(\xi)$ contains an open neighborhood of ξ in $\mathcal{N}_V$. After possibly shrinking $U(\xi)$, (**) follows. □

1.5.F The Cone Theorem

Let X be a smooth complex projective variety, and K_X a canonical divisor on X. In his seminal paper [438], Mori proved that the closed cone of curves $\overline{\mathrm{NE}}(X)$ has a surprisingly simple structure on the subset of $N_1(X)_{\mathbf{R}}$ of classes having negative intersection with K_X. He showed moreover that this has important structural implications for X. We briefly summarize here some of these results, and state some simple consequences, but we don't give proofs. Chapter 1 of the book [368] contains an excellent introduction to this circle of ideas, and full proofs appear in Chapters 1 and 3 of that book. See also [114] for an account aimed at novices, and [419] for a very detailed exposition in the spirit of [368].

As above, let X be a smooth projective variety. Given any divisor D on X write

$$\overline{\mathrm{NE}}(X)_{D \geq 0} \;=\; \overline{\mathrm{NE}}(X) \,\cap\, D_{\geq 0}$$

for the subset of $\overline{\mathrm{NE}}(X)$ lying in the non-negative half-space determined by D. Mori [438] first of all proved

Theorem 1.5.33. (Cone Theorem). *Assume that* $\dim X = n$ *and that* K_X *fails to be nef.*

(i). *There are countably many rational curves* $C_i \subseteq X$*, with*

$$0 \;\leq\; -\big(C_i \cdot K_X\big) \;\leq\; n+1,$$

that together with $\overline{\mathrm{NE}}(X)_{K_X \geq 0}$ *generate* $\overline{\mathrm{NE}}(X)$*, i.e.*

$$\overline{\mathrm{NE}}(X) \;=\; \overline{\mathrm{NE}}(X)_{K_X \geq 0} \;+\; \sum_i \mathbf{R}_+ \cdot [C_i].$$

(ii). *Fix an ample divisor* H*. Then given any* $\varepsilon > 0$*, there are only finitely many of these curves — say* $C_1, \ldots, C_t$ *— whose classes lie in the region* $(K_X + \varepsilon \cdot H)_{\leq 0}$*. Therefore*

$$\overline{\mathrm{NE}}(X) \;=\; \overline{\mathrm{NE}}(X)_{(K_X + \varepsilon H) \geq 0} \;+\; \sum_{i=1}^{t} \mathbf{R}_+ \cdot [C_i].$$

The theorem was illustrated in the example of $\mathbf{P}^2$ blown up at ten or more points: see Figure 1.7.

Example 1.5.34. (Fano varieties). Let X be a Fano variety, i.e. a smooth projective variety such that $-K_X$ is ample. Then $\overline{\mathrm{NE}}(X) \subseteq N_1(X)_{\mathbf{R}}$ is a finite rational polytope, spanned by the classes of rational curves. (Apply statement (ii) of the cone theorem.) □

Example 1.5.35. (Adjoint bundles). Let X be a smooth projective variety of dimension n, and H any ample integer divisor on X. Then $K_X + (n+1)H$ is nef, and $K_X + (n+2)H$ is ample. More generally, if D is any ample divisor such that

$$\big(D \cdot C\big) \geq n+1 \quad \text{(respectively } \big(D \cdot C\big) \geq n+2\text{)}$$

for every irreducible curve $C \subseteq X$, then $K_X + D$ is nef (respectively $K_X + D$ is ample). (This follows from the inequalities on the intersection numbers $\big(C_i \cdot K_X\big)$ appearing in statement (i) and the fact that the intersection numbers $\big(C_i \cdot H\big)$ are positive integers.) See Example 1.8.23 and Section 10.4.A for some remarks on the interest in statements of this sort. □

Mori proved Theorem 1.5.33 via his "bend and break" method. This in turn has found a multitude of other applications, for example to the circle of ideas involving rational connectedness. We refer to Kollár's book [363] for an extensive survey of some of these developments. An alternative cohomological approach to the cone theorem — which works also on mildly singular varieties, but does not recognize the curves C_i as rational — was developed by Kawamata and Shokurov following ideas of Reid ([317], [534], [517]). We again refer to [368, Chapter 3] for details.

The rational curves C_i appearing in the cone theorem generate extremal rays of $\overline{\mathrm{NE}}(X)$ in the sense of Definition 1.4.32. Mori showed that if $\dim X = 3$, and if $\mathbf{r}$ is an extremal ray in $\overline{\mathrm{NE}}(X)_{(K+\varepsilon H)\leq 0}$, then there is a mapping

$$\mathrm{cont}_{\mathbf{r}} : X \longrightarrow \overline{X}$$

that contracts every curve whose class lies in $\mathbf{r}$. This was extended to smooth projective varieties of all dimensions, as well as to varieties with mild singularities, by Kawamata and Shokurov, again following ideas of Reid. This contraction theorem is of great importance, because it opens the door to the possibility of constructing minimal models for varieties of arbitrary dimension. Specifically, given a projective variety X whose canonical bundle is not nef, these results guarantee that X carries an extremal curve which contracts under the corresponding morphism $\mathrm{cont}_{\mathbf{r}} : X \longrightarrow \overline{X}$. Then one would like to repeat the process starting on $\overline{X}$. Unfortunately, this naive idea runs into problems coming from the singularities of $\overline{X}$. However this *minimal model program* has led to many important developments in recent years. Besides the book [368] of Kollár and Mori, the reader can consult [326], [114], or [419] for an account of some of this work.

1.6 Inequalities

In recent years, generalizations and analogues of the classical Hodge index inequality have arisen in several contexts, starting with work of Khovanskii and Teissier (cf. [334], [566], [567], [420], [411], [126]). This section is devoted to a presentation of some of this material. We start with some inequalities of Hodge type among the intersection numbers of nef divisors on a complete variety, and then discuss briefly some related results of Teissier for the mixed multiplicities of $\mathfrak{m}$-primary ideals.

1.6.A Global Results

The basic global result is a generalization of the Hodge index theorem on surfaces. The statement was known (at least by experts) to follow from inequalities of Khovanskii, Matsusaka, and Teissier (Example 1.6.4); it achieved wide circulation in Demailly's paper [124]. The direct argument we give is due to Beltrametti and Sommese [50, Chapter 2.5], and independently to Fulton and Ein (see [205, p. 120]).

Theorem 1.6.1. (Generalized inequality of Hodge type). *Let X be an irreducible complete variety* (*or scheme*) *of dimension n, and let*

$$\delta_1, \ldots, \delta_n \ \in \ N^1(X)_{\mathbf{R}}$$

be nef classes. Then

$$\Big(\delta_1 \cdot \ldots \cdot \delta_n\Big)^n \ \geq \ \Big((\delta_1)^n\Big) \cdot \ldots \cdot \Big((\delta_n)^n\Big). \tag{1.23}$$

Proof. We can assume first of all that X is reduced since in general its cycle satisfies $[X] = a \cdot [X_{\mathrm{red}}]$ for some $a > 0$. By Chow's lemma (Remark 1.4.3), we can suppose also that X is projective. In this case, it suffices to prove the theorem under the additional assumption that the δ_i are ample. In fact, if the stated inequality holds for $\delta_i \in \mathrm{Amp}(X)$, then by continuity it holds also for $\delta_i \in \mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ (Theorem 1.4.23).

We now argue by induction on $n = \dim X$. If X is a smooth projective surface, then the stated inequality is a version of the Hodge index theorem (cf. [280], Exercise V.1.9). As in the previous paragraph, once one knows (1.23) for ample classes, it follows also for nef ones. This being said, if X is singular, one deduces (1.23) by passing to a resolution. We assume henceforth that $n \geq 3$, and that (1.23) is already known on all irreducible varieties of dimension $n-1$.

We next claim that given any ample classes

$$\beta_1, \ldots, \beta_{n-1}, h \ \in \ N^1(X)_{\mathbf{R}},$$

one has the inequality

$$\Big(\beta_1 \cdot \ldots \cdot \beta_{n-1} \cdot h\Big)^{n-1} \ \geq\ \Big((\beta_1)^{n-1} \cdot h\Big) \cdot \ldots \cdot \Big((\beta_{n-1})^{n-1} \cdot h\Big). \tag{1.24}$$

In fact, it suffices by continuity to prove this when h and the β_i are rational ample classes. So we may assume that they are represented by very ample divisors $B_1, \ldots, B_{n-1}$ and H on X, and the issue is to verify the inequality

$$\Big(B_1 \cdot \ldots \cdot B_{n-1} \cdot H\Big)^{n-1} \ \geq\ \Big((B_1)^{n-1} \cdot H\Big) \cdot \ldots \cdot \Big((B_{n-1})^{n-1} \cdot H\Big).$$

We suppose that H is an irreducible projective scheme of dimension $n-1$, and that each B_i meets H properly. Denoting by $\overline{B}_i$ the restriction of B_i to H, the inequality in question is equivalent term by term to the relation

$$\Big(\overline{B}_1 \cdot \ldots \cdot \overline{B}_{n-1}\Big)^{n-1} \ \geq\ \Big((\overline{B}_1)^{n-1}\Big) \cdot \ldots \cdot \Big((\overline{B}_{n-1})^{n-1}\Big)$$

of intersection numbers on H. But this follows by applying the induction hypothesis to H.

We now show that the desired inequality (1.23) follows formally from (1.24). So let

$$\delta_1, \ldots, \delta_n \in N^1(X)_{\mathbf{R}}$$

be n ample classes on X.[20] Fix some index $j \in [1, n]$ and apply (1.24) with $h = \delta_j$ and $\beta_1, \ldots, \beta_{n-1}$ the remaining δ_i. One finds

$$\Big(\delta_1 \cdot \ldots \cdot \delta_n\Big)^{n-1} \ \geq\ \prod_{i \neq j} \Big(\delta_i^{n-1} \cdot \delta_j\Big).$$

Taking the product over j yields

$$\Big(\delta_1 \cdot \ldots \cdot \delta_n\Big)^{n(n-1)} \ \geq\ \prod_j \prod_{i \neq j} \Big(\delta_i^{n-1} \cdot \delta_j\Big). \tag{1.25}$$

But now apply (1.24) with $h = \beta_1 = \ldots = \beta_{n-2} = \delta_i$ and $\beta_{n-1} = \delta_j$ to obtain

$$\Big(\delta_i^{n-1} \cdot \delta_j\Big)^{n-1} \ \geq\ \Big(\delta_i^n\Big)^{n-2} \Big(\delta_i \cdot \delta_j^{n-1}\Big).$$

Therefore

$$\begin{aligned}\prod_j \prod_{i \neq j} \Big(\delta_i^{n-1} \cdot \delta_j\Big)^{n-1} \ &\geq\ \prod_j \prod_{i \neq j} \Big(\delta_i^n\Big)^{n-2} \cdot \Big(\delta_i \cdot \delta_j^{n-1}\Big) \\ &= \left(\prod_i \Big(\delta_i^n\Big)^{(n-1)(n-2)}\right) \left(\prod_j \prod_{i \neq j} \Big(\delta_j \cdot \delta_i^{n-1}\Big)\right).\end{aligned}$$

[20] The amplitude of the δ_i guarantees that all of the intersection numbers appearing in the computations that follow are positive. Therefore we can cancel and take roots without further thought.

