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Introductory Operations Research

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INTRODUCTORY OPERATIONS RESEARCH

Theory and Applications

With 62 Figures

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Preface

This book is an outcome of the lectures delivered by the authors for engineering and management students at Birla Institute of Technology and Science, Pilani, India. However, the text started when author shifted to Thapar Institute of Engineering and Technology, Patiala and coauthor shifted to Indian Institute of Technology, Kanpur. During the teaching of this course the authors realized a need of a good text on "Operations Research" and its applications which may give comprehensive idea of various concepts and function as a companion for problem solving techniques.

The primary purpose of this text is to bring this new mathematical formalism into the education system, not merely for its own sake, but as a basic framework for characterizing the full scope of the concept of modern approach. The authors have tried all contents of this text utilizing four hours a week in one semester as a core course. The level of this text assumes that the reader is well acquainted with elementary calculus and linear algebra. Being a text book, we have taken enough care so that reader may attempt different type of problems.

Any one who aspires to some managerial assignment or who is part of decision making body will find an understanding of optimization techniques very useful. The book is applied in orientation with concentration on engineering and management problems. Each concept has been discussed with sufficient mathematical back ground supported by constructing examples. A set of problems has been added in the end of every chapter.

Because of the imposed restriction of writing a relatively brief text on an extensive subject area, a number of choices had to be made relating to the inclusion and exclusion of certain topics. No obvious way of resolving this problem exists. Even though a wide selection of topics has been achieved. The basic thinking is centralized about the theme how the reader may continue to read advanced level texts by self-study to develop research oriented thoughts. Hence, the basic concepts and fundamental techniques have been emphasized while highly specialized topics and methods should be relegated to secondary one.

During the course of writing we have received remarkable encouragement from our colleagues Prof. S. R. Yadava and Dr. S. P. Yadava at BITS, Pilani, India.

H. S. Kasana and K. D. Kumar

Contents

Chapter 1

Formulation

We start with the introduction of linear programming and illustrate the preliminary concepts which form the basic foundation of optimization. The concentration will remain focus on formulation of linear programming problems. In the end some nonlinear programming problems have also been formulated.

1.1 The Scope of Optimization

Optimization means the mathematical process through which best possible results are obtained under the given set of conditions. Initially, the optimization methods were restricted to the use of calculus based techniques. Cauchy made the first attempt by applying steepest descent method for minimizing a function over the domain of definition. A contribution but very little was made by Newton, Leibniz and Lagrange in this direction. Also, as early as 1939, L. V. Kantorovich pointed out the practical significance of a restricted class of linear programming models for production planning, and proposed an algorithm for their solution. Unfortunately, Kantorovich's work remain neglected in the USSR, and unknown elsewhere until after programming had been well established by G. B. Dantzig and others.

During the second world war the subject, *'Optimization Techniques'* in the name of *'Operations Research'* gained a momentum. The development of famous simplex method for solving the linear programming problems was first conceived by Dantzig in 1947, while he was working as mathematical adviser to the United States. This method gave a real boost to the subject. In 1975, the topic came to public attention when the Royal Swedish Academy of Sciences awarded the noble prize in economic science to L. V. Kantorovich and T. C. Koopmans. Based on simplex algorithms various linear systems have been studied in detail. In 1979, Khachian proved the ellipsoid method of Shor which ultimately exhibited polynomial-time performance. The time performance of simplex method is exponential. The theoretically superior, ellipsoid method could not be popular in practical use, even though its time performance is better than simplex method. In 1984, a real breakthrough came from N. Karmarkar's "projective scaling algorithm" for linear programming. The new algorithm not only outperforms the simplex method in theory but also shows its enormous potential for solving large scale practical problems. Karmarkar's algorithm is radically different from simplex method-it approaches an optimal solution from the interior of the feasible region. However, this research was limited to linear optimization problems.

The pioneer work by Karush, Kuhn and Tucker in 1951 on the necessary and sufficient conditions for the optimal solution laid the foundation for researchers to work on nonlinear systems. In 1957, the emergence of dynamic programming by Bellman brought a revolution in the subject and consequently, linear and nonlinear systems have been studied simultaneously. Although no universal techniques have been established for nonlinear systems, the researches by Fiacco and McCornik proved to be significant. Geometric programming was developed by Zener, Duffin and Peterson in 1961. Later on, Dantzig and Charnes developed Stochastic programming.

The process of optimizing more than one objective led to the development of multi-objective programming. During the meantime problems on network analysis essentially useful for management control came into existence. Well known games theory has been successfully applied for different programming problems. Multi-objective problems with specified goals in the name of Goal programming has been the topic of recent interest. At the moment fuzzy logic is being extensively used for studying various linear and nonlinear systems.

The subject has been fully exploited to solve various engineering, scientific, economics and management problems. We mention a few as

1. Design of aircrafts and aerospace structures for tolerating environment resistance.

2. Setting up of pipelines and reservoirs for flow of different items

at required points.

3. Decision making for maximizing industrial outputs.

4. Selection of machining conditions in different industrial processes to minimize the production cost.

5. Optimal production planning, controlling and scheduling of various projects.

6. Optimal designing of chemical processing equipment and plants.

7. Shortest route problems under varying conditions.

8. Planning the best strategies to obtain maximum profit.

9. Design of pumps, electric machines, computers etc. , for minimizing the cost.

10. Transportation of materials from places of manufacture to places of requirement so that the cost of transportation is minimized.

11. How the jobs should be assigned to workers so to have optimal efficiency of the system.

12. Allocation of resources and services among several activities to maximize the profit.

13. Inventory problem deals with the demands at specific time, here we have to decide how much and what to order.

14. Queueing problems deal with customers at service stations. The direct increase in service stations increases the service cost but waiting time in queue is reduced. However, waiting time also involves cost. In such type of problems we seek optimal number of services so that cost of service and waiting time is minimized.

Least to say, in every walk of life, optimization techniques are being extensively applied in day to day practice. Operations Research Society, USA defined OR as

> *"Operations research is the systematic applications of quantitative methods, techniques, tools to the analysis of problems involved in the operation of systems"*

1.2 Introduction

The mathematical formulation of our thoughts to optimize profit, loss, production etc., under given set of conditions is called mathematical programming.

The mathematical programming problem (MPP) is written as

$$\begin{aligned} \text{opt} \quad &= f(X) \\ \text{subject to} \quad & g_i(X) \geq,\ =,\ \leq\ 0, \quad i = 1, 2, \ldots, m \qquad (1.1) \\ & X \in \mathbb{R}^n, \qquad (1.2) \end{aligned}$$

where $X = (x_1, x_2, \ldots, x_n)^T$ is the column vector in n-dimensional real linear space $\mathbb{R}^n$.

Thus, $X^T = (x_1, x_2, \ldots, x_n)$ is the row vector. In the text, column vectors and row vectors will be represented by a column matrix and row matrix, respectively.

