Introduction to Part Two

The three chapters in this Part Two deal with amplitude for vector bundles of higher rank.

Starting in the early 1960s, several mathematicians — notably Grauert [232], Griffiths [245], [246], [247] and Hartshorne [274] — undertook the task of generalizing to vector bundles the theory of positivity for line bundles. One of the goals was to extend to the higher-rank setting as many as possible of the beautiful cohomological and topological properties enjoyed by ample divisors. It was not initially clear how to achieve this, and the literature of the period is marked by a certain terminological chaos as authors experimented with various definitions and approaches. With the passage of time, however, it has become apparent that Hartshorne's definition — which involves the weakest notion of positivity — does in fact lead to most of the basic results one would like. The idea is simply to pass to the associated projective bundle, where one reduces to the rank one case. This approach is by now standard, and it is the one we adopt here.

After the foundational work of the late sixties and early seventies, the geometric consequences of positivity were substantially clarified during the later seventies and the eighties. These same years brought many new examples and applications of the theory. While aspects of this story have been surveyed on several occasions — e.g. [533, Chapter V], [15, Chapter 5, §1] and [352, Chapter 3, §6] — there hasn't been a systematic exposition of the theory taking these newer developments into account. Our aim in these chapters is to help fill this gap. Although we certainly make no claims to completeness, we hope that the reader will take away the picture of a mature body of work touching on a considerable range of questions.

We start in Chapter 6 with the basic formal properties of ample and nef bundles, after which we dwell at length on examples and constructions. Chapter 7 deals with geometric properties of ample bundles: we discuss higher-rank generalizations of the Lefschetz hyperplane and Kodaira vanishing theorems, as well as the effect of positivity on degeneracy loci. Finally, we take up in Chapter 8 the numerical properties of ample bundles.

In contrast to our focus in the case of line bundles, we say very little about the various notions of "generic amplitude" for bundles that appear in the literature. Some of these — for example the weak positivity of Viehweg or Miyaoka's generic semipositivity — have led to extremely important developments. However these concepts are tied to particular applications lying beyond the scope of this volume. Our feeling was that they are best understood in vivo rather than through the sort of in vitro presentation that would have been possible here. So for the most part we (regretfully) pass over them.

6

Ample and Nef Vector Bundles

This chapter is devoted to the basic theory of ample and nef vector bundles. We start in Section 6.1 with the "classical" material from [274]. In Section 6.2 we develop a formalism for twisting bundles by **Q**-divisor classes, which is used to study nefness. The development parallels — and for the most part reduces to — the corresponding theory for line bundles. The next two sections constitute the heart of the chapter. In the extended Section 6.3 we present numerous examples of positive bundles arising "in nature," as well as some methods of construction. Finally, we study in Section 6.4 the situation on curves, where there is a close connection between amplitude and stability: following Gieseker [224] one obtains along the way an elementary proof of the tensorial properties of semistability for bundles on curves.

6.1 Classical Theory

We start by fixing notation and assumptions. Throughout this section, unless otherwise stated X is a projective algebraic variety or scheme defined over $\mathbf{C}$. Given a vector bundle E on X, $S^m E$ is the m^{th} symmetric product of E, and

$$\pi : \mathbf{P}(E) \longrightarrow X$$

denotes the projective bundle of one-dimensional quotients of E. On occasion, when it seems desirable to emphasize that $\mathbf{P}(E)$ is a bundle over X, we will write $\mathbf{P}_X(E)$ in place of $\mathbf{P}(E)$. As usual, $\mathcal{O}_{\mathbf{P}(E)}(1)$ is the Serre line bundle on $\mathbf{P}(E)$, i.e. the tautological quotient of $\pi^* E$: thus $S^m E = \pi_* \mathcal{O}_{\mathbf{P}(E)}(m)$. We refer to Appendix A for a review of basic facts about projective bundles.

As in Chapter 1 (Definition 1.1.15), we denote by

$$N^1\big(\mathbf{P}(E)\big) \;=\; \mathrm{Div}\big(\mathbf{P}(E)\big)\,/\,\mathrm{Num}\big(\mathbf{P}(E)\big)$$

the Néron–Severi group of numerical equivalence classes of divisors on $\mathbf{P}(E)$. Since X is projective the Serre line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$ is represented by a divisor, and we write

$$\xi = \xi_E \in N^1\big(\mathbf{P}(E)\big)$$

for its numerical equivalence class: in other words, ξ_E corresponds to the first Chern class of $\mathcal{O}_{\mathbf{P}(E)}(1)$.[1] Finally, given a finite-dimensional vector space V, $V_X = V \otimes_{\mathbf{C}} \mathcal{O}_X$ denotes the trivial vector bundle on X with fibres modeled on V.

6.1.A Definition and First Properties

In the case of line bundles, essentially all notions of positivity turn out to be equivalent, but this is no longer true for vector bundles of higher rank. Consequently the early literature in the area has an experimental flavor, involving competing definitions of positivity. By the 1980s, however, the situation had stabilized, with the approach adopted by Hartshorne [274] generally accepted as the most useful.

Hartshorne's basic idea is to reduce the definition of amplitude for a bundle E to the corresponding notion for divisors by passing to $\mathbf{P}(E)$:

Definition 6.1.1. (Ample and nef vector bundles). A vector bundle E on X is *ample* if the Serre line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$ is an ample line bundle on the projectivized bundle $\mathbf{P}(E)$. Similarly, E is *numerically effective* (or *nef*) if $\mathcal{O}_{\mathbf{P}(E)}(1)$ is so. □

Observe that if $E = L$ is a line bundle, then $\mathbf{P}(E) = X$ and $\mathcal{O}_{\mathbf{P}(E)}(1) = L$, so that at least this definition generalizes the rank one case. We will see that 6.1.1 leads to the formal properties of amplitude for which one would hope. In this subsection we focus on results that follow directly from corresponding facts for line bundles.

The first statement of the following proposition reflects the principle that "positivity increases in quotients." The second includes the fact that the restriction of an ample bundle to a closed subvariety (or subscheme) is ample.

Proposition 6.1.2. (Quotients and finite pullbacks). *Let E be a vector bundle on the projective variety or scheme X.*

(i). *If E is ample (or nef), then so is any quotient bundle Q of E.*

(ii). *Let $f : X \longrightarrow Y$ be a finite mapping. If E is ample (or nef), then the pullback f^*E is an ample (or nef) bundle on X.*

Proof. A surjection $E \twoheadrightarrow Q$ determines an embedding

$$\mathbf{P}(Q) \subseteq \mathbf{P}(E) \qquad \text{with} \qquad \mathcal{O}_{\mathbf{P}(Q)}(1) = \mathcal{O}_{\mathbf{P}(E)}(1) \,|\, \mathbf{P}(Q),$$

[1] As indicated in Remark 1.1.22, one can work on arbitrary complete schemes provided that one understands $N^1\big(\mathbf{P}(E)\big)$ to be numerical equivalence classes of line bundles.

and (i) follows from the fact that the restriction of an ample (or a nef) bundle is ample (or nef). As for (ii), f gives rise to a finite map $F : \mathbf{P}(f^*E) \longrightarrow \mathbf{P}(E)$ such that $\mathcal{O}_{\mathbf{P}(f^*E)}(1) = F^*\mathcal{O}_{\mathbf{P}(E)}(1)$, so again we reduce to the corresponding statement (Proposition 1.2.13) for line bundles. $\square$

Example 6.1.3. (Bundles on $\mathbf{P}^1$). Let E be a vector bundle on the projective line $\mathbf{P}^1$. According to a celebrated theorem of Grothendieck [488, Chapter 1, Section 2.1], E is a direct sum of line bundles, say

$$E \;=\; \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbf{P}^1}(a_e).$$

Then E is ample if and only if $a_i > 0$ for every $1 \le i \le e$. (If each a_i is positive, then $\mathcal{O}_{\mathbf{P}(E)}(1)$ embeds $\mathbf{P}(E)$ as a rational normal scroll.) This is a special case of Proposition 6.1.13. $\square$

Example 6.1.4. (Pullbacks of negative bundles). Let $f : Y \longrightarrow X$ be a surjective morphism of projective varieties, with $\dim X \ge 1$, and let E be an ample bundle on X. Then $H^0\big(Y, f^*E^*\big) = 0$. (It is enough to show that if $C \subseteq Y$ is a general curve obtained as the complete intersection of very ample divisors on Y then $\mathrm{Hom}\big(f^*E \,|\, C\,,\, \mathcal{O}_C\big) = 0$. One can assume that C maps finitely to X, and then the assertion follows from 6.1.2 upon normalizing C.) $\square$

As another illustration, we analyze the case of globally generated bundles:

Example 6.1.5. (Globally generated bundles). Let V be a finite-dimensional vector space, and let E be a quotient

$$V_X \longrightarrow E \longrightarrow 0$$

of the trivial vector bundle V_X modeled on V. This gives rise to a morphism

$$\phi : \mathbf{P}_X(E) \longrightarrow \mathbf{P}(V) \tag{6.1}$$

from the projective bundle $\mathbf{P}_X(E) = \mathbf{P}(E)$ to the projective space of one dimensional quotients of V, defined as the composition

$$\mathbf{P}_X(E) \;\subseteq\; \mathbf{P}(V_X) \;=\; \mathbf{P}(V) \times X \xrightarrow{\mathrm{pr}_1} \mathbf{P}(V).$$

By construction, $\mathcal{O}_{\mathbf{P}(E)}(1) = \phi^*\mathcal{O}_{\mathbf{P}(V)}(1)$.

(i). E is ample if and only if ϕ is finite.

(ii). Given $x \in X$, denote by $E(x)$ the fibre of E at x, so that $E(x)$ is a quotient of V, and let

$$\mathbf{P}(E(x)) \;\subseteq\; \mathbf{P}(V)$$

be the corresponding linear subspace of $\mathbf{P}(V)$. Then E is ample if and only if there are only finitely many $x \in X$ such that $\mathbf{P}(E(x))$ passes through any given point of $\mathbf{P}(V)$. $\square$

Example 6.1.6. (Tautological bundle on the Grassmannian). Let $\mathbf{G} = \mathrm{Grass}(V, k)$ be the Grassmannian of k-dimensional quotients of a vector space V, and denote by E the tautological rank-k quotient bundle of $V_{\mathbf{G}}$. If $k \geq 2$ then E is nef but *not* ample. (Use 6.1.5 (ii).) In particular, it would be incorrect to try by analogy with Definition 1.2.1 to define amplitude naively in terms of embeddings into Grassmannians. □

For globally generated bundles, amplitude also is equivalent to the absence of trivial quotients along curves:

Proposition 6.1.7. (Gieseker's lemma). *Suppose that E is a globally generated bundle on an irreducible projective variety X. Then E fails to be ample if and only if there is a curve $C \subseteq X$ such that the restriction $E \mid C$ admits a trivial quotient.*

Proof. If E is ample, then it follows from Proposition 6.1.2 that no restriction of E can have a trivial quotient. Conversely, suppose that E fails to be ample. Set $V = H^0\big(X, E\big)$, and consider the morphism $\phi : \mathbf{P}(E) = \mathbf{P}_X(E) \longrightarrow \mathbf{P}(V)$ appearing in (6.1). By 6.1.5 (i), ϕ fails be finite, i.e. there exists a curve $C \subset \mathbf{P}(E)$ contracted by ϕ to a point. But ϕ is in any event an embedding on each fibre of $\pi : \mathbf{P}(E) \longrightarrow X$, and hence C maps isomorphically to its image in X. Moreover

$$\mathcal{O}_{\mathbf{P}(E)}(1) \mid C \ = \ \phi^*\mathcal{O}_{\mathbf{P}(V)}(1) \mid C \ ,$$

and since $\mathcal{O}_{\mathbf{P}(E)}(1)$ is a quotient of π^*E, we arrive at a trivial quotient of $E \,|\, C$. □

We refer to Section 6.3 for more substantial examples and methods of construction. Here we return to developing the general theory.

As in the case of line bundles, the first two parts of the following proposition allow one in practice to focus attention on reduced and irreducible varieties:

Proposition 6.1.8. (Additional formal properties of amplitude). *Let E be a vector bundle on a projective variety or scheme X.*

(i). *E is ample (or nef) if and only if the restriction E_{red} of E to X_{red} is ample (or nef).*

(ii). *E is ample (or nef) if and only if its restriction to each irreducible component of X is ample (or nef).*

(iii). *If $f : Y \longrightarrow X$ is a surjective finite map, and if f^*E is an ample vector bundle on Y, then E is ample.*

(iv). *If $f : Y \longrightarrow X$ is an arbitrary surjective mapping, and if f^*E is a nef bundle on Y, then E itself is nef.*

Proof. By passing to $\mathbf{P}(E)$, these statements again follow immediately from the corresponding facts (1.2.16, 1.2.28, 1.4.4) for line bundles. □

The next result expresses the strong open nature of amplitude in families:

Proposition 6.1.9. (Amplitude in families). *Let $f : X \longrightarrow T$ be a proper surjective mapping, and suppose that E is a vector bundle on X. Given $t \in T$, denote by X_t the fibre of X over t, and by $E_t = E \mid X_t$ the restriction of E to X_t. If there is a point $0 \in T$ such that E_0 is ample, then there is an open neighborhood $U \subset T$ of 0 such that E_t is ample for all $t \in U$.*

Proof. Since $\mathbf{P}(E_t)$ is the fibre of the evident map $\mathbf{P}(E) \longrightarrow T$, this yet again follows directly from the corresponding statement (Theorem 1.2.17) for line bundles. □

6.1.B Cohomological Properties

We next analyze the asymptotic cohomological properties of ample vector bundles. The following theorem of Hartshorne [274] is the analogue for vector bundles of the theorem of Cartan–Serre–Grothendieck (Theorem 1.2.6).

Theorem 6.1.10. (Cohomological characterization of ample vector bundles). *Let E be a vector bundle on the projective variety or scheme X. The following are equivalent:*

(i). *E is ample.*

(ii). *Given any coherent sheaf $\mathcal{F}$ on X, there is a positive integer $m_1 = m_1(\mathcal{F})$ such that*

$$H^i\big(X, S^m E \otimes \mathcal{F}\big) \;=\; 0 \quad \textit{for} \quad i > 0\ ,\ m \geq m_1.$$

(iii). *Given any coherent sheaf $\mathcal{F}$ on X, there is a positive integer $m_2 = m_2(\mathcal{F})$ such that $S^m E \otimes \mathcal{F}$ is globally generated for all $m \geq m_2$.*

(iv). *For any ample divisor H on X, there is a positive integer $m_3 = m_3(H)$ such that if $m \geq m_3$ then $S^m E$ is a quotient of a direct sum of copies of $\mathcal{O}_X(H)$.*

(iv)*. *The statement of* (iv) *holds for some ample divisor H.*

Proof. (i) ⇒ (ii). Denoting as above by $\pi : \mathbf{P}(E) \longrightarrow X$ the bundle map, assume that E is ample, so that $\mathcal{O}_{\mathbf{P}(E)}(1)$ is an ample line bundle on $\mathbf{P}(E)$, let $\mathcal{F}$ be a coherent sheaf on X. Then by Serre's criterion (Theorem 1.2.6) there is an integer $m_1 = m_1(\mathcal{F})$ such that

$$H^i\big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(m) \otimes \pi^*\mathcal{F}\big) \;=\; 0 \quad \text{for} \quad i > 0\ ,\ m \geq m_1. \qquad (*)$$

Now suppose for the moment that $\mathcal{F}$ is locally free. Then the projection formula implies that

$$\pi_*\big(\mathcal{O}_{\mathbf{P}(E)}(m) \otimes \pi^*\mathcal{F}\big) \;=\; \pi_*\big(\mathcal{O}_{\mathbf{P}(E)}(m)\big) \otimes \mathcal{F} \;=\; S^m E \otimes \mathcal{F},$$

and by the same token the higher direct images vanish provided that $m \geq 0$. Therefore

$$H^i\big(X, S^m E \otimes \mathcal{F}\big) \quad = \quad H^i\big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(m) \otimes \pi^*\mathcal{F}\big),$$

and the required vanishings follow from (*). For an arbitrary coherent sheaf $\mathcal{F}$ we can find a (possibly non terminating) resolution

$$\ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

of $\mathcal{F}$ by locally free sheaves (Example 1.2.21). If $\dim X = n$, it is sufficient in view of Proposition B.1.2 from Appendix B to establish that

$$H^i\big(X, S^m E \otimes F_n\big) = \cdots = H^i\big(X, S^m E \otimes F_0\big) \; = \; 0 \;\; \text{for } i > 0 \text{ and } m \gg 0,$$

and this follows from the case already treated.[2]

(ii) $\Rightarrow$ (iii). Given $\mathcal{F}$, fix a point $x \in X$, with maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_X$. Consider the subsheaf $\mathfrak{m}_x \mathcal{F} \subset \mathcal{F}$ of germs of sections of $\mathcal{F}$ that vanish at x, so that one has the exact sequence

$$0 \longrightarrow \mathfrak{m}_x\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathfrak{m}_x\mathcal{F} \longrightarrow 0.$$

By (ii) there exists a positive integer $m_2(\mathcal{F}, x)$ such that $H^1(X, S^m E \otimes \mathfrak{m}_x \mathcal{F}) = 0$ for $m \geq m_2(\mathcal{F}, x)$, and so we see using the sequence above that $S^m E \otimes \mathcal{F}$ is generated by its sections at x. The same therefore holds in a Zariski open neighborhood of x, and consequently by quasi-compactness we can find a single integer $m_2 = m_2(\mathcal{F})$ such that $S^m E \otimes \mathcal{F}$ is globally generated when $m \geq m_2$.

(iii) $\Rightarrow$ (iv). Apply (iii) with $\mathcal{F} = \mathcal{O}_X(-H)$.

(iv) $\Rightarrow$ (iv)*. Tautological.

(iv)* $\Rightarrow$ (i). It follows from (iv)* that there is an ample divisor H on X and a positive integer $m > 0$ such that we can write $S^m E$ as a quotient of $U = \mathcal{O}_X(H)^{\oplus N+1}$ for some N. Note first that U is ample.[3] In fact, $\mathbf{P}(U)$ is isomorphic to the product $\mathbf{P}^N \times X$ in such a way that

$$\mathcal{O}_{\mathbf{P}(U)}(1) = \mathrm{pr}_1^* \mathcal{O}_{\mathbf{P}^N}(1) \otimes \mathrm{pr}_2^* \mathcal{O}_X(H),$$

and this is an ample line bundle on $\mathbf{P}^N \times X$. Since $S^m E$ is a quotient of U, it follows that $S^m E$ is ample. But one has a Veronese embedding

$$\mathbf{P}(E) \hookrightarrow \mathbf{P}(S^m E) \quad \text{with} \quad \mathcal{O}_{\mathbf{P}(S^m E)}(1) \,|\, \mathbf{P}(E) \; = \; \mathcal{O}_{\mathbf{P}(E)}(m).$$

Therefore $\mathcal{O}_{\mathbf{P}(E)}(m)$ — and hence also $\mathcal{O}_{\mathbf{P}(E)}(1)$ — is ample, as required. □

[2] One can avoid this step by observing directly that

$$R^i \pi_*\big(\mathcal{O}_{\mathbf{P}(E)}(m) \otimes \pi^* \mathcal{F}\big) \; = \; R^i \pi_*\big(\mathcal{O}_{\mathbf{P}(E)}(m)\big) \otimes \mathcal{F}$$

for any coherent sheaf $\mathcal{F}$ on X: see [274, Lemma 3.1].

[3] Compare the next proposition for a more general statement.

Remark 6.1.11. The theorem shows in effect that amplitude of a bundle E is detected by geometric properties of a high symmetric power S^mE. However in contrast to the situation for line bundles — where for instance one can use covering constructions (e.g. Theorem 4.1.10) to replace a divisor by a multiple — it is often not obvious how to pass from information about S^mE to a statement for E itself. In practice, this is the essential difficulty in working with amplitude for bundles of higher rank. □

Remark 6.1.12. The equivalence of statements (i), (ii), and (iii) remains valid on an arbitrary complete (but possibly non-projective) scheme over $\mathbf{C}$: see [274], (2.1), (3.2) and (3.3). On a non-complete scheme, (iii) is taken as the definition of amplitude. □

Turning to direct sums and extensions, there is the useful

Proposition 6.1.13. (Direct sums and extensions). *Let E_1 and E_2 be vector bundles on X.*

(i). *The direct sum $E_1 \oplus E_2$ is ample if and only if both E_1 and E_2 are.*

(ii). *Suppose that F is an extension of E_2 by E_1:*

$$0 \longrightarrow E_1 \longrightarrow F \longrightarrow E_2 \longrightarrow 0. \tag{6.2}$$

If E_1 and E_2 are ample, then so is F.

