CHAPTER 1

FUNDAMENTALS OF INTERNAL-MODEL-BASED CONTROL THEORY

1.1 Introduction

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories and/or asymptotic rejection of disturbances is a central problem in control theory. There are essentially three different possibilities to approach the problem: tracking by dynamic inversion, adaptive tracking, tracking via internal models. Tracking by *dynamic inversion* consists in computing a precise initial state and a precise control input (or equivalently a reference trajectory of the state), such that, if the system is accordingly initialized and driven, its output exactly reproduces the reference signal. The computation of such control input, however, requires "perfect knowledge" of the entire trajectory which is to be tracked as well as "perfect knowledge" of the model of the plant to be controlled. Thus, this type of approach is not suitable in the presence of large uncertainties on plant parameters as well as on the reference signal. *Adaptive* tracking consists in tuning the parameters of a control input computed via dynamic inversion in such a way as to guarantee asymptotic convergence to zero of a tracking error. This method can successfully handle parameter uncertainties, but still presupposes the knowledge of the entire trajectory which is to be tracked (to be used in the design of the adaptation algorithm) and therefore an approach of this kind is not suited to the problem of tracking unknown trajectories. Of course, one might consider the problem of tracking a slowly varying reference trajectory as a stabilization problem in the presence of a slowly varying unknown parameter, but this would, in most cases, yield a very conservative solution. *Internal-model-based* tracking, on the other hand, is able to handle simultaneously uncertainties in plant parameters as well as in the trajectory which is to be tracked. It has been proven that, if the trajectory to be tracked belongs to the set of all trajectories generated by some fixed dynamical system, a controller which incorporates an internal model of such a system is able to secure asymptotic decay to zero of the tracking error for every possible trajectory in this set and does it robustly with respect to parameter uncertainties. This is in sharp contrast with the two approaches mentioned above, where in lieu of the assumption that a signal is within a class of signals generated by an exogenous system, one instead needs to assume complete knowledge of the past, present and future time history of the trajectory to be tracked. It is for this reason that the internal-model-based approach seems to be the best suited to problems of tracking unknown reference trajectories or rejecting unknown disturbances. The purpose of this chapter is to present the fundamentals of the so-called internal-model-based design methods.

1.2 Asymptotic Tracking and Disturbance Attenuation

A central problem in control theory is the design of feedback controllers so as to have certain outputs of a given plant *to track* prescribed reference trajectories. In any realistic scenario, this control goal has to be achieved in spite of a good number of phenomena which would cause the system to behave differently from that expected. These phenomena could be endogenous, for instance parameter variations, or exogenous, such as additional undesired inputs affecting the behaviour of the plant.

If the plant can be modeled as a linear, finite-dimensional, time-invariant system, the problem in question can be formally cast as follows. Suppose the model of the plant is a set of first-order linear differential equations, written in the form

$$\begin{aligned} \dot{x} &= Ax + B_1 u + B_2 w \\ z &= C_1 x + D_{11} u + D_{12} w \\ y &= C_2 x + D_{21} u + D_{22} w \,, \end{aligned} \tag{1.1}$$

in which x is a vector of state variables, u is a vector of inputs to be used for *control* purposes, w is a vector of inputs which cannot be controlled and thus are viewed as undesired external *disturbances*, z is the vector of outputs that need to be *controlled* and y is a vector of outputs that are available for *measurement*, hence used to feed the device that supplies the control action. Let $z_{\rm ref}(t)$ denote the prescribed behavior, in time, that the controlled output $z(t)$ of (1.1) is required to reproduce. A way to address the design problem described above is to seek a controller, which receives $y(t)$ as input and produces $u(t)$ as output, able to guarantee that, in the resulting closed-loop system, $z(t)$ *asymptotically tracks* $z_{\rm ref}(t)$, i.e.,

$$\lim_{t\to\infty} \|z(t) - z_{\rm ref}(t)\| = 0\,. \tag{1.2}$$

Of course, as a generally accepted prerequisite to this specific design goal, as well as to any other design goal, the controller must also be able to secure a "proper behavior" of all the internal (state) variables which characterize the closed-loop system, not just the components of the controlled output z. A way to express this prerequisite is to impose that all these variables remain *bounded* when $w(t)$ and $z_{\rm ref}(t)$ are bounded, which in turn is automatically guaranteed (the system being linear) by the property of *asymptotic stability.*