Now, we define

(i) The function $f(X)$ to be optimized is termed objective function;

(ii) The relations in (1.1) are constraints;

(iv) Variables $x_1, x_2, \ldots, x_n$ are decision variables.

(v) The terminology opt (optimize) stands for minimize or maximize.

The symbols $\geq, =, \leq$ mean that one and only one of these is involved in each constraint.

The mathematical programming problem (MPP) is further classified into two classes, viz.,

1. Linear programming problem. If, in a mathematical programming problem the objective function $f(X)$ and all the constraints $g_i(X)$ are linear, we call the problem as a linear programming problem (LPP).

2. Nonlinear programming problem. If, in a mathematical programming problem objective function $f(X)$ or at least one of the constraints $g_i(X)$ or both are nonlinear functions then the problem is termed nonlinear programming problem (NLPP).

Remarks. Integer programming problem is a particular case of LPP or NLPP in which some or all the decision variables $x_1, x_2, \ldots, x_n$ are integers. Quadratic programming problem is also a particular NLPP in which objective function $f(X)$ is quadratic but all the constraints $g_i(X)$ are linear functions.

Let us discuss the linear programming problems in detail. Any LPP has the general form:

$$\begin{aligned}
\text{opt}\quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.}\quad & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq,\ =,\ \leq b_i, \quad i = 1, 2, \ldots, m \\
& x_1, x_2, \ldots, x_n \geq 0,
\end{aligned}$$

where $c_k,\ k = 1, 2, \ldots, n$ and $b_i,\ i = 1, 2, \ldots, m$ are real numbers (may be negative).

The conditions in $x_1, x_2, \ldots, x_n \geq 0$ are nonnegative restrictions. Note that in an LPP some of the variables may be unrestricted in sign, i.e., may take any real value.

From now onward "s.t." stands for "subject to" in the whole text.

Standard form of linear programme. The standard form of an LPP is written as

$$\begin{aligned}
\text{opt}\quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.}\quad & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i, \quad i = 1, 2, \ldots, m \\
& x_1, x_2, \ldots, x_n \geq 0,\ b_1, b_2, \ldots, b_m \geq 0
\end{aligned}$$

or

$$\begin{aligned}
\text{opt}\quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.}\quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
& x_1, x_2, \ldots, x_n \geq 0,\ b_1, b_2, \ldots, b_m \geq 0
\end{aligned}$$

or, in matrix form:

$$\begin{aligned}
\text{opt}\quad & z = C^TX \\
\text{s.t.}\quad & AX = b \\
& X \geq 0,\ b \geq 0,
\end{aligned}$$

where

$$\begin{aligned} C &= (c_1, c_2, \ldots, c_n)^T \text{ (cost vector)},\\ X &= (x_1, x_2, \ldots, x_n)^T,\\ A &= (a_{ij})_{m\times n}, \text{ the coefficient matrix of order } m \text{ by } n \text{ and}\\ b &= (b_1, b_2, \ldots, b_m)^T. \end{aligned}$$

Converting to standard form. The standard form of a linear programme deals with nonnegative decision variables and linear equality constraints. Here we explain the means how to convert the linear programme into standard form in case any or both of these conditions are not available in the LPP.

Linear inequalities. A linear inequality can easily be converted into an equation by introducing slack and surplus variables. If the ith constraint has the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i,$$

we can add a nonnegative variable $s_i \geq 0$ to have

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_i = b_i.$$

Here, variable s_i is called the *slack variable.*

Similarly, if ith constraint has the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

a nonnegative variable $s_i \geq 0$ is subtracted to have

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - s_i = b_i.$$

This time s_i is termed the *surplus variable.*

Note that $b_i \geq 0$ in the above inequalities, if not, multiply by -1 before introducing the slack or surplus variables.

Restricted and unrestricted variables. If a variable x is restricted, i.e., for $x \geq p$, this implies $x - p \geq 0$. Taking, $x' = x - p$ implies $x' \geq 0$. So, we replace x by $x' + p$, and in a similar way, for the case $x \leq p$, replace x by $-x' + p$ to have $x' \geq 0$.

However, if a variable x is unrestricted in sign, i.e., $x \in \mathbb{R}$ (may be positive or negative), we write $x = x^+ - x^-$, where x^+ and x^- are defined by

$$x^+ = \begin{cases} x, & x \geq 0, \\ 0, & x \leq 0, \end{cases} \quad \text{and} \quad x^- = \begin{cases} 0, & x \geq 0, \\ -x, & x \leq 0. \end{cases}$$

Obviously, for each real number x we can find nonnegative real number u and v such that $|x| = u + v$ and $x = u - v$. Here, u and v play the role of x^+ and x^-, respectively.

Example 1. Write the following programme into standard form of LPP:

$$\begin{aligned} \text{opt} \quad & z = x_1 + 2x_2 - x_3 - 2 \\ \text{s.t.} \quad & -x_1 + 2x_2 + 3x_3 \geq -4 \\ & 2x_1 + 3x_2 - 4x_3 \geq 5 \\ & x_1 + x_2 + x_3 = 2 \\ & x_1 \geq 0,\ x_2 \geq 1 \text{ and } x_3 \text{ is unrestricted in sign.} \end{aligned}$$

Here, x_2 and x_3 are restricted and unrestricted variables, respectively. Replacing x_2 by $x_2' + 1$ and x_3 by $x_3^+ - x_3^-$, the above LPP is written in standard form as

$$\begin{aligned} \text{opt} \quad & z = x_1 + 2x_2' - x_3^+ + x_3^- \\ \text{s.t.} \quad & x_1 - 2x_2' - 3x_3^+ + 3x_3^- + s_1 = 6 \\ & 2x_1 + 3x_2' - 4x_3^+ + 4x_3^- - s_2 = 2 \\ & x_1 + x_2' + x_3^+ - x_3^- = 1 \\ & x_1, x_2', x_3^+, x_3^-, s_1, s_2 \geq 0. \end{aligned}$$

Note that x_1, x_2', x_3^+ and x_3^- are now the decision variables, s_1 slack variable and s_2 surplus variable when the LPP has been written in standard form.

From now onward it will be understood that the slack or surplus variable s_i means it is associated with the ith constraint.

The above discussion reveals that, in general k unrestricted variables produce $2k$ nonnegative variables to write the problem in standard form. This will substantially increase the size of the problem. However, under certain conditions we develop a better technique in which k unrestricted variable can be replaced by $k+1$ nonnegative variables to express the LPP into standard form.

Remark. In case any variable is missing in the nonnegative restrictions of a problem, it is, of course understood to be of unrestricted in sign.

Theorem. In an LPP, let k variables out of n variables be unrestricted in sign and are bounded (below). Then the problem can be converted into standard form by using $k+1$ nonnegative variables in place of these k unrestricted variables.

