Preface

This book aims to be a course in Lie groups that can be covered in one year with a group of good graduate students. I have attempted to address a problem that anyone teaching this subject must have, which is that the amount of *essential* material is too much to cover.

One approach to this problem is to emphasize the beautiful representation theory of compact groups, and indeed this book can be used for a course of this type if after Chapter 25 one skips ahead to Part III. But I did not want to omit important topics such as the Bruhat decomposition and the theory of symmetric spaces. For these subjects, compact groups are not sufficient.

Part I covers standard general properties of representations of compact groups (including Lie groups and other compact groups, such as finite or p-adic ones). These include Schur orthogonality, properties of matrix coefficients and the Peter-Weyl Theorem.

Part II covers the fundamentals of Lie groups, by which I mean those subjects that I think are most urgent for the student to learn. These include the following topics for compact groups: the fundamental group, the conjugacy of maximal tori (two proofs), and the Weyl character formula. For noncompact groups, we start with complex analytic groups that are obtained by complexification of compact Lie groups, obtaining the Iwasawa and Bruhat decompositions. These are the reductive complex groups. They are of course a special case, but a good place to start in the noncompact world. More general noncompact Lie groups with a Cartan decomposition are studied in the last few chapters of Part II. Chapter 31, on symmetric spaces, alternates examples with theory, discussing the embedding of a noncompact symmetric space in its compact dual, the boundary components and Bergman-Shilov boundary of a symmetric tube domain, and Cartan's classification. Chapter 32 constructs the relative root system, explains Satake diagrams and gives examples illustrating the various phenomena that can occur, and reproves the Iwasawa decomposition, formerly obtained for complex analytic groups, in this more general context. Finally, Chapter 33 surveys the different ways Lie groups can be embedded in one another.

Part III returns to representation theory. The major unifying theme of Part III is Frobenius-Schur duality. This is the correspondence, originating in Schur's 1901 dissertation and emphasized by Weyl, between the irreducible representations of the symmetric group and the general linear groups. The correspondence comes from decomposing tensor spaces over both groups simultaneously. It gives a dictionary by which problems can be transferred from one group to the other. For example, Diaconis and Shahshahani studied the distribution of traces of random unitary matrices by transferring the problem of their distribution to the symmetric group. The plan of Part III is to first use the correspondence to simultaneously construct the irreducible representations of both groups and then give a series of applications to illustrate the power of this technique. These applications include random matrix theory, minors of Toeplitz matrices, branching formulae for the symmetric and unitary groups, the Cauchy identity, and decompositions of some symmetric and exterior algebras. Other thematically related topics topics discussed in Part III are the cohomology of Grassmannians, and the representation theory of the finite general linear groups.

This plan of giving thematic unity to the "topics" portion of the book with Frobenius-Schur the unifying theme has the effect of somewhat overemphasizing the unitary groups at the expense of other Lie groups, but for this book the advantages outweigh this disadvantage, in my opinion. The importance of Frobenius-Schur duality cannot be overstated.

In Chapters 48 and 49, we turn to the analogies between the representation theories of symmetric groups and the finite general linear groups, and between the representation theory of the finite general linear groups and the theory of automorphic forms. The representation theory of $\mathrm{GL}(n, \mathbb{F}_q)$ is developed to the extent that we can construct the cuspidal characters and explain Harish-Chandra's "Philosophy of Cusp Forms" as an analogy between this theory and the theory of automorphic forms. It is a habit of workers in automorphic forms (which many of us learned from Piatetski-Shapiro) to use analogies with the finite field case systematically.

The three parts have been written to be somewhat independent. One may thus start with Part II or Part III and it will be quite a while before earlier material is needed. In particular, either Part II or Part III could be used as the basis of a shorter course. Regarding the independence of Part III, the Weyl character formula for the unitary groups is obtained independently of the derivation in Part II. Eventually, we need the Bruhat decomposition but not before Chapter 47. At this point, the reader may want to go back to Part II to fill this gap.

Prerequisites include the Inverse Function Theorem, the standard theorem on the existence of solutions to first order systems of differential equations and a belief in the existence of Haar measures, whose properties are reviewed in Chapter 1. Chapters 17 and 50 assume some algebraic topology, but these chapters can be skipped. Occasionally algebraic varieties and algebraic groups are mentioned, but algebraic geometry is not a prerequisite. For affine algebraic varieties, only the definition is really needed.

The notation is mostly standard. In $\mathrm{GL}(n)$, I or I_n denotes the $n \times n$ identity matrix and if g is any matrix, ${}^t g$ denotes its transpose. Omitted entries in a matrix are zero. The identity element of a group is usually denoted 1 but also as I, if the group is $\mathrm{GL}(n)$ (or a subgroup), and occasionally as e when it seemed the other notations could be confusing. The notations $\subset$ and $\subseteq$ are synonymous, but we mostly use $X \subset Y$ if X and Y are known to be unequal, although we make no guarantee that we are completely consistent in this. If X is a finite set, $|X|$ denotes its cardinality.

One point where we differ with some of the literature is that the root system lives in $\mathbb{R} \otimes X^*(T)$ rather than in the dual space of the Lie algebra of the maximal torus T as in much of the literature. This is of course the right convention if one takes the point of view of algebraic groups, and it is also arguably the right point of view in general since the real significance of the roots has to do with the fact that they are characters of the torus, not that they can be interpreted as linear functionals on its Lie algebra.

To keep the book to a reasonable length, many standard topics have been omitted, and the reader may want to study some other books at the same time. Cited works are usually recommended ones.

Acknowledgments. The proofs in Chapter 36 on the Jacobi-Trudi identity were worked out years ago with Karl Rumelhart when he was still an undergraduate at Stanford. Very obviously, Chapters 40 and 41 owe a great deal to Persi Diaconis, and Chapter 43 on Cauchy's identity was suggested by a conversation with Steve Rallis. I would like to thank my students in Math 263 for staying with me while I lectured on much of this material.

This book was written using TeXmacs, with further editing of the exported LaTeX file. The utilities patch and diff were used to maintain the differences between the automatically generated and the hand-edited TeX files. The figures were made with MetaPost. The weight diagrams in Chapter 24 were created using programs I wrote many years ago in Mathematica based on the Freudenthal multiplicity formula.

This work was supported in part by NSF grant DMS-9970841.

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