Proof. (i). If $E_1 \oplus E_2$ is ample, then so are its quotients E_1 and E_2. Conversely, assume that E_1 and E_2 are ample. The plan is to apply criterion (iii) from 6.1.10, so fix a coherent sheaf $\mathcal{F}$ on X. We need to show that $S^m(E_1 \oplus E_2) \otimes \mathcal{F}$ is globally generated for $m \gg 0$. But $S^m(E_1 \oplus E_2) = \sum_{p+q=m} S^pE_1 \otimes S^qE_2$, so the question is to find an integer m_0 such that

$$S^pE_1 \otimes S^qE_2 \otimes \mathcal{F} \quad \text{is globally generated for} \quad p+q \geq m_0. \tag{*}$$

To this end, first use the amplitude of E_1 and E_2 to choose $t_1 > 0$ such that

$$S^tE_1 \ \ , \ \ S^tE_2 \otimes \mathcal{F} \quad \text{are globally generated for} \quad t \geq t_1.$$

Next, for $k = 0, 1, \ldots, t_1$ apply 6.1.10 (iii) to each of the coherent sheaves $S^kE_2 \otimes \mathcal{F}$ and $S^kE_1 \otimes \mathcal{F}$ to produce t_2 such that

$$S^tE_1 \otimes S^kE_2 \otimes \mathcal{F} \ , \ S^kE_1 \otimes S^tE_2 \otimes \mathcal{F} \quad \text{are globally generated for} \quad t \geq t_2.$$

We claim that then (*) holds with $m_0 = t_1 + t_2$. In fact, suppose that $p + q \geq t_1 + t_2$. If $p, q \geq t_1$ then S^pE_1 and $S^qE_2 \otimes \mathcal{F}$ are globally generated, and hence so is their tensor product. If $p \leq t_1$ then $q \geq t_2$, and so (*) holds by choice of t_2, and similarly if $q \leq t_1$. This proves (i).

For (ii) we apply 6.1.10 (ii), so again fix a coherent sheaf $\mathcal{F}$. Now S^mF has a filtration whose quotients are $S^p(E_1) \otimes S^q(E_2)$ with $p + q = m$. So (using Lemma B.1.7) it is enough to prove the vanishings

$$H^i(X, S^pE_1 \otimes S^qE_2 \otimes \mathcal{F}) = 0 \ \text{ for } i > 0 \text{ and } p + q = m \gg 0. \qquad (**)$$

But by part (i) of the proposition we already know that $E_1 \oplus E_2$ is ample, and then (**) follows upon applying 6.1.10 (ii) to this bundle. □

Example 6.1.14. One can also deduce statement (ii) from (i) via Proposition 6.1.9. In fact, by scaling the extension class defining (6.2), the bundle F is realized as a small deformation of $E_1 \oplus E_2$, and then 6.1.9 applies. □

At least in characteristic zero, the tensorial properties of amplitude flow from the next result.

Theorem 6.1.15. (Symmetric products). *A vector bundle E on X is ample if and only if S^kE is ample for some — or equivalently, for all — $k \geq 1$.*

Proof. Supposing first that S^kE is ample, the amplitude of E follows via the Veronese embedding $\mathbf{P}(E) \subseteq \mathbf{P}(S^kE)$ as in the argument that (iv)* ⇒ (i) in 6.1.10. Conversely, assume that E is ample. We first note that S^mE is ample for all $m \gg 0$. In fact, according to Theorem 6.1.10 (iv), for large m the bundle in question is a quotient of a direct sum of ample line bundles. It remains to deduce that S^kE is ample for any fixed $k \geq 1$. To this end, we again use a Veronese-type morphism. Specifically, we assert that (in characteristic zero!) there exists for any $\ell \geq 1$ a finite mapping

$$\nu_{\ell,k} \, : \, \mathbf{P}(S^kE) \longrightarrow \mathbf{P}(S^{k\ell}E) \qquad (6.3)$$

with $\mathcal{O}_{\mathbf{P}(S^kE)}(\ell) = \nu_{\ell,k}^*\mathcal{O}_{\mathbf{P}(S^{\ell k}E)}(1)$. Grant this for the moment. For $\ell \gg 0$ we already know that $\mathcal{O}_{\mathbf{P}(S^{k\ell}E)}(1)$ is ample, and hence so is $\mathcal{O}_{\mathbf{P}(S^kE)}(\ell)$. As before, this implies that S^kE is ample, as required.

It remains to prove the existence of $\nu_{\ell,k}$. Recall to begin with that there are canonical homomorphisms

$$S^{\ell k}E \xhookrightarrow{i_{\ell,k}} S^\ell\big(S^kE\big) \xrightarrow{m_{\ell,k}} S^{\ell k}E,$$

with $m_{\ell,k}$ given by multiplication. The composition $m_{\ell,k} \circ i_{\ell,k}$ is multiplication by a fixed non-zero integer $C_{\ell,k}$, and — since we are in characteristic zero — is therefore a homothety. Now write $\pi_k : \mathbf{P}(S^kE) \longrightarrow X$ for the bundle map. Composing $\pi_k^*(i_{\ell,k})$ with the ℓ^{th} symmetric power of the canonical quotient $\pi_k^*S^kE \longrightarrow \mathcal{O}_{\mathbf{P}(S^kE)}(1)$ gives a vector bundle morphism

$$\pi_k^* \, S^{\ell k}E \longrightarrow \mathcal{O}_{\mathbf{P}(S^kE)}(\ell). \qquad (*)$$

We will show that (*) is surjective: it then defines the required mapping $\nu_{\ell,k}$.

The surjectivity of (*) can be checked fibre by fibre over X, so we assume that E is a vector space. Given a non-zero functional $\phi : S^kE \longrightarrow \mathbf{C}$, and its ℓ^{th} symmetric power $S^\ell\phi : S^\ell(S^kE) \longrightarrow \mathbf{C}$, the issue is to show that

$$\mathrm{im}(i_{\ell,k}) \not\subseteq \ker(S^\ell \phi). \qquad (**)$$

Write $N = \ker(\phi)$, choose a one-dimensional subspace $L \subseteq S^k E$ splitting ϕ, and denote by $L^\ell \subseteq S^{\ell k}E$ the image of $S^\ell L$ under the multiplication $m_{\ell,k}$. Then

$$\ker(S^\ell \phi) = S^\ell N + S^{\ell-1}N \cdot L + \ldots + N \cdot S^{\ell-1}L.$$

In particular, L^ℓ is not in the image of $\ker(S^\ell\phi)$ under $m_{\ell,k}$. By the remark at the beginning of the previous paragraph, this implies that $S^\ell\phi \circ i_{\ell,k}$ does not vanish on L^ℓ, and (**) is established, giving the surjectivity of (*). A similar analysis shows that $\nu_{\ell,k}$ is one-to-one. This completes the proof. □

Corollary 6.1.16. (Amplitude of tensor products). *Let E and F be vector bundles on a projective variety or scheme X.*

(i). *If E and F are ample, then so is $E \otimes F$. Consequently any tensor power $T^q(E)$ of E is ample.*

(ii). *If E is ample, then so are all of its exterior powers. More generally, for any Young diagram λ denote by $\Gamma^\lambda E$ the bundle deduced from E via the representation of the general linear group corresponding to λ.*[4] *If E is ample, then so is $\Gamma^\lambda E$.*

Proof. Since E and F are ample, so is $E \oplus F$ and consequently also $S^2(E \oplus F)$ (6.1.13, 6.1.15). But $E \otimes F$ is a summand of $S^2(E \oplus F)$, and hence is ample. The amplitude of $T^q(E)$ follows by induction. The exterior products $\Lambda^i E$ and more generally $\Gamma^\lambda E$ are quotients of $T^q(E)$ for suitable q, and (ii) follows. □

Remark 6.1.17. (Positive characteristics). All the results that have appeared so far remain valid for varieties defined over an algebraically closed field of arbitrary characteristic. The proof of Theorem 6.1.15 requires characteristic zero. However Barton [34] proved by other methods that the tensor product of ample vector bundles is ample in arbitrary characteristic, so 6.1.15 and 6.1.16 remain true. (In 6.1.16 (ii) one should define $\Gamma^\lambda E$ as in [207, Chapter 8.1], so that it is a quotient of a tensor power of E.) □

6.1.C Criteria for Amplitude

We now indicate a few criteria for amplitude and nefness parallel to some of the statements for line bundles from Sections 1.2 and 1.4.

To begin with, nefness and amplitude can be tested by pulling back to curves:

Proposition 6.1.18. (Barton–Kleiman criterion). *Let E be a vector bundle on X.*

[4] See [210, §15.5] or Section 7.3.B.

(i). *E is nef if and only if the following condition is satisfied:*

> *Given any finite map $\nu : C \longrightarrow X$ from a smooth irreducible projective curve C to X, and given any quotient line bundle L of ν^*E, one has*
> $$\deg L \;\; \geq \;\; 0.$$

(ii). *Fix an ample divisor class $h \in N^1(X)$ on X. Then E is ample if and only if there exists a positive rational number $\delta = \delta_h > 0$ such that*

$$\deg L \;\; \geq \;\; \delta \big(C \cdot \nu^* h \big) \qquad (*)$$

for any $\nu : C \longrightarrow X$ and $\nu^* E \twoheadrightarrow L$ as above.

Note that in (ii) the constant δ is required to be independent of C, ν, and L.

Proof. Recall that giving a line bundle quotient $\nu^*E \twoheadrightarrow L$ is the same as giving a map $\mu : C \longrightarrow \mathbf{P}(E)$ commuting with the projections to X:

$$\begin{array}{ccc} C & \xrightarrow{\;\;\mu\;\;} & \mathbf{P}(E) \\ & {\scriptstyle \nu}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X, & \end{array}$$

with $L = \mu^* \mathcal{O}_{\mathbf{P}(E)}(1)$. So the property stated in (i) is evidently equivalent to the nefness of $\mathcal{O}_{\mathbf{P}(E)}(1)$. For (ii), assuming that (*) holds we use Corollary 1.4.10 to establish the amplitude of E. In fact, as usual write

$$\xi \;=\; c_1\big(\mathcal{O}_{\mathbf{P}(E)}(1)\big) \;\in\; N^1\big(\mathbf{P}(E)\big).$$

Then (*) is equivalent to requiring that the $\mathbf{Q}$-divisor class

$$\xi \,-\, \delta \cdot \pi^* h \;\in\; N^1\big(\mathbf{P}(E)\big)_{\mathbf{Q}}$$

be nef. On the other hand, since $\mathcal{O}_{\mathbf{P}(E)}(1)$ is relatively ample with respect to π, the class $a\xi + \delta \cdot \pi^* h$ is ample for $0 < a \ll 1$ (Proposition 1.7.10). Therefore

$$\big(\xi - \delta \cdot \pi^* h\big) \,+\, \big(a\xi + \delta \cdot \pi^* h\big) \;=\; \big(1 + a\big) \cdot \xi$$

is the sum of a nef and an ample class, and hence is ample. Thus $\mathcal{O}_{\mathbf{P}(E)}(1)$ is likewise ample. We leave the converse to the reader. □

The proposition shows that twisting by numerically trivial line bundles does not affect amplitude:

Corollary 6.1.19. (Numerically trivial twists). *Let P be a numerically trivial line bundle on X. Then given any vector bundle E on X, E is ample or nef if and only if $E \otimes P$ is so.* □

(Of course one can also argue directly. In fact, $\mathbf{P}(E \otimes P) \cong \mathbf{P}(E)$ by an isomorphism under which $\mathcal{O}_{\mathbf{P}(E\otimes P)}(1)$ corresponds to $\mathcal{O}_{\mathbf{P}(E)}(1) \otimes \pi^* P$ on $\mathbf{P}(E)$. But under the stated hypothesis on P, the latter bundle is numerically equivalent to $\mathcal{O}_{\mathbf{P}(E)}(1)$.)

We conclude by sketching some other criteria.

Example 6.1.20. (Analogue of Seshadri's criterion). Given a reduced irreducible curve C, write $m(C)$ for the maximum of the multiplicities of C at all points $P \in C$. Now suppose that E is a vector bundle on the projective variety X. Then E is ample if and only if the following condition is satisfied:

> There is a real number $\varepsilon > 0$ such that for every non-constant map
>
> $$\nu : C \longrightarrow X$$
>
> from a reduced and irreducible (but possibly singular) curve C to X, and for any quotient line bundle L of $\nu^* E$, one has
>
> $$\frac{\deg_C L}{m(C)} \ \geq \ \varepsilon. \tag{*}$$

(To show that (*) implies the amplitude of E, fix any curve $C \subseteq \mathbf{P}(E)$ finite over X. As in the proof of 6.1.18, (*) implies that

$$\deg\big(\mathcal{O}_{\mathbf{P}(E)}(1)\,|\,C\big) \ \geq \ \varepsilon \cdot m(C).$$

On the other hand, if C lies in a fibre of $\pi : \mathbf{P}(E) \longrightarrow X$, then

$$\deg\big(\mathcal{O}_{\mathbf{P}(E)}(1)\,|\,C\big) \ \geq \ m(C).$$

Therefore it follows from Seshadri's criterion (Theorem 1.4.13) that $\mathcal{O}_{\mathbf{P}(E)}(1)$ is ample.) □

Example 6.1.21. (Generalization of Gieseker's lemma). Let E be a vector bundle on X. Then E is ample if and only if the following two conditions are satisfied:

> (i). There is a positive integer $m_0 = m_0(E)$ such that $S^m E$ is globally generated for every $m \geq m_0$.
>
> (ii). There is no reduced irreducible curve $C \subset X$ such that $E|C$ admits a trivial quotient.

(Assuming the conditions are satisfied, it follows from (i) that $\mathcal{O}_{\mathbf{P}(E)}(m_0)$ is globally generated and hence defines a morphism

$$\phi = \phi_{m_0} : \mathbf{P}(E) \longrightarrow \mathbf{P} = \mathbf{P}H^0(S^{m_0}E).$$

The issue is to show that ϕ is finite. If not there is a curve $C \subset \mathbf{P}(E)$ that ϕ contracts to a point. As in the proof of 6.1.7, C must map isomorphically to its image in X, and it remains only to show that $\mathcal{O}_{\mathbf{P}(E)}(1)|C$ is trivial. But by construction $\mathcal{O}_{\mathbf{P}(E)}(m_0)|C$ is trivial, and hence

$$\mathcal{O}_{\mathbf{P}(E)}(1)|C \quad = \quad \mathcal{O}_{\mathbf{P}(E)}(m_0+1)|C$$

is a degree-zero line bundle on C that (thanks to (i) for $m = m_0 + 1$) is generated by its global sections.) □

Example 6.1.22. (Grauert's criterion). Let E be a vector bundle on X, and denote by $\mathbf{F}$ the total space of the dual of E, so that $\mathbf{P}(E)$ is the projective bundle of one-dimensional subspaces of $\mathbf{F}$. Then E is ample if and only if the zero section $\mathbf{0}_{\mathbf{F}}$ can be blown down to a point. (In fact, let $\mathbf{L}$ denote the total space of the line bundle $L = \mathcal{O}_{\mathbf{P}(E)}(-1)$. One reduces to the case of line bundles by noting that the complement of the zero section in $\mathbf{F}$ is isomorphic to the complement of the zero section in $\mathbf{L}$. See [274, (3.5)] for details.) □

Example 6.1.23. (Big vector bundles). A vector bundle E on an irreducible projective variety X is *big* if $\mathcal{O}_{\mathbf{P}(E)}(1)$ is a big line bundle on $\mathbf{P}(E)$ (Definition 2.2.1). The characterization of big divisors as sums of ample and effective classes (Corollary 2.2.7) generalizes to the bundle setting in a natural manner. Specifically, assume that $H^0\big(X, S^mE\big) \neq 0$ for some $m \geq 1$. Then for any ample line bundle A on X, the bundle $E \otimes A$ is big. Conversely, if E is big, then given any line bundle B on X, $H^0\big(X, S^mE \otimes B\big) \neq 0$ for $m \gg 0$. (For the first statement, the hypothesis implies that $\xi = c_1\big(\mathcal{O}_{\mathbf{P}(E)}(1)\big)$ is an effective class. Then argue as in the proof of Proposition 6.1.18 or 6.2.11 that given any ample class $\alpha \in N^1(X)_{\mathbf{R}}$, $\xi + \pi^*\alpha$ is a big class on $\mathbf{P}(E)$. The second assertion follows quickly from Kodaira's Lemma 2.2.6 applied on $\mathbf{P}(E)$.) Analogous statements — which we leave to the reader — hold in the setting of $\mathbf{Q}$-twisted bundles (Section 6.2). □

6.1.D Metric Approaches to Positivity of Vector Bundles

In the case of line bundles, amplitude is equivalent to the existence of a metric of positive curvature, and it is natural to attempt to generalize this approach to higher ranks. Griffiths [247] defined a differential-geometric notion of positivity that seems reasonably close to amplitude, although it is unknown whether the two concepts actually coincide. The present section is devoted to a very brief sketch of such metric definitions of positivity for bundles. We refer to [248, Chapter 0, §5], [126, §3], [381, Chapter 5], [352, Chapter III, §6] or [533, Chapter VI] for background and more details. We follow the notation and presentation of [126].[5]

Let X be a complex manifold of dimension n, and E a holomorphic vector bundle of rank e on X equipped with a Hermitian metric h. The Hermitian

[5] Concerning the various sign conventions in the literature, see [381, p. 136].

bundle (E, h) determines a unique Hermitian connection D_E compatible with the complex structures on X and E, and D_E in turn gives rise to a curvature tensor

$$\Theta(E, h) \;\in\; \mathcal{C}^\infty\big(X, \Lambda^{1,1}T_X^* \otimes \operatorname{Hom}(E, E)\big),$$

a $\operatorname{Hom}(E, E)$-valued $(1, 1)$-form on X. If $z_1, \ldots, z_n$ are local analytic coordinates on X, and if $(e_\lambda)_{(1 \leq \lambda \leq e)}$ is a local orthogonal frame on E, then one can write

$$i \cdot \Theta(E, h) \;=\; \sum_{\substack{1 \leq j,k \leq n \\ 1 \leq \lambda,\mu \leq e}} c_{jk\lambda\mu} \cdot dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu,$$

where $\bar{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$. This curvature tensor gives rise to a Hermitian form θ_E on the bundle $T_X \otimes E$, given locally by

$$\theta_E \;=\; \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} \cdot (dz_j \otimes e_\lambda^*) \otimes \overline{(dz_k \otimes e_\mu^*)}.$$

Definition 6.1.24. (i). E is *Nakano-positive* if θ_E is a positive-definite Hermitian form on $TX \otimes E$.

(ii). E is *Griffiths-positive* if θ_E is positive on all simple tensors in $TX \otimes E$, i.e. if at every point $x \in X$,

$$\theta_E(\xi \otimes s, \xi \otimes s) \;>\; 0 \quad \text{for all} \;\; 0 \neq \xi \in T_xX \text{ and } 0 \neq s \in E(x).$$

Nakano and Griffiths semipositivity, negativity, and seminegativity are defined analogously. □

If we fix a holomorphic tangent vector $\xi \in T_xX$, then we can view

$$\theta_{E,\xi} \;=\; \theta_E(\xi \otimes \bullet, \xi \otimes \bullet)$$

as a Hermitian form on $E(x)$. The condition of Griffiths positivity is that $\theta_{E,\xi}$ should be positive definite for every non-zero ξ. Note that if E has rank one, then the Nakano and Griffiths definitions both coincide with positivity in the sense of Kodaira.

The connection between these notions is given by

Theorem 6.1.25. *Let (E, h) be a Hermitian vector bundle on a complex projective manifold X.*

(i). *If (E, h) is Nakano-positive, then (E, h) is Griffiths-positive.*

(ii). *If (E, h) is Griffiths-positive, then E is ample.*

Statement (i) of the theorem is clear from the definitions, and we refer to [533, Theorem 6.30] for a proof of (ii). The idea, naturally enough, is to consider the projective bundle $\pi : \mathbf{P}(E) \longrightarrow X$. The given Hermitian metric h on E determines one on π^*E, which in turn induces a natural Hermitian metric k

on its quotient $\mathcal{O}_{\mathbf{P}(E)}(1)$. Then one uses the principle that curvature increases in quotients to show that the corresponding curvature form $\Theta(\mathcal{O}_{\mathbf{P}(E)}(1), k)$ is positive, and hence that this line bundle is ample.

The Nakano condition is known to be considerably stronger than Griffiths positivity and amplitude (cf. Example 7.3.18). It is an interesting open question whether or not Griffiths-positivity is equivalent to amplitude, i.e. whether every ample bundle carries a Griffiths-positive metric. Fulton [203, p. 28] points out that a weaker condition would have many of the same consequences.

Remark 6.1.26. (Garrity's theorem). Garrity has given an alternate metric approach to amplitude coming from Proposition 6.1.18. A detailed account appears in [381, Chapter V, §4]. □

6.2 Q-Twisted and Nef Bundles

As we saw in Chapter 1, the language of **Q**-divisors greatly facilitates discussions of positivity for line bundles. In the present section, we start by developing an analogous formalism involving twists of vector bundles by **Q**-divisor classes. This device was initiated by Miyaoka [429], and its utility was re-emphasized by some related constructions in [133]. As in the rank-one case, the formalism will allow us first of all to treat nef bundles as limits of ample ones. Beyond that, it absorbs many of the branched covering arguments that arise in previous developments of the theory. In the second subsection, we apply the formalism to establish the basic properties of nef bundles.