Proof. Let $x_1, x_2, \ldots, x_n$ be n variables of an LPP. Given that k of these variables are unrestricted in sign. Without loss of generality we may assume $x_1, x_2, \ldots, x_k$ are unrestricted in sign.

Define $y = |\min\{x_1, x_2, \ldots, x_k\}|$. Obviously, the minimum exists exists. Then, we observe that

$$\begin{aligned} y_1 &= x_1 + y \geq 0 \\ y_2 &= x_2 + y \geq 0 \\ &= \vdots \\ y_k &= x_k + y \geq 0 \\ y &\geq 0 \end{aligned}$$

This implies, $x_1 = y_1 - y,\ x_2 = y_2 - y, \cdots, x_k = y_k - y$, and the constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq, =, \geq b_i$$

are converted into

$$\begin{aligned} &a_{i1}y_1 + \cdots + a_{ik}y_k - (a_{i1} + \cdots + a_{ik})y \\ &\quad + a_{i,k+1}x_{k+1} + \cdots + a_{in}x_n \pm s_i = b_i \\ &y_1, y_2, \ldots, y_k, y, x_{k+1}, \ldots, x_n \geq 0. \end{aligned}$$

Here, $i = 1, 2, \ldots, m$ and s_i is a slack or surplus variable and in case of equality constraint $s_i = 0$.

Example 2. Illustrate the above theorem by taking a particular LPP:

$$\begin{aligned} \max \quad & z = x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 \leq 5 \\ & 2x_1 - x_2 + 2x_3 \geq 7 \\ & x_1 - x_2 - 3x_3 \leq 9 \\ & x_3 \geq 0 \text{ and } x_1, x_2 \text{ are unrestricted in sign.} \end{aligned}$$

Let $y = |\min\{x_1, x_2\}|$. Then $y_1 = x_1 + y \geq 0,\ y_2 = x_2 + y \geq 0$, and we have

$$\begin{aligned} \max \quad & z = y_1 + y_2 + x_3 - 2y \\ \text{s.t.} \quad & y_1 - y_2 + x_3 + s_1 = 5 \\ & 2y_1 - y_2 + 2x_3 - y - s_2 = 7 \\ & y_1 - y_2 - 3x_3 + s_3 = 9 \\ & y_1, y_2, y, x_3, s_1, s_2, s_3 \geq 0. \end{aligned}$$

Example 3. Linearize the following objective function:

$$\max\ z = \min\{|2x_1 + 5x_2|, |7x_1 - 3x_2|\}.$$

Let $y = \min\{|2x_1 + 5x_2|, |7x_1 - 3x_2|\}$. Hence

$$y \leq |2x_1 + 5x_2| \Rightarrow u_1 + v_1 \geq y \text{ for some variables } u_1, v_1 \geq 0.$$

and

$$y \leq |7x_1 - 3x_2| \Rightarrow u_2 + v_2 \geq y \text{ for some variables } u_2, v_2 \geq 0.$$

Combining the above inequalities, the given objective function can be written in the form of LPP as

$$\begin{aligned} \max \quad & z = y \\ \text{s.t.} \quad & u_1 + v_1 - y \geq 0 \\ & u_2 + v_2 - y \geq 0 \\ & u_1, v_1, u_2, v_2, y \geq 0. \end{aligned}$$

Note that $2x_1 + 5x_2$ and $7x_1 - 3x_2$ may be nonnegative or nonpositive, since we are silent about the nature of x_1 and x_2.

1.3 Formulation of Models

Learning to formulate the mathematical programming problem using the given data is the first step for optimizing any system. If we fail at this stage then it bears no fruitful results. The modelling of the problem includes

(i) Decision variables that we seek to determine.

(ii) Construction of the objective function to be optimized.

(iii) Constrains that satisfy various conditions.

(iv) Nonnegative restrictions and their nature.

The proper definition of the decision variables is the most sensitive part toward the development of a model. Once decision variables are defined the construction of objective function and constraints from the given data is not laborious.

For incorporating $\geq, =, \leq$ in constraints one has to be careful about the phrases: at least or minimum, exactly satisfied, at most or maximum or no longer than, etc.

In this section we have formulated various problems which are in common use.

Linear models. Here, we formulate some well known problems as linear programming problems.

Diet problem. A medical practitioner recommends the constituents of a balanced diet for a patient which satisfies the daily minimum requirements of Proteins P units, Fats F units, and Carbohydrates C units at a minimum cost. Choice from five different types of foods can be made. The yield per unit of these foods are given by

Food type	Protein	Fats	Carbohydrates	Cost/unit
1	p_1	f_1	c_1	d_1
2	p_2	f_2	c_2	d_2
3	p_3	f_3	c_3	d_3
4	p_4	f_4	c_4	d_4
5	p_5	f_5	c_5	d_5

How the patient should select the items so that he has to pay minimum.

Suppose x_i = the number of units of the ith food which the patient selects. The objective function is

$$\min\ z = d_1x_1 + d_2x_2 + \cdots + d_5x_5,$$

and the constraints are

$$\begin{aligned}
&p_1x_1 + p_2x_2 + \cdots + p_5x_5 \geq P\\
&f_1x_1 + f_2x_2 + \cdots + f_5x_5 \geq F\\
&c_1x_1 + c_2x_2 + \cdots + c_5x_5 \geq C\\
&x_i \geq 0,\ i = 1, 2, \ldots, 5.
\end{aligned}$$

Product mix problem. A manufacturing process requires three different inputs viz., A, B and C. A sandal soap of first type requires 30 gm of A, 20 gm of B and 6 gm of C, while this data for the second type of soap is 25, 5 and 15, respectively. The maximum availability of A, B and C are 6000, 3000 and 3000 gm, respectively. The selling price of the sandal soap of the first and second type are \$14 and \$15, respectively. The profit is proportional to the amount of soaps manufactured. How many soaps of first and second kind should be manufactured to maximize the profit. Assume that the market has unlimited demand.

Let us put the data in tabular form

Type	Inputs/unit			Selling price/unit
	A	B	C	
I	30	20	6	14
II	25	5	15	15
Max availability	6000	3000	3000	

Let x_1 and x_2 be the number of the first and second type of soaps to be manufactured. The profit from selling is given by $z = 14x_1 + 15x_2$. This is subjected to the availability constraints given by $30x_1 + 25x_2 \leq 6000$, $20x_1 + 5x_2 \leq 3000$, $6x_1 + 15x_2 \leq 3000$. The decision variables are $x_1, x_2 \geq 0$, and in addition, these must be integers.