The second term on the right cancels against the left-hand side, and taking $(n-2)^{\mathrm{nd}}$ roots one arrives at

$$\prod_{j}\prod_{i\neq j}\Big(\delta_i^{n-1}\cdot\delta_j\Big) \;\geq\; \prod_i \Big(\delta_i^n\Big)^{(n-1)}.$$

The inequality (1.23) follows by plugging this into (1.25) and taking $(n-1)^{\mathrm{st}}$ roots. □

We record some variants and special cases. To begin with, one has the following:

Variant 1.6.2. Let X be an irreducible complete variety or scheme of dimension n, and fix an integer $0 \leq p \leq n$. Let

$$\alpha_1,\ldots,\alpha_p,\ \beta_1,\ldots,\beta_{n-p} \;\in\; N_1(X)_{\mathbf{R}}$$

be nef classes. Then

$$\Big(\alpha_1\cdot\ldots\cdot\alpha_p\cdot\beta_1\cdot\ldots\cdot\beta_{n-p}\Big)^p \\ \geq\; \Big(\alpha_1^p\cdot\beta_1\cdot\ldots\cdot\beta_{n-p}\Big)\cdot\ldots\cdot\Big(\alpha_p^p\cdot\beta_1\cdot\ldots\cdot\beta_{n-p}\Big). \quad (1.26)$$

Indication of Proof. Following the argument leading to the inequality (1.24) in the proof of the theorem, one applies 1.6.1 to the complete intersection $B_1\cap\ldots\cap B_{n-p}$ of suitable very ample divisors on X. □

The case of two classes is particularly useful:

Corollary 1.6.3. (Inequalities for two classes). *Let X be an irreducible complete variety or scheme of dimension n, and let $\alpha,\ \beta\in N_1(X)_{\mathbf{R}}$ be nef classes on X. Then the following inequalities are satisfied:*

(i). *For any integers* $0\leq q\leq p\leq n$,

$$\Big(\alpha^q\cdot\beta^{n-q}\Big)^p \;\geq\; \Big(\alpha^p\cdot\beta^{n-p}\Big)^q\cdot\Big(\beta^n\Big)^{p-q}. \quad (1.27)$$

(ii). *For any* $0\leq i\leq n$,

$$\Big(\alpha^i\cdot\beta^{n-i}\Big)^n \;\geq\; \Big(\alpha^n\Big)^i\cdot\Big(\beta^n\Big)^{n-i}. \quad (1.28)$$

(iii).

$$\Big(\big(\alpha+\beta\big)^n\Big)^{1/n} \;\geq\; \Big(\big(\alpha^n\big)\Big)^{1/n} + \Big(\big(\beta^n\big)\Big)^{1/n}. \quad (1.29)$$

Proof. For (i), take $\alpha_1=\ldots=\alpha_q=\alpha$ and $\alpha_{q+1}=\ldots=\alpha_p=\beta_1=\ldots=\beta_{n-p}=\beta$ in Variant 1.6.2. Statement (ii) is the special case $q=i$ and $p=n$ of (i). For (iii) expand out $(\alpha+\beta)^n$, apply (ii), and take n^{th} roots. □

Example 1.6.4. (Inequalities of Khovanskii and Teissier). In the situation of 1.6.3, put $s_i = \big(\alpha^i \cdot \beta^{n-i}\big)$. Then for all $1 \leq i \leq n-1$,

$$s_i^2 \ \geq \ s_{i-1} \cdot s_{i+1}.$$

In other words, the function $i \mapsto \log s_i$ is concave. (Apply 1.6.2 with $p = 2$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\beta_1 = \ldots = \beta_{i-1} = \alpha$, and $\beta_i = \ldots = \beta_{n-p} = \beta$.) See [363, VI.2.15.8] for some applications due to Matsusaka. The papers [253] and [489] of Gromov and Okounkov have some interesting ideas on concavity statements of this sort. Okounkov also establishes an inequality [489, (3.6)] closely related to (1.29). □

Remark 1.6.5. (Positive characteristics). The material in this section remains valid for varieties defined over an algebraically closed field of arbitrary characteristic. □

1.6.B Mixed Multiplicities

We now sketch some local analogues originating with Teissier. Let X be a variety or scheme of pure dimension n, and let $x \in X$ be a (closed) point with maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_X$. We suppose given $k \leq n$ ideal sheaves

$$\mathfrak{a}_1, \ldots, \mathfrak{a}_k \ \subseteq \ \mathcal{O}_X \ \text{ with } \ \mathrm{Supp}(\mathcal{O}_X/\mathfrak{a}_i) = \{x\},$$

i.e. we assume that the $\mathfrak{a}_i$ are $\mathfrak{m}$-primary. Fix also non-negative integers $d_1, \ldots, d_k$ with $\sum d_i = n$. Then one can define the *mixed multiplicity*

$$e\big(\mathfrak{a}_1^{[d_1]}\,;\, \mathfrak{a}_2^{[d_2]}\,;\, \ldots\,;\, \mathfrak{a}_k^{[d_k]}\big) \in \mathbf{N}$$

of the $\mathfrak{a}_i$, for instance by the property that the lengths

$$\mathrm{length}_{\mathcal{O}_X}\big(\mathcal{O}_X/\mathfrak{a}_1^{t_1} \cdot \ldots \cdot \mathfrak{a}_k^{t_k}\big)$$

are given for $t_i \gg 0$ by a polynomial of the form

$$\sum_{d_1+\ldots+d_k=n} \frac{n!}{d_1! \cdot \ldots \cdot d_k!}\, e\big(\mathfrak{a}_1^{[d_1]}\,;\, \mathfrak{a}_2^{[d_2]}\,;\, \ldots\,;\, \mathfrak{a}_k^{[d_k]}\big) \cdot t_1^{d_1} \cdot \ldots \cdot t_k^{d_k} \\ + \Big(\text{ lower degree terms }\Big).$$

So for example $e(\mathfrak{a}^{[n]}) = e(\mathfrak{a})$ is the classic Samuel multiplicity of $\mathfrak{a}$ (viewed as an ideal in the local ring $\mathcal{O} = \mathcal{O}_{X,x}$). When X is affine, and one takes d_i "general" elements $g_{i,1}, \ldots, g_{i,d_i} \in \mathfrak{a}_i$,[21] then $e\big(\mathfrak{a}_1^{[d_1]}\,;\, \mathfrak{a}_2^{[d_2]}\,;\, \ldots\,;\, \mathfrak{a}_k^{[d_k]}\big)$ computes the intersection multiplicity at x of the corresponding divisors. More geometrically, let

[21] For instance, one can take general $\mathbf{C}$-linear combinations of a system of generators of the ideal in question: see Definition 9.2.27.

$$\mu : X' \longrightarrow X$$

be a proper birational map that dominates the blow-up of each $\mathfrak{a}_i$, so that $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-f_i)$ for some effective Cartier divisor f_i on X' contracting to x. Then

$$e\big(\mathfrak{a}_1^{[d_1]}\,;\,\mathfrak{a}_2^{[d_2]}\,;\,\ldots\,;\,\mathfrak{a}_k^{[d_k]}\big) \;=\; (-1)\cdot\Big((-f_1)^{d_1}\cdot\ldots\cdot(-f_k)^{d_k}\Big).$$

Note that by allowing repetitions, it is enough to study the multiplicities $e(\mathfrak{a}_1\,;\,\ldots\,;\,\mathfrak{a}_n)$ associated to n (possibly non-distinct) ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_n \subseteq \mathcal{O}_X$. We refer to [565], [566], [567], [514], [519], [349], and [350] for fuller accounts and further developments.

Example 1.6.6. (Samuel multiplicity of a product). Given $\mathfrak{m}$-primary ideals $\mathfrak{a},\mathfrak{b} \subseteq \mathcal{O}_X$ as above, the Samuel multiplicity $e(\mathfrak{a}\mathfrak{b})$ of their product has the expression

$$e\big(\mathfrak{a}\mathfrak{b}\big) \;=\; \sum_{i=0}^{n}\binom{n}{i}\,e\big(\mathfrak{a}^{[i]}\,;\,\mathfrak{b}^{[n-i]}\big)$$

in terms of the mixed multiplicities of $\mathfrak{a}$ and $\mathfrak{b}$. (This follows immediately from the definition.) $\square$

Teissier [565], [566] and Rees–Sharp [514] proved some inequalities among these mixed multiplicities that one can view as local analogues of the global statements appearing in the previous subsection.

Theorem 1.6.7. (Inequalities for mixed multiplicities). *Fix $x \in X$ as above.*

(i). *For any $\mathfrak{m}$-primary ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_n \subseteq \mathcal{O}_X$, one has*

$$e(\mathfrak{a}_1\,;\,\ldots\,;\,\mathfrak{a}_n)^n \;\le\; e(\mathfrak{a}_1)\cdot\ldots\cdot e(\mathfrak{a}_n).$$

(ii). *Given $\mathfrak{m}$-primary ideals $\mathfrak{a},\mathfrak{b} \subseteq \mathcal{O}_X$, and any integers $0 \le q \le p \le n$,*

$$e\big(\mathfrak{a}^{[q]}\,;\,\mathfrak{b}^{[n-q]}\big)^p \;\le\; e\big(\mathfrak{a}^{[p]}\,;\,\mathfrak{b}^{[n-p]}\big)^q\cdot e\big(\mathfrak{b}\big)^{p-q}.$$

(iii). *With $\mathfrak{a},\mathfrak{b} \subseteq \mathcal{O}_X$ as in* (ii), *and $0 \le i \le n$,*

$$e\big(\mathfrak{a}^{[i]}\,;\,\mathfrak{b}^{[n-i]}\big)^n \;\le\; e\big(\mathfrak{a}\big)^i\cdot e\big(\mathfrak{b}\big)^{n-i}.$$

(iv). *In the situation of* (iii), *set $m_i = e\big(\mathfrak{a}^{[i]}\,;\,\mathfrak{b}^{[n-i]}\big)$. Then for $1 \le i \le n-1$,*

$$m_i^2 \;\le\; m_{i-1}\cdot m_{i+1}.$$

Remark 1.6.8. (Reversal of inequalities). Note that compared to the global statements, the direction of the inequalities here is reversed. This may be explained by noting that the global results finally come down to the fact that the intersection form has signature $(1,-1)$ on a two-dimensional space of ample classes on a surface. One can view the local inequalities, on the other hand, as ultimately springing from the negativity of the intersection form on the space spanned by the exceptional curves in a birational map of surfaces. Of course, the classical Hodge index theorem is at work in both cases. □

Example 1.6.9. (An inequality of Teissier). Given $\mathfrak{m}$-primary ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ as in the theorem, the Samuel multiplicities of $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{ab}$ satisfy the inequality

$$e\big(\mathfrak{ab}\big)^{1/n} \;\le\; e\big(\mathfrak{a}\big)^{1/n} + e\big(\mathfrak{b}\big)^{1/n}. \quad □$$

Example 1.6.10. Teissier gives the following nice example in [566, Example A]. Consider polynomials $f_1, \ldots, f_n \in \mathbf{C}[z_1, \ldots, z_n]$ defining a finite map $f : \mathbf{C}^n \longrightarrow \mathbf{C}^n$, and let $g_1, \ldots, g_{n-1}$ be general elements in the ideal generated by the f_i. Assume that $f_1, \ldots, f_n$ all vanish at the origin, so that $f(0) = 0$. The g_i cut out a curve $\Gamma \subseteq \mathbf{C}^n$ which is typically singular at 0. But the multiplicity of Γ at 0 is bounded by the degree of f:

$$\operatorname{mult}_0(\Gamma) \;\le\; \big(\deg\, f\,\big)^{1-\frac{1}{n}}.$$

(Take $\mathfrak{a} = (z_1, \ldots, z_n)$, $\mathfrak{b} = (f_1, \ldots, f_n)$, and $i = 1$ in (iii).) One can replace the polynomials in question by convergent power series. □

Remark 1.6.11. It will follow from the proof that the inequalities in the theorem hold for $\mathfrak{m}$-primary ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subseteq \mathcal{O}$ in any Noetherian n-dimensional local ring $(\mathcal{O}, \mathfrak{m})$ having infinite residue field. □

Sketch of Proof of Theorem 1.6.7. We focus on (i). When $n = 2$ the assertion is that $e(\mathfrak{a};\mathfrak{b})^2 \le e(\mathfrak{a}) \cdot e(\mathfrak{b})$: this was established by Teissier [565] for normal surfaces and by Rees and Sharp [514], Theorem 2.2, for $\mathfrak{m}$-primary ideals in any local Noetherian ring $(\mathcal{O}, \mathfrak{m})$ of dimension two.[22] As in the cited papers, we proceed by induction on $n = \dim X$. Specifically, Teissier ([565], p. 306) shows[23] that if $f \in \mathfrak{a}_n$ is sufficiently general, and if one denotes by

$$Y \;=\; \operatorname{div}(f) \;\subseteq\; X$$

the hypersurface cut out by f, and by $\overline{\mathfrak{a}}_i \subseteq \mathcal{O}_Y$ the ideals determined by the $\mathfrak{a}_i$, then

$$e_X\big(\mathfrak{a}_1\,;\,\ldots\,;\,\mathfrak{a}_n\big) \;=\; e_Y\big(\overline{\mathfrak{a}}_1\,;\,\ldots\,;\,\overline{\mathfrak{a}}_{n-1}\big), \tag{*}$$

[22] In the geometric setting, the idea roughly speaking is to resolve the singularities of the blow-up of $\mathfrak{ab}$, and to use the fact that the intersection form on the exceptional fibre is negative.