Throughout this section, X denotes a complex projective variety or scheme. Concerning divisors, we adhere to the notation and conventions introduced in Sections 1.1 and 1.3. In particular, $\mathrm{Div}(X)$ denotes the group of Cartier divisors on X, and by a **Q**-divisor we understand an element of $\mathrm{Div}_{\mathbf{Q}}(X) = \mathrm{Div}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

6.2.A Twists by Q-Divisors

Whereas the definition of a **Q**-divisor did not present any difficulties, it is not really clear what one might mean by a **Q**-vector bundle. However, for many purposes — notably for questions of positivity and numerical properties — there is a natural way to make sense of twisting a bundle by a **Q**-divisor (class).

We start by defining formally the objects with which we shall deal:

Definition 6.2.1. (Q-twisted bundles). A **Q**-*twisted vector bundle*

$$E{<}\delta{>}$$

on X is an ordered pair consisting of a vector bundle E on X, defined up to isomorphism, and a numerical equivalence class $\delta \in N^1(X)_{\mathbf{Q}}$. If $D \in \mathrm{Div}_{\mathbf{Q}}(X)$ is a **Q**-divisor, we write $E{<}D{>}$ for the twist of E by the numerical equivalence class of D. □

So in other words, $E{<}D{>}$ and $E{<}\delta{>}$ are just formal symbols, but the notation is intended to suggest that we are twisting E by the **Q**-divisor D or class δ. This leads to a natural notion of equivalence among such pairs:

Definition 6.2.2. (Q-isomorphism). We define **Q-isomorphism** of **Q**-twisted bundles to be the equivalence relation generated by declaring that $E{<}A+D{>}$ be equivalent to

$$\big(E \otimes \mathcal{O}_X(A)\big){<}D{>}$$

whenever A is an integral Cartier divisor on X and $D \in \mathrm{Div}_{\mathbf{Q}}(X)$. □

We generally identify **Q**-isomorphic **Q**-twisted bundles without explicit mention. Therefore we will take care that further definitions respect this equivalence relation.

A "classical" (untwisted) bundle E may be considered in the natural way as a **Q**-twisted bundle, viz. as $E{<}0{>}$. Note that then E_1 and E_2 determine the same **Q**-twisted bundle — i.e. $E_1{<}0{>}$ and $E_2{<}0{>}$ are **Q**-isomorphic — if $E_1 = E_2 \otimes P$ for a numerically trivial line bundle P. But as long as we are dealing with positivity or numerical properties, this shouldn't cause undue confusion.

We have already had occasion to note that if E is a vector bundle and A is a line bundle on X, then $\mathbf{P}(E \otimes A) \cong \mathbf{P}(E)$ by an isomorphism under which $\mathcal{O}_{\mathbf{P}(E\otimes A)}(1)$ corresponds to the bundle

$$\mathcal{O}_{\mathbf{P}(E)}(1) \otimes \pi^* A$$

on $\mathbf{P}(E)$, where $\pi : \mathbf{P}(E) \longrightarrow X$ denotes as usual the bundle map. This motivates

Definition 6.2.3. (Ample and nef Q-twisted bundles). A **Q**-twisted vector bundle $E{<}\delta{>}$ is *ample* (or *nef*) if

$$\xi_E + \pi^*\delta \ \in \ N^1\big(\mathbf{P}(E)\big)_{\mathbf{Q}}$$

is an ample (or nef) **Q**-divisor class on $\mathbf{P}(E)$. Here $\xi_E = c_1\big(\mathcal{O}_{\mathbf{P}(E)}(1)\big)$ is the class of a divisor representing the Serre line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$. □

Observe that E is ample or nef in the usual sense if and only if it so considered as the **Q**-twisted bundle $E{<}0{>}$. The remarks preceding the definition show that the notions of amplitude and nefness just introduced respect **Q**-isomorphism.

The same is true for the natural definitions of tensor products and pullbacks:

Definition 6.2.4. (Tensorial operations and pullbacks). Let X be a projective variety or scheme, and let E, E_1, E_2 be vector bundles on X.

(i). Tensor, symmetric, and exterior powers of **Q**-twisted bundles are defined via the rules

$$E_1{<}\delta_1{>} \otimes E_2{<}\delta_2{>} \;=\; \big(E_1 \otimes E_2\big){<}\delta_1 + \delta_2{>}, \tag{6.4a}$$

$$S^m\big(E{<}\delta{>} \big) \;=\; \big(S^m E\big){<}m\delta{>}, \tag{6.4b}$$

$$\Lambda^m\big(E{<}\delta{>}\big) \;=\; \big(\Lambda^m E\big){<}m\delta{>} . \tag{6.4c}$$

(ii). If $f : Y \longrightarrow X$ is a morphism, then we define the pullback of a **Q**-twisted bundle $E{<}\delta{>}$ on X to be the **Q**-twisted bundle

$$f^*\big(E{<}\delta{>}\big) \;=\; \big(f^*E \;\big){<}f^*\delta{>} \tag{6.5}$$

on Y. □

Note that it follows from the definition that pullback commutes with each of the products in (i).

There is also a natural notion of quotients, sums, and ranks:

Definition 6.2.5. (Quotients, sums, and ranks). Keep notation as above.

(i). A *quotient* of a **Q**-twisted bundle $E{<}\delta{>}$ is a **Q**-twisted bundle $Q{<}\delta{>}$, where Q is a quotient of E. Sub-bundles and locally free subsheaves of $E{<}\delta{>}$ are defined similarly.

(ii). The *direct sum* of two **Q**-twisted bundles $E_1{<}\delta{>}$ and $E_2{<}\delta{>}$ with the same twisting class is the **Q**-twisted bundle $\big(E_1 \oplus E_2\big){<}\delta{>}$. Extensions are defined similarly.

(iii). The *rank* of $E{<}\delta{>}$ is simply $\mathrm{rank}(E)$. □

Remark 6.2.6. Note that we do not attempt to define direct sums or extensions of two vector bundles twisted by different numerical equivalence classes. □

We will discuss Chern classes of **Q**-twisted bundles later in Section 8.1.A. However a special case will be useful before then:

Definition 6.2.7. (Degrees of Q-twists on curves). If $E{<}\delta{>}$ is a **Q**-twisted bundle of rank e on a curve X, then its *degree* is the rational number

$$\deg E + e \cdot \deg \delta. \quad \square$$

The basic facts about **Q**-twisted bundles are summarized in the next lemma. The first two statements allow one to pass to covers to reduce to the case of integral divisors.

Lemma 6.2.8. (Formal properties of Q-twists). *Let E be a vector bundle on X and $\delta \in N^1(X)_{\mathbf{Q}}$ a numerical equivalence class.*

(i). *The definitions above respect the relation of **Q**-isomorphism. In particular, if A is an integral Cartier divisor on X, then $E{<}A{>}$ is ample (or nef) if and only if $E \otimes \mathcal{O}_X(A)$ is.*

(ii). *If $f : Y \longrightarrow X$ is a finite surjective map, then $f^*\big(E{<}\delta{>}\big)$ is ample (or nef) on Y if and only if $E{<}\delta{>}$ is ample (or nef) on X.*

(iii). *$E{<}\delta{>}$ is ample if and only if $S^k\big(E{<}\delta{>}\big)$ is ample for some — or equivalently, for all — integers $k \geq 1$.*

(iv). *If $E_1{<}\delta_1{>}$ and $E_2{<}\delta_2{>}$ are ample **Q**-twisted bundles on X, then*

$$E_1{<}\delta_1{>} \otimes E_2{<}\delta_2{>}$$

is also ample.

(v). *If $E_1{<}\delta{>}$ and $E_2{<}\delta{>}$ are ample, then so too is their direct sum (or any extension of one by the other).*

(vi). *Suppose that $E{<}\delta{>}$ is ample, and let δ' be any **Q**-divisor class on X. Then $E{<}\delta + \varepsilon \cdot \delta'{>}$ is ample for all sufficiently small rational numbers $0 < \varepsilon \ll 1$.*

Proof. The first assertion has already been observed. The second statement follows from the fact that amplitude or nefness for **Q**-divisor classes can be tested after pulling back by a finite surjective map (1.2.28, 1.4.4 (ii)). For (iii), use the covering constructions from Section 4.1 (e.g. Theorem 4.1.10) to form a branched covering $f : Y \longrightarrow X$ such that $f^*\delta$ is an integral class on Y. By (ii) it is equivalent to prove the statement after pulling back to Y. But thanks to (i), here we are reduced to Theorem 6.1.15. For (iv) and (v) one reduces in a similar manner to the case in which δ_1 and δ_2 are integral, and then 6.1.16 and 6.1.13 apply. Finally, (vi) is a consequence of the fact (1.3.7) that the ample cone in $N^1\big(\mathbf{P}(E)\big)_{\mathbf{Q}}$ is open. □

Remark 6.2.9. After the evident modifications, Proposition 6.1.18 and Example 6.1.20 extend to **Q**-twisted bundles. We leave the statements to the reader. □

Remark 6.2.10. (R-twists). With some extra care, it would be possible to define and work out the elementary properties of **R**-twisted bundles. However, except at one point in Chapter 8 — viz. Theorem 8.2.1, where we will proceed ad hoc — we do not require this formalism, and it seemed simplest to bypass it. □

6.2.B Nef Bundles

With the formalism of **Q**-twists in hand, the basic properties of nef bundles follow easily from the corresponding facts for amplitude.

We start with the analogue of Corollary 1.4.10, showing that a bundle is nef if and only if it is a limit of ample **Q**-twisted bundles:

Proposition 6.2.11. (Ample twists of nef bundles). *A vector bundle E on X is nef if and only if the $\mathbf{Q}$-twisted bundle $E{<}h{>}$ is ample for every ample class $h \in N^1(X)_{\mathbf{Q}}$. Similarly, if $E{<}\delta{>}$ is a $\mathbf{Q}$-twisted bundle, then $E{<}\delta{>}$ is nef if and only if $E{<}\delta + h{>}$ is ample for any such h.*

Proof. If $E{<}h{>}$ is ample for every ample class h, then $\xi = \xi_E$ is the limit as $h \to 0$ of the ample classes $\xi + \pi^* h$, and hence is nef. Assuming conversely that E is nef, we argue as in the proof of 6.1.18. Specifically, since ξ is ample for π, $a\xi + \pi^* h$ is an ample class on $\mathbf{P}(E)$ for $0 < a \ll 1$ (Proposition 1.7.10). Therefore

$$\xi \,+\, \pi^* h \;=\; \big((1-a)\xi\big) \,+\, \big(a\xi + \pi^* h\big),$$

being the sum of a nef and an ample class, is ample (Corollary 1.4.10). Hence $E{<}h{>}$ is ample. The statement for **Q**-twists is similar. □

It is now immediate to establish the analogues for nef bundles of the formal properties of ample bundles given earlier. We state them all explicitly for ease of reference.

Theorem 6.2.12. (Formal properties of nef vector bundles). *Let X be a projective variety or scheme.*

(i). *Quotients and arbitrary pullbacks of nef bundles on X are nef. Given a vector bundle E on X, and a surjective morphism $f : Y \longrightarrow X$ from a projective variety (or scheme)) to X, if f^*E is a nef bundle on Y, then E is nef.*

(ii). *Direct sums and extensions of nef bundles are nef.*

(iii). *A vector bundle E on X is nef if and only if $S^k E$ is nef for any — or equivalently for all — $k \geq 1$.*

(iv). *Any tensor or exterior product of nef bundles is nef. If E is nef and F is ample, then $E \otimes F$ is ample.*

(v). *All of these statements hold if the "classical" bundles involved are replaced by $\mathbf{Q}$-twists (provided in* (ii) *that the sum or extension is defined).*

Proof. Statement (i) follows immediately from the corresponding facts for nef line bundles. For the first assertion in (ii), suppose that E_1 and E_2 are nef, and fix any ample **Q**-divisor class $h \in N^1(X)_{\mathbf{Q}}$. By the previous proposition, it suffices to prove that $\big(E_1 \oplus E_2\big){<}h{>}$ is an ample **Q**-twisted bundle, and this follows from Lemma 6.2.8 (v). Extensions are treated in the same manner. Turning to (iii), note that

$$\big(S^k E\big)<\delta> \;=\; S^k\big(E<\tfrac{1}{k}\delta>\big),$$

and so the amplitude of $\big(S^k E\big) < h >$ for all ample $\mathbf{Q}$-divisor classes h is equivalent to the amplitude of $S^k\big(E<h'>\big)$ for all ample classes h'. Therefore we are reduced by the previous proposition to Lemma 6.2.8 (iii). The first assertion of (iv) is similar, and in characteristic zero the nefness of tensor powers implies in the usual way the nefness of exterior products. For the last statement in (iv), suppose that E is nef and F is ample, and let h be any ample class on X. Then $E<\varepsilon h>$ is ample for all $\varepsilon > 0$ by 6.2.11, and $F<-\varepsilon h>$ is ample for $0 < \varepsilon \ll 1$ thanks to 6.2.8 (vi). Therefore 6.2.8 (iv) applies to show that

$$E \otimes F \;=\; E<\varepsilon h> \;\otimes\; F<-\varepsilon h>$$

is ample. We leave the extensions to $\mathbf{Q}$-twisted bundles to the reader. □

Example 6.2.13. (Fixed twists of large symmetric powers). Let E be a vector bundle and B an ample divisor on X. If $S^m E \otimes \mathcal{O}_X(B)$ is nef for all $m \gg 0$, then E itself is nef. Similarly, the nefness of the m-fold tensor power $T^m E \otimes \mathcal{O}_X(B)$ for every $m \gg 0$ implies the nefness of E. (Replacing B by $2B$, there is no loss in generality in supposing that $S^m E \otimes \mathcal{O}_X(B)$ is ample for all $m \gg 0$. But considered as a $\mathbf{Q}$-twisted bundle, $S^m E \otimes \mathcal{O}_X(B)$ is $\mathbf{Q}$-isomorphic to

$$S^m\big(E<\tfrac{1}{m}B>\big).$$

Therefore $E<\frac{1}{m}B>$ is ample for all $m \gg 0$, and hence E is nef. The analogous assertion for tensor powers follows from this since in characteristic zero $S^m E$ is a summand of $T^m E$.) □

Example 6.2.14. (The Barton invariant of a bundle). Let E be a vector bundle and h an ample divisor class on the projective variety X. Define the *Barton invariant* of E with respect to h to be the real number

$$\delta(X, E, h) \;=\; \sup\big\{ t \in \mathbf{Q} \;\big|\; E<-t \cdot h> \text{ is nef } \big\}.$$

(i). E is ample if and only if $\delta(X, E, h) > 0$, and E is nef if and only if $\delta(X, E, h) \geq 0$. (Compare Proposition 6.1.18.)

(ii). For every integer $m > 0$, $\delta(X, E, m \cdot h) = \frac{1}{m}\,\delta(X, E, h)$.

(iii). If $f : Y \longrightarrow X$ is a finite surjective mapping, then

$$\delta(Y, f^*E, f^*h) = \delta(X, E, h).$$

Given an ample divisor D or line bundle bundle B, we write $\delta(X, E, D)$ or $\delta(X, E, B)$ for the quantity defined by using D or B in place of h. □

Example 6.2.15. (Irrational Barton invariants). It is possible for the Barton invariant to be an irrational number. For a simple example, take $X = C \times C$ to be the product of an elliptic curve with itself, so that $\mathrm{Nef}(X)$ is a circular cone (Example 1.5.4). Given ample divisors A and H on X, put $E = \mathcal{O}_X(A)$. Then the Barton invariant $\delta(X, E, H)$ of E with respect to H is the smallest root of the quadratic polynomial $s(t) = \big((A - tH)^2 \big)$, and for general choices of A and H this will be irrational (compare Section 2.3.B). Peternell (private communication) has constructed more interesting examples involving bundles of higher rank. □

Remark 6.2.16. (Positive characteristics). Except for the second assertion of Example 6.2.13, all of the material appearing so far in this section remains valid for varieties defined over an algebraically closed field of arbitrary characteristic. □

By analogy with the corresponding notion for line bundles (Section 2.1.B), it is natural to define semiamplitude for vector bundles.

Definition 6.2.17. (Semiample vector bundles). A vector bundle E on a complete variety or scheme is *semiample* if $\mathcal{O}_{\mathbf{P}(E)}(m)$ is globally generated for some $m > 0$. □

The condition is satisfied for example if $S^m(E)$ is globally generated for some $m > 0$. Evidently a semiample bundle is nef.

Remark 6.2.18. (k-amplitude). Sommese [551] has introduced a quantitative measure of how close a semiample bundle comes to being ample. Specifically, let E be a semiample bundle – so that $\mathcal{O}_{\mathbf{P}(E)}(m)$ is free for some $m > 0$ – and let

$$\phi = \phi_m : \mathbf{P}(E) \longrightarrow \mathbf{P}\big(H^0\big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(m)\big)\big)$$

be the corresponding map. Sommese defines E to be *k-ample* if every fibre of ϕ has dimension $\leq k$. Thus E is ample if and only if it is 0-ample. Many of the basic facts about amplitude extend with natural modifications to this more general setting. We shall point out some of these as we go along, and the reader can consult [551] for a more complete survey. □

Example 6.2.19. (Some formal properties of k-amplitude). We indicate how some of the results of this and the previous section extend to k-ample bundles on a projective scheme X (Remark 6.2.18).

(i). Any quotient of a k-ample bundle is k-ample.

(ii). If E is a k-ample bundle on X and $f : Y \longrightarrow X$ is a projective morphism all of whose fibres have dimension $\leq m$, then f^*E is $(k + m)$-ample.

(iii). If E is k-ample then for any coherent sheaf $\mathcal{F}$ on X there is an integer $m(\mathcal{F})$ such that

$$H^i\big(X, S^m E \otimes \mathcal{F}\big) \;=\; 0 \quad \text{for} \;\; m \geq m(\mathcal{F}) \;\; \text{and} \;\; i > k;$$

the converse holds provided that E is semiample.

(iv). If E and F are globally generated and k-ample, then $E \otimes F$ and $E \oplus F$ and any extension of E by F are k-ample.

(See [551, §1, (1.7), (1.9), and (1.10)].) □

6.3 Examples and Constructions

In order to add substance to the general theory, we present in this section several examples and constructions of ample and nef vector bundles. Our hope is to convey a sense of some of the many settings in which positivity of bundles arises "in nature," and to illustrate a few of the methods that have been used to detect and exploit it.

In the first two subsections, we discuss the geometric consequences of positivity conditions on tangent, cotangent, and normal bundles. We then consider the Picard bundles on the Jacobian of a curve (and their analogues for irregular varieties of arbitrary dimension), and prove the positivity of a vector bundle associated to a branched covering of projective space. Direct images of canonical bundles are discussed briefly in 6.3.E, while in 6.3.F we indicate some methods of construction. On several occasions we call on results to be established later, but we felt that the value of presenting early on some non-trivial illustrations of the theory outweighs any lapses in strict logical development.

6.3.A Normal and Tangent Bundles

Historically, an important motivation for developing the theory of ample vector bundles was the desire to generalize to higher codimensions some of the positivity properties enjoyed by ample divisors. If M is a smooth projective variety, and if $X \subset M$ is a non-singular ample divisor, then of course its normal bundle $N_{X/M} = \mathcal{O}_M(X)|X$ is an ample line bundle on X. Given $X \subset M$ of arbitrary codimension, one would like to think of the amplitude of $N_{X/M}$ as reflecting the intuition that X is "positively embedded" in M. While this has never been made precise, we present here some of the basic examples and results supporting this heuristic. We start by considering subvarieties of projective space and abelian varieties, and then briefly discuss what is known on arbitrary smooth ambient varieties. Hartshorne's book [276] remains a valuable source of information on these matters.

Projective space and its subvarieties. The positivity of projective space is manifested in the amplitude of its tangent bundle and the normal bundles of all smooth subvarieties.