Thus, the required LPP is

$$\begin{aligned}
\max \quad & z = 14x_1 + 15x_2\\
\text{s.t.} \quad & 30x_1 + 25x_2 \leq 6000\\
& 20x_1 + 5x_2 \leq 3000\\
& 6x_1 + 15x_2 \leq 3000\\
& x_1, x_2 \geq 0 \text{ and are integers.}
\end{aligned}$$

Bus scheduling problem. IP Depo runs buses during the time period 5 AM to 1 AM. Each bus can operate for 8 hours successively, and then it is directed to workshop for maintenance and fuel. The minimum number of buses required fluctuate with the time intervals. The desired number of buses during different time interval are given in the following table:

Time intervals	Minimum number of buses required
5 AM–9 AM	5
9 AM–1 PM	13
1 PM–5 PM	11
5 PM–9 PM	14
9 PM–1 AM	4

The depo keeps in view the reduction of air pollution and smog problem. It is required to determine the number of buses to operate during different shifts that will meet the minimum requirement while minimizing the total number of daily buses in operation.

Let x_i be the number of buses starting at the beginning of the ith period, $i = 1$ to 5. Note that each bus operates during two consecutive shifts. Buses which join the crew at 5 AM and 9 AM will be in operation between 9 AM and 1 PM. As the minimum number of buses required in this interval is 13, we have $x_1 + x_2 \geq 13$, and similarly others.

The LPP formulation is

$$\begin{aligned}
\min \quad & z = x_1 + x_2 + x_3 + x_4 + x_5 \\
\text{s.t.} \quad & x_1 + x_2 \geq 13 \\
& x_2 + x_3 \geq 11 \\
& x_3 + x_4 \geq 14 \\
& x_4 + x_5 \geq 4 \\
& x_5 + x_1 \geq 5 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and are integers.}
\end{aligned}$$

The warehousing problem. A warehouse has a capacity of 2000 units. The manager of the warehouse buys and sells the stock of potatoes over a period of 6 weeks to make profit. Assume that in the jth week the same unit price p_j holds for both purchase and sale. In addition, there is unit cost \$15 as weekly expenses for holding stock. The warehouse is empty at the beginning and is required to be empty after the sixth week. How should the manager operate?

The major activities involve buying, selling, and holding the stock for a week. Define the variables

$x_j =$ the level of the stock at the beginning of the jth week;

$y_j =$ the amount bought during the jth week;

$z_j =$ the amount sold during the jth week.

Then the manager tries to maximize

$$\sum_{j=1}^{6} p_j(z_j - y_j) - 15x_j$$

subject to the stock balance constraints

$$x_{j+1} = x_j + y_j - z_j, \quad j = 1, 2, \ldots, 5$$

the warehouse capacity constraints

$$x_j \leq 2000, \; j = 1, 2, \ldots, 6$$

the boundary conditions

$$x_1 = 0, \quad x_6 + y_6 - z_6 = 0$$

and the nonnegative restrictions

$$x_j \geq 0, \; y_j \geq 0, \; z_j \geq 0, \quad j = 1, 2, \ldots, 6.$$

Caterer problem. Thapar institute has to organize its annual cultural festival continuously for next five days. There is an arrangement of dinner for every invited team. The requirement of napkins during these five days is

Days	1	2	3	4	5
Napkins required	80	50	100	80	150.

Accordingly, a caterer has been requested to supply the napkins according to the above schedule. After the festival is over caterer has no use of napkins. A new napkin costs \$2. The washing charges for a used napkin is \$0.5 by ordinary services and Re 1, if express service is used. A napkin given for washing by ordinary service is returned third day, while under express service it is return next day. How the caterer should meet the requirement of the festival organizers so that the total cost is minimized.

Define the decision variables as

$x_i =$ number of napkins purchased on the ith day, $i = 1$ to 5.

$y_j =$ number of napkins given for washing on jth day under express service, $j = 1$ to 4.

$z_k =$ number of napkins given for washing on kth day under ordinary service, $k = 1$ to 3.

$v_\ell =$ number of napkins left in the stock on ℓth day after the napkins have been given for washing, $\ell = 1$ to 5.

The data is tabulated as

Type	Number of napkins required on days				
	1	2	3	4	5
New napkins	x_1	x_2	x_3	x_4	x_5
Express service	–	y_1	y_2	y_3	y_4
Ordinary service	–	–	z_1	z_2	z_3
Napkins required	80	50	100	80	150

We have to minimize

$$2(x_1 + x_2 + x_3 + x_4 + x_5) + y_1 + y_2 + y_3 + y_4 + 0.5(z_1 + z_2 + z_3).$$

From the table:

$$x_1 = 80,\ x_2 + y_1 = 50,\ x_3 + y_2 + z_1 = 100,$$
$$x_4 + y_3 + z_2 = 80,\ x_5 + y_4 + z_3 = 150.$$

Also, there is another set of constraints which shows the total number of napkins which may be given for washing and some napkins which were not given for washing just on the day these have been used. These constraints are: $y_1 + z_1 + v_1 = 80$, $y_2 + z_2 + v_2 = 50 + v_1$, $y_3 + z_3 + v_3 = 100 + v_2$, $y_4 + v_4 = 80 + v_3$, $v_5 = 150 + v_4$.

Thus the desired LPP model is

$$
\begin{aligned}
\min \quad & z = 160 + 2(x_2 + x_3 + x_4 + x_5) + y_1 + y_2 + y_3 + y_4 \\
& \qquad + 0.5(z_1 + z_2 + z_3) \\
\text{s.t.} \quad & x_2 + y_1 = 50 \\
& x_3 + y_2 + z_1 = 100 \\
& x_4 + y_3 + z_2 = 80 \\
& x_5 + y_4 + z_3 = 150 \\
& y_1 + z_1 + v_1 = 80 \\
& y_2 + z_2 + v_2 - v_1 = 50 \\
& y_3 + z_3 + v_3 - v_2 = 100 \\
& y_4 + v_4 - v_3 = 80 \\
& v_5 - v_4 = 150 \\
& \text{all var } \geq 0.
\end{aligned}
$$

Trim-loss problem. Paper cutting machines are available to cut standard news print rolls into the subrolls. Each standard roll is of 180 cm width and a number of them must be cut to produce smaller subrolls at the current orders for 30 of width 70 cm, 60 of width 50 cm and 40 of width 30 cm. Formulate the problem so as to minimize the amount of wastes. Ignoring the recycling or other uses for the trim, assume that the length of each required subroll is the same as that of the standard roll.

A standard roll may be cut according to the following patterns.

Widths ordered (in cm)	Number of subrolls cut on different patterns							
	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
30	6	4	3	2	2	1	1	0
50	0	1	0	1	2	0	3	2
70	0	0	1	1	0	2	0	1
Trim loss	0	10	20	0	20	10	0	10

Let x_i be the number of the standard news print rolls pieces to cut on the pattern p_i, $i = 1, 2, \ldots, 8$.