[23] Teissier proves this assuming only that the residue field of $(\mathcal{O}, \mathfrak{m})$ is infinite.

where we are using subscripts to indicate the scheme on which the multiplicities are computed. By induction, one finds that

$$\begin{aligned} e_X\big(\mathfrak{a}_1\,;\,\ldots\,;\,\mathfrak{a}_n\big)^{n-1} &= e_Y\big(\overline{\mathfrak{a}}_1\,;\,\ldots\,;\,\overline{\mathfrak{a}}_{n-1}\big)^{n-1} \\ &\le e_Y\big(\overline{\mathfrak{a}}_1\big)\cdot\ldots\cdot e_Y\big(\overline{\mathfrak{a}}_{n-1}\big) \\ &= e_X\big(\mathfrak{a}_1^{[n-1]}\,;\,\mathfrak{a}_n\big)\cdot\ldots\cdot e_X\big(\mathfrak{a}_{n-1}^{[n-1]}\,;\,\mathfrak{a}_n\big), \end{aligned}$$

where in the last line we have used (*) again to express the multiplicity of $\overline{\mathfrak{a}}_i$ on Y as a mixed multiplicity on X. The resulting inequality

$$e\big(\mathfrak{a}_1\,;\,\ldots\,;\,\mathfrak{a}_n\big)^{n-1} \;\le\; e\big(\mathfrak{a}_1^{[n-1]}\,;\,\mathfrak{a}_n\big)\cdot\ldots\cdot e\big(\mathfrak{a}_{n-1}^{[n-1]}\,;\,\mathfrak{a}_n\big)$$

of mixed multiplicities on X is the analogue of equation (1.24) in the proof of Theorem 1.6.1, and as in that argument the inequality appearing in (i) now follows formally. □

1.7 Amplitude for a Mapping

In this section we outline the basic facts concerning amplitude relative to a mapping. We follow the conventions of Grothendieck in [255], which differ slightly from those adopted by Hartshorne in [280].

By way of preparation, consider a proper mapping $f : X \longrightarrow T$ of schemes, and a coherent sheaf $\mathcal{F}$ on X. Then $f_*\mathcal{F}$ is a coherent sheaf on T, and so one can form the T-scheme

$$\mathbf{P}(\mathcal{F}) \;=_{\text{def}}\; \operatorname{Proj}_{\mathcal{O}_T}\big(\operatorname{Sym}(f_*\mathcal{F})\big) \longrightarrow T$$

whose fibre over a given point $t \in T$ is the projective space of one-dimensional quotients of the fibre $f_*(\mathcal{F})\otimes\mathbf{C}(t)$. This is the analogue in the relative setting of the projective space of sections of a sheaf on a fixed complete variety. Moreover, there is a natural mapping $f^*f_*\mathcal{F} \longrightarrow \mathcal{F}$ whose surjectivity is the analogue of the global generation of $\mathcal{F}$ in the absolute situation.

These remarks motivate

Definition 1.7.1. (Amplitude for a map). Let $f : X \longrightarrow T$ be a proper mapping of algebraic varieties or schemes, and let L be a line bundle on X.

(i). L is *very ample relative to* f, or f-*very ample*, if the canonical map

$$\rho : f^*f_*L \longrightarrow L$$

is surjective and defines an embedding

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathbf{P}(f_*L) \arrow[dl] \\ & T & \end{tikzcd}$$

of schemes over T.

(ii). L is *ample relative to* f, or f*-ample*, if $L^{\otimes m}$ is f-very ample for some $m > 0$.

A Cartier divisor D on X is very ample for f if the corresponding line bundle is so, and f-amplitude for Cartier $\mathbf{Q}$-divisors is defined by clearing denominators. □

Example 1.7.2. If E is a vector bundle on a scheme T, then the Serre line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$ on $X = \mathbf{P}(E)$ is ample for the natural mapping $\pi : \mathbf{P}(E) \longrightarrow T$. □

We start by recording several useful observations.

Remark 1.7.3. (Amplitude is local on the base). Observe that both properties in 1.7.1 are local on T. In other words, either condition holds for $f : X \longrightarrow T$ if and if only it holds for each of the restrictions $f_i : X_i = f^{-1}(U_i) \longrightarrow U_i$ of f to the inverse images of the members of an open covering $\{U_i\}$ of T. □

Remark 1.7.4. (Equivalent condition for f-very ample). The condition in Definition 1.7.1 (i) is equivalent to the existence of a coherent sheaf $\mathcal{F}$ on T, plus an embedding $i : X \hookrightarrow \mathbf{P}(\mathcal{F})$ over T, such that $L = \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1) \,|\, X$. In fact, such an embedding gives rise to a surjection $\tau : f^*\mathcal{F} \longrightarrow L$, which determines a homomorphism $\sigma : \mathcal{F} \longrightarrow f_*L$ together with a factorization

$$\begin{tikzcd} f^*\mathcal{F} \arrow[rr, two heads, "\tau"] \arrow[dr, "f^*\sigma"'] & & L \\ & f^*f_*L \arrow[ur, "\rho"'] & \end{tikzcd}$$

of τ. It follows in the first place that ρ is surjective. Moreover, the given embedding i is realized as the composition of the morphism $j : X \longrightarrow \mathbf{P}(f_*L)$ arising from ρ with a linear projection

$$\Big(\mathbf{P}(f_*L) - \mathbf{P}(\operatorname{coker}\sigma)\Big) \longrightarrow \mathbf{P}(\mathcal{F})$$

of T-schemes. This shows that j is an embedding. (See [255, II.4.4.4] for details.)

It follows in particular that if T is *affine* — so that f_*L is globally generated — then L is very ample for f if and only if there is an embedding $j : X \hookrightarrow \mathbf{P}^N \times T$ such that $L = j^*\mathcal{O}_{\mathbf{P}^N \times T}(1)$. This is taken as the definition of very ample relative to a mapping in [280]. □

Remark 1.7.5. The condition in 1.7.1 (ii) does not appear as the definition of f-amplitude in [255], but it is equivalent to that definition by virtue of [255, II.4.6.11]. □

An analogue of Theorem 1.2.6 holds in the present setting:

Theorem 1.7.6. *Let $f : X \longrightarrow T$ be a proper morphism of schemes, and L a line bundle on X. Then the following are equivalent:*

(i). *L is ample for f.*

(ii). *Given any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_1 = m_1(\mathcal{F})$ such that*

$$R^i f_*\big(\mathcal{F} \otimes L^{\otimes m}\big) \;=\; 0 \qquad \text{for all } \; i > 0 \;,\; m \geq m_1(\mathcal{F}).$$

(iii). *Given any coherent sheaf $\mathcal{F}$ on X, there is a positive integer $m_2 = m_2(\mathcal{F})$ such that the canonical mapping*

$$f^* f_*\big(\mathcal{F} \otimes L^{\otimes m}\big) \longrightarrow \mathcal{F} \otimes L^{\otimes m}$$

is surjective whenever $m \geq m_2$.

(iv). *There is a positive integer $m_3 > 0$ such that $L^{\otimes m}$ is f-very ample for every $m \geq m_3$.*

References for Proof. As in the proof of Theorem 1.2.6, for (i) $\Longrightarrow$ (ii) one reduces first to the case in which L is very ample for f. The assertion being local on T, one can suppose that T is affine, and then [280, III.5.2] applies. Note that if T is affine, then the condition in (iii) is equivalent to asking that $\mathcal{F} \otimes L^{\otimes m}$ be globally generated. This being said, (ii) $\Leftrightarrow$ (iii) follows (at least when f is projective) from [280, III.5.3], and under the same hypothesis (iii) $\Rightarrow$ (iv) is a consequence of [280, II.7.6]. For an arbitrary proper mapping f, and the remaining implications, consult [255, II.4.6.8, II.4.6.11] and [256, III.2.6]. □

Example 1.7.7. Suppose given a diagram

$$\begin{tikzcd} Y \arrow[rr, "\mu"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & T & \end{tikzcd}$$

of schemes over T, with μ finite. If L is an f-ample line bundle on X, then $\mu^* L$ is ample relative to g. □

The following useful result allows one in practice to reduce to the absolute setting:

Theorem 1.7.8. (Fibre-wise amplitude). *Let $f : X \longrightarrow T$ be a proper morphism of schemes, let L be a line bundle on X, and for $t \in T$ set*

$$X_t \;=\; f^{-1}(t) \quad , \quad L_t = L \mid X_t.$$

Then L is ample for f if and only if L_t is ample on X_t for every $t \in T$.

Proof. If L is ample for f, then the previous Example 1.7.7 shows that each of the restrictions L_t is ample on X_t. Conversely, suppose that L_t is ample for every $t \in T$. As noted in Remark 1.2.18, the proof of Theorem 1.2.17 shows that every point $t \in T$ has a neighborhood U with the property that $L|f^{-1}(U) = \phi^*\big(\mathcal{O}_{\mathbf{P}\times T}(1)\big)$ for a finite mapping $X_U = f^{-1}(U) \longrightarrow \mathbf{P}^r \times U$ of schemes over U. Recalling that f-amplitude is local on T, it follows from Example 1.7.7 that L is indeed ample with respect to f. □

Corollary 1.7.9. (Nakai's criterion for a mapping). *In the setting of the previous theorem, a $\mathbf{Q}$-divisor D on X is ample with respect to f if and only if $\big(D^{\dim V} \cdot V\big) > 0$ for every irreducible subvariety $V \subset X$ of positive dimension that maps to a point in T.* □

The next result summarizes the connection between relative and global amplitude.

Proposition 1.7.10. *Consider a morphism*

$$f : X \longrightarrow T$$

of projective schemes. Let L be a line bundle on X, and let A be an ample line bundle on T. Then L is f-ample if and only if $L \otimes f^*\big(A^{\otimes m}\big)$ is an ample line bundle on X for all $m \gg 0$.

Proof. Assume that L is f-ample. Replacing L by a high power, we can suppose that $f^*f_*L \longrightarrow L$ is surjective. Since A is ample, $f_*(L) \otimes A^{\otimes p}$ is globally generated if p is sufficiently large. Therefore its pullback $f^*f_*(L) \otimes f^*A^{\otimes p}$ is likewise generated by its global sections, and choosing generators gives rise to a morphism

$$\phi : X \longrightarrow \mathbf{P} \times T$$

of schemes over T with the property that $L \otimes f^*(A^{\otimes p}) \;= \phi^*\mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}}(1)$. Moreover, ϕ is finite since it is evidently so on each fibre of f. Therefore

$$L \otimes f^*(A^{\otimes p+1}) \;=\; \phi^*\big(\mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}}(1) \otimes \mathrm{pr}_2^*A\big)$$

is the pullback of an ample line bundle under a finite map, and consequently is ample. The converse follows from 1.7.8. □

Finally, we say a few words about nefness for a mapping. Here one takes the analogue of 1.7.8 as the definition:

Definition 1.7.11. (Nefness relative to a mapping). Given a proper morphism $f : X \longrightarrow T$ as above, a line bundle L on X is *nef relative to* f if the restriction $L_t = L \,|\, X_t$ of L to each fibre is nef, or equivalently if $\big(c_1(L) \cdot C\big) \geq 0$ for every curve $C \subseteq X$ mapping to a point under T. □

Note that the analogue of Proposition 1.7.10 need not be true.

Example 1.7.12. The following example is taken from [368, Example 1.46]. Let $X = E \times E$ be the product of an elliptic curve with itself and $f : X \longrightarrow E$ the first projection. Let $D = E \times \{\text{point}\}$ and denote by $\Delta \subseteq E \times E$ the diagonal. Then $D - \Delta$ is f-nef. However, for any divisor A on E the divisor

$$D - \Delta + f^*(A)$$

has self-intersection -2, and hence cannot be nef. □

Remark 1.7.13. (Fujita vanishing for a mapping). In his paper [331], Keeler extends Fujita's vanishing theorem 1.4.35 to the relative setting. □

Remark 1.7.14. (Other ground fields). Everything in this section goes through to varieties defined over an algebraically closed field of arbitrary characteristic. □

1.8 Castelnuovo–Mumford Regularity

The Cartan–Serre–Grothendieck theorems imply that all the cohomological subtleties that may be associated to a coherent sheaf $\mathcal{F}$ on a projective space $\mathbf{P}$ disappear after twisting by a sufficiently high multiple of the hyperplane line bundle. Specifically, for $m \gg 0$:

- the higher cohomology groups of $\mathcal{F}(m)$ vanish;
- $\mathcal{F}(m)$ is generated by its global sections;
- the maps $H^0(\mathbf{P}, \mathcal{F}(m)) \otimes H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k)) \longrightarrow H^0(\mathbf{P}, \mathcal{F}(m+k))$ are surjective for every $k > 0$.

