Introduction

The aim of this book is to provide a survey of algebraic methods useful in the investigation of the structure of graded rings and their modules. The concept of gradation is strongly linked to the notion of degree; for example, graded rings may be viewed as rings generated by certain elements having a "degree" and this degree is a natural number or an integer.

The mother of all graded rings, a polynomial ring in one variable over a field, is graded by $\mathbb{Z}$. But when considering polynomials in more than one, say n variables one has the choice of using the total degree in $\mathbb{Z}$, or a new multi-degree in $\mathbb{Z}^n$ extending the idea of exponent of a variable. In modern language this multi-degree is an element of the ordered group $\mathbb{Z}^n$ with the lexicographical order; this concept plays an essential role in the proof of the fundamental theorem for symmetric polynomials. The concept of degree then extends to other classes of rings, commutative but even noncommutative, and in fact it also underlies the modern treatment of projective varieties in Algebraic Geometry. Another important generalization consists in allowing the degree to take values in abstract groups, for example finite groups, usually not embeddeable into groups of numbers. Then group theory and representation theory of groups enter the picture.

You do not have to know a group to see it act. Perhaps this statement summarizes adequately one of the basic principles of representation theory of groups. Indeed, even before the definition of abstract group had been given, arguments in classical geometry often referred to actions of specific groups, usually symmetry groups of a geometric configuration. The best testament of the faith mathematicians had in the power of group theory within geometry is worded in Klein's "Erlangen Programm". But group actions continued to be successful even on the more abstract side, e.g. Galois Theory featured group actions on abstract algebraic structures in terms of automorphism just to solve down to earth problems related to polynomial equations. The success of the abstract approach induced new ideas concerning continuous groups or Lie groups, and since then the devil of abstract algebra, as in A. Weil's famous quote, is indeed invading the soul of many disciplines in mathematics, most often using group theory to open the door ! The formal treatment of group theory and group representations allowed other algebraic methods to enter the picture, e.g. Ring Theory. Indeed the representation theory of finite groups amounts to the study of modules over the group algebra KG of the group G over the field K. The KG-module structure allows to encode precisely most properties of a G-action on a K-vector space in ring theoretical data. The fact that KG is graded by the group G may be omnipresent in representation theory, the basic theory can be developed completely without stressing that point ... up to a point. That happens when a subgroup H of G is being considered and representations of H and G have to be compared. Hence, in Clifford Theory, certain aspects of graded algebra are more dominantly present. In particular, when H is not normal in G and passing from G to G/H does not really fit into the framework of group theory, then the graded objects e.g. gradation by G-sets etc ..., turn out to be useful exactly because of their generality. It is even possible to keep the philosophy of “action” because a gradation by G may be viewed as an action of the dual algebra $(KG)^*$, a Hopf algebra dual to the Hopf algebra KG for a finite group G. Hence a G-action will be a KG-action and a G-gradation will be a $(KG)^*$-action, or equivalently, a KG-coaction. In the shadow of “Quantum Groups”, abstract Hopf algebras also gained popularity in recent years and therefore several techniques developed in Hopf algebra theory, independent of their equivalent in graded ring theory. A clear case is presented by the use of smash products, originating in a Hopf algebra setting but most effective for the graded theory because the presence of a group allows more concrete interpretations.

Chapter 7 is devoted to the introduction of smash product constructions inspired by the general concept of smash product associated to a Hopf module algebra. In our case the Hopf algebra considered will be the group algebra $k[G]$ over a commutative ring k with respect to an arbitrary group G. Several constructions of the smash product exist, for example depending on the fact whether G is a finite group or not. We have chosen to follow the approach of M. Cohen, S. Montgomery in the first case and D. Quin's in the second, cf. [43], [174] resp. The main idea behind the introduction of smash products associated to graded rings is that the smash product defines a new ring such that the category of graded modules over the graded ring becomes isomorphic to a closed subcategory of modules over the smash product.

At this point let us state that the philosophy underlying graded ring theory is almost contrary to the one of representation theory. Indeed a polynomial ring over a field, is graded by $\mathbb{Z}$ but what will this tell us about $\mathbb{Z}$? The graded methods are not aiming to obtain information about the grading group G, on the contrary the existence of a gradation is used to relate R and R_e, e the unit element of G, or graded to ungraded properties of R. Where possible, knowledge about the structure of G is used.

We may formulate the basic problems in graded algebra as the relational problems between a trio of categories. Indeed, for a graded ring R with respect to a group G we consider the three important categories

i. R-gr, the category of graded left modules

ii. R-mod, the category of left modules

iii. R_e-mod, the category of left modules over the subring R_e of R consisting of the homogeneous elements of degree e where e is the neutral element of G.

These categories are connected by several functors :

Ind : R_e-mod $\to$ R-gr, the induction

Coind : R-gr $\to$ R_e-mod, the coindiction

U : R-gr $\to$ R-mod, the forgetful functor

F : R-mod $\to$ R-gr, the right adjoint of U

Observe that the functors Ind and Coind stem from representation theory; the induction functor Ind is a left adjoint of the functor $(-)_e$: R-gr $\to$ R_e-mod, associating to a graded odule M then R_e-module M_e which is the part of M consisting of elements of degree e, the coinduction functor is a right adjoint adjoint to the same functor. A large part of this book deals with problems relating to the "transfer of structure" via the functors introduced alone. A typical example is presented by the problem of identifying properties of ungraded nature implied by similar (or slightly modified) graded properties.

The material in this book is aimed to have a general applicability and therefore we stress "methods" and avoid specific technical structure theory. Since we strove to make the presentation as self contained as possible, this should make the text particularly useful for graduate students or beginning researchers; however, it should be an asset if some knowledge of a graduate course on general algebra is present e.g. several chapters in P. Cohn's book [45]. For full detail on the category theoretical aspects of Ring Theory we refer to the book by B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975, [181]. For classical notions concerning riings and modules we recommend the reading of F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, Berlin, 1992, [6].

We have chosen to present these methods in suitable category theoretical settings, more specific applications have often been listed as exercises (but with extensive hints of how to solve them, or even with complete solution included). Typical topics include: category of graded rings and graded modules, the structure of simple or injective objects in the category of graded modules, Green theory for strongly graded rings, graded Clifford theory, internal and external homogenization, smash products and related functors, localization theory for graded rings, ...

For more detail on the contents of each separate chapter, we refer to the extra section “Comments and References” at the end of each chapter.