Thus, the required LPP is

$$\begin{aligned} \min \quad & z = 10x_2 + 20x_3 + 20x_5 + 10x_6 + 10x_8 \\ \text{s.t.} \quad & 6x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 + x_6 + x_7 = 40 \\ & x_2 + x_4 + 2x_5 + 3x_7 + 2x_8 = 60 \\ & x_3 + x_4 + 2x_6 + x_8 = 30 \\ & x_i \geq 0,\ i = 1, 2, \ldots, 8 \text{ and are integers.} \end{aligned}$$

Here, in the constraints the equality is desired due to the fact any thing left is of no use.

Example 4. Two alloys, A and B are made from four different metals, I, II, III, and IV, according to the following specifications:

Alloy	Specifications	Selling price ($)/ton
A	at most 80% of I	200
	at least 30% of II	
	at least 50% of IV	
B	between 40% & 60% of II	300
	at least 30% of III	
	at most 70% of IV	

The four metals, in turn, are extracted from three different ores with the following data:

Ore	Max Quantity (in tons)	Constituents % (percentage)					Purchase Price $ per tone
		I	II	III	IV	others	
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

How much of each alloy should be produced to maximize the profit. Formulate the problem as LP model.

Define

x_{ij} = tons of ore i allocated to alloy j; $i = 1, 2, 3$; j = A, B

w_j = tons of alloy j produced

$$\max\ z = 200w_A + 300w_B - 30(x_{1A} + x_{1B}) - 40(x_{2A} + x_{2B}) - 50(x_{3A} + x_{3B})$$

Specification constraints:

$$\begin{aligned}
0.2x_{1A} + 0.1x_{2A} + 0.05x_{3A} &\le 0.8w_A \\
0.1x_{1A} + 0.2x_{2A} + 0.05x_{3A} &\ge 0.3w_A \\
0.3x_{1A} + 0.3x_{2A} + 0.2x_{3A} &\ge 0.5w_A \\
0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} &\ge 0.4w_B \\
0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} &\le 0.6w_B \\
0.3x_{1B} + 0.3x_{2B} + 0.7x_{3B} &\ge 0.3w_B \\
0.3x_{1B} + 0.3x_{2B} + 0.2x_{3B} &\le 0.7w_B
\end{aligned}$$

Ore constraints:

$$\begin{aligned}
x_{1A} + x_{1B} &\le 1000 \\
x_{2A} + x_{2B} &\le 2000 \\
x_{3A} + x_{3B} &\le 3000
\end{aligned}$$

Alloy constraints:

$$\begin{aligned}
&x_{1A} + x_{2A} + x_{3A} \ge w_A \\
&x_{1B} + x_{2B} + x_{3B} \ge w_B \\
&x_{iA} \ge 0, x_{iB}, w_j \ge 0 \quad i = 1, 2, 3,\ j = \text{A, B}.
\end{aligned}$$

Nonlinear models. The formulation of nonlinear problems requires little more efforts in comparison to linear models. In this section, we formulate some nonlinear programming problems.

Gambler problem. A gambler has \$24000 to play a game. In the game there are three places for stake. He divides his total money among three choices. There are three outcomes in the game. The return per unit deposited at each choice can be read from the table:

	Gain or loss per dollar at choice		
Out-comes	1	2	3
1	-5	1	1
2	-7	6	10
3	13	-2	6

The probabilities of different outcomes are not known. The gambler wants least risk as far as loss is concerned. He decides to divide his money among three choices in such a way, if there is any loss, then it is least. Any way he maximizes the minimum return.

Suppose that x_1, x_2, x_3 dollars are invested by the gambler on the choices 1, 2, 3, respectively. Then, the returns depending upon outcomes 1,2,3 are

$$-5x_1 + x_2 + x_3, \qquad -7x_1 + 6x_2 + 10x_3, \qquad 13x_1 - 2x_2 + 6x_3.$$

The problem is formulated as

$$\begin{aligned} \max \ \ & z = \min\{-5x_1 + x_2 + x_3,\ -7x_1 + 6x_2 + 10x_3,\ 13x_1 - 2x_2 + 6x_3\} \\ \text{s.t.} \ \ & x_1 + x_2 + x_3 = 24000 \\ & x_1, x_2, x_3 \geq 0 \text{ and are integers.} \end{aligned}$$

Remark. This is a nonlinear programming problem (NLPP). However, it may be converted into an LPP as follows:

Let $y = \min\{-5x_1 + x_2 + x_3,\ -7x_1 + 6x_2 + 10x_3,\ 13x_1 - 2x_2 + 6x_3\}$. Then

$$\begin{aligned} y &\leq -5x_1 + x_2 + x_3 \\ y &\leq -7x_1 + 6x_2 + 10x_3 \\ y &\leq 13x_1 - 2x_2 + 6x_3 \end{aligned}$$

The required LPP is

$$\begin{aligned} \max \ \ & z = y \\ \text{s.t.} \ \ & 5x_1 - x_2 - x_3 + y \leq 0 \\ & 7x_1 - 6x_2 - 10x_3 + y \leq 0 \\ & -13x_1 + 2x_2 - x_3 + y \leq 0 \\ & x_1 + x_2 + x_3 = 24000 \\ & x_1, x_2, x_3, y \geq 0 \text{ and are integers.} \end{aligned}$$

Production planning problem. The municipal corporation of Patiala decides to clean the water system of open wells in urban areas, and for this purpose a medicated product is made by mixing two parts of potassium permagnate and three parts of bleaching powder. These products are processed in departments of Chemical, Chemistry and

Biotechnology operating in Ranbaxy laboratories. Departments have limited number of production hours available, viz., , 100, 150 and 200, respectively. The production rate of potassium permagnate and bleaching powder in each department is given in the following table.

	Production rate(no. of units/hour)	
Department	Potassium permagnate	Bleaching powder
Chemical	20	25
Chemistry	25	20
Biotechnology	20	5

The objective is to determine the number of hours to be assigned to each department to maximize the completed units of the medicated product. Formulate the appropriate model.

Writing the given data in tabular form as

Department	Production rate(no. of units/hour)		Limited
	Potassium permagnate	Bleaching powder	number of hours
Chemical	20	25	100
Chemistry	25	20	150
Biotechnology	20	5	200

Let x_{ij} be the number of hours assigned to ith department for jth part, $i = 1, 2, 3$ and $j = a, b$, where suffixes 1 = Chemical, 2 = Chemistry, 3 = Biotechnology and a = Potassium permagnate and b = bleaching powder.

Total number of parts of a manufactured $= 20x_{1a} + 25x_{2a} + 20x_{3a}$.

Total number of parts of b manufactured $= 25x_{1b} + 20x_{2b} + 5x_{3b}$.