Castelnuovo–Mumford regularity gives a quantitative measure of how much one has to twist in order that these properties take effect. It then governs the algebraic complexity of a coherent sheaf, and for this reason has been the focus of considerable recent activity. As we shall see, regularity is also well adapted to arguments involving vanishing theorems.

In the first subsection, we give the definition and basic properties, and describe some variants. The complexity-theoretic meaning is indicated in the second, and in the third we survey without proof several results giving bounds on regularity. Section 1.8.D gives a brief overview — also without proof — of a circle of ideas surrounding syzygies of algebraic varieties.

1.8.A Definitions, Formal Properties, and Variants

Fix a complex vector space V of dimension $r+1$, and denote by $\mathbf{P} = \mathbf{P}(V)$ the corresponding r-dimensional projective space. We start with the definition of m-regularity of a coherent sheaf on $\mathbf{P}$.

Definition 1.8.1. (Castelnuovo–Mumford regularity of a coherent sheaf). Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbf{P}$, and let m be an integer. One says that $\mathcal{F}$ is *m-regular in the sense of Castelnuovo–Mumford* if

$$H^i\big(\mathbf{P}, \mathcal{F}(m-i)\big) \ = \ 0 \quad \text{for all } i > 0.$$

As long as the context is clear, we usually speak simply of an m-regular sheaf.

Example 1.8.2. (i). The line bundle $\mathcal{O}_{\mathbf{P}}(a)$ is $(-a)$-regular.

(ii). The ideal sheaf $\mathcal{I}_L \subseteq \mathcal{O}_{\mathbf{P}}$ of a linear subspace $L \subseteq \mathbf{P}$ is 1-regular.

(iii). If $X \subseteq \mathbf{P}$ is a hypersurface of degree d, then its structure sheaf $\mathcal{O}_X$ — viewed via extension by zero as a coherent sheaf on $\mathbf{P}$ — is $(d-1)$-regular. □

While the formal definition may seem rather non intuitive, a result of Mumford gives a first indication of the fact that Castelnuovo–Mumford regularity measures the point at which cohomological complexities vanish.

Theorem 1.8.3. (Mumford's theorem, I). *Let $\mathcal{F}$ be an m-regular sheaf on $\mathbf{P}$. Then for every $k \geq 0$:*

(i). *$\mathcal{F}(m+k)$ is generated by its global sections.*

(ii). *The natural maps*

$$H^0(\mathbf{P}, \mathcal{F}(m)) \ \otimes \ H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k)) \longrightarrow H^0(\mathbf{P}, \mathcal{F}(m+k))$$

are surjective.

(iii). *$\mathcal{F}$ is $(m+k)$-regular.*

Proof. Since $\mathcal{F}(m+\ell)$ is in any event globally generated for $\ell \gg 0$ (Theorem 1.2.6), the surjectivities in (ii) imply that $\mathcal{F}(m)$ itself must already be generated by its global sections. The same is then true of $\mathcal{F}(m+k)$ whenever $k \geq 0$. Hence we need only prove (ii) and (iii), and thanks to (iii) it suffices to treat the case $k = 1$.

For this we consider the canonical Koszul complex of bundles on $\mathbf{P} = \mathbf{P}(V)$ (see Appendix B.2). Denote by $V_{\mathbf{P}} = V \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}}$ the trivial vector bundle on $\mathbf{P}$ with fibre V. Starting with the surjective bundle map

$$V_{\mathbf{P}}(-1) = V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}},$$

form the exact sequence

$$0 \longrightarrow \Lambda^{r+1} V_{\mathbf{P}}(-r-1) \longrightarrow \ldots \longrightarrow \Lambda^2 V_{\mathbf{P}}(-2) \longrightarrow V_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0. \tag{$K_\bullet$}$$

Twisting through by $\mathcal{F}(m+1)$ yields an exact complex

$$\begin{aligned} \ldots \longrightarrow \Lambda^3 V_{\mathbf{P}} \otimes \mathcal{F}(m-2) \longrightarrow \Lambda^2 V_{\mathbf{P}} \otimes \mathcal{F}(m-1) \longrightarrow V_{\mathbf{P}} \otimes \mathcal{F}(m) \\ \longrightarrow \mathcal{F}(m+1) \longrightarrow 0. \end{aligned} \tag{*}$$

The m-regularity of $\mathcal{F}$ implies that $H^i\big(\mathbf{P}, \Lambda^{i+1} V_{\mathbf{P}} \otimes \mathcal{F}(m-i)\big) = 0$, and it follows by chasing through (*) that the map

$$H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \otimes H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)\big) = H^0\big(\mathbf{P}, V_{\mathbf{P}} \otimes \mathcal{F}(m)\big) \to H^0\big(\mathbf{P}, \mathcal{F}(m+1)\big)$$

is surjective. This proves (ii). For (iii) one twists $(K_\bullet)$ by $\mathcal{F}(m)$ and argues similarly. □

Theorem 1.8.3 is equivalent to a variant involving globally generated ample line bundles on any projective variety. Since we frequently use it in this form, we give the relevant definition and statement explicitly:

Definition 1.8.4. (Regularity with respect to a globally generated ample line bundle). Let X be a projective variety and B an ample line bundle on X that is generated by its global sections. A coherent sheaf $\mathcal{F}$ on X is *m-regular with respect to B* if

$$H^i\big(X, \mathcal{F} \otimes B^{\otimes(m-i)}\big) = 0 \quad \text{for } i > 0. \quad \square$$

Then 1.8.3 becomes:

Theorem 1.8.5. (Mumford's theorem, II). *Let $\mathcal{F}$ be an m-regular sheaf on X with respect to B. Then for every $k \geq 0$:*

(i). *$\mathcal{F} \otimes B^{\otimes(m+k)}$ is generated by its global sections.*

(ii). *The natural maps*

$$H^0\big(X, \mathcal{F} \otimes B^{\otimes m}\big) \otimes H^0\big(X, B^{\otimes k}\big) \longrightarrow H^0\big(X, \mathcal{F} \otimes B^{\otimes(m+k)}\big)$$

are surjective.

(iii). *$\mathcal{F}$ is $(m+k)$-regular with respect to B.*

Proof. One repeats the proof of Theorem 1.8.3 with $\mathcal{O}_{\mathbf{P}}(1)$ replaced by B and V replaced by $H^0\big(X, B\big)$. Alternatively, one can apply 1.8.3 to the direct image of $\mathcal{F}$ under the finite map $X \longrightarrow \mathbf{P}$ determined by B. □

For most purposes, dealing with sheaves on projective space involves little loss in generality. This is therefore the context in which we shall work — unless stated otherwise — for the remainder of this section.

A number of additional concrete examples are worked out in the next subsection. Here we continue by presenting some formal properties of regularity.

Example 1.8.6. (Extensions). An extension of two m-regular sheaves on the projective space $\mathbf{P}$ is itself m-regular. □

Example 1.8.7. Suppose that a coherent sheaf $\mathcal{F}$ on $\mathbf{P}$ is resolved by a long exact sequence

$$\ldots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

of coherent sheaves on $\mathbf{P}$. If $\mathcal{F}_i$ is $(m+i)$-regular for every $i \geq 0$, then $\mathcal{F}$ is m-regular. Moreover, in this case the mapping $H^0\big(\mathbf{P}, \mathcal{F}_0(m)\big) \longrightarrow H^0\big(\mathbf{P}, \mathcal{F}(m)\big)$ is surjective. □

A partial converse of Example 1.8.7 provides a useful characterization of m-regularity.

Proposition 1.8.8. (Linear resolutions). *Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbf{P}$. Then $\mathcal{F}$ is m-regular if and only if $\mathcal{F}$ is resolved by a long exact sequence*

$$\ldots \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-m-2) \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-m-1) \longrightarrow \oplus\, \mathcal{O}_{\mathbf{P}}(-m) \longrightarrow \mathcal{F} \longrightarrow 0 \tag{1.30}$$

whose terms are direct sums of the indicated line bundles.

Proof. Given a long exact sequence (1.30), the m-regularity of $\mathcal{F}$ follows from Example 1.8.7 (or equivalently by reading off the vanishings $H^i\big(\mathbf{P}, \mathcal{F}(m-i)\big) = 0$ for $i > 0$). Conversely, supposing that $\mathcal{F}$ is m-regular, we construct the resolution (1.30) step by step. To this end, recall first from Proposition 1.8.3 (i) that $\mathcal{F}(m)$ is globally generated. Setting $W = H^0\big(\mathbf{P}, \mathcal{F}(m)\big)$, one therefore has a surjective sheaf homomorphism $e : W_{\mathbf{P}}\mathcal{O}_{\mathbf{P}}(-m) \longrightarrow \mathcal{F}$. Let $\mathcal{F}_1 = \ker e$:

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow W_{\mathbf{P}} \otimes \mathcal{O}_{\mathbf{P}}(-m) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{*}$$

We will show momentarily that $\mathcal{F}_1$ is $(m+1)$-regular. Granting this, set $W' = H^0\big(\mathbf{P}, \mathcal{F}_1(m+1)\big)$ and map the corresponding trivial bundle $W'_{\mathbf{P}}$ to $\mathcal{F}_1(m+1)$ to construct an exact sequence

$$W'_{\mathbf{P}} \otimes \mathcal{O}_{\mathbf{P}}(-m-1) \longrightarrow W_{\mathbf{P}} \otimes \mathcal{O}_{\mathbf{P}}(-m) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Then continue until one arrives finally at (1.30). As for the $(m+1)$-regularity of $\mathcal{F}_1$, it is almost immediate. In fact, by construction the homomorphism $H^0\big(\mathbf{P}, W_{\mathbf{P}}\big) \longrightarrow H^0\big(\mathbf{P}, \mathcal{F}(m)\big)$ determined by (*) is surjective, and therefore $H^1\big(\mathbf{P}, \mathcal{F}_1(m)\big) = 0$. On the other hand, it follows from (*) and the m-regularity of $\mathcal{F}$ that

$$\begin{aligned} H^{i+1}\left(\mathbf{P}, \mathcal{F}_1\big((m+1)-(i+1)\big)\right) &= H^{i+1}\left(\mathbf{P}, \mathcal{F}_1(m-i)\right) \\ &= H^{i}\left(\mathbf{P}, \mathcal{F}(m-i)\right) \\ &= 0 \end{aligned}$$

for $i > 0$. Therefore $\mathcal{F}_1$ is $(m+1)$-regular, as required. □

We next use Proposition 1.8.8 to show that at least for vector bundles, regularity has pleasant tensorial properties.

