Preface

Using stochastic differential equations (SDEs), we can successfully model systems that function in the presence of random perturbations. Such systems are among the basic objects of modern control theory, signal processing and filtering, physics, biology and chemistry, economics and finance, to mention just a few. However the very importance acquired by SDEs lies, to a large extent, in the strong connection they have with the equations of mathematical physics. It is well known that problems of mathematical physics involve "damned dimensions", often leading to severe difficulties in solving boundary value problems. A way out is provided by the employment of probabilistic representations together with Monte Carlo methods. As a result, a complex multi-dimensional problem for partial differential equations (PDEs) reduces to the Cauchy problem for a system of SDEs. The last system can naturally be regarded as one-dimensional, since it contains only one independent variable; it arises as a characteristic system of the considered problem for PDEs. The importance of this approach, while enabling the reduction of a multi-dimensional boundary value problem to the one-dimensional Cauchy problem, cannot be underestimated for computational mathematics.

Two books:

Milstein G.N., Numerical Integration of Stochastic Differential Equations. Kluwer, 1995 (English translation from Russian 1988),
Kloeden P.E. and Platen E., Numerical Solution of Stochastic Differential Equations. Springer, 1992,

present a systematic treatment of mean-square and weak numerical schemes for SDEs. These approximations represent two fundamental aspects in the contemporary theory of numerical integration of SDEs. Mean-square methods are useful for direct simulation of stochastic trajectories which, for instance, can give information on general behavior of a stochastic model. They are the basis for construction of weak methods which are important for many practical applications. Weak methods are sufficient for evaluation of mean values and solving problems of mathematical physics by the Monte Carlo technique, and they are simpler than mean-square ones.

In the present book, numerical integration of SDEs receives a large developmental effort in two directions: a lot of new special schemes are constructed for a number of stochastic systems which are important for applications and for the first time numerical methods for SDEs in bounded domains are proposed. The second part of the book is devoted to construction of stochastic numerical algorithms for solving complicated problems for PDEs, both linear and nonlinear.

The first two chapters contain essentially revised material of the previously mentioned book by the first author, with broad supplements. For instance, a number of effective numerical methods for systems with colored noise are included in these chapters. Another example of the new material is construction of fully implicit mean-square schemes for SDEs with multiplicative noise.

Many difficulties arise with realizing numerical methods for SDEs of a general form. At the same time methods adapted to specific systems can be more efficient than general methods. Very often fluctuations, which affect a physical system, are small. Fortunately, as shown in Chap. 3, in the case of stochastic systems with small noise, it is possible to construct special numerical methods. The errors of these methods are estimated in terms of products $h^i\varepsilon^j$, where h is the step-size of discretization and ε is a small noise parameter. Usually, the global errors in these methods have the form $O(h^j+\varepsilon^k h^l)$, where $j>l,\ k>0$. Thanks to the fact that the accuracy order l of such methods is comparatively small, they are not too complicated, while due to the large j and the small factor ε^k at h^l, their errors are fairly low. This allows us to construct effective (high-exactness) methods with low time-step order which nevertheless have small errors.

In Chap. 4, specific methods for stochastic Hamiltonian systems and Langevin type equations are proposed. Stochastic Hamiltonian systems, like deterministic Hamiltonian systems, possess the property of preserving symplectic structure (symplecticness). For instance, Hamiltonian systems with additive noise are a rather wide and important class of equations having this property. It is well known from deterministic numerical analysis that an effective numerical solution of deterministic Hamiltonian systems on long time intervals requires symplectic methods. It turns out that symplectic methods for stochastic Hamiltonian systems, which are proposed in the first part of Chap. 4, have significant advantages over standard schemes for SDEs.

In the second part of Chap. 4 we construct special numerical methods (we call them quasi-symplectic) for Langevin type equations which have widespread occurrence in models from physics, chemistry, and biology. The proposed methods are such that they degenerate to symplectic methods when the system degenerates to a Hamiltonian one and their law of phase volume contractivity is close to the exact one. The presented numerical tests of both symplectic and quasi-symplectic methods clearly demonstrate superiority of the proposed methods over very long time intervals in comparison with standard ones.