Proposition 6.3.1. (i). *The tangent bundle $T\mathbf{P}^n$ of n-dimensional projective space is ample.*

(ii). *If $X \subseteq \mathbf{P}^n$ is any smooth subvariety, then the normal bundle $N_{X/\mathbf{P}}$ to X in $\mathbf{P} = \mathbf{P}^n$ is ample.*

Proof. Let V be a vector space of dimension $n+1$, so that $\mathbf{P}(V) = \mathbf{P}^n$. Then one has the Euler sequence [280, II.8.13]

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow V^* \otimes_k \mathcal{O}_{\mathbf{P}^n}(1) \longrightarrow T\mathbf{P}^n \longrightarrow 0. \tag{6.6}$$

So $T\mathbf{P}^n$ is a quotient of the $(n+1)$-fold direct sum of copies of $\mathcal{O}_{\mathbf{P}}(1)$, and hence is ample. The second statement then follows from the normal bundle sequence

$$0 \longrightarrow TX \longrightarrow T\mathbf{P}^n|X \longrightarrow N_{X/\mathbf{P}} \longrightarrow 0. \tag{6.7}$$

In fact, the restriction $T\mathbf{P}^n|X$ is ample, and hence so it its quotient $N_{X/\mathbf{P}}$. □

Remark 6.3.2. (Mori's theorem). A very fundamental theorem of Mori [437] states that projective space is the only smooth projective variety with ample tangent bundle. This had been conjectured by Hartshorne [276], and Siu and Yau [542] proved a complex geometric analogue at essentially the same time as Mori's theorem. In the course of his argument, Mori introduced several spectacular ideas that soon led to a flowering of higher-dimensional geometry. We refer to [135] for an excellent overview of Mori's proof, and to [363] for an account of the developments growing in part out of his techniques. The papers [80], [81], [6], [91], and [603] contain some results and conjectures concerning other characterizations of projective space involving positive vector bundles. □

Remark 6.3.3. (Nef tangent bundles). Continuing the train of thought of the previous remark, it is natural to ask whether one can characterize complex varieties whose tangent bundles satisfy weaker positivity properties. The first results in this direction were obtained by Mok [435], who classified all compact Kähler manifolds with semi-positive bisectional curvature. Following work of Campana and Peternell, [79], Demailly, Peternell and Schneider [133] studied compact Kähler manifolds X with nef tangent bundles.[6] They showed that X admits a finite étale cover $\tilde{X} \longrightarrow X$ having the property that the Albanese mapping $\tilde{X} \longrightarrow \mathrm{Alb}(\tilde{X})$ is a smooth fibration whose fibres are Fano manifolds with nef tangent bundles. It is conjectured in [79] that a complex Fano variety

[6] On possibly non-algebraic Kähler manifolds, nefness of a bundle E is defined by asking that $\mathcal{O}_{\mathbf{P}(E)}(1)$ carry a metric satisfying the condition of Remark 1.4.7. See [133, §1B].

with nef tangent bundle must be a rational homogeneous space G/P. If true, this would give a complete picture, up to étale covers, of Kähler manifolds whose tangent bundles are nef. An amusing numerical property of varieties with nef tangent bundles appears in Corollary 8.4.4. □

Remark 6.3.4. (Bundles of differential operators). In a somewhat related direction, Ran and Clemens [512] use very interesting considerations of positivity and stability for sheaves of differential operators to study the geometry of Fano manifolds of Picard number one. □

While every smooth subvariety of projective space has ample normal bundle, it was observed in [211, Remark 7.5] that a twist of that bundle carries additional geometric information:

Proposition 6.3.5. (Amplitude of N(-1)). *Given a smooth subvariety $X \subset \mathbf{P} = \mathbf{P}^n$ not contained in any hyperplane, the twisted normal bundle $N_{X/\mathbf{P}}(-1)$ is ample if and only if every hyperplane $H \subset \mathbf{P}$ that is tangent to X is tangent at only finitely many points.*

The condition is equivalent to asking that for every hyperplane H, $X \cap H$ have at most finitely many singular points.

Proof of 6.3.5. Restricting the Euler sequence to X, and combining it with the normal bundle sequence, one arrives at a bundle surjection

$$V_X^* = V^* \otimes \mathcal{O}_X \longrightarrow N_{X/\mathbf{P}}(-1) \longrightarrow 0.$$

This gives rise to an embedding

$$\mathbf{P}\big(N_{X/\mathbf{P}}(-1)\big) \ \subseteq \ \mathbf{P}\big(V_X^*\big) = X \times \mathbf{P}(V)^*,$$

and in $X \times \mathbf{P}(V)^*$, $\mathbf{P}\big(N_{X/\mathbf{P}}(-1)\big)$ is identified with the locus

$$\big\{ \ (x, H) \ \big| \ x \in X, \ H \subset \mathbf{P}^n \ \text{ a hyperplane tangent to } X \text{ at } x \ \big\}.$$

So the condition in the proposition is equivalent to the finiteness of the projection $\mathbf{P}(N_{X/\mathbf{P}}(-1)) \longrightarrow \mathbf{P}(V)^*$, which in turn is equivalent to the amplitude of $N_{X/\mathbf{P}}(-1)$. □

Example 6.3.6. (Tangencies to complete intersections). Let $X \subseteq \mathbf{P}^n$ be a smooth complete intersection of hypersurfaces of degrees ≥ 2. Then a hyperplane can be tangent to X at only a finite number of points. □

Example 6.3.7. Given a smooth non-degenerate subvariety $X \subseteq \mathbf{P} = \mathbf{P}^n$, the twisted normal bundle $N_{X/\mathbf{P}}(-1)$ is k-ample in the sense of Sommese (Remark 6.2.18) if and only if no hyperplane is tangent to X along a subset of dimension $\geq k+1$. It follows from Zak's theorem on tangencies (Theorem 3.4.17) that in fact this always holds with $k = \operatorname{codim}(X, \mathbf{P}) - 1$. □

Example 6.3.8. (Tangencies along hypersurfaces). Proposition 6.3.5 admits a partial generalization to other twists. Specifically let $X \subseteq \mathbf{P} = \mathbf{P}^n$ be a smooth subvariety, and let $S = S_d \subseteq \mathbf{P}^n$ be a non-singular hypersurface of degree d, not containing X. If the twisted normal bundle $N_{X/\mathbf{P}}(-d)$ is ample, then S cannot be tangent to X along a curve.[7] This applies for example when X is the complete intersection of hypersurfaces of degrees $> d$. (In fact, suppose to the contrary that there exists a reduced and irreducible curve $C \subset X \cap S$ such that $TX|C \subseteq TS|C$. This gives rise to a surjective homomorphism

$$N_{X/\mathbf{P}}|C \longrightarrow N_{S/\mathbf{P}}|C$$

and hence a surjection $N_{X/\mathbf{P}}(-d)|C \twoheadrightarrow \mathcal{O}_C$. But an ample bundle on a curve does not admit a trivial quotient.) This result appears in [186, Proposition 4.3.6]. □

Remark 6.3.9. (Normal bundles to local complete intersection subvarieties). If $X \subset \mathbf{P}^n$ is a singular local complete intersection subvariety, then the normal bundle $N_{X/\mathbf{P}}$ is still defined, but one no longer has the normal bundle sequence (6.7). Therefore one can no longer conclude the amplitude of $N_{X/\mathbf{P}}$. This is discussed in Fritzsche's paper [189, §3] □

Subvarieties of abelian varieties. Let A be an abelian variety of dimension n. The tangent bundle TA of A is trivial, so the first interesting question is the amplitude of normal bundles. This was analyzed by Hartshorne [277]:

Proposition 6.3.10. *Let $X \subseteq A$ be a smooth subvariety, and denote by $N = N_{X/A}$ the normal bundle to X in A. Then:*

(i). *N is ample if and only if for every regular one-form $\omega \in \Gamma(A, \Omega^1_A)$, the restriction*

$$\omega \,|\, X \;\in\; \Gamma(X, \Omega^1_X)$$

of ω to X vanishes on at most a finite set.

(ii). *If N fails to be ample, then there is a reduced and irreducible curve $C \subseteq X$ that lies in a proper abelian subvariety of A.*

Corollary 6.3.11. (Subvarieties of simple abelian varieties). *If A is simple, i.e. if it contains no proper abelian subvarieties, then every smooth subvariety of A has ample normal bundle.* □

Proof of Proposition 6.3.10. Write $TA = V \otimes \mathcal{O}_A$ where $V = T_0A$ is a vector space of dimension n, which is canonically identified with the tangent space to A at the origin $0 \in A$. By Gieseker's Lemma 6.1.7, N fails to be ample if and only if there is a curve $C \subseteq X$ such that $N \,|\, C$ admits a trivial quotient. The normal bundle sequence

[7] By definition, S is tangent to X at a point $x \in S \cap X$ if $T_xX \subset T_xS$.

$$0 \longrightarrow TX \longrightarrow V_X = TA|X \longrightarrow N \longrightarrow 0$$

shows that this is equivalent to the existence of a one-form $\omega \in V^* = \Gamma\big(A, \Omega^1_A\big)$ whose restriction $\omega \,|\, X \in \Gamma\big(X, \Omega^1_X\big)$ vanishes along C. This proves the first assertion. Assuming such a curve exists, let C' be its normalization, and denote by $\nu : C' \longrightarrow X$ the natural (finite) mapping. Then $\nu^*(\omega) = 0 \in \Gamma\big(C', \nu^*\Omega^1_X\big)$ and consequently

$$(d\nu)^*\omega \;=\; 0 \;\in\; \Gamma\big(C', \Omega^1_{C'}\big).$$

But this implies that the Jacobian $\mathrm{Jac}(C')$ — and hence also C' itself — maps to a proper abelian subvariety of A. □

Example 6.3.12. The converse of the second statement of Proposition 6.3.10 is not true in general. (For instance, let A be an abelian variety of dimension $n \geq 3$ that contains an elliptic curve $C \subseteq A$. Then there exist smooth ample divisors $X \subset A$ with $C \subset X$, but of course $N_{X/A} = \mathcal{O}_X(X)$ is ample.) □

General ambient manifolds. We now survey some of the geometric consequences of the amplitude of normal bundles in a general ambient manifold, referring to [276] for more information. Throughout this discussion, M denotes a non-singular complex quasi-projective complex variety of dimension n, and $X \subseteq M$ is a smooth irreducible subvariety of dimension d. Unless otherwise stated we do not assume that M is complete, but we always suppose that X is projective. We denote by $N = N_{X/M}$ the normal bundle of X in M.

As suggested above, the basic goal is to give some substance to the intuition that the amplitude of N reflects the fact that X is "positively embedded" in M. A first idea in this direction is to study the ring of formal functions along X, or more generally the sections of the formal completion along X of a locally free sheaf on M. Along these lines one has:

Proposition 6.3.13. (Formal functions along ample subvarieties). *Assume that $d = \dim X \geq 1$ and that $N = N_{X/M}$ is ample, and consider the formal completion $\widehat{M} = \widehat{M}_{/X}$ of M along X.*

(i). *The only formal holomorphic functions along X are constants, i.e.*

$$H^0\big(\widehat{M}, \mathcal{O}_{\widehat{M}_{/X}}\big) \;=\; \mathbf{C}.$$

(ii). *Given a locally free sheaf E on M, denote by $\widehat{E}$ its completion along X. Then $H^0\big(\widehat{M}, \widehat{E}\big)$ is finite dimensional.*

Recall that if $U \supseteq X$ is any connected neighborhood of X in M, then the natural map

$$H^0\big(U, E|U\big) \longrightarrow H^0\big(\widehat{M}, \widehat{E}\big)$$

is injective. The same is true if U is a connected neighborhood in the classical topology and the group on the left is replaced by the space $H^0\big(U_{an}, E_{an}\big)$ of holomorphic sections of E on U. So it follows that both these spaces of sections are finite dimensional. We refer to [276, Chapter 5] for a discussion of some related results of Hartshorne, Hironaka, Matsumura, and others concerning formal rational functions.

Remark 6.3.14. (Contractible subvarieties). To appreciate why this sort of statement is suggestive of the positivity of X in M, consider by contrast the "opposite" case, when X contracts. Specifically, suppose that X is the fibre over a point $p \in N$ of a surjective mapping $f : M \longrightarrow N$ of M onto a variety N of dimension ≥ 1. Then by the theorem on formal functions [280, III.11] the group appearing in 6.3.13 (i) is isomorphic to the completion of the stalk at p of the sheaf $f_*\mathcal{O}_M$:

$$H^0\big(\widehat{M}, \mathcal{O}_{\widehat{M}/X}\big) \;=\; \big(\widehat{f_*\mathcal{O}_X}\big)_p.$$

In particular, the group in question is infinite dimensional. Similarly, if $E = f^*F$ for some bundle F on N, the finiteness in (ii) will also fail. □

Proof of Proposition 6.3.13. We focus on statement (ii). Let $\mathcal{I}$ denote the ideal sheaf of X in M, and let X_k be the k^{th} infinitesimal neighborhood of X in M, i.e. the subscheme of M defined by $\mathcal{I}^k$. Then

$$H^0(\widehat{M}, \widehat{E}) \;=\; \varprojlim H^0(M, E \otimes \mathcal{O}_{X_k}).$$

For every $k \geq 1$ there is an exact sequence

$$0 \longrightarrow E \otimes \mathcal{I}^k/\mathcal{I}^{k+1} \longrightarrow E \otimes \mathcal{O}_{X_{k+1}} \longrightarrow E \otimes \mathcal{O}_{X_k} \longrightarrow 0,$$

and one has the isomorphism $\mathcal{I}^k/\mathcal{I}^{k+1} = S^kN^*$ of $\mathcal{O}_X$-modules. It is enough for (ii) to show that $H^0(X, S^kN^* \otimes E) = 0$ if $k \gg 0$: for then $H^0(\widehat{M}, \widehat{E})$ injects into $H^0(X_k, E \otimes \mathcal{O}_{X_k})$ for some fixed $k \gg 0$, and hence is finite-dimensional. But since we are in characteristic zero, $S^kN^* = (S^kN)^*$. So for $k \gg 0$, $S^kN^* \otimes E$ is the dual of an ample bundle and hence has no non-vanishing sections, as required. By the same token, $H^0(X, S^kN^*) = 0$ for all $k \geq 1$, which in a similar fashion yields (i). □

Remark 6.3.15. Since we only need to control H^0 in the argument just completed, Proposition 6.3.13 holds under considerably weaker conditions than the amplitude of $N = N_{X/M}$. For example, it suffices to assume the following:

> If $C \subseteq X$ is a curve arising as a general complete intersection of very ample divisors on X, then the restriction $N \,|\, C$ of N to C is ample.[8]

[8] This is analogous to the condition of generic semipositivity figuring in the theorem of Miyaoka described in Remark 6.3.34.

For then, keeping notation as in the previous proof, one has

$$H^0\big(C, (S^k N^* \otimes E) \mid C\big) \;=\; 0$$

for $k \gg 0$. Since this holds for the general member of a family of curves that covers X, it follows that $H^0\big(X, S^k N^* \otimes E\big) = 0$. □

Remark 6.3.16. (Finiteness of formal cohomology). By a similar argument, Hartshorne [275, Theorem 5.1] proves in the situation of the Proposition that the formal cohomology groups $H^i\big(\widehat{M}, \widehat{E}\big)$ are finite dimensional for all $i < d$. Combining this with formal duality, he deduces [275, Corollary 5.5] that if in addition M is projective, then for every coherent sheaf $\mathcal{F}$ on M, the cohomology group $H^i\big(M - X, \mathcal{F}\big)$ is finite-dimensional whenever $i \geq n - d$. □

There have been a number of attempts to find global geometric consequences of the amplitude of the normal bundle $N = N_{X/M}$. Hartshorne [276, III.4.2] showed that if X is a smooth divisor in M whose normal bundle $\mathcal{O}_X(X)$ is ample, then some large multiple of X moves in a free linear series, and hence meets any curve with ample normal bundle (Example 1.2.30). This led him to make two conjectures [276, Conjectures 4.4 and 4.5] concerning what one might expect in higher codimension:

> **Hartshorne's Conjecture A.** If $X \subseteq M$ is a smooth subvariety with ample normal bundle, then a sufficiently high multiple of $[X]$ should move (as a cycle) in a large algebraic family.
>
> **Hartshorne's Conjecture B.** Let X , $Y \subseteq M$ be smooth complete subvarieties having ample normal bundles. If $\dim X + \dim Y \geq \dim M$, then X and Y must meet in M.

It was observed by Fulton and the author in [213] that Conjecture A would imply B: see Remark 6.3.18 or Corollary 8.4.3. Conjecture B remains open, although it has been verified in some special cases (cf. [213], [25], [27], [26]). However, it was also shown in [213] that Conjecture A can fail:

Example 6.3.17. (Counterexample to Conjecture A). There exists for $d \gg 0$ a rank-2 ample vector bundle E on $\mathbf{P}^2$ sitting in an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-d)^2 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-1)^4 \longrightarrow E \longrightarrow 0. \tag{*}$$

Such bundles were originally constructed by Gieseker [223] via a reduction to characteristic p; we exhibit them as a special case of a general construction in Example 6.3.67 below. For our example we take $M = \mathbf{E}$ to be the total space of E, and $X \subseteq M$ to be the zero-section. Thus $X = \mathbf{P}^2$ and $N_{X/M} = E$. We will show that the only projective surface $Y \subseteq M$ is the zero-section X itself, and so in particular no multiple of X can move. In fact, let $Y' \longrightarrow Y$ be a resolution of singularities, and denote by $f : Y' \longrightarrow \mathbf{P}^2$ the natural map. The inclusion $Y \hookrightarrow M$ gives rise to a mapping $Y' \longrightarrow M$, which in turn determines a "tautological" section $s \in H^0\big(Y', f^*E\big)$: for $y \in Y'$, $s(y) \in E\big(f(y)\big)$ is the point of the fibre of $\mathbf{E} \longrightarrow \mathbf{P}^2$ over $f(y)$ to which y maps. But it follows from vanishing for the big and nef bundle $f^*\mathcal{O}_{\mathbf{P}}(d)$ (Theorem 4.3.1) that

$$H^1\big(Y', f^*\mathcal{O}_{\mathbf{P}}(-d)\big) \ = \ 0,$$

and by pulling back (*) we deduce that $H^0\big(Y', f^*E\big) = 0$. Therefore Y' must map to the zero-section, as claimed. □

Remark 6.3.18. (Numerical consequences of positive normal bundles). With $X \subseteq M$ as above, the positivity of the normal bundle $N = N_{X/M}$ has numerical consequences. Specifically, if N is nef then for every subvariety Y of dimension complementary to X, the intersection number of X and Y satisfies $\big(X \cdot Y\big) \geq 0$. If N is ample then strict inequality holds provided that Y is homologous to an effective algebraic cycle that meets X. This appears as Corollary 8.4.3. □

Finally, in the spirit of the Lefschetz hyperplane theorem, one can attempt to compare the topology of X and M. Assume now that M is projective. Napier and Ramachandran [473] used L^2 methods to prove that if $N = N_{X/M}$ is ample, then the image of the map

$$\pi_1(X) \longrightarrow \pi_1(M)$$

on fundamental groups has finite index in $\pi_1(M)$. To give a taste of the argument, we will establish a somewhat weaker algebro-geometric assertion:

Theorem 6.3.19. (Analogue of theorem of Napier–Ramachandran). *Let M be a connected complex projective manifold, and let*

$$X \ \subseteq \ M$$

be a smooth irreducible subvariety with $N = N_{X/M}$ ample. Then there is a positive constant $\ell = \ell(X, M)$ depending only on X and M with the following property:

> *If $f : M' \longrightarrow M$ is any finite connected étale covering that admits a section over X, then $\deg f \leq \ell$.*

Equivalently, there do not exist subgroups of arbitrarily large finite index in $\pi_1(M)$ which contain the image of $\pi_1(X)$.

Remark 6.3.20. One can deduce this from the results of Hironaka, Matsumura, et al. on formal rational functions (see [276, Chapter V]). However, we prefer an argument based on the very nice approach of Napier and Ramachandran. In fact, the proof will show — thanks to Remark 6.3.15 — that it is sufficient to assume that the restriction of N to a general complete intersection curve $C \subset M$ is ample. A related cohomological result due to Sommese appears later as Proposition 7.1.12. □

Proof of Theorem 6.3.19. Suppose to the contrary that one can find an infinite sequence $f_k : M_k \longrightarrow M$ of connected (and hence irreducible) étale coverings, with $\deg f_k \to \infty$, each of which admits a section $\mu_k : X \longrightarrow M_k$ over X:

$$\begin{array}{ccc} & & M_k \\ & \overset{\mu_k}{\nearrow} & \big\downarrow{\scriptstyle f_k} \\ X & \hookrightarrow & M. \end{array}$$

We view X as a subvariety both of M and of M_k. Observe that μ_k extends to an isomorphism $U \cong U_k$ between small (classical) neighborhoods of X in M and M_k respectively. It follows in particular that the formal completions $\widehat{M} = \widehat{M_{/X}}$ and $\widehat{M_k} = \widehat{M_{k/X}}$ of M and M_k respectively along X are isomorphic.

Now fix any line bundle L on M, and set $L_k = f_k^*L$. Then $L \mid U \cong L_k \mid U_k$ (as holomorphic line bundles) and therefore

$$H^0\big(\widehat{M_k}, \widehat{L}_k\big) \;\cong\; H^0\big(\widehat{M}, \widehat{L}\big)$$

(since by GAGA we can compute these completions analytically). But quite generally $H^0\big(M_k, L_k\big)$ injects into $H^0\big(\widehat{M_k}, \widehat{L}_k\big)$, and so we conclude that

$$h^0\big(M_k, L_k\big) \;\leq\; h^0\big(\widehat{M}, \widehat{L}\big),$$

the right hand side being finite thanks to Proposition 6.3.13 and the hypothesis on $N_{X/M}$. So to get the required contradiction, it is sufficient to exhibit any line bundle L on M such that the dimension $h^0\big(M_k, f_k^*L\big)$ goes to infinity with k.