Complete units of the final product are

$$y = \min\left\{\frac{20x_{1a} + 25x_{2a} + 20x_{3a}}{2}, \frac{25x_{1b} + 20x_{2b} + 5x_{3b}}{3}\right\}.$$

Thus

$$\begin{aligned} \max \quad & z = y \\ \text{s.t.} \quad & x_{1a} + x_{1b} \leq 100 \\ & x_{2a} + x_{2b} \leq 150 \\ & x_{3a} + x_{3b} \leq 200 \\ & x_{ij} \geq 0,\ i = 1,2,3;\ j = a,b, \text{ and are integers.} \end{aligned}$$

The problem is formulated as nonlinear programming problem.

Remark. The formulation of this problem can be done as LPP, see Problem 19, Problem set 1.

Example 5. A firm produces two products A and B using two limited resources. The maximum amount of Resource 1 available per week is 3000, while for Resource 2 is 2500. The production of one unit of A requires 3 units of Resource 1 and 1 unit of of Resource 2, and the production of B requires 2 units of Resource 1 and 2 units of Resource 2. The unit cost of Resource 1 is $(1.5 - .001u_1)$, where u_1 is the number of units of Resource 1 used. The unit cost of Resource 2 is $(2 - .004u_2)$, where u_2 is the number of units of Resource 2 is used. The selling price per unit of A and B are fixed as

$$p_A = 8 - .001x_A - .005x_B,$$

$$p_B = 9 - .002x_A - .004x_B,$$

where x_A and x_B are the number of units sold for product A and B, respectively. Assuming that how much has been manufactured is disposed off, formulate the above problem to maximize the profit over a week.

Let x_A and x_B be number of units of the products A and B produced per week. The requirement of Resource 1 per week is $(3x_A + 2x_B)$, while that of Resource 2 is $(x_A + 2x_B)$ and the constraints on the resources availability are $3x_A + 2x_B \leq 3000$ and $x_A + 2x_B \leq 2500$.

The total cost of Resource 1 and 2 per week is

$$(3x_A + 2x_B)[1.5 - .001(3x_A + 2x_B)] + (x_A + 2x_B)[2 - .004(x_A + 2x_B)]$$

and the total return per week from selling of A and B is

$$x_A(8 - .001x_A - .005x_B) + x_B(9 - .002x_A - .004x_B).$$

As the total profit is the difference of total cost and total return, the formulation of model is

$$\begin{aligned} \max \quad & z = .012x_A^2 + .016x_B^2 + .021x_Ax_B + 1.5x_A + 2x_B \\ \text{s.t.} \quad & 3x_A + 2x_B \le 3000 \\ & x_A + 2x_B \le 2500 \\ & x_A, x_B \ge 0 \end{aligned}$$

This is a quadratic programming problem.

Problem Set 1

1. Write the following problem in standard form of LPP

$$\begin{aligned} \text{opt} \quad & z = 2x_1 + x_2 - x_3 - 1 \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 \le 5 \\ & -3x_1 + 2x_2 + 3x_3 \ge -3 \\ & x_1 - 3x_2 + 4x_3 \ge 2 \\ & x_1 + x_2 + x_3 = 4 \\ & x_1 \ge 0, x_2 \ge 1 \text{ and } x_3 \text{ is unrestricted in sign.} \end{aligned}$$

2. Write the following problem in standard LPP:

$$\begin{aligned} \max \quad & z = 2x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 - x_2 + 2x_3 \ge 2 \\ & |2x_1 + x_2 - x_3| \le 4 \\ & 3x_1 - 2x_2 - 7x_3 \le 3 \\ & x_1, x_3 \ge 0, x_2 \le 0. \end{aligned}$$

3. Write the following problem in standard form of LPP with slack variables:

$$\begin{aligned} \text{opt} \quad & z = x_1 + 2x_2 - x_3 - 2p \\ \text{s.t.} \quad & x_1 + x_2 - x_3 \le 5 \\ & -x_1 + 2x_2 + 3x_3 \ge -4 \\ & 2x_1 + x_2 - x_3 \ge -1 \\ & x_1 + x_2 + x_3 = 2 \\ & x_1 \ge 0,\ x_2 \ge p \text{ and } x_3 \text{ is unrestricted in sign.} \end{aligned}$$

Mention the range of p so that the standard form remains intact.

4. Consider the following optimization problem:

$$\begin{aligned} \min \quad & z = |x_1| + 2|x_2| - x_3 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 \leq 9 \\ & x_1 - 2x_2 + 3x_3 = 11 \\ & x_3 \geq 0 \end{aligned}$$

(a) Is this a linear programming problem?

(b) Can you convert it into an LPP? If yes, write the standard form.

5. Convert the following problem into standard linear programme by using only three nonnegative variables in place of x_1 and x_2.

$$\begin{aligned} \min \quad & z = x_1 + x_2 - 1 \\ \text{s.t.} \quad & x_1 + x_2 \leq 7 \\ & x_1 - 2x_2 \geq 4. \end{aligned}$$

Suggestion. Replace x_1 by $x_1 + 1$ and then use the theorem in Section 1.2.

6. Convert the following problem into an equivalent linear model

$$\begin{aligned} \max \quad & \frac{-3 + 2x_1 + 4x_2 - 5x_3}{6 + 3x_1 - x_2} \\ \text{s.t.} \quad & x_1 - x_2 \geq 0 \\ & 7x_1 + 9x_2 + 10x_3 \leq 30 \\ & x_1 \geq 0, x_2 \geq 1, x_3 \geq 0 \end{aligned}$$

Suggestion. This is a linear fractional programming problem. First, replace x_2 by $x_2 + 1$. Assume $r = 5 + 3x_1 - x_2$ and define

$$\frac{x_1}{r} = y_1 \geq 0, \ \frac{x_2}{r} = y_2 \geq 0, \ \frac{x_3}{r} = y_3 \geq 0, \ \frac{1}{r} = u \geq 0.$$

7. Suppose n different food items are available at the market and the selling price for the jth food is c_j per unit. Moreover, there are m basic nutritional ingredients for the human body and minimum b_i units of the ith ingredient are required to achieve a balanced diet for good health. In addition, a study shows that each unit of the jth food contains a_{ij} units of the ith ingredients. A dietitian of a large group may face a problem of determining the most economical diet that satisfies the basic nutritional requirement for good health. Formulate the problem so that problem of dietitian is solved.

8. A small manufacturing plant produces two products, A and B. Each product must be worked on by a bank of CNC lathe machines and then, in succession, by a group of CNC milling machines. Product A requires 1 hr on CNC lathe machines and 3 hrs on CNC milling machines. Product B requires 5 hrs on CNC lathe machines and 1 hr on CNC milling machines. A total of 10000 hrs is available per week on CNC lathe machines and 7000 hrs on CNC milling machines. The net profit is \$0.5 per unit for product A and \$0.1 per unit for product B. Formulate the problem so as to maximize the weekly profit. Assume that all the quantities manufactured are disposed off.