Our probabilistic methods for solving boundary value problems ensure that the proposed approximations of solutions of the corresponding SDEs belong to a bounded domain. Such mean-square approximations are considered

in Chap. 5. A numerical method for simulation of an autonomous diffusion process in a space bounded domain is based on a space discretization using a random walk over small spheres. The algorithm gives points which are close to the points of the real phase trajectory for SDEs. To realize the algorithm, the exit point of the Wiener process from a d-dimensional ball has to be constructed at each step. Due to independence of the first exit time and the first exit point of the Wiener process from the ball, they can be simulated separately. It is known that the exit point is distributed uniformly on the sphere, but simulation of the exit time is a fairly laborious problem. Consequently, the algorithm gives only the phase component of the approximate trajectory without modelling the corresponding time component. The space-time point lies on the d-dimensional lateral surface of a semicylinder with spherical base in the $(d+1)$-dimensional semispace $[t_0, \infty) \times R^d$. The algorithm ensures smallness of the phase increments at each step, but the nonsimulated time increments can take arbitrary large values with some probability. "Ordinary" mean-square methods from Chap. 1, intended to solve SDEs on a finite time interval, are based on a time discretization. The space-time point, corresponding to an "ordinary" one-step approximation constructed at a time point t_k, lies on the d-dimensional plane $t = t_k$, which belongs to the $(d+1)$-dimensional semispace $[t_0, \infty) \times R^d$. The "ordinary" mean-square methods give both time and phase components of the approximate trajectory. They ensure smallness of time increments at each step, but space increments can take arbitrary large values with some probability. In Chap. 5 we also introduce mean-square approximations which control boundedness of both space increments and time increments. In addition they give approximate values for both phase and time components of the space-time diffusion in the space-time bounded domain. The space-time point lies on a bounded d-dimensional manifold. This problem is solved in a constructive manner by the implementation of a space-time discretization with a random walk over boundaries of small space-time parallelepipeds.

Chapter 6 is devoted to random walks related to linear Dirichlet and Neumann boundary value problems for PDEs of elliptic and parabolic type. These random walks are Markov chains. Using them together with the Monte Carlo technique, complex multi-dimensional problems for linear PDEs can be solved. The random walks are constructed on the basis of mean-square and weak approximations for the characteristic system of SDEs due to the corresponding probabilistic representations of the solution to the considered boundary value problem. As in Chap. 5, a certain boundedness of the simulated increments of the Markov chains is necessary here. The proposed algorithms are accompanied by convergence theorems and numerical tests.

Nonlinear PDEs are suggested as mathematical models of problems in many fields such as fluid dynamics, combustion, biochemistry, dynamics of populations, finance, etc. They are mostly investigated by numerical methods, which are traditionally based on deterministic approaches. A probabilistic approach to construction of new numerical methods for solving initial and boundary value problems for nonlinear parabolic PDEs is developed in Chaps. 7 and 8. The approach is based on making use of the well-known probabilistic

representations of solutions of linear PDEs and the idea of SDE numerical integration in the weak sense. Despite their probabilistic nature these methods are nevertheless deterministic. The probabilistic approach takes into account a coefficient dependence on the space variables and a relationship between diffusion and advection in an intrinsic manner. In particular, the layer methods derived allow us to avoid difficulties stemming from essentially changing coefficients and strong advection. A lot of computer experiments were made using the numerical algorithms proposed in Chaps. 7 and 8. Among them are numerical tests on the Burgers equation with small viscosity and on the FKPP-equation. Their results are in a good agreement with the theory. We also present a comparison analysis of the layer methods and the well known finite-difference schemes demonstrating some of the advantages of the proposed methods.

Chapter 9 is devoted to applications of stochastic numerics. Among lots of possibilities, we select applications of constructed stochastic simulation algorithms to such models of stochastic dynamics as systems with stochastic resonance and stochastic ratchets. We demonstrate here both mean-square methods for simulating trajectories of the considered models and weak methods for solving a number of boundary value problems.

An overwhelming majority of the methods proposed in this book are brought to numerical algorithms. Then it only remains to write a computer program, which is usually not complicated, and to use the method in practice. We give some illustrations in the Appendix how our methods can be implemented.

The field of stochastic numerics and its applications is too broad for one book. For example, such important topics as numerical integration of stochastic partial differential equations and of backward stochastic differential equations, being close to interests of the authors, are not covered in the book. The authors do not aim to provide an exhaustive review of literature. As a rule, the references are cited in the course of presentation. Some references are given without comments. On the whole, the content of the book is mainly based on the results obtained by the authors.

Throughout the book we use the hierarchical numbering system: the k-th equation (or the k-th theorem, figure, table, etc) in Sect. j of Chap. i is labelled at the place, where it occurs, and is cited within Chap. i as $(j.k)$ (or Theorem $j.k$, etc); it is cited as $(i.j.k)$ (or Theorem $i.j.k$, etc) outside Chap. i. The equation (theorem, figure, etc) counter is reset at the beginning of each section. The only exception is listings in the Appendix: the k-th listing is labelled and cited as Listing A.k.

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