But this is easily achieved. In fact, fix an ample divisor H on M and choose any positive integer $b \gg 0$ large enough so that

$$\chi\big(M, \mathcal{O}_M(K_M + bH)\big) \;=\; h^0\big(M, \mathcal{O}_M(K_M + bH)\big) \;\neq\; 0$$

(the equality coming from the Kodaira vanishing theorem). Then

$$f_k^*\mathcal{O}_M\big(K_M + bH\big) \;=\; \mathcal{O}_{M_k}\big(K_{M_k} + f_k^*(bH)\big)$$

since f is étale, and

$$h^0\big(M_k\ ,\ f_k^*\mathcal{O}_M\big(K_M + bH\big)\big) \;=\; \chi\big(M_k\ ,\ f_k^*\mathcal{O}_M\big(K_M + bH\big)\big)$$

thanks again to Kodaira vanishing. But since Euler characteristics are multiplicative in étale covers (Proposition 1.1.28) we conclude that

$$h^0\big(M_k, f_k^*\mathcal{O}_M(K_M + bH)\ \big) \;=\; \deg f_k \cdot h^0\big(M, \mathcal{O}_M(K_M + bH)\big),$$

so setting $L = \mathcal{O}_M(K_M + bH)$ we are done. □

Example 6.3.21. (Hironaka's example). A construction of Hironaka shows that one cannot expect a surjection $\pi_1(X) \twoheadrightarrow \pi_1(M)$ on fundamental groups in the setting of Theorem 6.3.19. Let $f : M' \longrightarrow M$ be a non-trivial connected étale covering between smooth projective varieties of dimension ≥ 3. If $Y \subset M'$ is a sufficiently general complete intersection curve, then the restriction of f will determine an embedding of Y in M: let $X = f(Y) \subset M$ denote its image. But then f splits over X, since by construction $f^{-1}(X)$ contains Y as a connected component. □

Remark 6.3.22. (Concavity and convexity of complements). From an analytic viewpoint, a natural way to measure the positivity properties of an embedding $X \subseteq M$ is to study the (pseudo)-concavity or convexity of the complement $M - X$ in the sense of Andreotti and Grauert [8]. Precise definitions and statements would take us too far afield here: see e.g. [276, Chapter 6, §1 (iv)] for a quick overview. Suffice it to say that Barth [29] obtained some estimates on the concavity and convexity of $\mathbf{P}^r - X$ when $X \subseteq \mathbf{P}^r$ is a closed submanifold: as explained in [276], these are related to Barth's Theorem 3.2.1 on the cohomology of low-codimensional subvarieties of projective space. Sommese [550], [554] extended some of these results to subvarieties of other homogeneous varieties. Assuming that M is projective, Sommese [554, Corollary 1.4] also proves a convexity estimate for $M - X$ when $X \subseteq M$ is any smooth subvariety whose normal bundle is ample and globally generated. □

6.3.B Ample Cotangent Bundles and Hyperbolicity

We now consider smooth projective varieties with ample cotangent bundles. Such varieties are hyperbolic, and the theme is that they exhibit strong forms of properties known or expected for hyperbolic varieties. In the first part of this subsection we summarize some of the basic geometric facts. In the second, we discuss methods of construction.

Geometric properties. We begin by recalling two notions of hyperbolicity:

Definition 6.3.23. (Hyperbolicity). Let X be a smooth complex projective variety, and let $h \in N^1(X)$ be an ample divisor class on X.

(i). X is *algebraically hyperbolic* if there is a positive real number $\varepsilon > 0$ with the following property:

For every finite map $\nu : C \longrightarrow X$ from a smooth curve C to X one has the inequality

$$\big(2g(C) - 2\big) \;\geq\; \varepsilon \;\cdot \big(C \cdot \nu^* h\,\big), \tag{6.8}$$

where as usual $g(C)$ denotes the genus of C.

(ii). Viewed as a complex manifold, X is *Kobayashi hyperbolic* if there are no non-constant entire holomorphic mappings $g : \mathbf{C} \longrightarrow X$. □

Algebraic hyperbolicity was introduced and studied by Demailly in [127]. He actually requires (6.8) to hold only for the normalizations of embedded curves, but it is easily seen using Riemann–Hurwitz that this is equivalent to the condition stated above. Similarly, the absence of entire holomorphic mappings is usually not taken as the definition of hyperbolicity, but for compact targets it is equivalent to the standard definition thanks to a theorem of Brody. (See [127, Corollary 1.2] or [381, Chapter III].)

Example 6.3.24. (Properties of hyperbolic varieties). Keep assumptions as above.

(i). The definition of algebraic hyperbolicity is independent of the ample class h.

(ii). If X is algebraically hyperbolic, then X does not contain any rational or elliptic curves.

(iii). If X is algebraically hyperbolic, then there are no non-constant maps $f : A \longrightarrow X$ from an abelian variety A to X.

(iv). If X is Kobayashi hyperbolic, then X is algebraically hyperbolic.

(The first two statements are clear. For (iii), suppose that $f : A \longrightarrow X$ is non-constant, and consider $\nu_k = f \circ m_k$, where $m_k : A \longrightarrow A$ is multiplication by k. If $C \subset A$ is (say) a general complete intersection curve, then as $k \to \infty$, ν_k will eventually violate (6.8). For (iv), see [127, Theorem 2.1].) □

Remark 6.3.25. (Hypersurfaces of large degree). Let $X \subseteq \mathbf{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2(n+1)$. Then X is algebraically hyperbolic. This is proved (but not explicitly stated) by Voisin [597, §1], building on earlier work of Clemens [93] and Ein [143]. □

It was established by Kobayashi [351] that compact manifolds with negative tangent bundles are hyperbolic:

Theorem 6.3.26. (Kobayashi's theorem). *Let X be a smooth projective variety whose cotangent bundle Ω^1_X is ample. Then X is algebraically hyperbolic. In fact, X is hyperbolic in the sense of Kobayashi.*

Partial Proof. For the Kobayashi hyperbolicity we refer to [127, (3.1)], or [381, III.3]. We prove the first statement using results from Section 6.4 concerning amplitude of bundles on curves. Assuming then that Ω^1_X is ample, fix an ample class h on X and a positive number $\varepsilon > 0$ sufficiently small so that the $\mathbf{Q}$-twisted bundle $\Omega^1_X\langle -\varepsilon h\rangle$ remains ample (Lemma 6.2.8.vi). Given a finite mapping $\nu : C \longrightarrow X$ from a smooth curve to X, the pullback $\nu^*\big(\Omega^1_X\langle -\varepsilon h\rangle\big)$ is then an ample $\mathbf{Q}$-twisted bundle on C. On the other hand, the derivative of ν determines a generically surjective homomorphism $\nu^*\Omega^1_X \longrightarrow \Omega^1_C$, and it then follows from Example 6.4.17 below that $\Omega^1_C{<}-\varepsilon h{>}$ is also ample. Therefore

$$\big(2g(C)-2\big) - \varepsilon\ \big(C\cdot \nu^* h\big) \ = \ \deg\big(\Omega^1_C{<}-\varepsilon h{>}\big)\ >\ 0$$

thanks to Lemma 6.4.10. □

Remark 6.3.27. The converse of Theorem 6.3.26 can easily fail. For example, if B is a curve of genus ≥ 2 then $X = B\times B$ is Kobayashi hyperbolic since it is uniformized by the product of two discs. But Ω^1_X is evidently not ample, since its restriction to $B\times\{\mathrm{pt}\}$ admits a trivial quotient.

Example 6.3.28. (Subvarieties). Let X be a smooth complex projective variety with ample cotangent bundle. Then every irreducible subvariety of X is of general type. (In fact, let $Y_0\subset X$ be an irreducible subvariety of dimension d, and let $\mu : Y \longrightarrow Y_0$ be a resolution of singularities. Then there is a generically surjective homomorphism $\mu^*\Omega^d_X \longrightarrow \Omega^d_Y = \mathcal{O}_Y(K_Y)$. Since Ω^d_X is ample, this implies upon taking symmetric powers after twisting by a small negative multiple of an ample class that $\mathcal{O}_Y(K_Y)$ is big.) It is conjectured by Lang (cf. [127, (3.8)]) that X is hyperbolic if and only if every subvariety of X (including of course X itself) is of general type. □

There are a number of interesting finiteness theorems in the literature for mappings to varieties with ample cotangent bundles: a nice survey appears in [618]. Here we use some ideas from Part One to prove a general boundedness statement for mappings into algebraically hyperbolic varieties.

Theorem 6.3.29. (Boundedness of regular mappings). *Let X be an algebraically hyperbolic variety, and let Y be any irreducible projective variety of dimension $m>0$. Then the set $\mathrm{Hom}(Y,X)$ of morphisms from Y to X forms a bounded family, i.e. all such morphisms are parameterized by finitely many irreducible varieties.*

Sketch of Proof. In fact, fix very ample divisors H and D on Y and X respectively, and let $f : Y \longrightarrow X$ be any morphism. By standard finiteness results, it is sufficient to show that there is a positive integer $a>0$ such that the degree of the graph $\Gamma_f\subset Y\times X$ with respect to the ample divisor $\mathrm{pr}_1^* aH + \mathrm{pr}_2^* D$ is bounded independent of f, i.e. we need to bound from above the intersection number

$$\int_Y c_1\big(\mathcal{O}_Y(aH+f^*D)\big)^m \tag{*}$$

independently of f. To this end, we may assume that f is non-constant. In this case the intersection number $\big(H^{m-1}\cdot f^*D\big)$ computes the degree (with respect to D) of the image in X of a curve obtained as the complete intersection of $(m-1)$ divisors in the linear series $|H|$. Therefore the algebraic hyperbolicity of X implies that there is a uniform upper bound on $\big(H^{m-1}\cdot f^*D\big)$. Then we can fix a positive integer $a\gg 0$, independent of f, such that

$$\big((aH)^m\big) \ > \ m\big((aH)^{(m-1)} \cdot f^*D\big).$$

As H and f^*D are nef, it follows from Theorem 2.2.15 that $(aH - f^*D)$ is big. Therefore some large multiple (possibly depending on f) of $(aH - f^*D)$ is effective. Thanks again to the nefness of H and f^*D, one concludes that

$$\Big((aH - f^*D) \cdot (aH)^{(m-1-i)} \cdot (f^*D)^i\Big) \ \geq \ 0$$

for all $0 \leq i \leq n-1$. This in turn leads to the inequalities

$$a^m\big(H^m\big) \ \geq \ a^{m-1}\big(H^{m-1} \cdot f^*D\big) \ \geq \ a^{m-2}\big(H^{m-2} \cdot f^*D^2\big) \geq \dots \\ \geq \big(f^*D^m\big) \ \geq 0, \quad (6.9)$$

from which it follows that the quantity in (*) is bounded above by $2^m a^m\big(H^m\big)$. □

As a consequence, we get a quick proof of a result of Kalka, Shiffman, and Wong from [309, Corollary 4]:

Corollary 6.3.30. (Finiteness of regular mappings). *Let X be a smooth projective variety whose cotangent bundle Ω^1_X is ample, and let Y be any irreducible projective variety. Then the set* $\mathrm{Hom}_*(Y,X)$ *of non-constant morphisms from Y to X is finite.*

Proof. The amplitude of Ω^1_X implies by 6.1.4 that $H^0(Y, f^*TX) = 0$, so in any event the Hom scheme in question is discrete (cf. [114, Proposition 2.4]). On the other hand, it follows from the previous result that the set of all maps $f : Y \longrightarrow X$ is parametrized by finitely many irreducible varieties. Putting these facts together, it follows that there are only finitely many such maps. □

Remark 6.3.31. (Finiteness of rational mappings). A related result of Noguchi and Sunada [480] states that with X and Y as in Corollary 6.3.30, the set $\mathrm{Rat}_*(Y,X)$ of non-constant rational maps from Y to X is also finite. □

Remark 6.3.32. (Rational points over function fields). Another interesting avenue of investigation concerns the diophantine properties of varieties with ample cotangent bundles defined over function fields. In this setting, a number of authors have obtained Mordell-type statements. For example, suppose that L is an algebraic function field over an algebraically closed groundfield K of characteristic zero, and suppose that X is a smooth, projective, and geometrically integral variety over L with ample cotangent bundle $\Omega^1_{X/L}$. Inspired by theorems of Grauert [233] and Manin [415] in the one-dimensional case, Martin-Deschamps [418] proves that if the set of L-rational points of X is Zariski dense, then X is isotrivial over L. There are related results due to Noguchi [479] and Moriwaki [441]. □

Remark 6.3.33. (Rational points over number fields). If X is a smooth projective variety defined over a number field L that has ample cotangent bundle, then it is a conjecture of Lang [380] that the set of L-rational points of X is finite. Moriwaki [442] remarks that this follows from work of Faltings [178], [179] if one assumes in addition that the cotangent bundle of X is globally generated. □

Remark 6.3.34. (Miyaoka's theorem on generic semipositivity). A basic theorem of Miyaoka [430] shows that the cotangent bundle of a projective variety satisfies a weak positivity property in very general circumstances. Specifically, let X be a smooth complex projective variety of dimension n, and let H be an ample divisor on X. Suppose that X is not uniruled, i.e. assume that X is not covered by rational curves. Miyaoka's theorem states that if $C \subset X$ is a sufficiently general curve arising as the complete intersection of $n-1$ divisors in the linear series $|mH|$ for $m \gg 0$, then the restriction of Ω^1_X to C is nef. We refer to [432] and Shepherd-Barron's exposition in [360, Chapter 9] for proofs and a discussion of some of the applications. □

Remark 6.3.35. (Varieties of general type). There are a number of very interesting results and conjectures concerning finiteness properties for varieties of general type. Bogomolov [61] proved that if X is a surface of general type satisfying the inequality $c_1(X)^2 > c_2(X)$, then the family of curves on X of fixed geometric genus is bounded. Martin-Deschamps gives a nice account of this work in [138]. One can view Bogomolov's theorem as going in the direction of conjectures of Lang concerning the diophantine and geometric properties of varieties of general type. These conjectures predict, for example, that if X is a projective variety of general type, then there exists a proper Zariski-closed subset $Z \subsetneqq X$ having the property that the image of any non-constant morphism $f : G \longrightarrow X$ from an algebraic group to X must lie in Z: in particular, Z must contain all rational curves on X. We refer to [382, Chapter 1] for a pleasant discussion of this circle of ideas. □

Constructions. Although one expects that varieties with ample cotangent bundle should be reasonably plentiful, until recently relatively few explicit constructions appeared in the literature except in the case of surfaces.

Construction 6.3.36. (Ball quotients). If X is a smooth complex projective variety that is uniformized by the ball $\mathbf{B}^n \subset \mathbf{C}^n$, then the Bergman metric on $\mathbf{B}^n$ descends to a metric on X with negative holomorphic sectional curvature, and hence Ω^1_X is ample (cf. [618, Example 2, p. 147]). □

Construction 6.3.37. (Surfaces). Yau raised the question of classifying all surfaces with positive cotangent bundles, and motivated in part by this several authors have given constructions of such surfaces.

- **Miyaoka's examples.** Building on ideas of Bogomolov, Miyaoka [428] showed that if X is a smooth complex projective surface of general type with $c_1(X)^2 > 2c_2(X)$, then the cotangent bundle Ω^1_X is "almost everywhere ample," which very roughly means that it fails to be ample only along finitely many curves. Using this, he deduces that if X_1 and X_2 are two such surfaces, then a complete intersection of two general sufficiently positive divisors in $X_1 \times X_2$ is a surface X with Ω^1_X ample.

- **Kodaira surfaces.** Martin-Deschamps [418] established the amplitude of the cotangent bundles of certain Kodaira surfaces, i.e. surfaces that admit a smooth map to a non-singular curve. A similar result was proved independently by Schneider and Tancredi [524], who also generalize Miyaoka's construction.
- **Hirzebruch–Sommese examples.** Hirzebruch found some interesting surfaces by desingularizing Kummer coverings of $\mathbf{P}^2$ branched over line arrangements, and Sommese [555] classified which of these have ample cotangent bundles. □

In higher dimensions, the most general constructions are due to Bogomolov and Debarre. We recommend Debarre's nice paper [115] for more information.

Construction 6.3.38. (Bogomolov's construction). Let $Y_1, \ldots, Y_m$ be smooth projective varieties of dimension $d \geq 1$, each having big cotangent bundle,[9] and let

$$X \subseteq Y_1 \times \ldots \times Y_m$$

be a general complete intersection of sufficiently high multiples of an ample divisor. Bogomolov proves that if

$$\dim X \leq \frac{d(m+1)+1}{2(d+1)},$$

then X has ample cotangent bundle. Bogomolov and Debarre deduce from this that there exists a projective variety X having ample cotangent bundle with the additional property that $\pi_1(X)$ can be any fixed group that arises as the fundamental group of a smooth projective variety: in particular, X can be simply connected. Wong [612] employed a similar construction in a differential-geometric context. A detailed description and verification of Bogomolov's construction appears in [115, §3]: we will work through an elementary special case in Construction 6.3.42. □

Construction 6.3.39. (Complete intersections in abelian varieties). Debarre [115] recently proved that if X is the complete intersection of $e \geq n$ sufficiently ample general divisors in a simple abelian variety of dimension $n + e$, then the cotangent bundle Ω^1_X is ample. □

[9] Recall from Example 6.1.23 that a vector bundle E on a projective variety V is big if $\mathcal{O}_{\mathbf{P}(E)}(1)$ is a big line bundle on $\mathbf{P}(E)$. Bogomolov shows that if Y is a surface of general type satisfying $c_1(Y)^2 > c_2(Y)$, then the cotangent bundle of Y is big.

Remark 6.3.40. (Complete intersections in projective space). Debarre [115, §2.2] conjectures that if $X \subseteq \mathbf{P}^r$ is the complete intersection of $e \geq r/2$ hypersurfaces of sufficiently high degree, then the cotangent bundle of X is ample. □

Remark 6.3.41. (Nef cotangent bundles). It is also very interesting to ask for examples of projective manifolds whose cotangent bundles are numerically effective. The class of all such is evidently closed under taking products, subvarieties, and finite unramified covers, and it includes smooth subvarieties of abelian varieties. A nice theorem of Kratz [372, Theorem 2] states that if X is a complex projective variety whose universal covering space is a bounded domain in $\mathbf{C}^n$ or in a Stein manifold, then Ω^1_X is nef. We refer to Theorem 7.2.19 for a result about projective embeddings of such varieties. □

As Miyaoka suggested, a special case of Bogomolov's construction is particularly elementary. We devote the rest of this subsection to working this case out explicitly.

Construction 6.3.42. (Complete intersections in curve products). Start with smooth projective curves $T_1, \ldots, T_{n+e}$ of genus ≥ 2, and set $T = T_1 \times \ldots \times T_{n+e}$, with projections $p_i : T \longrightarrow T_i$. Fix next very ample line bundles A_i on T_i and for each $d > 0$ put

$$A \ = \ p_1^* A_1 \otimes \ldots \ \otimes p_{n+e}^* A_{n+e} \quad , \quad L \ = \ L_d \ = \ A^{\otimes d}.$$

Choose finally e general divisors $D_1, \ldots, D_e \in |L_d|$, and set

$$X \ = \ D_1 \cap \cdots \cap D_e \ \subseteq \ T.$$

Assuming that $e \geq 2n-1$ and $d \geq n$, we will now verify that X is a smooth projective n-fold whose cotangent bundle Ω^1_X is ample.

Sketch of Verification of Construction 6.3.42. The fact that X is a smooth n-fold is clear, and the issue is to establish the amplitude of its cotangent bundle. To this end, we will consider projections of X onto various products of the T_i. As a matter of notation, for any multi-index $I = \{i_1, \ldots, i_k\}$ ($1 \leq i_1 < \cdots < i_k \leq n+e$), write

$$T_I = T_{i_1} \times \cdots \times T_{i_k},$$

and denote by $p_I : T \longrightarrow T_I$ the corresponding projection. Somewhat abusively, we will also write p_I for the restriction of this projection to subvarieties of T. The first point is to check that one can arrange by choosing the D_i generally enough that X satisfies two genericity conditions:

(i). For every I of length $2n$, the projection $p_I : X \longrightarrow T_I$ is unramified.

(ii). For every J of length n, the projection $p_J : X \longrightarrow T_J$ is finite.

Property (i) is verified by a standard dimension count as in [280, II.8.18]. The second follows from a general finiteness statement (Lemma 6.3.43) formulated and proved at the end of this subsection: this is where the hypothesis $d \geq n$ is used.

Assuming that we have arranged for X to satisfy the two properties just discussed, we verify that Ω^1_X is ample. Since Ω^1_X — being a quotient of $\Omega^1_T \mid X$ — is globally generated, it is enough by Gieseker's lemma (Proposition 6.1.7) to show that it does not admit a trivial quotient along any curve. Suppose then that $C \subseteq X$ is a reduced and irreducible curve. It follows from property (ii) that there can be at most $n-1$ indices $i \in [1, n+e]$ such that the projection $p_i : X \longrightarrow T_i$ maps C to a point. Since $n + e \geq 3n - 1$ we may assume after reindexing that $p_i|C$ is finite for $1 \leq i \leq 2n$. Set $I_0 = \{1, 2, \ldots, 2n\}$, and for any $J \subset I_0$ of length n denote by $R_J \subset X$ the ramification divisor of the branched covering $p_J : X \longrightarrow T_J$. It follows by a simple argument from property (i) that

$$\bigcap_{\substack{J \subset I_0 \\ |J|=n}} \operatorname{supp}\big(R_J\big) \ = \ \varnothing. \tag{*}$$

Therefore we can choose $J \subset I_0$ so that $C \not\subset \operatorname{supp}(R_J)$. Then the derivative dp_J gives rise to an exact sequence

$$0 \longrightarrow p_J^*\left(\Omega^1_{T_J}\right)|C \longrightarrow \Omega^1_X|C \longrightarrow \tau \longrightarrow 0,$$

where τ is a torsion sheaf supported on $C \cap \operatorname{supp}(R_J)$. But the bundle on the left is ample, and it follows right away that $\Omega^1_X|C$ has no trivial quotients, as required. □

Finally, we state and prove the finiteness lemma that was used in the course of the argument just completed. We will have occasion to refer to it also in Section 6.3.F.

