Preface

(A) These notes provide an introduction to some topics in orbit equivalence theory, a branch of ergodic theory. One of the main concerns of ergodic theory is the structure and classification of measure preserving (or more generally measure-class preserving) actions of groups. By contrast, in orbit equivalence theory one focuses on the equivalence relation induced by such an action, i.e., the equivalence relation whose classes are the orbits of the action. This point of view originated in the pioneering work of Dye in the late 1950's, in connection with the theory of operator algebras. Since that time orbit equivalence theory has been a very active area of research in which a number of remarkable results have been obtained.

Roughly speaking, two main and opposing phenomena have been discovered, which we will refer to as *elasticity* (not a standard terminology) and *rigidity*. To explain them, we will need to introduce first the basic concepts of orbit equivalence theory.

In these notes we will only consider countable, discrete groups Γ. If such a group Γ acts in a Borel way on a standard Borel space X, we denote by E^X_Γ the corresponding equivalence relation on X:

$$xE^X_\Gamma y \Leftrightarrow \exists \gamma \in \Gamma(\gamma \cdot x = y).$$

If μ is a probability (Borel) measure on X, the action *preserves* μ if $\mu(\gamma \cdot A) = \mu(A)$, for any Borel set $A \subseteq X$ and $\gamma \in \Gamma$. The action (or the measure) is *ergodic* if every Γ-invariant Borel set is null or conull.

Suppose now Γ acts in a Borel way on X with invariant probability measure μ and Δ acts in a Borel way on Y with invariant probability measure ν. Then these actions are *orbit equivalent* if there are conull invariant Borel sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism $\pi : A \to B$ which sends μ to ν (i.e., $\pi_* \mu = \nu$) and for $x, y \in A$:

$$xE^X_\Gamma y \Leftrightarrow \pi(x) E^Y_\Delta \pi(y).$$

We can now describe these two competing phenomena:

(I) Elasticity: For amenable groups there is exactly one orbit equivalence type of non-atomic probability measure preserving ergodic actions. More precisely, if Γ, Δ are amenable groups acting in a Borel way on X, Y with non-atomic, invariant, ergodic probability measures μ, ν, respectively, then these two actions are orbit equivalent. This follows from a combination of Dye's work with subsequent work of Ornstein-Weiss in the 1980's. Thus the equivalence relation induced by such an action of an amenable group does not "encode" or "remember" anything about the group (beyond the fact that it is amenable). For example, any two free, measure preserving ergodic actions of the free abelian groups $\mathbb{Z}^m, \mathbb{Z}^n$ $(m \neq n)$ are orbit equivalent.

(II) Rigidity: As originally discovered by Zimmer in the 1980's, for many non-amenable groups Γ we have the opposite situation: The equivalence relation induced by a probability measure preserving action of Γ "encodes" or "remembers" a lot about the group (and the inducing action). For example, a recent result of Furman, strengthening an earlier theorem of Zimmer, asserts that if the canonical action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ $(n \geq 3)$ is orbit equivalent to a free, non-atomic probability measure preserving, ergodic action of a countable group Γ, then Γ is isomorphic to $\mathrm{SL}_n(\mathbb{Z})$ and under this isomorphism the actions are also Borel isomorphic (modulo null sets). Another recent result, due to Gaboriau, states that if the free groups F_m, F_n $(1 \leq m, n \leq \aleph_0)$ have orbit equivalent free probability measure preserving Borel actions, then $m = n$. (This should be contrasted with the result mentioned in (I) above about $\mathbb{Z}^m, \mathbb{Z}^n$.)

(B) These notes are divided into three chapters. The first, very short, chapter contains a quick introduction to some basic concepts of ergodic theory.

The second chapter is primarily an exposition of the "elasticity" phenomenon described above. Some topics included here are: amenability of groups, the concept of hyperfiniteness for equivalence relations, Dye's Theorem to the effect that hyperfinite equivalence relations with non-atomic, invariant ergodic probability measures are Borel isomorphic (modulo null sets), quasi-invariant measures and amenable equivalence relations, and the Connes-Feldman-Weiss Theorem that amenable equivalence relations are hyperfinite a.e. We also include topics concerning amenability and hyperfiniteness in the Borel and generic (Baire category) contexts, like the result that finitely generated groups of polynomial growth always give rise to hyperfinite equivalence relations (without neglecting null sets), that generically (i.e., on a comeager set) every countable Borel equivalence relation is hyperfinite, and finally that, also generically, a countable Borel equivalence relation admits no invariant Borel probability measure, and therefore all generically aperiodic, non-smooth countable Borel equivalence relations are Borel isomorphic modulo meager sets.

The third chapter contains an exposition of the theory of costs for equivalence relations and groups, originated by Levitt, and mainly developed by

Gaboriau, who used this theory to prove the rigidity results about free groups mentioned in (II) above.

(C) In order to make it easier for readers who are familiar with the material in Chapter II but would like to study the theory of costs, we have made Chapter III largely independent of Chapter II. This explains why several basic definitions and facts, introduced in Chapter II (or even in Chapter I), are again repeated in Chapter III. We apologize for this redundancy to the reader who starts from the beginning.

Chapter II grew out of a set of rough notes prepared by the first author in connection with teaching a course on orbit equivalence at Caltech in the Fall of 2001. It underwent substantial modification and improvement under the input of the second author, who prepared the current final version. Chapter III is based on a set of lecture notes written-up by the first author for a series of lectures at the joint Caltech-UCLA Logic Seminar during the Fall and Winter terms of the academic year 2000-2001. Various revised forms of these notes have been available on the web since that time.

(D) As the title of these notes indicates, this is by no means a comprehensive treatment of orbit equivalence theory. Our choice of topics was primarily dictated by the desire to keep these notes as elementary and self-contained as possible. In fact, the prerequisites for reading these notes are rather minimal: a basic understanding of measure theory, functional analysis, and classical descriptive set theory. Also helpful, but not necessary, would be some familiarity with the theory of countable Borel equivalence relations (see, e.g., Feldman-Moore [FM], Dougherty-Jackson-Kechris [DJK], and Jackson-Kechris-Louveau [JKL]).

Since this is basically a set of informal lecture notes, we have not attempted to present a detailed picture of the historical development of the subject nor a comprehensive list of references to the literature.

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