9. A company makes two kinds of leather belts. Belt A is a high quality belt, and belt B is of lower quality. Each belt of type A requires twice as much time as a belt of type B, and if, all belts were of type B, the company could make 1500 per day. The supply of leather is sufficient for only 1000 belts per day (both A and B combined). Belt A requires a fancy buckle, and only 500 per day are available. There are only 800 buckles a day available for belt B. The profits in belt A and B are \$3 and \$2 per belt, respectively. Formulate the linear programming problem to maximize the profit.

10. The New Delhi Milk Corporation (NDMC) has two plants each of which produces and supplies two products: Milk and Butter. Plants can each work up to 16 hours a day. In Plant-I, it takes 3 hours to prepare from powder and pack 1000 litres of milk and 1 hour to prepare and pack 100 kg of butter. In Plant-II, it takes 2 hours to prepare and pack 1000 litres of milk and 1.5 hours to prepare and pack 100 kg of butter. In Plant-I, it costs \$15000 to prepare and pack 1000 litres of milk and \$28000 to prepare and pack 100 kg of butter, whereas these costs are \$18000 and \$26000, respectively for Plant-II. The NDMC is obliged to produce daily at least 10,000 litres of milk and 800 kg of butter. Formulate this as LPP to find as to how should the company organize its production so that the required amount of the products be obtained at minimum cost.

Suggestion. Let m_i = units of milk produced in ith plant per day and b_i = units of butter produced in ith plant per day, $i = 1, 2$. 1000 litres = one unit and 100 kg = one unit. Note that one unit of milk is produced by Plant-I in 3 hours and hence m_1 is produced in $3m_1$ hours and so on.

11. A farmer has to plant two kinds of trees, say A and B on a land with 4400 sq m area. Each A tree requires at least 25 sq m of land, and B requires 40 sq m. The annual water requirement of tree A is 30 units and that of B is 15 units, while at most 3300 units water is available. It is estimated that the ratio of the number of B trees to the number of A trees should not be less than 6/19 and not be more that 17/8. The return from one B tree is \$ 50, while from one A tree is one and a half times that of return from B. Describe the plantation project of the farmer in terms of LPP so that the return is maximum.

12. A metal slitting company cuts master rolls with width 200 cm into subrolls of small width. Customers specify that they need subrolls of different widths given in the following table

Width of subrolls (in cm)	Number required
35	200
80	90
90	350
120	850

The objective is to use a minimum number of master rolls to satisfy a set of customers' orders. Formulate the problem as LPP.

13. The Materials Science Division of TIET needs circular metallic plates of diameters 3 cm and 6 cm to perform experiments on heat treatment studies, and requirement of these plates are 2500 and 1500, respectively. These are to be cut from parent metallic sheets of dimension 6×15 cm^2. Formulate the problem as linear programming problem so that the minimum number of parent metallic sheets are used.

14. Martin furniture company manufactures tables and chairs using wood and labour only. Wood required for one table is 30 units and for one chair is 20 units, and the labour spent on table is 10 units and for chair is 5 units. Total units of wood available are 381 and of labour are 117. The unit profit for table is \$9 and for chair is \$6. How many tables and chairs should be made to get maximum profit?

15. IBM produces two kinds of memory chips (Chip-1 and Chip-2) for memory usage. The unit selling price is \$15 for Chip-1 and \$25 for Chip-2. To make one Chip-1, IBM has to invest 3 hours of skilled labour, 2 hours of unskilled labour and 1 unit of raw material. To make one Chip-2, it takes 4 hours of skilled labour, 3 hours of unskilled labour, and 2 units of raw material. The company has 100 hours of skilled labour, 70 hours of unskilled labour and 30 units of raw material available, and is interested to utilize the full potential of skilled labour. The sales contract signed by IBM requires that at least 3 units of chip-2 have to be produced and any fractional quantity is acceptable. Formulate a LP model to help IBM determine its optimal product mix.

16. A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw material (A and B) of which 2000 and 3000 units are available, respectively. The raw material requirement can be read from the following table

Raw material	Requirement per unit of given model		
	I	II	III
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force can produce the equivalent of 700 units of model I. A market survey indicates that the minimum demand of three models are 200, 200 and 150 units, respectively. Formulate the LPP to determine the number of units of each product which will maximize the profit. Assume that the profit per unit of models I, II, III are \$30, \$20, and \$60, respectively.

17. There are m machines and n products, and the time a_{ij} is required to process one unit of product j on machine i. The x_{ij} is the number of units of product j produced on machine i and c_{ij} is the respective cost of processing them. The b_i is the total time available on machine i whereas d_j is the number of units of product j which must be processed. Formulate the problem with an objective of minimizing the total cost.

18. A ship has three cargo loads, forward, after and centre; the capacity limits are

Placement	Weight (tonnes)	Capacity in m^3
Forward	2000	100,000
Centre	3000	135,000
After	1500	30,000

The following cargoes are offered, the ship owner may accept all or any part of each commodity:

Commodity	Weight (in tonnes)	Volume per ton in m^3	Profit per ton in \$
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship, the weight in each load must be proportional to the capacity. The objective is to maximize the profit. Formulate the linear programming model for this problem.

19. Convert the nonlinear problem obtained in production planning problem of Section 1.3 into linear programming problem.

20 Reformulate the LPP of Problem 16 with the modification: "The labour time for each unit of model-I is twice that of model-II and labour time for each unit of model-II is thrice that of model-III". The remaining data is same as given in Problem 16.

21. A company manufactures a product which consists of n ingredients that are being produced in m departments. Each department has limited number of production hours, viz., , the ith $(i = 1, 2, \dots, m)$ department has b_i hours available. The production rate of jth $(j = 1, 2, \dots, n)$ ingredient is a_{ij} units per hour in the ith department. The final product is made just by mixing one part of the first ingredient, two parts of the second ingredient, and so on n parts of the nth ingredients. The objective is to determine the number of hours of each department to be assigned to each ingredient to maximize the completed units of the product. Formulate the problem as NLPP.

Suggestion. x_{ij} = the number of hours assigned to ith department for the production of jth ingredient. This is a generalization of the production planning problem of Section 1.3.

22. A canteen of an institute which remains functional only for five days in a week has to recruit waiters. A waiter has to work continuously for three days and have two days off. The minimum number of waiters required on individual days are

Days	1	2	3	4	5
Number of waiters required	25	35	40	30	20

Not more than 30 waiters can be recruited on any day. Formulate the LPP model to minimize the number of waiters recruited.

23. A transporter company assigns three types of buses to four routes according to the following data:

Bus type	Capacity passengers	Number of buses	Weekly trips on route 1	2	3	4
1	100	5	3	2	2	1
2	70	8	4	3	3	2
3	50	10	5	5	4	2
Number of tourists			1000	2000	900	1200

The associated costs, including the penalties for losing customers because of space limitation, are

	Operating cost ($) per trip on route			
Bus type	1	2	3	4
1	1000	1100	1200	1500
2	900	1000	1100	1200
3	700	900	900	1000
Penalty in ($) per lost customer	40	50	45	70

Formulate the LPP model that determines the optimum allocation of buses to different cities and the associated number of trips.