Lemma 6.3.43. (Finiteness lemma). *Let Y and T be irreducible projective varieties of dimensions e and n respectively. Let A and B be very ample line bundles on Y and T, and for $d \geq 1$ set*

$$L \ = \ L_d \ = \ \mathrm{pr}_1^* A^{\otimes d} \otimes \mathrm{pr}_2^* B.$$

Consider e general divisors $D_1, \ldots, D_e \ \in \ |L_d|$ in the indicated linear series on $Y \times T$. If $d \geq n$, then the intersection $D_1 \cap \cdots \cap D_e$ is finite over T.

Proof. Fix some $0 \leq k < e$ and consider

$$X \ = \ D_1 \cap \cdots \cap D_k \ \subseteq \ Y \times T.$$

(If $k = 0$ take $X = Y \times T$.) We assume inductively that every fibre of the projection $X \longrightarrow T$ has pure dimension $e - k$, and we will show that one can arrange for every fibre of $X \cap D_{k+1} \longrightarrow T$ to have pure dimension $e - (k + 1)$.

To this end, fix $t \in T$, and denote by X_t the fibre of X over t. Consider the set of "bad" divisors at t:

$$Z_t = \big\{ D \in |L_d| \ \big| \ D \text{ contains one or more components of } X_t \ \big\}.$$

We claim:

$$Z_t \text{ has codimension} > d \text{ in the linear series } |L_d|. \qquad (*)$$

Granting this, it follows that the set

$$Z = \big\{ D \in |L_d| \ \big| \ \text{ some fibre of } X \cap D \longrightarrow T \text{ has dim} \geq e - k \ \big\}$$

has codimension $> d - n$ in $|L_d|$. Hence if $d \geq n$ we can find $D_{k+1} \notin Z$.

Turning to (*), since $H^0\big(Y \times T, L_d\big)$ maps surjectively onto the fibre-wise space of sections $H^0\big(Y, A^{\otimes d}\big)$ over t, and since A is very ample, it is enough to verify the following assertion:

> Let $V \subseteq \mathbf{P}$ be any algebraic subset of positive dimension in some projective space $\mathbf{P}$, and let
>
> $$Z_V = \big\{ E \in |\mathcal{O}_{\mathbf{P}}(d)| \ \big| \ E \text{ contains } V \big\}.$$
>
> Then Z_V has codimension $> d$ in $|\mathcal{O}_{\mathbf{P}}(d)|$.

But this follows from the elementary and well-known fact that any $d+1$ points on V impose independent conditions on hypersurfaces of degree d in $\mathbf{P}$. □

6.3.C Picard Bundles

When C is a smooth projective curve of genus $g \geq 1$, the Jacobian of C carries some interesting bundles, whose projectivizations are the symmetric products of C. It was established by Fulton and the author in [212] that these so-called Picard bundles are negative, a fact that was used there to study the varieties of special divisors on C. Here we follow the same arguments to prove the negativity of the analogous bundles on the Picard variety of any irregular smooth projective variety. The application to special divisors appears as Theorem 7.2.12.

Convention 6.3.44. In the present subsection, it is most natural to deal with the projective bundle of one-dimensional **sub**-bundles of a given bundle F on a variety Y. We denote this projectivization by $\mathbf{P}_{\text{sub}}(F)$, with $\pi : \mathbf{P}_{\text{sub}}(F) \longrightarrow Y$ the projection. So $\mathbf{P}_{\text{sub}}(F) = \mathbf{P}(F^*)$. On $\mathbf{P}_{\text{sub}}(F)$ one has the tautological line sub-bundle $\mathcal{O}_{\mathbf{P}_{\text{sub}}}(-1) \subseteq \pi^* F$. We say that F is *negative* if F^* is ample. Thus F is negative if and only if the tautological line bundle $\mathcal{O}_{\mathbf{P}_{\text{sub}}}(1)$ on $\mathbf{P}_{\text{sub}}(F)$ is ample. □

We start by constructing the Picard bundles. Throughout this subsection, X is a smooth projective variety of dimension n. Fix an algebraic equivalence class λ on X and denote by $\mathrm{Pic}^{\lambda}(X)$ the component of the Picard variety parameterizing bundles in the chosen class. Thus $\mathrm{Pic}^{\lambda}(X)$ is a torus of dimension $q(X) = \dim H^1(X, \mathcal{O}_X)$. Given a point $t \in \mathrm{Pic}^{\lambda}(X)$ we denote by L_t the corresponding line bundle on X. Choosing a base point $0 \in X$, there is a Poincaré line bundle $\mathcal{L}$ on $X \times \mathrm{Pic}^{\lambda}(X)$, characterized by the properties

$$\begin{aligned} \mathcal{L} \mid \big(X \times \{t\}\big) &= L_t \quad \forall\, t \in \mathrm{Pic}^{\lambda}(X); \\ \mathcal{L} \mid \big(\{0\} \times \mathrm{Pic}^{\lambda}(X)\big) &= \mathcal{O}_{\mathrm{Pic}^{\lambda}(X)}. \end{aligned}$$

Our object is to realize the groups $H^0(X, L_t)$ for $t \in \mathrm{Pic}^{\lambda}(X)$ as the fibres of a vector bundle E_{λ} on $\mathrm{Pic}^{\lambda}(X)$. In order for this to work smoothly we will suppose that the class λ is sufficiently positive so that

$$H^i(X, L_t) = 0 \ \text{ for all } i > 0 \text{ and all } t \in \mathrm{Pic}^{\lambda}(X), \tag{6.10}$$

$$H^0(X, L_t) \neq 0 \ \text{ for all } t \in \mathrm{Pic}^{\lambda}(X). \tag{6.11}$$

It follows from (6.10) by the theorems on cohomology and base-change that the direct image

$$E_{\lambda} =_{\mathrm{def}} \mathrm{pr}_{2,*}\big(\mathcal{L}\big)$$

under the second projection $\mathrm{pr}_2 : X \times \mathrm{Pic}^{\lambda}(X) \longrightarrow \mathrm{Pic}^{\lambda}(X)$ is a vector bundle on $\mathrm{Pic}^{\lambda}(X)$, which we call the *Picard bundle* corresponding to the class λ. It is non-zero by (6.11). Furthermore, push-forwards of $\mathcal{L}$ commute with base-change, so in particular one has a canonical isomorphism

$$E_{\lambda}(t) = H^0\big(X, L_t\big) \tag{6.12}$$

of the fibres of E_{λ} with the corresponding cohomology groups on X.

As in the case of curves the key to analyzing the properties of these Picard bundles is to interpret their projectivizations as spaces of divisors. Specifically, let $\mathrm{Div}^{\lambda}(X)$ be the Hilbert scheme parameterizing all effective divisors in the algebraic equivalence class λ, and denote by

$$u : \mathrm{Div}^{\lambda}(X) \longrightarrow \mathrm{Pic}^{\lambda}(X)$$

the Abel–Jacobi mapping that sends a divisor to its linear equivalence class. Given a point $s \in \mathrm{Div}^{\lambda}(X)$ we denote by D_s the corresponding divisor on X.

Lemma 6.3.45. *Still assuming that λ satisfies* (6.10) *and* (6.11), *one has a canonical isomorphism*

$$\mathbf{P}_{\mathrm{sub}}(E_{\lambda}) = \mathrm{Div}^{\lambda}(X)$$

under which the bundle projection $\pi : \mathbf{P}_{\mathrm{sub}}(E_{\lambda}) \longrightarrow \mathrm{Pic}^{\lambda}(X)$ corresponds to the Abel–Jacobi mapping u.

Idea of Proof. The essential point is simply that one has natural identifications

$$\pi^{-1}(t) \;=\; \mathbf{P}_{\mathrm{sub}}\big(E_\lambda(t)\big) \;=\; \mathbf{P}_{\mathrm{sub}}\big(H^0(X, L_t)\big) \;=\; u^{-1}(t)$$

coming from (6.12). We leave it to the reader to use the universal property of Div^λ to construct the stated isomorphism globally. □

Remark 6.3.46. When X is a smooth curve, the Picard bundles have been intensively studied. For instance, their Chern classes are given by a formula of Poincaré ([15, I.5] and [208, 4.3.3]), and they are stable with respect to the canonical polarization on $\mathrm{Jac}(X)$ ([150]). □

Homomorphisms defined by restriction to subsets also play an important role. Let Z be a fixed finite subscheme of X. Then the evaluation maps

$$\sigma_t : H^0(X, L_t) \longrightarrow H^0(X, L_t \otimes \mathcal{O}_Z)$$

globalize to a morphism $\sigma = \sigma_Z : E_\lambda \longrightarrow \Sigma_Z$ of vector bundles on $\mathrm{Pic}^\lambda(X)$. In fact, consider the subscheme $Z \times \mathrm{Pic}^\lambda(X) \subseteq X \times \mathrm{Pic}^\lambda(X)$, and define

$$\Sigma_Z \;=\; \mathrm{pr}_{2,*}\big(\mathcal{L} \;\big|\; Z \times \mathrm{Pic}^\lambda(X)\;\big).$$

Taking direct images of the restriction $\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_{Z \times \mathrm{Pic}^\lambda(X)}$ gives rise to σ. Observe that if $Z = \{x_1, \ldots, x_w\}$ is a reduced scheme consisting of w distinct points of X, then

$$\Sigma_Z = P_{x_1} \oplus \cdots \oplus P_{x_w},$$

where for $x \in X$, $P_x = \mathcal{L} \,|\, \big(\{x\} \times \mathrm{Pic}^\lambda(X)\big)$ is a line bundle on $\mathrm{Pic}^\lambda(X)$ that is a deformation of the trivial bundle $P_0 = \mathcal{O}_{\mathrm{Pic}^\lambda(X)}$. For an arbitrary finite subscheme $Z \subset X$, Σ_Z has a filtration whose quotients are line bundles of this type.

The maps σ_Z have a simple meaning in terms of the isomorphism in 6.3.45. In fact, consider on $\mathbf{P}_{\mathrm{sub}}(E_\lambda)$ the composition

$$\begin{array}{ccc} \mathcal{O}_{\mathbf{P}_{\mathrm{sub}}(E_\lambda)}(-1) & \hookrightarrow & \pi^* E_\lambda \\ & \searrow^{s_Z} & \downarrow {\scriptstyle \pi^*\sigma_Z} \\ & & \pi^*\Sigma_Z \end{array}$$

defining s_Z. Viewing s_Z as a section

$$s_Z \;\in\; \Gamma\Big(\,\mathbf{P}_{\mathrm{sub}}(E_\lambda)\,,\, \pi^*\Sigma_Z \otimes \mathcal{O}_{\mathbf{P}_{\mathrm{sub}}(E_\lambda)}(1)\,\Big),$$

it is immediate to verify

Lemma 6.3.47. *Under the identification* $\mathbf{P}_{\mathrm{sub}}(E_\lambda) = \mathrm{Div}^\lambda(X)$*, the zero locus of* s_Z *consists of all* $s \in \mathrm{Div}^\lambda(X)$ *such that the corresponding divisor* D_s *contains* Z*.* □

A basic fact is that Picard bundles are negative:[10]

Theorem 6.3.48. (Negativity of the Picard bundle). *Let* λ *be any class satisfying* (6.10) *and* (6.11). *Then the Picard bundle* E_λ *on* $\mathrm{Pic}^\lambda(X)$ *is negative, i.e. its dual* E_λ^* *is ample. Moreover, if* $Z \subseteq X$ *is any finite subscheme, then the bundle*

$$\mathrm{Hom}(E_\lambda, \Sigma_Z) \ = \ E_\lambda^* \otimes \Sigma_Z$$

is ample.

Proof. For the first statement, we apply Nakai's criterion to establish the amplitude of the tautological bundle $\mathcal{O}_{\mathbf{P}_{\mathrm{sub}}(E_\lambda)}(1)$ on $\mathbf{P}_{\mathrm{sub}}(E_\lambda)$. Suppose then that $V \subseteq \mathbf{P}_{\mathrm{sub}}(E_\lambda)$ is any irreducible subvariety of dimension $k \geq 1$, and let ξ denote the numerical equivalence class of $c_1\big(\mathcal{O}_{\mathbf{P}_{\mathrm{sub}}(E_\lambda)}(1)\big)$. The positivity of $\int_V \xi^k$ will follow by induction if we show that $\xi \cap [V]$ is represented (in numerical equivalence) by a non-zero effective $(k-1)$-cycle. To this end, fix a general point $x \in X$, and consider in $\mathrm{Div}^\lambda(X)$ the divisor

$$I_x \ = \ \{ s \in \mathrm{Div}^\lambda(X) \mid D_s \ni x \}.$$

It follows from Lemma 6.3.47 that under the identification $\mathrm{Div}^\lambda(X) = \mathbf{P}_{\mathrm{sub}}(E_\lambda)$, I_x arises as the zeroes of a section of $\mathcal{O}_{\mathbf{P}_{\mathrm{sub}}(E_\lambda)}(1) \otimes P_x$, where P_x is the deformation of $\mathcal{O}_{\mathrm{Pic}^\lambda(X)}$ introduced above. Hence $I_x \equiv_{\mathrm{num}} \xi$. So it suffices to show that for general x, I_x meets V in a non-empty proper subset of V. But this is clear: given any positive-dimensional family of effective divisors, those passing through a given general point form a non-empty proper subfamily. Turning to the amplitude of $E_\lambda^* \otimes \Sigma_Z$ we focus on the case in which Z is the reduced subscheme consisting of distinct points $\{x_1, \ldots, x_w\}$, leaving the general assertion to the interested reader. Then the Hom bundle in question is a direct sum of the bundles $E_\lambda^* \otimes P_{x_i}$, P_{x_i} being a deformation of $\mathcal{O}_{\mathrm{Pic}^\lambda(X)}$. But by Corollary 6.1.19, the amplitude of $E_\lambda^* \otimes P_{x_i}$ is equivalent to that of E_λ^*, which we have just treated. □

6.3.D The Bundle Associated to a Branched Covering

Following the author's paper [387] we discuss a vector bundle that is associated to a branched covering of smooth varieties, and establish in particular that it is ample for coverings of projective space. This will be used in Section 7.1.C to prove a Barth-type theorem for such coverings.

Let X and Y be smooth varieties of dimension n, with Y irreducible, and let

$$f : X \longrightarrow Y$$

[10] The second statement of the theorem will be needed later, in Section 7.2.C.

be a branched covering (i.e. a finite surjective mapping) of degree d. Then f is flat, and consequently the direct image sheaf $f_*\mathcal{O}_X$ is locally free of rank d on Y. Moreover, as we are in characteristic zero the natural inclusion $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ splits via the trace

$$\mathrm{Tr}_{X/Y} : f_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y.$$

Let $F = \ker \mathrm{Tr}_{X/Y}$, so that F is a bundle of rank $d-1$ on Y that appears in a canonical decomposition $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus F$. We consider (for reasons that will become apparent) the dual bundle

$$E \ = \ E_f \ = \ F^*,$$

which we call the *bundle associated to the covering* f.

Proposition 6.3.49. *Denote by* $\mathbf{E}$ *the total space of* E*, and by* $p : \mathbf{E} \longrightarrow Y$ *the bundle projection. Then the covering* f *canonically factors through an embedding of* X *into* $\mathbf{E}$:

$$\begin{array}{ccc} X & \subseteq & \mathbf{E} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y. & \end{array}$$

Proof. Recall that

$$\mathbf{E} \ = \ \mathrm{Spec}_{\mathcal{O}_Y}\ \mathrm{Sym}(E^*) \ = \ \mathrm{Spec}_{\mathcal{O}_Y}\ \mathrm{Sym}(F).$$

The natural inclusion $F \subseteq f_*\mathcal{O}_X$ gives rise to a surjection $\mathrm{Sym}(F) \longrightarrow f_*\mathcal{O}_X$ of $\mathcal{O}_Y$-algebras, which in turn defines the required embedding of X into $\mathbf{E}$ over Y. □

Example 6.3.50. (Double covers). Let L be a line bundle on Y with the property that there exists a smooth divisor $D \in |2L|$. In Proposition 4.1.6 we constructed a degree-two cyclic covering $f : X \longrightarrow Y$ branched over D, realizing X as a subvariety of the total space of L. Then $E_f = L$, and the embedding of X into L is a special case of Proposition 6.3.49. □

Remark 6.3.51. (Triple covers). Miranda [425] has given a quite complete description of the data involved in specifying a triple cover with given bundle. □

Example 6.3.52. Let $f : X \longrightarrow \mathbf{P}^n$ be the degree $(n+1)$ covering constructed in Example 3.4.13. Then E_f is isomorphic to the tangent bundle $T\mathbf{P}^n$. □

Example 6.3.53. There is a canonical isomorphism

$$\big(f_*\mathcal{O}_X\big)^* \ = \ f_*\,\omega_{X/Y},$$

where $\omega_{X/Y} = \mathcal{O}_X\big(K_X - f^*K_Y\big)$ is the relative canonical bundle of X over Y. (This is a special case of duality for a finite flat morphism: cf. [280, Exercises III.6.10, III.7.2].) □

Example 6.3.54. (Branch divisor). The algebra structure on $f_*\mathcal{O}_X$ defines a mapping $f_*\mathcal{O}_X \otimes f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X$ of bundles on Y, which composed with the trace determines a homomorphism $f_*\mathcal{O}_X \otimes f_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ and hence

$$\delta : \big(f_*\mathcal{O}_X\big) \longrightarrow \big(f_*\mathcal{O}_X\big)^*.$$

Then δ drops rank precisely on the branch divisor $B \subset Y$ of f, and in particular

$$-2 \cdot c_1\big(f_*\mathcal{O}_X\big) \ \equiv_{\text{lin}} \ [B].$$

(Locally $\det \delta$ is the classical discriminant of $f : Y \longrightarrow X$, cf. [3, Chapt. 6, §3]. Note that the divisor structure on B is determined by taking it to be the divisor-theoretic push-forward $f_*[R]$ of the ramification divisor $R \subset X$.) □

We now turn to branched coverings of projective space. Here the associated bundles E_f satisfy a very strong positivity property:

Theorem 6.3.55. (Coverings of projective space). *Let $f : X \longrightarrow \mathbf{P}^n$ be a branched covering of projective space by a smooth irreducible complex projective variety X, and let $E = E_f$ be the corresponding bundle on $\mathbf{P}^n$. Then $E(-1)$ is globally generated. In particular, E is ample.*

Proof. We use the basic criterion of Castelnuovo–Mumford regularity that if $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^n$ such that $H^i\big(\mathbf{P}^n, \mathcal{F}(-i)\big) = 0$ for all $i > 0$, then $\mathcal{F}$ is globally generated (Theorem 1.8.3). The plan is to apply this to the bundle $E(-1)$.

Example 6.3.53 implies that

$$E \oplus \mathcal{O}_{\mathbf{P}^n} \ = \ \big(f_*\mathcal{O}_X\big)^* \ = \ \big(f_*\,\omega_X\big)(n+1). \qquad (*)$$

Since f is finite, one has isomorphisms

$$H^i\big(\mathbf{P}^n, f_*\,\omega_X(k)\big) \ = \ H^i\big(X, \omega_X \otimes f^*\mathcal{O}_{\mathbf{P}^n}(k)\big) \quad \text{for all} \ \ i\ ,\ k.$$

It then follows from (*) and Kodaira vanishing (Theorem 4.2.1) that

$$H^i\big(\mathbf{P}^n, E(-1-i)\big) \ = \ 0 \quad \text{for} \ \ 1 \le i \le n-1.$$

On the other hand, $H^n\big(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-n-1)\big) = H^n\big(X, \omega_X\big) = \mathbf{C}$, so it also follows from (*) that $H^n\big(\mathbf{P}^n, E(-n-1)\big) = 0$. Thus $E(-1)$ satisfies the required vanishings, and hence is globally generated. □

Example 6.3.56. (Consequences of amplitude). Let $f : X \longrightarrow Y$ be a branched covering of smooth projective varieties of dimension n, and $E = E_f$ the corresponding bundle on Y. The amplitude of E has some interesting geometric consequences analogous to those deduced when $Y = \mathbf{P}^n$ from the Fulton–Hansen connectedness theorem (Section 3.4).

(i). Let S be a (possibly singular) reduced and irreducible projective variety of dimension ≥ 1, and let $g : S \longrightarrow Y$ be a finite morphism. If E_f is ample, then the fibre product $Z = X \times_Y S$ is connected.

(ii). Denote by $e_f(x)$ the local degree of f at a point $x \in X$ (Definition 3.4.7). Then there exists at least one point $x \in X$ at which $e_f(x) \geq \min\{\deg f\ ,\ n+1\}$.