Proposition 1.8.9. (Regularity of tensor products). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}$, and let E be a locally free sheaf on $\mathbf{P}$. If $\mathcal{F}$ is m-regular and E is ℓ-regular, then $E \otimes \mathcal{F}$ is $(\ell + m)$-regular.*

Corollary 1.8.10. (Wedge and symmetric products). *If E is an m-regular locally free sheaf, then the p-fold tensor power $T^p E$ is (pm)-regular. In particular, $\Lambda^p E$ and $S^p E$ are likewise (pm)-regular.* □

Proof of Proposition 1.8.9. Starting with the resolution (1.30) of $\mathcal{F}$ appearing in Proposition 1.8.8, tensor through by E to obtain a complex

$$\ldots \longrightarrow \oplus E(-m-2) \longrightarrow \oplus E(-m-1) \longrightarrow \oplus E(-m) \longrightarrow E \otimes \mathcal{F} \longrightarrow 0.$$

Since tensoring by a locally free sheaf preserves exactness, this sequence is in fact exact. But $E(-m-i)$ is $(\ell + m + i)$-regular thanks to the ℓ-regularity of E, and so the $(\ell + m)$-regularity of $E \otimes \mathcal{F}$ follows from Example 1.8.7. □

Remark 1.8.11. This proof shows that it suffices for Proposition 1.8.9 to assume that at every point of $\mathbf{P}$ either E or $\mathcal{F}$ is locally free. □

Example 1.8.12. Chardin informs us that he has found examples for which Proposition 1.8.9 fails when the sheaves in question are not locally free along a set of dimension ≥ 2. □

Example 1.8.13. In the situation of Proposition 1.8.9, the natural map

$$H^0\big(\mathbf{P}, \mathcal{F}(m)\big) \otimes H^0\big(\mathbf{P}, E(\ell)\big) \longrightarrow H^0\big(\mathbf{P}, \mathcal{F} \otimes E(m+\ell)\big)$$

is surjective. □

Remark 1.8.14. (Regularity in positive characteristics). Everything we have said so far except Corollary 1.8.10 goes through without change for varieties defined over an algebraically closed field of arbitrary characteristic. (In positive characteristics, the symmetric and alternating products appearing in 1.8.10 may no longer be direct summands of the tensor product.) □

Example 1.8.15. (Green's theorem). Let $W \subseteq H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d)\big)$ be a subspace of codimension c giving a free linear series. Then the map

$$s_k : W \otimes H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k)\big) \longrightarrow H^0\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d+k)\big)$$

determined by multiplication of polynomials is surjective for $k \geq c$. In other words, any homogeneous polynomial of degree $\geq d+c$ lies in the ideal spanned by W. (Let M_d be the vector bundle on $\mathbf{P} = \mathbf{P}(V)$ arising as the kernel of the evaluation map on forms of degree d:

$$0 \longrightarrow M_d \longrightarrow S^d V \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_{\mathbf{P}}(d) \longrightarrow 0.$$

Then M_d is 1-regular, and so $\Lambda^k M_d$ is k-regular thanks to 1.8.10. On the other hand, W determines an analogous bundle M_W,

$$0 \longrightarrow M_W \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{O}_{\mathbf{P}}(d) \longrightarrow 0,$$

and the two bundles in question sit in a sequence $0 \longrightarrow M_W \longrightarrow M_d \longrightarrow \mathcal{O}_{\mathbf{P}}^c \longrightarrow 0$. One then uses the Eagon–Northcott complex (EN_1) from Appendix B to show that M_W is $(c+1)$-regular, which implies that s_k is surjective as soon as $k \geq c$.) This result and its generalizations have interesting applications to infinitesimal computations in Hodge theory: see [236], [237], or [83, Chapter 7]. □

Remark 1.8.16. (Warning on generalized regularity). Owing to the fact that a globally generated ample line bundle B on a projective variety X need not itself be (-1)-regular, one should not expect Propositions 1.8.8 and 1.8.9 or Corollary 1.8.10 to extend to the setting of Definition 1.8.4. For example, $\mathcal{F} = \mathcal{O}_X$ has a (trivial) linear resolution, but it is not in general 0-regular. However, Arapura [14, Corollary 3.2] observes that if $R = \max\{1, \mathrm{reg}_B(\mathcal{O}_X)\}$, then $\mathcal{F} \otimes B^{\otimes m}$ has an "R-linear" resolution. □

It is often useful to know the best possible regularity of a sheaf:

Definition 1.8.17. (Regularity of a sheaf). The *Castelnuovo–Mumford regularity* $\mathrm{reg}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on $\mathbf{P}$ is the least integer m for which $\mathcal{F}$ is m-regular (or $-\infty$ if $\mathcal{F}$ is supported on a finite set, and hence m-regular for all $m \ll 0$). □

We conclude this subsection by discussing some variants.

Example 1.8.18. (Regularity with respect to a vector bundle). Let X be an irreducible projective variety of dimension n, and let U be a vector bundle on X having the property that for every point $x \in X$, there is a section of U whose zero locus is a finite set containing x. (Example: $U = B \oplus \cdots \oplus B$ (n times), where B is a globally generated ample line bundle on X.) If $\mathcal{F}$ is a coherent sheaf on X such that

$$H^i\big(X, \Lambda^i U^* \otimes \mathcal{F}\big) \ = \ 0 \quad \text{for} \ \ i > 0,$$

then $\mathcal{F}$ is globally generated. (Let $s \in \Gamma(X, U)$ be a section vanishing on a finite subscheme $Z \subset X$. Form the Koszul complex $K_\bullet$ determined by the resulting map $U^* \longrightarrow \mathcal{O}_X$ and suppose for the moment that $K_\bullet$ is exact (which will be the case e.g. if X is smooth and $\mathrm{rank}\, U = \dim X$) and that $\mathcal{F}$ is locally free. Tensoring through by $\mathcal{F}$, the given vanishings imply that the restriction map

$$H^0\big(X, \mathcal{F}\big) \longrightarrow H^0\big(X, \mathcal{F} \otimes \mathcal{O}_Z\big)$$

is surjective, and hence that $\mathcal{F}$ is globally generated at every point of Z. Thanks to Example B.1.3 in Appendix B, the same argument works even if $K_\bullet$ is not exact or $\mathcal{F}$ is not locally free, since in any event $K_\bullet \otimes \mathcal{F}$ is exact off the finite set Z.) □

Example 1.8.19. (Global generation on Grassmannians). Let $\mathbf{G} = \mathbf{G}(k,m)$ be the Grassmannian of k-dimensional quotients of an m-dimensional vector space, and denote by Q the tautological rank k quotient bundle on $\mathbf{G}$. Suppose that $\mathcal{F}$ is a coherent sheaf on $\mathbf{G}$ satisfying the condition that for every $i > 0$,

$$H^i\big(\mathbf{G}, \Lambda^{i_1}Q^* \otimes \cdots \otimes \Lambda^{i_{m-k}}Q^* \otimes \mathcal{F}\big) = 0 \quad \text{whenever} \quad i_1 + \cdots + i_{m-k} = i.$$

Then $\mathcal{F}$ is globally generated. (Given any point $x \in \mathbf{G}$, the $(m-k)$-fold direct sum $V = Q \oplus \cdots \oplus Q$ has a section vanishing precisely at x.) This result is due to M. Kim [335], who used it to study branched coverings of Grassmannians. (See Sections 6.3.D and 7.1.C.) □

Remark 1.8.20. (Regularity on Grassmannians). A general theory of regularity on Grassmannians has been developed and studied by Chipalkatti [90]. □

Remark 1.8.21. (Regularity on abelian varieties). Pareschi and Popa have introduced some very interesting notions of regularity on an abelian variety X with respect to an arbitrary ample line bundle L on X [496], [497]. For example, let $\mathcal{F}$ be a coherent sheaf on X, and assume that $\mathcal{F}$ satisfies the vanishing

$$H^i\big(X, \mathcal{F} \otimes L^{-1} \otimes P\big) \;=\; 0 \quad \text{for all} \;\; i > 0 \;\; \text{and} \;\; P \in \operatorname{Pic}^0(X).$$

It is established in [496] that then $\mathcal{F}$ is globally generated. This generalizes a useful lemma of Kempf. Pareschi and Popa establish similar statements under weaker hypotheses, and use them to prove several striking results about the geometry of abelian varieties and their subvarieties, as well as the equations defining their projective embeddings. □

Just as regularity can be valuable for proving that a sheaf is globally generated, it is also useful for establishing that certain line bundles are very ample:

Example 1.8.22. (Criterion for very ample bundles). Let X be an irreducible projective variety of dimension n, and B an ample line bundle that is generated by its global sections. One has then the following

> **Proposition.** Let N be a line bundle on X that is 0-regular with respect to B in the sense of Definition 1.8.4. Then $N \otimes B$ is very ample.

(As $N \otimes B$ is free, it is enough to show that the sheaf $N \otimes B \otimes \mathfrak{m}_x$ is globally generated for any point $x \in X$, $\mathfrak{m}_x$ being the maximal ideal of x. As in Example 1.8.18 this in turn will follow if we show that given any $x \in X$, there is a finite scheme $Z = Z(x)$ containing x such that $N \otimes B \otimes \mathcal{I}_Z$ is 0-regular with respect to B, for this implies the 0-regularity of $N \otimes B \otimes \mathfrak{m}_x$. Having fixed x, take Z to be the zero-scheme cut out by n general sections of B that vanish at x. As in Example 1.8.18, form the resulting Koszul complex and twist by $B \otimes N$: one reads off the required 0-regularity of $B \otimes N \otimes \mathcal{I}_Z$ from the vanishings giving the 0-regularity of N.) $\square$

The next example shows that Kodaira-type vanishing theorems are well suited to regularity statements.

Example 1.8.23. (Adjoint bundles). Let X be a smooth complex projective variety of dimension n and let D be an ample divisor on X. As usual denote by K_X a canonical divisor on X, and let P be an arbitrary nef divisor on X. There has been a great deal of interest recently in adjoint-type bundles of the form

$$L_k \ = \ \mathcal{O}_X(K_X + kD + P).$$

One can view these as the analogues of line bundles of large degree on curves (see Section 10.4.A). Under various hypotheses on D, divisors of this shape have also been extensively studied by Sommese and his school (see [50]), as well as by Fujita (see [197]).

Assume now that D is *free* (and ample). Then the divisor L_k above is free when $k \geq n+1$ and very ample when $k \geq n+2$. (The Kodaira vanishing theorem 4.2.1 asserts that if A is any ample divisor on X, then $H^i\big(X, \mathcal{O}_X(K_X + A)\big) = 0$ for $i > 0$. One uses this as input to Theorem 1.8.5 and Example 1.8.22.) These results for L_k are elementary special cases of a celebrated conjecture of Fujita, which asserts that the same statements should hold — at least when $P = 0$ — assuming only that D is ample. Fujita's conjecture is discussed in more detail in Section 10.4. See also Theorem 1.8.60. $\square$

We conclude by outlining a relative notion of regularity.

Example 1.8.24. (Regularity with respect to a mapping). Let $f : X \longrightarrow Y$ be a proper surjective mapping of varieties (or schemes). We suppose given a line bundle A on X satisfying:

(a). A is ample for f;

(b). The canonical mapping $f^*f_*A \longrightarrow A$ is surjective.

For example, condition (b) holds if A is globally generated. If Y is normal and X is the normalized blowing-up of a sheaf of ideals $\mathfrak{a} \subseteq \mathcal{O}_Y$, with exceptional divisor $E \subseteq \mathcal{O}_X$, then (a) and (b) hold with $A = \mathcal{O}_X(-E)$. (For (b), use that $\mathfrak{a} \subseteq f_*\mathcal{O}_X(-E)$.)

Given a coherent sheaf $\mathcal{F}$ on X, we define $\mathcal{F}$ to be m-regular with respect to A and f if

$$R^if_*\big(\mathcal{F} \otimes A^{\otimes(m-i)}\big) = 0 \quad \text{for} \quad i > 0.$$

Then the natural analogue of Theorem 1.8.3 holds in this setting. Specifically, assume that $\mathcal{F}$ is m-regular with respect to A and f. Then for every $k \geq 0$:

(i). The homomorphism

$$f^*f_*\big(\mathcal{F}\otimes A^{\otimes(m+k)}\big)\longrightarrow \mathcal{F}\otimes A^{\otimes(m+k)}$$

is surjective;

(ii). The map

$$f_*\big(\mathcal{F}\otimes A^{\otimes m}\big)\otimes f_*\big(A^{\otimes k}\big)\longrightarrow f_*\big(\mathcal{F}\otimes A^{\otimes(m+k)}\big)$$

is surjective;

(iii). $\mathcal{F}$ is $(m+k)$-regular with respect to A and f.