24. A retired employee wants to invest $2,00,000 which he received as provident fund. He was made acquainted with two schemes. In scheme-A he is ensured that for each dollar invested will earn $0.60 a year, and in scheme-B each dollar will earn $1.4 after two years. In scheme-A investments can be made annually, while in scheme-B investments are allowed for periods that are multiples of two years only. How should the employee invest his hard earn money to maximize the earnings at the end of three years? Formulate the LP model for the problem.

25. A factory is to produce two products P_1 and P_2. The product requires machining on two machines M_1 and M_2. Product P_1 requires 5 hours on machine M_1 and 3 hours on machine M_2. Product P_2 requires 4 hours on machine M_1 and 6 hours on machine M_2. Machine M_1 is available for 120 hours per week during regular working hours and 50 hours on overtime. Weekly machine hours on M_2 are limited to 150 hours on regular working hours and 40 hours on overtime. Product P_1 earns a unit profit of $8 if produced on regular time and $6, if produced on regular time on M_1 and on overtime on M_2, and $4 if produced on overtime on both the machines. Product P_2 earns a unit profit of $10 if produced on regular time and $9, if produced on regular time on M_1 and on overtime on M_2, and $8 if produced on overtime on both the machines. Formulate an LPP model for designing an optimum production schedule for maximizing the profit.

Suggestion. Define the variables

$x_1 =$ number of units of P_1 made on regular time

$x_2 =$ number of units of P_2 made on regular time

$x_3 =$ number of units of P_1 made on overtime

$x_4 =$ number of units of P_2 made on overtime

$x_5 =$ number of units of P_1 made on regular time on M_1 and overtime on M_2

$x_6 =$ number of units of P_2 made on regular time on M_1 and overtime on M_2

The objective function is $8x_1 + 10x_2 + 4x_3 + 8x_4 + 6x_5 + 9x_6$ and to find the constraints, construct the table:

Machine type	Product type		Available time	
	Time for P_1	Time for P_2	Regular time	Overtime
M_1	5	4	120	50
M_2	3	6	150	40

$$5x_1 + 4x_2 + 5x_5 + 4x_6 \leq 120 \text{ (regular time of } M_1)$$
$$5x_3 + 4x_4 \leq 50 \text{ (overtime of } M_1)$$
$$3x_1 + 6x_2 \leq 150 \text{ (regular time on } M_2)$$
$$3x_3 + 6x_4 + 3x_5 + 6x_6 \leq 40 \text{ (overtime on } M_2)$$
$$x_j \geq 0,\ j = 1, 2, \ldots, 6.$$

26. A chemical company has been requested by its state government to install and employ antipollution devices. The company makes two products; for each of these products, the manufacturing process yields excessive amount of irritant gases and particulates (airborne solids). The table shows the daily emission, in pounds, of each pollutant for every 1000 gallons of product manufactured. The company is prohibited from emitting more than G_1, G_2 and P_1 pounds of gas CM, gas SD, and Particulates, respectively. The profit for each thousand gallons of Products 1 and 2 manufactured per day is p_1 and p_2, respectively.

Type of Pollutant	Pounds of pollution emitted Per 1000 gallons of Product 1	Per 1000 gallons of Product 2
Gas CM	24	36
Gas SD	8	13
Particulates	100	50

The production manager has approved the installations of two antipollution devices. The first device removes 0.75 of gas CM, 0.5 of gas SD and 0.9 of the Particulates, regardless of the product made. The second device removes 0.33 of gas CM, none of gas SD, and 0.6 of the Particulates for Product 2. The first device reduces profit per thousand gallons manufactured daily by c_1, regardless of the product; similarly, the second device reduces profit by c_2 per thousand gallons manufactured, regardless of the product. Sales commitments dictate that at least R_1 thousand gallons of Product 1 be produced per day, and R_2 thousand of gallons of Product 2. Formulate the appropriate optimization model.

Suggestion. Define the decision variables as $x_1 =$ 1000 gallons of Product 1 made per day without using any control device; $x_{11} =$ 1000 gallons of Product 1 made per day using the first control device; $x_{12} =$ 1000 gallons of Product 1 made per day using the second device. Define the similar variables y_1, y_{11}, y_{12} for Product 2.

27. A company produces two products P_1 and P_2. The sales volume for P_1 is at least 40% of the total sales of both P_1 and P_2. The market survey ensures that it is not possible to sell more than 100 unit of P_1 per day. Both product use one raw material whose availability to 120 lb a day. The usage rates of the raw material are 1 lb per unit for P_1 and 2 lb for per unit for P_2. The unit prices for P_1 and P_2 are \$10 and \$30, respectively. Formulate the LPP model to optimize the product mix for the company.

28. A farming organization operates three farms of comparable productivity. The output of each farm is limited both by the usable acreage and the water available for irrigation. The data for the upcoming season are

Farm	Usable acreage	Water available (acre feet)
1	400	1500
2	600	2000
3	300	900

The organization is interested in three crops for planting which differ primarily in their expected profit per acre and their consumption of water. Furthermore, the total acreage that can be devoted to each of the crops is limited by the amount of appropriate harvesting equipment available

Crop	Maximum acreage	Water consumption in acre feet	Expected profit per acre ($)
A	700	5	4000
B	800	4	3000
C	3000	3	1000

In order to maintain a uniform workload among farms, the policy of the organization is that the percentage of the usable acreage planted be the same at each farm. However, any combination of the crops may be grown at any of the farms. The organization wishes to know how much each crop should be planted at the respective farms to maximize the expected profit.

Suggestion. x_{ij} = number of acres of $i$the farm to be alloted to jth crop, $i = 1, 2, 3$ and $j = A, B, C$.

29. Weapons of three types are to be assigned to 8 different targets. Upper limits on available weapons and lower limits on weapons to be assigned are specified. The characteristics of the three weapons type are as follows

(a) W_1: Fighter bombers
(b) W_2: Medium-range ballistic missiles
(c) W_3: Intercontinental ballistic missiles

The following table 1 gives the values of the parameters needed for the model: probabilities that target will be undamaged by weapons, minimum number of weapons to be assigned (b_j), and military value of targets (u_j).

Targets	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
W_1	1.00	0.90	1.00	0.95	1.00	0.90	1.00	0.95
W_2	0.85	0.85	0.90	1.00	0.95	1.00	0.95	0.90
W_3	0.95	1.00	0.95	0.90	0.90	0.95	0.85	1.00
b_j	30	100	50	40	60	70	50	10
u_j	60	50	75	80	40	200	100	150

The targets and the total number of weapons available for each targets are tabulated as

Targets	Weapons available
W_1	200
W_2	100
W_3	300

Formulate the model for maximizing expected target damage value.