(For (i), the amplitude of E_f implies that $H^0\big(S, g^*E^*\big) = 0$. Writing $f' : Z \longrightarrow S$ for the induced map, it follows from this that $H^0\big(S, f'_*\mathcal{O}_Z\big) = \mathbf{C}$. Statement (ii) is then deduced as in the proof of Theorem 3.4.8. See [387, Proposition 1.3] for details.) □

Example 6.3.57. Given a branched covering $f : X \longrightarrow Y$ of smooth projective varieties, it is not true in general that the associated bundle E_f is ample or even nef. Examples may be constructed for instance by observing that if f is the double covering associated to a smooth divisor $D \in |2L|$ as in 6.3.50, then $E_f = L$ is ample or nef if and only if L is. On the other hand, if $C \subset Y$ is any curve not contained in the branch locus of f, then $E \mid C$ is nef. (It is enough to test this after pulling back by any cover $C' \longrightarrow C$, and then one reduces to the case of a branched covering $D' \longrightarrow C'$ of curves with the property that every irreducible component of D' maps isomorphically to C'. See [504, Appendix] for details.)

Remark 6.3.58. (Rational homogeneous spaces and other varieties). Kim and Manivel [336], [335], [337], [417] have studied the bundle associated to a branched covering $f : X \longrightarrow Y$ for certain rational homogeneous spaces Y. In the cases they treat, they prove that for any X and f, the bundle E_f is always spanned, and even ample when $b_2(Y) = 1$. They conjecture that this is true for any rational homogeneous space Y with $b_2(Y) = 1$. Some other results concerning these bundles — notably when Y is a curve or a Del Pezzo (or more general Fano) manifold — appear in the papers [504], [503] of Peternell and Sommese. □

Example 6.3.59. (Coverings of abelian varieties). Let

$$f : X \longrightarrow A$$

be a branched covering of an abelian variety A by a smooth (but possibly disconnected) projective variety X of dimension n. Then E_f is nef. This is a result of Peternell–Sommese [504], extending earlier work of Debarre [109]. (Choose a very ample divisor B on A, and argue first as in the proof of Theorem 6.3.55 that $\big(f_*\,\omega_X\big) \otimes \mathcal{O}_A\big((n+1)B\big)$ is globally generated. This gives a lower bound on the Barton invariant (Example 6.2.14) of $f_*\,\omega_X$:

$$\delta(A, f_*\,\omega_X, B)\ \geq\ -(n+1). \qquad (*)$$

Now fix $k > 0$ and consider the map $\nu = \nu_k : A \longrightarrow A$ given by multiplication by k. Applying (*) to the pulled-back covering $f' : X' = X \times_A A \longrightarrow A$ one finds that

$$\begin{aligned}\delta\big(A\ ,\ f_*\,\omega_X\ ,\ B\big) \;&=\; \delta\big(A\ ,\ \nu^* f_*\,\omega_X\ ,\ \nu^* B\big)\\ &=\; \delta\big(A\ ,\ \nu^* f_*\,\omega_X\ ,\ k^2\cdot B\big)\\ &=\; \tfrac{1}{k^2}\cdot\delta\big(A\ ,\ \nu^* f_*\,\omega_X\ ,\ B\big)\\ &=\; \tfrac{1}{k^2}\cdot\delta\big(A\ ,\ f'_*\,\omega_{X'}\ ,\ B\big)\\ &\geq\; \frac{-(n+1)}{k^2}.\end{aligned}$$

Letting $k \to \infty$ it follows that $\delta(A, f_*\,\omega_X, B) \geq 0$, as required.) Debarre conjectures that if $f : X \longrightarrow A$ is a non-trivial branched covering of a *simple* abelian variety A by a smooth irreducible variety, then E_f is ample provided that f does not factor through an étale covering of A. □

6.3.E Direct Images of Canonical Bundles

Here we discuss very briefly one more instance where positivity properties of vector bundles have proven to be of great importance. Our modest intention is to convey something of the flavor of a large and imposing body of work through a couple of highly oversimplified statements.

In 1978, Fujita [192] proved an important and suggestive result about the direct images of the relative canonical bundles of fibre spaces over curves:[11]

Theorem. (Fujita's theorem). *Let X be a smooth projective variety of dimension n, and suppose given a surjective mapping $f : X \longrightarrow C$ with connected fibres from X to a smooth projective curve. Denote by $\omega_{X/C}$ the relative canonical bundle of X over C. Then $f_*\omega_{X/C}$ is a nef vector bundle on C.* □

While we do not attempt to reproduce the calculations here, the rough strategy of Fujita's proof is easily described. Specifically, at least away from the finitely many points $t \in C$ over which $F_t = f^{-1}(t)$ is singular, there is a natural identification of the fibres of the bundle in question:

$$\big(f_*\omega_{X/C}\big)(t) \;=\; H^0\big(F_t, \omega_{F_t}\big) \;=\; H^{n-1,0}(F_t).$$

The space on the right carries a natural Hermitian metric defined by integration, which in fact extends over the singular fibres to define a Hermitian metric on the bundle $f_*\omega_{X/C}$. Fujita then deduces the statement by an explicit curvature calculation.

The interest in such a result is that it can be used to study the geometry of f, which is the simplest example of the sort of fibre space that arises frequently in the approach to birational geometry pioneered by Iitaka and his school. For example, Fujita ([192], Corollary 4.2) uses it to re-prove a statement of Ueno concerning additivity of Kodaira dimension in the case at hand:

[11] Recall (Definition 2.1.11) that for a surjective mapping between smooth projective varieties to be a fibre space means simply that it has connected fibres.

Example 6.3.60. (Fibre spaces over curves). In the setting of Fujita's theorem, assume that C has genus $g \geq 2$ and that a general fibre F of f has positive geometric genus $p_g(F) =_{\text{def}} h^{n-1,0}(F) > 0$. Then

$$\kappa(X) \ = \ \kappa(F) + 1.$$

(The inequality $\kappa(X) \leq \kappa(F) + 1$ holds quite generally, so the issue is to show that $\kappa(X) \geq \kappa(F) + 1$. To this end, observe first that $f_*\omega_{X/C} \neq 0$ thanks to the hypothesis on $p_g(F)$. Moreover, since $g(C) \geq 2$ it follows from Fujita's theorem that

$$f_*\omega_X = f_*\omega_{X/C} \otimes \omega_C$$

is ample. Fixing a very ample divisor H on C, this implies that $S^m\big(f_*\omega_X\big) \otimes \mathcal{O}_C(-H)$ is globally generated for all $m \gg 0$. Using the natural map $S^m\big(f_*\omega_X\big) \longrightarrow f_*\big(\omega_X^{\otimes m}\big)$, one then deduces that

$$H^0\big(X, \omega_X^{\otimes m} \otimes f^*\mathcal{O}_C(-H)\big) \ = \ H^0\big(C, f_*(\omega_X^{\otimes m})(-H)\big) \ \neq \ 0,$$

and hence $H^0\big(X, \mathcal{O}_X(f^*H)\big)$ is realized as a subspace of $H^0\big(X, \mathcal{O}_X(mK_X)\big)$ for all $m \gg 0$. This implies that f factors as a composition of rational maps:

$$X \overset{\rho}{\dashrightarrow} V \dashrightarrow C,$$

where ρ is the Iitaka fibration of X associated to the canonical bundle $\mathcal{O}_X(K_X)$ (Section 2.1.C). In particular, if $G \subset X$ is a general fibre of ρ, then $\kappa(G, K_X|G) = \kappa(G, K_F|G) = 0$, from which it follows that $\kappa(F) \leq \dim F - \dim G = \kappa(X) - 1$. See [192, Propositions 1 and 2] for details.) $\square$

In the years since [192], these ideas have been greatly developed by a number of authors, notably Viehweg [588], [590], [591], [592], [593], Kawamata [315], and Kollár [358], [356], [357], to study the positivity properties of direct images of dualizing sheaves for fibre spaces $f : X \longrightarrow Y$ of projective varieties under various smoothness hypotheses. We refer to [315], [358], and [594] for further references and precise statements of the results that have been obtained, which are necessarily somewhat involved and technical. Besides consequences for the geometry of fibre spaces (e.g. questions involving additivity of Kodaira dimension), this machine has found important applications to proving projectivity or quasi-projectivity of moduli spaces. Kawamata [321] [322] has recently applied positivity theorems for direct images of canonical bundles to study linear series on higher-dimensional varieties: see Remark 10.4.9.

As in the paper of Fujita, the work of Kawamata [315] and Kollár [358] analyzed the direct image bundles via metrics arising from Hodge theory. However, Kollár showed in [356, 357] that one could replace some of these arguments with vanishing theorems. To give the flavor, we conclude this subsection with a "toy" special case of [356, Corollary 3.7]:

Proposition 6.3.61. *Let $f : X \longrightarrow Y$ be a morphism between smooth projective varieties, and assume that f is smooth, i.e. that the derivative of f is everywhere surjective. Then $f_*\omega_{X/Y}$ is nef.*

It goes without saying that the hypothesis that f is smooth is unrealistic in practice, and the assumptions in [356] are much weaker.

Proof of Proposition 6.3.61. We will use the theorem of Kollár stated in 4.3.8 that if $\pi : V \longrightarrow W$ is any surjective projective mapping with V smooth and projective, and if L is any ample line bundle on W, then

$$H^j(W, L \otimes R^i\pi_*\omega_V) \ = \ 0 \text{ for any } \ i \geq 0 \text{ and } j > 0. \qquad (*)$$

Fix $s > 0$, and consider the s-fold fibre product $f^{(s)} : X^{(s)} = X \times_Y \cdots \times_Y X \longrightarrow Y$ of X over Y. The smoothness hypothesis on f guarantees that $X^{(s)}$ is still smooth, and one has

$$f_*^{(s)}\omega_{X^{(s)}/Y} \ = \ \big(f_*\omega_{X/Y}\big)^{\otimes s}.$$

Now suppose that $\dim Y = d$, and let B be a very ample line bundle on Y that is sufficiently positive so that $B \otimes \omega_Y^*$ is ample. Applying Kollár's vanishing theorem (*) to $f^{(s)}$, we deduce that

$$\begin{aligned} H^j\big(Y, (f_*\omega_{X/Y})^{\otimes s} \otimes B^{\otimes(d+1-j)}\big) \ &= \ H^j\big(Y, f_*^{(s)}(\omega_{X^{(s)}}) \otimes \omega_Y^* \otimes B^{\otimes(d+1-j)}\big) \\ &= \ 0 \end{aligned}$$

for $j > 0$. By Castelnuovo–Mumford regularity (Theorem 1.8.5), this implies that the vector bundle

$$\big(f_*\omega_{X/Y}\big)^{\otimes s} \otimes B^{\otimes(d+1)}$$

is globally generated. But since this holds for all $s > 0$, the bundle in question must be nef thanks to 6.2.13. □

6.3.F Some Constructions of Positive Vector Bundles

We conclude by presenting a couple of methods of construction of ample bundles.

Pulling back bundles on $\mathbf{P}^n$. We discuss an "amplification" process for bundles on $\mathbf{P}^n$ suggested by Barton's use of the Frobenius in [34, Proposition 3.1]. Fix for each $k \geq 1$ a branched covering

$$\nu_k : \mathbf{P}^n \longrightarrow \mathbf{P}^n \quad \text{with} \quad \nu_k^*\mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{O}_{\mathbf{P}^n}(k).$$

For example, one might take a Fermat-type covering $\nu_k([a_0, \ldots, a_n]) = [a_0^k, \ldots, a_n^k]$, but the next proposition holds for any choice of ν_k.

Proposition 6.3.62. *Let E be an ample vector bundle on $\mathbf{P}^n$, and let F be an arbitrary bundle on $\mathbf{P}^n$. Then there is a positive integer $k_0 = k_0(E, F)$ such that*

$$\nu_k^* E \otimes F \quad \textit{is ample for all} \;\; k \;\geq\; k_0.$$

Proof. For suitable $a \in \mathbf{Z}$ we can realize F as a quotient of a direct sum of copies of $\mathcal{O}_{\mathbf{P}^n}(a)$. So it suffices to treat the case $F = \mathcal{O}_{\mathbf{P}^n}(a)$. Let $\delta = \delta(\mathbf{P}^n, E, \mathcal{O}_{\mathbf{P}^n}(1))$ be the Barton invariant of E with respect to $\mathcal{O}_{\mathbf{P}^n}(1)$ (Example 6.2.14). In other words,

$$\delta = \sup \big\{ t \in \mathbf{Q} \;\big|\; E<-tH> \quad \text{is nef} \;\big\},$$

where H denotes the hyperplane divisor on $\mathbf{P}^n$. Then by part (iii) of the example just cited,

$$\begin{aligned} \delta(\mathbf{P}^n, \nu_k^* E, \mathcal{O}_{\mathbf{P}^n}(1)) \;&=\; k \cdot \delta(\mathbf{P}^n, \nu_k^* E, \mathcal{O}_{\mathbf{P}^n}(k)) \\ &=\; k \cdot \delta(\mathbf{P}^n, \nu_k^* E, \nu_k^* \mathcal{O}_{\mathbf{P}^n}(1)) \\ &=\; k \cdot \delta(\mathbf{P}^n, E, \mathcal{O}_{\mathbf{P}^n}(1)) \\ &=\; k \cdot \delta. \end{aligned}$$

Therefore

$$\delta(\mathbf{P}^n, \nu_k^* E \otimes \mathcal{O}_{\mathbf{P}^n}(a), \mathcal{O}_{\mathbf{P}^n}(1)) \;=\; k\delta + a. \qquad (*)$$

But $\delta > 0$ since E is ample, so if $k \gg 0$ the right-hand side of (*) is positive, and $\nu_k^* E \otimes \mathcal{O}_{\mathbf{P}^n}(a)$ is ample. □

Example 6.3.63. An analogous statement holds if A is an abelian variety, and $\nu_k : A \longrightarrow A$ is the isogeny determined by multiplication by k. (Compare Example 6.3.59.) □

Generic cokernels. In [223] Gieseker used a reduction to characteristic $p > 0$ to produce some interesting ample bundles on $\mathbf{P}^2$. In this paragraph we construct analogous bundles on an arbitrary projective variety.

Proposition 6.3.64. (Generic cokernels, I). *Let X be an irreducible projective variety of dimension n, let H be a very ample divisor on X, and let V be a vector space of dimension $n + e$ with $e \geq n$. Then for $d \geq n + e$ the cokernel of a general vector bundle map*

$$\mathcal{O}_X(-dH)^{\oplus n} \xrightarrow{\;u\;} V_X$$

is an ample vector bundle of rank e on X.

Proof. The condition $e \geq n$ guarantees that a general map u has constant rank n on X, and hence that $E_u = \operatorname{coker}(u)$ is indeed a bundle of rank e. Since ampleness is an open condition in a family of bundles (Proposition 6.1.9), it is enough to show that E_u is ample for some u. Now u is defined by an $n \times (n+e)$

matrix of sections in $\Gamma(X, \mathcal{O}_X(dH))$, and we are then free to assume that these are pulled back from $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$ under a branched covering $f : X \longrightarrow \mathbf{P}^n$ with $f^*\mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{O}_X(H)$. Then u and E_u are themselves pulled back from $\mathbf{P}^n$, and since amplitude is preserved under finite coverings (Proposition 6.1.8), we are reduced to the case $X = \mathbf{P}^n$.

Set $r = n + e - 1$. Fixing u, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-d)^{\oplus n} \xrightarrow{u} V_{\mathbf{P}^n} \longrightarrow E_u \longrightarrow 0,$$

and the amplitude of E_u is equivalent to the assertion that the natural map

$$\mathbf{P}(E_u) \longrightarrow \mathbf{P}(V) = \mathbf{P}^r$$

is finite. But $\mathbf{P}(E_u) \subset \mathbf{P}(V_{\mathbf{P}^n}) = \mathbf{P}^n \times \mathbf{P}^r$ is the complete intersection of n divisors in the linear series $|\mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}^n}(d) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}^r}(1)|$ determined by the vanishing of the composition

$$\mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}^n}(-d)^{\oplus n} \xrightarrow{\mathrm{pr}_1^* u} V_{\mathbf{P}^n \times \mathbf{P}^r} \longrightarrow \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}^r}(1),$$

the second map being the pullback under the projection pr_2 of the evaluation $V_{\mathbf{P}^r} \longrightarrow \mathcal{O}_{\mathbf{P}^r}(1)$. The required finiteness is then a consequence of the finiteness lemma (Lemma 6.3.43) established above. □

The previous proposition generalizes to quotients of an arbitrary bundle:

Theorem 6.3.65. (Generic cokernels, II). *Let X be an irreducible projective variety of dimension n, let H be a very ample divisor on X, and let F_0 be any vector bundle on X of rank $n + f$ with $f \geq n$. Then for any $d \gg 0$ the cokernel of a sufficiently general vector bundle map*

$$u : \mathcal{O}_X(-dH)^{\oplus n} \longrightarrow F_0$$

is an ample vector bundle of rank f.

Remark 6.3.66. Demailly informs us that he has proven a similar result via a metric argument. □

Example 6.3.67. ([223]). Taking $X = \mathbf{P}^2$, for $d \gg 0$ there is an ample vector bundle E having a presentation of the form

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-d)^2 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-1)^4 \longrightarrow E \longrightarrow 0. \quad \square$$

Proof of Theorem 6.3.65. We start with some reductions analogous to those in the proof of Proposition 6.3.64. Specifically, in the first place we will require that d be large enough so that the bundle $\mathrm{Hom}(\mathcal{O}_X(-dH)^n, F_0)$ is globally generated. This guarantees that the general cokernel is at least a vector bundle of the stated rank. Note also that as before, it is enough to establish the amplitude in question for some u having a locally free cokernel.

We next argue that we can reduce to the case in which F_0 is of the form $V_X \otimes O_X(-mH)$ for some vector space V of rank $\geq 2n + f$. In fact, choose $m_0 \gg 0$ so that $F_0(mH)$ is globally generated for every $m \geq m_0$. Then we can fix for each $m \geq m_0$ a surjective map $v_m : V_X \otimes O_X(-mH) \longrightarrow F_0$. Now for $d \geq m$ consider a general homomorphism

$$\tilde{u}_{d,m} : \mathcal{O}_X(-dH)^n \longrightarrow V_X \otimes \mathcal{O}_X(-mH),$$

and set $u_{d,m} = v_m \circ \tilde{u}_{d,m}$:

$$\begin{array}{ccc} \mathcal{O}_X(-dH)^n & \xrightarrow{\tilde{u}_{d,m}} & V_X \otimes \mathcal{O}_X(-mH) \\ & \searrow^{u_{d,m}} & \downarrow v_m \\ & & F_0. \end{array}$$

Then $\operatorname{coker}(u_{d,m})$ is a quotient of $\operatorname{coker}(\tilde{u}_{d,m})$, so it suffices to establish the amplitude of the latter. Moreover, as in the proof of 6.3.64 we may further restrict attention to the case when $\tilde{u}_{d,m}$ is pulled back from $\mathbf{P}^n$ under a branched covering $f : X \longrightarrow \mathbf{P}^n$. So finally we are reduced to considering $X = \mathbf{P}^n$, and a trivial vector bundle $V_{\mathbf{P}^n}$ of fixed large rank $\geq 2n$. Given an integer m_0, we need to show that for every sufficiently large $d \gg 0$ there is some $m \geq m_0$ such that the cokernel of a general map $\mathcal{O}_{\mathbf{P}^n}(-d)^n \longrightarrow V_{\mathbf{P}^n}(-m)$ is ample.

To this end, first apply Proposition 6.3.64 to choose a natural number d_0, depending only on n and $\dim V$, plus a map $u_0 : \mathcal{O}_{\mathbf{P}^n}(-d_0)^n \longrightarrow V_{\mathbf{P}^n}$ such that $E_0 = \operatorname{coker} u_0$ is ample. By Proposition 6.3.62, there is a large integer k_0 such that whenever we pull back by a covering $\nu_k : \mathbf{P}^n \longrightarrow \mathbf{P}^n$ defined by $\mathcal{O}_{\mathbf{P}^n}(k)$ with $k \geq k_0$, then for $0 \leq i \leq d_0 - 1$ each of the bundles

$$E_{k,i} \ = \ \nu_k^* E_0 \otimes \mathcal{O}_{\mathbf{P}^n}(-m_0 - i)$$

is ample. Note that $E_{k,i}$ sits in an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-kd_0 - m_0 - i)^n \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}^n}(-m_0 - i) \longrightarrow E_{k,i} \longrightarrow 0.$$

But every $d \gg 0$ is of the form $kd_0 + m_0 + i$ for some $k \geq k_0$ and $0 \leq i \leq d_0 - 1$, so we are done. □

6.4 Ample Vector Bundles on Curves

The object of this section is to study ample vector bundles on smooth curves, and in particular to give Hartshorne's characterization (Theorem 6.4.15) of such bundles. It was observed already in [277] that amplitude on curves is closely related to the concept of stability, and we emphasize that connection here. In order to control the stability of tensor products, Hartshorne originally drew on results of Narasimhan and Seshadri giving an essentially analytic characterization of stability. Gieseker [224] and Miyaoka [429] later realized that one can in effect reverse the process, and we follow their approach. In particular, along the way to Hartshorne's theorem we will use the theory of ample bundles to recover in an elementary fashion the tensorial properties of stability (Corollary 6.4.14).

Throughout this section, C denotes a smooth irreducible complex projective curve.

6.4.A Review of Semistability

For the convenience of the reader we recall in this subsection the basic facts and definitions surrounding semistability of bundles on curves.