(The statement being local on Y, one can assume that Y is affine. Then there exists a finite-dimensional vector space $V\subseteq H^0\big(Y,f_*A\big)$ of sections that generate f_*A, giving rise to a surjective bundle map $V_Y=V\otimes\mathcal{O}_Y\longrightarrow f_*A$. Pulling back and composing with $f^*f_*A\longrightarrow A$, one arrives at a surjective map $V_X\longrightarrow A$ of bundles on X, and now one argues as in the proof of Theorem 1.8.3. Namely, form the corresponding Koszul complex and twist by $\mathcal{F}\otimes A^{\otimes m}$:

$$\ldots\longrightarrow \Lambda^2V_X\otimes\mathcal{F}\otimes A^{\otimes(m-1)}\longrightarrow V_X\otimes\mathcal{F}\otimes A^{\otimes m}\longrightarrow\mathcal{F}\otimes A^{\otimes(m+1)}\longrightarrow 0.$$

Chasing through the complex and using the hypothesis of m-regularity, one finds that the map $V_Y\otimes f_*\big(\mathcal{F}\otimes A^{\otimes m}\big)\longrightarrow f_*\big(\mathcal{F}\otimes A^{\otimes(m+1)}\big)$ is surjective. But this factors through

$$f_*A\otimes f_*\big(\mathcal{F}\otimes A^{\otimes m}\big)\longrightarrow f_*\big(\mathcal{F}\otimes A^{\otimes(m+1)}\big),$$

and (ii) follows. The proof of (iii) is similar, while for (i) one uses (ii) plus the fact that $f^*f_*\big(\mathcal{F}\otimes A^{\otimes(m+k)}\big)\longrightarrow\mathcal{F}\otimes A^{\otimes(m+k)}$ is surjective for $k\gg 0$ by virtue of the f-amplitude of A.) $\square$

Example 1.8.25. (Regularity on a projective bundle). Let X be a variety or scheme, and E a vector bundle on X, with projectivization $\pi:\mathbf{P}(E)\longrightarrow X$. A coherent sheaf $\mathcal{F}$ on $\mathbf{P}(E)$ is *m-regular with respect to* π if

$$R^i\pi_*\big(\mathcal{F}\otimes\mathcal{O}_{\mathbf{P}(E)}(m-i)\big)\ =\ 0$$

for $i>0$. If this condition holds, then:

(i). $\pi^*\pi_*\big(\mathcal{F}\otimes\mathcal{O}_{\mathbf{P}(E)}(m)\big)$ surjects onto $\mathcal{F}\otimes\mathcal{O}_{\mathbf{P}(E)}(m)$;

(ii). The mapping

$$\pi_*\big(\mathcal{F}\otimes\mathcal{O}_{\mathbf{P}(E)}(m)\big)\otimes\pi_*\mathcal{O}_{\mathbf{P}(E)}(k)\longrightarrow\pi_*\big(\mathcal{F}\otimes\mathcal{O}_{\mathbf{P}(E)}(m+k)\big)$$

is surjective for $k\geq 0$; and

(iii). $\mathcal{F}$ is $(m+1)$-regular for π.

(This follows from the previous example with $A=\mathcal{O}_{\mathbf{P}(E)}(1)$.) $\square$

1.8.B Regularity and Complexity

We now present some results suggesting that the regularity of a sheaf governs the complexity[24] of the algebraic objects associated with it.

Continuing to work on the projective space $\mathbf{P} = \mathbf{P}(V)$, denote by $S = \mathrm{Sym}(V)$ the homogeneous coordinate ring of $\mathbf{P}$, so that S is a polynomial ring in $r+1$ variables. Fix a coherent sheaf $\mathcal{F}$ on $\mathbf{P}$, and let

$$F \;=\; \bigoplus_k H^0\big(\mathbf{P}, \mathcal{F}(k)\big)$$

be the corresponding graded S-module. We assume for simplicity that

$$H^0\big(\mathbf{P}, \mathcal{F}(k)\big) \;=\; 0 \quad \text{for all} \;\; k \ll 0,$$

so that F is finitely generated.[25]

Like any finitely generated S-module, F admits a minimal graded free resolution $E_\bullet$:

$$0 \longrightarrow E_{r+1} \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow F \longrightarrow 0. \tag{1.31}$$

Here each E_p is a free graded S-module,

$$E_p \;=\; \bigoplus_i S(-a_{p,i}),$$

and minimality means that the maps of $E_\bullet$ are given by matrices of homogeneous polynomials containing no non-zero constants as entries. The integers $a_{p,i} \in \mathbf{Z}$ specifying the degrees of the generators of E_p are uniquely determined by F and hence $\mathcal{F}$. Set

$$a_p \;=\; a_p(\mathcal{F}) \;=\; \max_i \{a_{p,i}\},$$

so that a_p is the largest degree of a generator of the p^{th} module of syzygies of F. Algorithms for computing with homogeneous polynomials typically proceed degree by degree, so the integers $a_0, \ldots, a_{r+1}$ in effect serve as a measure of the algebraic complexity of F or the underlying sheaf $\mathcal{F}$.

From the present point of view, the basic meaning of regularity is that it is equivalent to an upper bound on all of the a_p.

Theorem 1.8.26. (Regularity and syzygies). *The sheaf $\mathcal{F}$ is m-regular if and only if each of the integers $a_p = a_p(\mathcal{F})$ satisfies the inequality*

$$a_p \;\leq\; p + m. \tag{1.32}$$

[24] We stress that we are using the term “complexity” in a non-technical sense.

[25] This is equivalent to the assumption that none of the associated primes of $\mathcal{F}$ have zero-dimensional support. If this condition is not satisfied, then one should work instead with a truncation $F_{\geq k_0} = \oplus_{k \geq k_0} H^0\big(\mathbf{P}, \mathcal{F}(k)\big)$ of F.

So knowing the regularity of a coherent sheaf is the same thing as having bounds on the degrees of generators of all the modules of syzygies of the corresponding module.

Example 1.8.27. (Complete intersections). Let $X \subseteq \mathbf{P}$ be the complete intersection of hypersurfaces of degrees $d_1, \ldots, d_e$, and let $\mathcal{I}_X$ be the ideal sheaf of X. Then the Koszul resolution of the homogeneous ideal I_X of X shows that

$$\operatorname{reg}(\mathcal{I}_X) \;=\; (d_1 + \cdots + d_e - e + 1).$$

In this example, the regularity is governed by the highest module of syzygies of I_X rather than by the degrees of its generators. □

Indication of Proof of Theorem 1.8.26. The argument parallels the proof of Proposition 1.8.8. If F admits a resolution satisfying the degree bound (1.32), then sheafifying yields a resolution of $\mathcal{F}$ to which Example 1.8.7 applies. Assume conversely that $\mathcal{F}$ is m-regular. Then it follows from statement (ii) of Mumford's theorem that all the generators of F occur in degrees $\leq m$. Choosing generators then gives rise to an exact sequence of S-modules

$$0 \longrightarrow F_1 \longrightarrow \oplus\, S(-a_{0,i}) \longrightarrow F \longrightarrow 0$$

with all $a_{0,i} \leq m$. The sheafification of F_1 is $(m+1)$-regular, and hence all generators of F_1 occur in degrees $\leq (m+1)$. Continuing the process step by step leads to the required resolution. □

A particularly interesting case occurs when $\mathcal{F}$ is the ideal sheaf of a subvariety (or subscheme) of projective space:

Definition 1.8.28. (Regularity of a projective subvariety). We say that a subvariety (or subscheme) $X \subseteq \mathbf{P}$ is m-regular if its ideal sheaf $\mathcal{I}_X$ is. The *regularity* of X is the regularity $\operatorname{reg}(\mathcal{I}_X)$ of its ideal. □

Thus if X is m-regular, then its saturated homogeneous ideal

$$I_X \;=\; \oplus\, H^0\big(\mathbf{P}, \mathcal{I}_X(k)\big)$$

is generated by forms of degrees $\leq m$, and the p^{th} syzygies among these generators appear in degrees $\leq m+p$. In the next subsection we will discuss the problem of bounding the regularity of X in terms of geometric data. Section 1.8.D centers around some more subtle invariants associated to syzygies.

We conclude this subsection with some examples. The first shows that in checking the regularity of a subvariety, some of the vanishings in the definition are automatic:

Example 1.8.29. If $X \subseteq \mathbf{P}$ has dimension n, then (for $m > 0$) X is m-regular if and only if $H^i\big(\mathbf{P}, \mathcal{I}_X(m-i)\big) = 0$ for $1 \leq i \leq n+1$. □

Example 1.8.30. (Regularity of finite sets). Suppose that $X \subseteq \mathbf{P}$ is a finite subset consisting of d distinct (reduced) points. Then X is d-regular, and if the points of X are collinear then X is not $(d-1)$-regular. The analogous statement holds if X is a finite scheme of length d. □

Example 1.8.31. (Regularity of some monomial curves). Suppose that $C \subset \mathbf{P}^r$ is a smooth rational curve, embedded by a possibly incomplete linear series. Then (for $m > 0$) C is m-regular if and only if hypersurfaces of degree $m-1$ cut out a complete linear series on C, i.e. the map

$$H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m-1)) \longrightarrow H^0(C, \mathcal{O}_C(m-1))$$

is surjective. If C is the image of the embedding

$$\mathbf{P}^1 \hookrightarrow \mathbf{P}^3 \quad , \quad [s,t] \mapsto [s^d, s^{d-1}t, st^{d-1}, t^d],$$

then C is $(d-1)$-regular but not $(d-2)$-regular. □

Example 1.8.32. (Regularity of a disjoint union). Let $X, Y \subset \mathbf{P}$ be disjoint subvarieties or subschemes. If X is m-regular and Y is ℓ-regular, then $X \cup Y$ is $(m+\ell)$-regular. (Use Remark 1.8.11.) Sidman proves some related results for homogeneous ideals in [535]. □

Remark 1.8.33. (Algebraic pathology). Castelnuovo–Mumford regularity does not behave very well with respect to natural algebraic operations, but examples are hard to come by. For example, given a homogeneous ideal $I \subseteq S$, Ravi [513] raised the question whether $\operatorname{reg}(\sqrt{I}) \leq \operatorname{reg}(I)$.[26] However counter-examples were only recently given, by Chardin and D'Cruz [86]. This paper also gives an example in which the regularity of an ideal increases after removing some positive-dimensional embedded components. In general, the absence of systematic techniques for constructing examples is one of the biggest lacunae in the current state of the theory. □

Remark 1.8.34. (Work of Bayer and Stillman). Bayer and Stillman [43] establish a more precise connection between regularity and complexity. Namely, suppose that $I \subseteq S$ is the saturated homogeneous ideal of a scheme $X \subseteq \mathbf{P}$. Most computational algorithms in algebraic geometry are based on choosing coordinates on $\mathbf{P}$, and working with Gröbner bases. Bayer and Stillman show that the regularity $\operatorname{reg}(X)$ of X is equal to the largest degree of a generator of the initial ideal $\operatorname{in}(I)$ of I with respect to the reverse-lex order on generic coordinates. In other words, the regularity of X is already detected as soon as one computes Gröbner bases. In the same paper [43], Bayer and Stillman give a computationally efficient method of calculating the regularity of an ideal. □

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[26] Concerning the meaning of regularity for an arbitrary homogeneous ideal, see the comments following Example 1.8.38.

1.8.C Regularity Bounds

The results of the previous section point to the interest in finding upper bounds on the regularity of a projective scheme $X \subseteq \mathbf{P}$ in terms of geometric data. Here we survey without proof some of the main results in this direction. While the picture is not yet complete, a rather fascinating dichotomy emerges, as emphasized in the influential survey [42] of Bayer and Mumford. On the one hand, for arbitrary schemes X — for which essentially best-possible bounds are known — the regularity can grow doubly exponentially as a function of the input parameters. On the other hand, the situation for "nice" varieties is very different: in particular, the regularity of non-singular varieties is known or expected to grow linearly in terms of geometric invariants. (The situation for reduced but possibly singular varieties remains somewhat unclear.)

Gotzmann's bound. The earliest results bounded regularity in terms of Hilbert polynomials. Given a projective scheme $X \subseteq \mathbf{P}$, with ideal sheaf $\mathcal{I} = \mathcal{I}_X \subseteq \mathcal{O}_{\mathbf{P}}$, write

$$Q(k) \;=\; \chi\big(X, \mathcal{O}_X(k)\big)$$

for the Hilbert polynomial of X. In the course of his construction of Grothendieck's Hilbert schemes, Mumford [445] bounded the regularity of X in terms of Q.[27] Although one could render the statements effective, Mumford's arguments were not intended to give sharp estimates. Brodmann has extended Mumford's boundedness theorem in various ways (see [72, Chapters 16 and 17] and the references therein).

Using a different approach, Gotzmann [228] subsequently found the optimal statement in this direction:

Theorem 1.8.35. (Gotzmann's regularity theorem). *There are unique integers*

$$a_1 \;\geq\; a_2 \;\geq\; \ldots \;\geq\; a_s \;\geq\; 0$$

such that $Q(k)$ can be expressed in the form

$$Q(k) \;=\; \binom{k+a_1}{a_1} + \binom{k+a_2-1}{a_2} + \ldots + \binom{k+a_s-(s-1)}{a_s},$$

and then $\mathcal{I}$ is s-regular. $\square$

We refer to [238], [240, §3], and [74] for discussion, proofs, and generalizations.

Example 1.8.36. (One-dimensional schemes). Suppose that $X \subseteq \mathbf{P}^r$ is a one-dimensional scheme of degree d and arithmetic genus $p = 1 - \chi\big(X, \mathcal{O}_X\big)$, so that

$$Q(k) \;=\; dk + (1-p).$$

[27] In fact, Mumford introduced Definition 1.8.1 and proved Theorem 1.8.3 in order to establish the boundedness of the family of all projective subschemes having given Hilbert polynomial.

Then the Gotzmann representation is obtained by taking

$$s \;=\; \binom{d}{2} \,+\, (1-p),$$
$$a_1 \;=\; \ldots \;=\; a_d \;=\; 1 \;\;,\quad a_{d+1} \;=\; \ldots \;=\; a_s \;=\; 0.$$

In particular, X is $\big(\binom{d}{2} + 1 - p\big)$-regular. One can contrast this statement with the Castelnuovo-type bound from [259] for reduced irreducible curves: if $X \subseteq \mathbf{P}^r$ is a non-degenerate curve of degree d, then X is $(d+2-r)$-regular (see 1.8.46). □

Bounds from defining equations. As Bayer remarks, in an actual computation — where a scheme is described by explicit equations — the degrees of generators of an ideal will be known or bounded from the given input data. So it is very natural to look for regularity bounds in terms of these degrees.

We start with a definition:

Definition 1.8.37. (Generating degree of an ideal). The *generating degree* $d(\mathcal{I})$ of an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}}$ is the least integer d such that $\mathcal{I}(d)$ is generated by its global sections. Similarly, if

$$J \;\subseteq\; S \;=\; \mathbf{C}[T_0, \ldots, T_r]$$

is a homogeneous ideal, the *generating degree* $d(J)$ of J is the largest degree of a minimal generator of J. □

Note that if $I \subseteq S$ is the saturated homogeneous ideal determined by an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{P}}$, then $d(I) \geq d(\mathcal{I})$.

Example 1.8.38. (Generating degree of a smooth variety). Let $X \subseteq \mathbf{P}^r$ be a smooth variety of dimension n, and write $\mathcal{I}_X$ for the ideal sheaf of X in $\mathbf{P}^r$. Then

$$d(\mathcal{I}_X) \;\leq\; \deg(X),$$

i.e. X is cut out scheme-theoretically by hypersurfaces of degree $\leq \deg(X)$. (Let $\Lambda \subseteq \mathbf{P}^r$ be a linear space of dimension $r-n-1$ disjoint from X. Then the cone $C_\Lambda(X)$ over X centered on Λ is a hypersurface of degree $d = \deg(X)$ passing through X. As Λ varies, these hypersurfaces generate $\mathcal{I}_X(d)$.) This result is due to Mumford: see [448] for details. □

The first bounds are most naturally stated for homogeneous ideals. One can develop the general theory in this context (cf. [42] or [164]), but for present purposes it is simplest to recall that a homogeneous ideal $I \subseteq S$ is m-regular if I is saturated in degrees $\geq m$ and if the corresponding ideal sheaf $\mathcal{I}$ is m-regular. Bayer and Mumford [42, Proposition 3.8] give a very elementary proof of an essentially doubly exponential bound:

Theorem 1.8.39. (Bound for arbitrary ideals). *If $I \subseteq \mathbf{C}[T_0, \ldots, T_r]$ is any homogeneous ideal, then*

$$\operatorname{reg}(I) \ \leq \ \big(2d(I)\big)^{r!}. \quad \square$$

They also observe [42, Theorem 3.7] that work of Giusti and Galligo leads to the stronger bound $\operatorname{reg}(I) \leq \big(2d(I)\big)^{2^{r-1}}$.

Quite surprisingly, the shape of these statements cannot be avoided: there are examples of ideals whose regularity actually grows doubly exponentially in the generating degree. Indeed, Bayer and Stillman [44] show that a construction used by Mayr and Meyer leads to the following remarkable fact:

> Set $r = 10n$ and fix an integer $d \geq 3$. Then there exists an ideal $I \subseteq \mathbf{C}[T_0, \ldots, T_r]$ with $d(I) = d + 2$ and
>
> $$\operatorname{reg}(I) \ \geq \ (d)^{2^{n-1}}.$$

Note that by introducing a new variable and working on $\mathbf{P}^{r+1}$ one can then get ideal sheaves whose regularity satisfies the analogous bounds. We remark that the example of Mayr–Meyer–Bayer–Stillman is rather combinatorial in nature: it would be interesting to have also more geometric constructions.

By contrast, for the ideals of *non-singular* varieties the picture is completely different. Specifically, a result of Bertram, Ein, and the author from [55] leads to a linear inequality in the generating degree:

Theorem 1.8.40. (Linear bound for smooth ideals). *Let $X \subset \mathbf{P}^r$ be a smooth irreducible complex projective variety of dimension n and codimension $e = r - n$, and set $d = d(\mathcal{I})$. Then*

$$H^i\big(\mathbf{P}^r, \mathcal{I}_X(k)\big) \ = \ 0 \quad \text{for} \quad i \geq 1 \quad \textit{and} \quad k \geq e \cdot d - r.$$

In particular, X is $(ed - e + 1)$-regular.

The proof — which is a very quick application of vanishing theorems — appears in Section 4.3.B below. A more algebraic approach has recently been given by Chardin and Ulrich [87].

Example 1.8.41. The bound in the theorem is achieved by the complete intersection of e hypersurfaces of degree d, so the statement is the best possible. In fact, these are the only borderline cases (Example 1.8.43). $\square$

Example 1.8.42. (Criterion for projective normality). As in Theorem 1.8.40, suppose that $X \subseteq \mathbf{P}^r$ is a smooth subvariety of dimension n and codimension e that is cut out by hypersurfaces of degree d. If $ed \leq r + 1$ then X is projectively normal, and if $ed \leq r$ then X is projectively Cohen–Macaulay. These inequalities apply for instance to n-folds $X \subseteq \mathbf{P}^{2n+1}$ or $X \subseteq \mathbf{P}^{2n}$ that are cut out by quadrics. $\square$

Example 1.8.43. (Borderline cases of regularity bound). In the situation of Theorem 1.8.40, X fails to be $(ed-e)$-regular if and only if it is the transversal complete intersection of e hypersurfaces of degree d. So we may say that complete intersections have the worst regularity among all smooth varieties cut out by hypersurfaces of a given degree. (The vanishings in Theorem 1.8.40 show that if X fails to be $(ed-e)$-regular, then necessarily

$$H^{n+1}\big(\mathbf{P}^r, \mathcal{I}_X(ed-e-n-1)\big) \ = \ H^n\big(X, \mathcal{O}_X(ed-r-1)\big) \neq 0,$$

or equivalently $H^0\big(X, \omega_X \otimes \mathcal{O}_X(r+1-ed)\big) \neq 0$. Assuming for simplicity that X has dimension ≥ 1, choose e general hypersurfaces $D_1, \ldots, D_e$ of degree d passing through X, so that

$$D_1 \cap \cdots \cap D_e \ = \ X \cup X'.$$

But $X \cap X'$ is an effective divisor on X representing the line bundle

$$\mathcal{O}_X(de) \otimes \det(N^*_{X/\mathbf{P}}) \ = \ \omega_X^{-1}(ed-r-1).$$

It follows that $X \cap X' = \varnothing$, and since $X \cup X'$ is in any event connected (e.g. by Theorem 3.3.3), we conclude that $X = D_1 \cap \cdots \cap D_e$. See [55] for details.) □

Remark 1.8.44. (Generators of different degrees). The result established in [55] takes into account generators of different degrees. Specifically, suppose that $X \subset \mathbf{P}^r$ is a non-singular variety of codimension e cut out scheme-theoretically by hypersurfaces of degrees $d_1 \geq d_2 \geq \cdots \geq d_m$. Then

$$H^i\big(\mathbf{P}^r, \mathcal{I}_X^a(k)\big) \ = \ 0 \quad \text{for} \ \ i > 0 \ \ \text{and} \ k \ \geq \ ad_1 + d_2 + \cdots + d_e - r,$$

and in particular X is $(d_1 + \cdots + d_e - e + 1)$-regular. Again this regularity statement is sharp (exactly) for complete intersections. □

Remark 1.8.45. (Regularity of singular subvarieties). It would be very interesting to know to what extent these results remain valid for reduced (but possibly singular) varieties $X \subseteq \mathbf{P}$. One can use multiplier ideals to construct sheaves $\mathcal{J} \subseteq \mathcal{I}_X$ that have the expected regularity, but $\mathcal{J}$ may differ from $\mathcal{I}_X$ along the singular locus of X. See Example 10.1.5 for further discussion. □

Castelnuovo-type bounds. Consider a subvariety $X \subseteq \mathbf{P}^r$ with ideal sheaf $\mathcal{I}_X$. For $i \geq 2$ (and $k \geq -r$) there is an isomorphism

$$H^i\big(\mathbf{P}, \mathcal{I}_X(k)\big) \ = \ H^{i-1}\big(X, \mathcal{O}_X(k)\big).$$

Therefore these groups depend only on the line bundle on X defining the embedding, and in practice they can often be handled relatively easily. Thus the essential point for regularity bounds is usually to control the groups $H^1\big(\mathbf{P}, \mathcal{I}_X(k)\big)$, which measure the failure of hypersurfaces of degree k to cut out a complete linear series on X. This leads to a connection with some classical results of Castelnuovo.

Specifically, let $C \subseteq \mathbf{P}^r$ be a smooth irreducible curve of degree d that doesn't lie in any hyperplanes. Castelnuovo proved that hypersurfaces of degrees $\geq d-2$ cut out a complete linear series on C. By the argument leading up to his celebrated bound on the genus of a space curve, this implies that C is $(d-1)$-regular. The optimal statement along these lines was established by Gruson, Peskine, and the author in [259]:

Theorem 1.8.46. (Regularity for curves). *Let $C \subseteq \mathbf{P}^r$ be an irreducible (but possibly singular) reduced curve of degree d. Assume that C is non-degenerate, i.e. that it doesn't lie in any hyperplanes. Then C is $(d+2-r)$-regular.*

Example 1.8.31 exhibits some borderline examples in $\mathbf{P}^3$. The paper [259] also treats the case of possibly reducible curves.

The natural extrapolation of 1.8.46 to smooth varieties of higher dimension occurred to several mathematicians at the time of [259].

Conjecture 1.8.47. (Castelnuovo-type regularity conjecture). Consider a smooth non-degenerate subvariety $X \subseteq \mathbf{P}^r$ of dimension n and degree d. Then X is $(d+n+1-r)$-regular.

This has been established for surfaces by Pinkham and the author in [505] and [391], and for threefolds by Ran in [511]. We refer to [375] for a nice survey and some extensions. Eisenbud and Goto conjecture in [165] that the bound should hold for any reduced and irreducible non-degenerate variety. For some evidence in this direction see the papers [499], [137] of Peeva–Sturmfels and Derksen–Sidman.

Example 1.8.48. (Mumford's bound). Suppose that $X \subseteq \mathbf{P}^r$ is a smooth subvariety of degree d. Then X is $\big((n+1)(d-1)+1\big)$-regular. (By taking a generic projection, one reduces to the case $r = 2n+1$ and $e = n+1$. Then use the fact that X is scheme-theoretically cut out by hypersurfaces of degree d (Example 1.8.38) and apply 1.8.40.) □

Asymptotic regularity of powers of an ideal. It is a basic principle in commutative algebra that the powers of an ideal often exhibit better behavior than the ideal itself. Recently it has become clear that this holds in particular for Castelnuovo–Mumford regularity.

Specifically, the regularity of powers of an ideal is studied in several papers ([55], [560], [220], [84], [100], [354], [99], [535]). Asymptotically the picture becomes very clean. Most notably, Cutkosky, Herzog, and Trung [100], and independently Kodiyalam [354] prove the appealing result:

Theorem 1.8.49. *Let $I \subset \mathbf{C}[T_0, \ldots, T_r]$ be an arbitrary homogeneous ideal. Then*

$$\lim_{k\to\infty} \frac{\operatorname{reg}(I^k)}{k} \ = \ \lim_{k\to\infty} \frac{d(I^k)}{k} \ \le \ d(I),$$

where as above, $d(J)$ is the generating degree of a homogeneous ideal J. □

These authors also show that the limit in question is an integer. The essential point is to exploit the finite generation of certain Rees rings. Analogous results for ideal sheaves — deduced from Fujita's vanishing theorem — were given in [99], and are reproduced in Section 5.4.