It is classical that much of the geometry associated to a line bundle on a curve is governed by the degree of the bundle. However, it was recognized early on that the degree is a less satisfactory invariant for vector bundles of higher rank. The pathologies stem from the fact that bundles of a given degree can become arbitrarily "unbalanced": for instance, if L_n is a line bundle of degree n on C, then $E_n = L_n \oplus L_n^*$ has degree zero, but most of the properties of E_n depend on n. The condition of semistability in effect rules out this sort of problem, and the Harder–Narasimhan filtration expresses an arbitrary bundle as a successive extension with semistable quotients.

Definition 6.4.1. (Slope and semistability). Let E be a vector bundle on the smooth projective curve C. The *slope* of E is the rational number

$$\mu(E) \;=\; \frac{\deg(E)}{\operatorname{rank}(E)},$$

where as usual the degree of E is the integer $\deg(E) = \int c_1(E)$. One says that E is *semistable* if

$$\mu(F) \;\leq\; \mu(E) \text{ for every sub-bundle } F \subseteq E. \tag{*}$$

E is *unstable* if it is not semistable. □

Thus the slope of E measures "degree per unit rank," and the condition of semistability means that E cannot have any inordinately positive sub-bundles. There is a related notion of *stability*, for which one requires strict inequality in (*), but we will not require this.

We collect some elementary but useful observations that the reader may check:

Example 6.4.2. (i). If $E = E_1 \oplus E_2$ is a direct sum of two bundles, then E is semistable if and only if E_1 and E_2 are semistable with $\mu(E_1) = \mu(E_2)$.

(ii). E is semistable if and only if $\mu(Q) \geq \mu(E)$ for all quotients Q of E.

(iii). In the definition of semistability, it is equivalent to work with sub-*sheaves* $F \subseteq E$ (so F is still locally free, but one can allow E/F to have torsion).

(iv). If E and F are any two vector bundles on C, then $\mu(E \otimes F) = \mu(E) + \mu(F)$.

(v). Given any divisor D on C, E is semistable iff $E \otimes \mathcal{O}_C(D)$ is. □

Facts (iv) and (v) of the previous example give two equivalent ways to extend these concepts to $\mathbf{Q}$-twisted bundles. Suppose that E is a vector bundle on C, and that δ is a $\mathbf{Q}$-divisor class on C. Recall that in Definition 6.2.7 we defined the degree of the $\mathbf{Q}$-twisted bundle $E{<}\delta{>}$ on C to be the rational number

$$\deg\big(E{<}\delta{>}\big) \ = \ \deg E \ + \ \mathrm{rank}(E) \cdot \deg \delta.$$

Recalling also that a sub-bundle of $E{<}\delta{>}$ is the $\mathbf{Q}$-twisted bundle $F{<}\delta{>}$ where $F \subseteq E$ is a sub-bundle of E, we are led to make

Definition 6.4.3. (i). The *slope* of $E{<}\delta{>}$ is defined to be

$$\mu\big(E{<}\delta{>}\big) \ = \ \frac{\deg\big(E{<}\delta{>}\big)}{\mathrm{rank}\big(E{<}\delta{>}\big)} \ = \ \mu(E) \ + \ \deg\,\delta.$$

(ii). $E{<}\delta{>}$ is *semistable* if $\mu\big(F{<}\delta{>}\big) \leq \mu\big(E{<}\delta{>}\big)$ for every $\mathbf{Q}$-twisted sub-bundle $F{<}\delta{>}$ of $E{<}\delta{>}$. In the contrary case, $E{<}\delta{>}$ is *unstable.* □

This definition evidently agrees with 6.4.1 in case δ is the class of an integral divisor, i.e. it respects $\mathbf{Q}$-isomorphism. As one expects in light of 6.4.2 (v):

Lemma 6.4.4. *The $\mathbf{Q}$-twisted bundle $E{<}\delta{>}$ is semistable if and only if E itself is.* □

Remark 6.4.5. All the statements of Example 6.4.2 remain valid for $\mathbf{Q}$-twisted bundles, with the proviso that in (i) one deals with two bundles twisted by the same class, so that their sum is defined. □

A basic fact for our purposes is that an unstable bundle has a canonical filtration with semistable graded pieces (Proposition 6.4.7). We start by establishing the following

Lemma 6.4.6. *Let E be a vector bundle on C. Then the set of slopes*

$$\big\{\mu(F) \mid F \subseteq E\big\}$$

of sub-bundles of E is bounded from above. Moreover, if E is unstable, there is a unique maximal sub-bundle $U \subseteq E$ of largest slope.

The sub-bundle $U \subseteq E$ is called the *maximal destabilizing sub-bundle* of E.

Proof of Lemma 6.4.6. The first statement is clear for sub-bundles of a trivial bundle $\mathcal{O}_C^{\oplus N}$, and hence also for sub-bundles of $\mathcal{O}_C(H)^{\oplus N}$ for any divisor H thanks to Example 6.4.2 (iv). But we can realize any bundle E as a sub-bundle of $\mathcal{O}_C(H)^{\oplus N}$ for some sufficiently positive H, and the first statement follows. For the second, let $E_1, E_2 \subset E$ be sub-bundles having maximal slope μ, and let $E_1 + E_2 \subseteq E$ be the sub-sheaf they generate. It is sufficient to prove that $\mu(E_1 + E_2) = \mu$. But this follows from the exact sequence

$$0 \longrightarrow E_1 \cap E_2 \longrightarrow E_1 \oplus E_2 \longrightarrow E_1 + E_2 \longrightarrow 0.$$

Indeed, by maximality $\mu(E_1 \cap E_2) \leq \mu$, and then it follows with a computation that $\mu(E_1 + E_2) \geq \mu(E_1 \oplus E_2) = \mu$. □

One then obtains:

Proposition 6.4.7. (Harder–Narasimhan filtration). *Any vector bundle E on C has a canonically defined filtration*

$$\mathrm{HN}_\bullet(E) \;:\; 0 = \mathrm{HN}_\ell(E) \subset \mathrm{HN}_{\ell-1}(E) \subset \ldots \subset \mathrm{HN}_1(E) \subset \mathrm{HN}_0(E) = E$$

by sub-bundles, characterized by the properties that if

$$\mathrm{Gr}_i \;=\; \mathrm{HN}_i(E) \;/\; \mathrm{HN}_{i+1}(E)$$

is the i^{th} associated graded bundle, then each of the bundles Gr_i is semistable, and

$$\mu(\mathrm{Gr}_{\ell-1}) \;>\; \ldots \;>\; \mu(\mathrm{Gr}_1) \;>\; \mu(\mathrm{Gr}_0).$$

Proof. In fact, if E is semistable, take $\ell = 1$. Otherwise, let $\mathrm{HN}_1(E) \subset E$ be the maximal destabilizing sub-bundle of E, and continue inductively. □

Remark 6.4.8. (Extension to Q-twists). Again the previous results extend in a natural way to **Q**-twisted bundles. In the situation of Definition 6.4.3, we define the maximal destabilizing sub-bundle of a **Q**-twisted bundle $E{<}\delta{>}$ to be the **Q**-twisted sub-bundle $U{<}\delta{>}$ of $E{<}\delta{>}$, where $U \subset E$ is the maximal destabilizing subsheaf of E. In a similar fashion, $E{<}\delta{>}$ has the Harder–Narasimhan filtration $\mathrm{HN}_\bullet\big(E{<}\delta{>}\big) = \mathrm{HN}_\bullet(E){<}\delta{>}$. □

Remark 6.4.9. In the present discussion, stability appears as a technical tool for studying positivity. However, it originally arose in connection with the construction of moduli spaces of bundles (see [477] for a very readable account and [47] for a survey of later developments). More recently, the concept of stability has proven to be very fundamental from many points of view; see for example [474], [139], [579], [180], [181]. □

6.4.B Semistability and Amplitude

The result for which we are aiming states that the positivity of a bundle E on a curve is characterized in terms of the degrees of E and its quotients. One direction is elementary:

Lemma 6.4.10. (i). *Let E be a vector bundle of rank e on C. If E is nef then* $\deg E \geq 0$ *and if E is ample then* $\deg E > 0$.

(ii). *The same statement holds more generally if E is replaced by a $\mathbf{Q}$-twisted bundle $E{<}\delta{>}$.*

Proof. We treat (ii). Consider the projective bundle $\pi : \mathbf{P}(E) \longrightarrow C$, and let

$$\xi \;=\; \xi_E \,+\, \pi^*\delta \;\in\; N^1\big(\mathbf{P}(E)\big)_{\mathbf{Q}},$$

where as usual ξ_E represents (the first Chern class of) $\mathcal{O}_{\mathbf{P}(E)}(1)$. Then

$$\deg\ E{<}\delta{>} \;=\; \int_{\mathbf{P}(E)} \xi^e.$$

But if $E{<}\delta{>}$ is ample then by definition ξ is an ample class on $\mathbf{P}(E)$, and consequently the degree in question is strictly positive. Similarly, Kleiman's theorem (Theorem 1.4.9) shows that it is non-negative if $E{<}\delta{>}$ is nef. □

We now wish to consider how to pass from numerical properties to statements about amplitude and stability. Let E be a vector bundle of rank e on the smooth curve C, and let Δ be a divisor representing $\det(E)$. It will be convenient to work with the $\mathbf{Q}$-twisted bundle

$$E_{\text{norm}} \;=\; E{<}-\tfrac{1}{e}\Delta{>}\ .$$

The point of this normalization is that it reduces one to the case of bundles of degree zero: it follows from Definition 6.2.7 that $\deg E_{\text{norm}} = 0$. Since by 6.4.4 E_{norm} is semistable if and only if E is, we see using Example 6.4.2 (ii) that E is semistable if and only if every quotient of E_{norm} has degree ≥ 0 in the sense that

$$\deg\big(Q{<}-\tfrac{1}{e}\Delta{>}\big) \;\geq\; 0$$

for any quotient bundle Q of E. Analogous remarks hold starting from a $\mathbf{Q}$-twisted bundle $E{<}\delta{>}$: indeed, if one mirrors the definition above one finds that in fact

$$\big(E{<}\delta{>}\big)_{\text{norm}} = E_{\text{norm}}.$$

The basic link between stability and positivity is then given by

Proposition 6.4.11. (Semistability and nefness). *E is semistable if and only if E_{norm} is nef.*

We start with a useful lemma:

Lemma 6.4.12. *Let E be a vector bundle on C, and let $f : C' \longrightarrow C$ be any branched covering of C by a smooth irreducible projective curve C'. Then E is semistable if and only if f^*E is semistable.*

Proof. If E is unstable, then certainly f^*E is as well since the pullback of a destabilizing sub-bundle $U \subseteq E$ will destabilize f^*E. Conversely, suppose for a contradiction that E is semistable but that f^*E is unstable. By what we have just observed the pullback of f^*E under a covering $C'' \longrightarrow C'$ will remain unstable, so we may assume without loss of generality that f is Galois, with group $G = \mathrm{Gal}(C'/C)$. Let $V \subseteq f^*E$ be the maximal destabilizing sub-bundle of f^*E. Now G acts in the natural way on f^*E and hence also on the collection of sub-bundles of f^*E. It follows from the uniqueness of the maximal destabilizing sub-bundle that V is G-stable. Hence $V = f^*U$ for a sub-bundle $U \subset E$, and one checks right away that U must destabilize E. □

Proof of Proposition 6.4.11. Suppose first that E_{norm} is nef. Then thanks to Theorem 6.2.12 (i), all of its quotients are nef, and hence have non-negative degree. Therefore by the remarks preceding the statement of Proposition 6.4.11, E is semistable. Conversely, suppose that E is semistable, but that E_{norm} is not nef. It follows from the Barton–Kleiman criterion (Proposition 6.1.18 and Remark 6.2.9) that there is a finite map $f : C' \longrightarrow C$ from a smooth irreducible curve C' to C such that f^*E_{norm} has a rank one quotient of negative degree. Then f^*E_{norm} is unstable. On the other hand, since C is smooth f must be a branched covering. Thus we arrive at a contradiction to 6.4.12, which completes the proof. □

Remark 6.4.13. It follows from 6.4.4 that Proposition 6.4.11 and Lemma 6.4.12 both remain valid if the "classical" bundle E is replaced by a $\mathbf{Q}$-twist $E{<}\delta{>}$. □

We pause to note that as an application one obtains the following fundamental result, which is traditionally established via the theorem of Narasimhan and Seshadri [474] characterizing stable bundles in terms of representations of the fundamental group of C. The present much more elementary approach is due to Gieseker [224], rendered particularly transparent via $\mathbf{Q}$-twists in [429].

Corollary 6.4.14. (Semistability of tensor products). *The tensor product of two semistable bundles on a smooth curve is semistable. Consequently, if E is semistable, then so is S^mE for every $m \geq 0$. The same statements hold for $\mathbf{Q}$-twisted bundles.*

Proof. Suppose that E and F are semistable. Then E_{norm} and F_{norm} are nef, and hence

$$\big(E \otimes F\big)_{\mathrm{norm}} = E_{\mathrm{norm}} \otimes F_{\mathrm{norm}}$$

is also nef thanks to Theorem 6.2.12 (iv) and (v). The semistability of $E \otimes F$ then follows from the previous proposition. By induction, if E is semistable then so is any tensor power $T^q(E)$. In characteristic zero, $S^m E$ is a summand of $T^m E$, so its semistability follows from 6.4.2 (i). The extension to **Q**-twists is evident. □

Finally, we turn to a theorem of Hartshorne [277] giving a very pleasant characterization of nef and ample vector bundles on a curve.

Theorem 6.4.15. (Hartshorne's theorem). *A vector bundle E on C is nef if and only if E and every quotient bundle of E has non-negative degree, and E is ample if and only if E and every quotient has strictly positive degree. The same statements hold if E is replaced by a* **Q**-*twisted bundle* $E{<}\delta{>}$.

Proof. One direction is immediate: if E is ample (or nef), then every quotient is ample (or nef) and hence has positive (or non-negative) degree thanks to Lemma 6.4.10. Conversely, the essential point will be to treat the case in which E is semistable:

> *Main Claim:* Let E be a semistable bundle on C. If E has non-negative degree then E is nef, and if E has positive degree then E is ample.

Granting this for the moment, we complete the proof.

Suppose then that E is not nef. We need to show that E itself or a quotient has negative degree. If $\deg E < 0$ there is nothing further to prove, so we may suppose that $\deg E \geq 0$. It then follows from the Claim that E must be unstable. Consider its Harder–Narasimhan filtration

$$\mathrm{HN}_\bullet(E) : \quad 0 \;=\; \mathrm{HN}_\ell(E) \;\subset\; \mathrm{HN}_{\ell-1}(E) \;\subset\; \ldots \;\subset\; \mathrm{HN}_1(E) \;\subset\; \mathrm{HN}_0(E) \;=\; E$$

and as before set $\mathrm{Gr}_i = \mathrm{HN}_i(E)/\mathrm{HN}_{i+1}(E)$. Since an extension of nef bundles is nef, it follows again from the Claim that at least one of these graded pieces — say Gr_k — must have negative degree. Therefore

$$\mu(\mathrm{Gr}_0) \;<\; \mu(\mathrm{Gr}_1) \;<\; \ldots \;<\; \mu(\mathrm{Gr}_k) \;<\; 0.$$

But then $\deg\big(E \,/\, \mathrm{HN}_{k+1}(E)\big) < 0$, and we have produced the desired quotient. A similar argument shows that if E is not ample, then E or some quotient has degree ≤ 0.

Turning to the claim, the point is to apply Proposition 6.4.11. In fact, assume that E is semistable. Then E_{norm} is nef. But

$$E \;=\; E_{\mathrm{norm}}{<}\tfrac{1}{e}\Delta{>} \;=\; E_{\mathrm{norm}} \otimes \mathcal{O}_C{<}\tfrac{1}{e}\Delta{>},$$

where as above Δ is a divisor class representing $\det(E)$. If $\deg(E) \geq 0$ then $\deg(\Delta) \geq 0$, whence $\mathcal{O}_C{<}\frac{1}{e}\Delta{>}$ is nef, and if $\deg(E) > 0$ then $\mathcal{O}_C{<}\frac{1}{e}\Delta{>}$ is ample. Therefore thanks to Theorem 6.2.12, E itself is nef in the first case and ample in the second, as asserted.

Finally, the extension to **Q**-twists presents no difficulties, and is left to the reader. □

Example 6.4.16. (Higher cohomology and amplitude). Suppose that C has genus $g \geq 2$. If E is a bundle on C such that $H^1\big(C, E\big) = 0$, then E is ample. (If Q is a quotient of E having degree ≤ 0, then it follows from Riemann–Roch that $H^1\big(C, Q\big) \neq 0$, and hence that $H^1\big(C, E\big) \neq 0$.) This result is due to Fujita [192, Lemma 3]. □

Example 6.4.17. (Generically surjective morphisms). Let E and F be vector bundles on C, and suppose that $u : E \longrightarrow F$ is a homomorphism that is surjective away from finitely many points of C. If E is ample then so too is F, and the analogous statements hold also for $\mathbf{Q}$-twists. (Any quotient Q of F gives rise to a quotient Q' of E with $\deg Q' \leq \deg Q$.) □

Example 6.4.18. (Singular curves). Fulton [201, Proposition 4] gave an example to show that the conclusion of Theorem 6.4.15 can fail on a singular curve C. In fact, fixing a non-singular point $P \in C$, one can construct a non-split extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C(P) \longrightarrow 0$$

which splits when pulled back to the normalization $\nu : C' \longrightarrow C$ of C. Then every quotient of E has positive degree, but $\nu^* E$ is not ample. Serre had used a similar construction to show that Hartshorne's theorem fails in positive characteristics. □

Remark 6.4.19. (Higher-dimensional varieties). Examples suggest that there cannot be a clean numerical criterion for the amplitude of bundles on higher-dimensional varieties analogous to Hartshorne's theorem: see Remark 8.3.14. □

Example 6.4.20. (Hartshorne's proof of Theorem 6.4.15). The original proof of Theorem 6.4.15 in [277] is quite interesting, and we indicate the idea. As in the argument above, the essential point is to show that if E is a semistable bundle of non-negative degree on C, then E is nef. Supposing this is false, there is a reduced irreducible curve $\varGamma \subset \mathbf{P}(E)$ such that

$$\int_{\varGamma} c_1(\mathcal{O}_{\mathbf{P}(E)}(1)) \ < \ 0. \tag{*}$$

Evidently $\varGamma$ cannot lie in a fibre of the projection $\pi : \mathbf{P}(E) \longrightarrow C$, and so $\varGamma$ is flat over C, say of degree d. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{\varGamma/\mathbf{P}(E)}(m) \longrightarrow \mathcal{O}_{\mathbf{P}(E)}(m) \longrightarrow \mathcal{O}_{\varGamma}(m) \longrightarrow 0.$$

For $m \gg 0$ — so that $R^1\pi_*\mathcal{I}_{\varGamma/\mathbf{P}(E)}(m) = 0$ — this gives rise to a surjection

$$S^m E \longrightarrow \pi_*\mathcal{O}_{\varGamma}(m) \longrightarrow 0.$$

But $\deg \mathcal{O}_{\varGamma}(m) \leq -m$ by virtue of (*), and it follows from Riemann–Roch on the normalization $\varGamma'$ of $\varGamma$ that $\chi(\varGamma, \mathcal{O}_{\varGamma}(m))$ becomes increasingly negative as

m grows. On the other hand, the bundle $G_m = \pi_* \mathcal{O}_\Gamma(m)$ has fixed rank d, and satisfies $h^i(C, G_m) = h^i(\Gamma, \mathcal{O}_\Gamma(m))$. Applying the Riemann–Roch formula

$$\chi(C, G_m) \;=\; \deg G_m + d \cdot (1 - \mathrm{genus}(C))$$

on C, we see that $\deg G_m \ll 0$ for $m \gg 0$. In other words, we have established that if $m \gg 0$, then $S^m E$ has a quotient of negative degree. But this is impossible: for E is semistable of non-negative slope, and hence so is $S^m E$. □

Notes

Section 6.1 largely follows [274], although the use of a Veronese mapping to establish the tensorial properties of ample bundles in characteristic zero (Theorem 6.1.15) — which simplifies earlier approaches — is new. The presentation of this argument follows some suggestions of Fulton. Gieseker's lemma (Proposition 6.1.7) appears in [223].

A formalism of twisting bundles by **Q**-divisors was initiated by Miyaoka in [429], and its utility was re-emphasized by an analogous construction in [133]. Nef vector bundles have come into focus in recent years with the flowering of higher-dimensional geometry, where nef line bundles play a critical role. The Barton invariant (Example 6.2.14) appears implicitly in [34], where Barton uses the Frobenius to establish the amplitude of tensor products in positive characteristics.

Section 6.3 draws on many sources, most of which are cited in the text and won't be repeated here. The proof of Theorem 6.3.19 is adapted from [473], while Theorem 6.3.29 is new. The discussion of Picard bundles closely follows [212], although it seems to have been overlooked that the arguments for curves work without change on irregular varieties of all dimensions. Section 6.3.D follows [387]. The proof of Proposition 6.3.61 was explained to me by Ein. The material in Section 6.3.F is new, although as noted Demailly has obtained some similar results by different methods.

As we have indicated, the approach of Section 6.4 originates with Gieseker [224] and Miyaoka [429]. We have drawn on the very nice exposition in [432, Lecture III, §2].