1.8.D Syzygies of Algebraic Varieties

There are a number of natural settings — particularly involving embeddings defined by complete linear series — in which the regularity of a variety carries only coarse geometric information (cf. Example 1.8.52). In his pioneering paper [235], Green observed that in such cases it is interesting to study more delicate algebraic invariants, involving syzygies. This circle of ideas has generated considerable work in recent years, and we present here a quick overview — entirely without proof — of some of the highlights. Eisenbud's forthcoming book [163] contains a detailed introduction from a more algebraic perspective.

Let X be an irreducible projective variety, and L a very ample line bundle on X defining an embedding

$$\phi_L : X \hookrightarrow \mathbf{P} \ = \ \mathbf{P}H^0\big(X, L\big).$$

Consider the graded ring $R_L = R(X,L) = \oplus H^0\big(X, L^{\otimes m}\big)$ determined by L (Definition 2.1.17), and write $S = \operatorname{Sym} H^0\big(X, L\big)$ for the homogeneous coordinate ring of $\mathbf{P}$. Viewed as an S-module, R_L admits as in (1.31) a minimal graded free resolution $E_\bullet$:

$$\begin{array}{ccccccccc}
 & & & & & & S & & \\
\cdots \longrightarrow & \oplus S(-a_{2,j}) & \longrightarrow & \oplus S(-a_{1,j}) & \longrightarrow & & \oplus & \longrightarrow R_L \longrightarrow 0. & \\
 & \| & & \| & & & \oplus S(-a_{0,j}) & & \\
 & \| & & \| & & & \| & & \\
 & E_2 & & E_1 & & & E_0 & &
\end{array}$$

Here the first summand in E_0 corresponds to the degree zero generator of R_L given by the constant function 1, and $a_{0,j} \ge 2$ for every j thanks to the fact that ϕ_L defines a linearly normal embedding. Similarly, a moment's thought reveals that $a_{i,j} \ge i+1$ for all $i \ge 1$ and every j. Observe that ϕ_L defines a projectively normal embedding of X if and only if $E_0 = S$, i.e. the summands $\oplus S(-a_{0,j})$ are not actually present. In this case, the remainder of $E_\bullet$ determines a resolution of the homogeneous ideal $I_{X/\mathbf{P}}$ of X in $\mathbf{P}$.

We ask when the first p terms in $E_\bullet$ are as simple as possible:

Definition 1.8.50. (Property ($\mathbf{N_p}$)). The embedding line bundle L satisfies *Property* (N_p) if $E_0 = S$, and

$$a_{i,j} \;=\; i+1 \quad \text{for all} \;\; j$$

whenever $1 \leq i \leq p$. A divisor D satisfies (N_p) if (N_p) holds for the corresponding line bundle $\mathcal{O}_X(D)$. □

Thus, very concretely:

(N_0) holds for L $\iff$ For every $m \geq 0$, the natural maps

$$S^m H^0\big(X, L\big) \longrightarrow H^0\big(X, L^{\otimes m}\big)$$

are surjective;

(N_1) holds for L $\iff$ (N_0) is satisfied, and the homogeneous ideal $I = I_{X/\mathbf{P}}$ of X in $\mathbf{P}$ is generated by quadrics;

(N_2) holds for L $\iff$ (N_1) is satisfied, and the first module of syzygies among quadratic generators $Q_\alpha \in I$ is spanned by relations of the form

$$\sum L_\alpha Q_\alpha \;=\; 0,$$

where the L_α are linear forms;

and so on. Properties (N_0) and (N_1) were studied by Mumford [448], who called them "normal generation" and "normal presentation" respectively. The present terminology was introduced in [241].

Example 1.8.51. (Rational and elliptic normal curves). A rational normal cubic $C \subseteq \mathbf{P}^3$ satisfies (N_2). An elliptic normal curve $E \subseteq \mathbf{P}^3$ of degree 4 satisfies (N_1) but not (N_2). (C is defined by the maximal minors of a 2×3 matrix of linear forms, and so admits an Eagon–Northcott resolution (EN_0) from Appendix B. Similarly, E is the complete intersection of two quadrics.) □

Example 1.8.52. (Regularity of curves of large degree). Let X be a smooth curve of genus $g \geq 1$, and let L be a line bundle of degree $d \geq 2g+1$. Then L defines an embedding $X \subseteq \mathbf{P}^{d-g}$ in which X is 3-regular but not 2-regular. This uniform behavior of Castelnuovo–Mumford regularity contrasts with a number of interesting results and questions relating the syzygies to the geometry of X and L: see 1.8.53, 1.8.54, and 1.8.58. □

Curves. Consider to begin with a non-singular projective curve X of genus g, and a line bundle L on X. A classical result of Castelnuovo, Mattuck [421] and Mumford [448] states that if $\deg L \geq 2g+1$ then L is normally generated, while Fujita [191] and Saint-Donat [521] showed that if $\deg L \geq 2g+2$ then

L is normally presented. Green [235] proved that these are special cases of a general result for higher syzygies:

Theorem 1.8.53. (Syzygies of curves of large degree). *Assume that* $\deg L \geq 2g + 1 + p$. *Then* (N_p) *holds for* L.

The crucial point for the proof is to interpret syzygies via Koszul cohomology. A quick formulation using vector bundles appears in [392].

Remark 1.8.54. (Borderline cases). The inequality in 1.8.53 is best possible. In fact, it is established in [243] that if L is a line bundle of degree $2g + p$, then (N_p) fails for L if and only if either X is hyperelliptic or X has a $(p + 2)$-secant p-plane in the embedding defined by L. □

The most interesting statements concern canonical curves, where $L = \mathcal{O}_X(K_K)$ is the canonical bundle. Here there are again two classical results: Noether's theorem that K_X defines a projectively normal embedding if X is non-hyperelliptic, and Petri's theorem that the homogeneous ideal of a canonical curve $X \subseteq \mathbf{P}^{g-1}$ is generated by quadrics unless X is trigonal or a smooth plane quintic (cf. [15, Chapter III, §2, §3]). Green realized that this should generalize to higher syzygies via the Clifford index:

Definition 1.8.55. (Clifford index). Let A be a line bundle on a smooth curve X. The *Clifford index* of A is

$$\mathrm{Cliff}(A) \ = \ \deg(A) - 2r(A),$$

where as usual $r(A) = h^0\big(X, A\big) - 1$. The Clifford index of X itself is

$$\mathrm{Cliff}(X) \ = \ \min \big\{ \mathrm{Cliff}(A) \mid h^0\big(X, A\big) \geq 2 \ , \ h^1\big(X, A\big) \geq 2. \big\}. \quad \square$$

Thus Clifford's theorem states that $\mathrm{Cliff}(X) \geq 0$, with equality if and only if X is hyperelliptic. Similarly, $\mathrm{Cliff}(X) = 1$ if and only if X is trigonal or a smooth plane quintic. If X is a general curve of genus g, then $\mathrm{Cliff}(X) = \big[\frac{g-1}{2}\big]$.

The natural extension of the theorems of Noether and Petri is contained in a celebrated conjecture of Green:

Conjecture 1.8.56. (Green's conjecture on canonical curves). The Clifford index $\mathrm{Cliff}(X)$ is equal to the least integer p for which Property (N_p) fails for the canonical divisor K_X.

One direction is elementary: it was established by Green and the author in [235, Appendix] that if $\mathrm{Cliff}(X) = e$, then (N_e) fails. What seems very difficult is to start with a syzygy and produce a line bundle. The first non-classical case $p = 2$ was treated by Schreyer [527] and Voisin [596].

The most significant progress to date on Green's conjecture is due to Voisin [598], [601], who proves that it holds for general curves:

Theorem 1.8.57. (Voisin's theorem on canonical curves). *If* X *is a general curve of genus* g, *then Green's conjecture holds for* X.

At least in the case of even genus, the idea is to study curves lying on a suitably generic $K3$ surface: it was established in [390] that such curves are Brill–Noether general, so this is a natural place to look. By a number of deep calculations, Voisin is able to establish a vanishing on the surface that implies 1.8.57. Cases of 1.8.57 had been obtained previously by Teixiodor.

Remark 1.8.58. (Further conjectures for curves). Green and the author proposed in [241] a more general conjecture. Specifically, suppose that L is a very ample line bundle with

$$\deg L \;\geq\; 2g + 1 + p - 2 \cdot h^1\big(X, L\big) - \mathrm{Cliff}(X).$$

Then (N_p) should hold for L unless ϕ_L embeds X with a $(p{+}2)$-secant p-plane. The case $p = 0$ is treated in [241]. In another direction, one can consider the full resolution of a curve X of large degree. Conjecture 3.7 of [241] asserts that its grading depends (in a precise way) only on the gonality of X. This has recently been established for general curves of large gonality by Aprodu and Voisin [12] using the ideas of Voisin from [598]. $\square$

Abelian varieties. Embeddings of abelian varieties were also considered classically from an algebraic perspective. Let X be an abelian variety of dimension g, and let L be an ample line bundle on X. It is a classical theorem of Lefschetz that $L^{\otimes m}$ is very ample provided that $m \geq 3$, and Mumford, Koizumi [355] and Sekiguchi [529] proved that $L^{\otimes m}$ is normally generated in this case. Mumford [448] and Kempf [333] proved that if $m \geq 4$, then X is cut out by quadrics under the embedding defined by $L^{\otimes m}$. The author remarked that these statements admit a natural extrapolation to higher syzygies, and the resulting conjecture was established by Pareschi [495]:

Theorem 1.8.59. (Pareschi's theorem). *Property (N_p) holds for $L^{\otimes m}$ as soon as $m \geq p + 3$.*

Pareschi's theorem has been systematized and extended by Pareschi and Popa through their work on regularity for abelian varieties ([496], [497]).

Varieties of arbitrary dimension. Inspired by Fujita's conjectures (Section 10.4.A), Mukai observed that one can rephrase Green's Theorem 1.8.53 as asserting that if A is an ample divisor on a curve C then $D = K_C + (p+3)A$ satisfies (N_p). Given an ample divisor A on a smooth projective variety X of dimension n, he remarked that it is then natural to wonder whether $D = K_X + (n + p + 2)A$ satisfies (N_p). At the moment this seems completely out of reach: even Fujita's conjecture that the divisor in question is very ample when $p = 0$ remains very much open as of this writing.

However, the situation becomes much simpler if one works with very ample instead of merely ample divisors. Specifically, Ein and the author [153] used vanishing theorems for vector bundles to establish:

Theorem 1.8.60. (Syzygies of "hyperadjoint" divisors). *Let X be a smooth projective variety of dimension n, let B be a very ample divisor on X, and let P be any nef divisor. Then*

$$D = K_X + (n+1+p)B + P$$

satisfies Property (N_p).

When $p = 0$ the assertion is that $K_X + (n+1)B$ defines a projectively normal embedding of X: a simple proof appears in Example 4.3.19. (Note that if $X = \mathbf{P}^n$, $p = 0$, and B is a hyperplane divisor, then D is trivial, so the statement needs to be properly interpreted. However in all other cases the divisor in question is actually very ample.) Syzygies of surfaces have been studied by Gallego and Purnaprajna [218], [217].

Notes

The material in Sections 1.1–1.4 is for the most part classical, although the contemporary outlook puts greater emphasis on nef bundles and **Q**-divisors than earlier perspectives. Chapter 1 of Hartshorne's notes [276] remains an excellent source for the basic theory of ample and nef divisors. We have also drawn on [363, Chapter VI] and [368, Chapter 1.5], as well as Debarre's presentation [114]. Theorem 1.4.40 (at least for locally free sheaves on smooth varieties) appears in [126].

The essential facts about Castelnuovo–Mumford regularity are present or implicit in [445] and [448]. Examples 1.8.18, 1.8.22, and 1.8.24 are generalized folklore, while Proposition 1.8.9 was noted by the author some years ago in response to a question from Ein. Special cases of Mumford's Theorem 1.8.5 have been rediscovered repeatedly in the literature in connection with vanishing theorems.