The ability to successfully address this problem very much depends on how much the controller is allowed to know about the external stimuli $w(t)$

and $z_{\rm ref}(t)$ and on their specific shape. In the ideal situation in which $w(t)$ and $z_{\rm ref}(t)$ are exactly known, ahead of time, the design problem indeed looks much simpler. This is, though, only an extremely optimistic situation which does not represent, in any circumstance, a realistic scenario. The other extreme situation is the one in which nothing is known about these stimuli, but some loose bounds which they are known to satisfy. In this, pessimistic, scenario the best one could hope for is to guarantee certain ultimate bounds for the distance between $z(t)$ and $z_{\rm ref}(t)$, and not the fulfilment of a sharp goal such as (1.2). A more comfortable, intermediate, situation is the one in which $w(t)$ and $z_{\rm ref}(t)$ are only known *to belong to a fixed family* of functions of time, for instance the family of all solutions obtained from a fixed differential equation as the corresponding initial conditions are allowed to vary on a given set. This situation is in fact sufficiently distant from the ideal but unrealistic case of perfect knowledge of $w(t)$ and $z_{\rm ref}(t)$ and from the realistic but conservative case of almost totally unknown $w(t)$ and $z_{\rm ref}(t)$. But, above all, this way of thinking of the external stimuli covers a number of cases of major practical relevance, as will be seen in the sequel. Once the components of $w(t)$ and $z_{\rm ref}(t)$ have been thought of in these terms, i.e., as members of a family of solutions obtained from a fixed differential equation, there is no reason to keep them separate in the model of the plant. In fact, they can be viewed as components of a larger vector of *exogenous* inputs, written

$$w^{\rm a} = \begin{pmatrix} w \\ z_{\rm ref} \end{pmatrix}.$$

Accordingly, in the model (1.1) the controlled output z can be replaced by the *tracking error*, i.e., by the difference

$$e(t) = z(t) - z_{\rm ref}(t)$$

which, as the equations above show, is itself a linear function

$$e = C_1^{\rm a} x + D_{11}^{\rm a} u + D_{12}^{\rm a} w^{\rm a} \tag{1.3}$$

of the state x, of the control input u and of the (augmented) disturbance $w^{\rm a}$. To say that the various components of the external stimuli can be viewed as members of a family of solutions of a fixed differential equation, which to begin with and to simplify matters is assumed to be a *linear* differential equation, is to say that

$$\dot{w}^{\rm a} = S^{\rm a} w^{\rm a} \tag{1.4}$$

in which $S^{\rm a}$ is a fixed matrix. In this context, system (1.4) is referred to as an *exosystem*. As a matter of fact, as its initial condition $w^{\rm a}(0)$ ranges on some prescribed set $\mathcal{W}$, this system provides a model of all possible exogenous signals to be taken into account in the design problem: reference outputs that the plant might be required to track, as well as disturbance inputs that might affect its behavior.

Cast in these terms, the design problem is that of finding a feedback controller such that, for all initial conditions in the state spaces of the plant and of the controller (if the latter has an internal dynamics), and *for all initial conditions in a prescribed subset* $\mathcal{W}$ *of the state space of the exosystem*, all trajectories of the resulting closed-loop system are bounded (if also are those of (1.4)) and

$$\lim_{t\to\infty} e(t) = 0\,.$$

In the above formulation we have not yet explicitly taken into account another, relevant, source of possible mismatch between the actual behavior of the controlled output $z(t)$ and its prescribed behavior $z_{\rm ref}(t)$: *plant parameter uncertainties.* A conventional, somewhat simplified but effective, way to think of plant parameter uncertainties is to assume that the coefficient matrices of the model (1.1) depend on a vector of *constant, but unknown,* parameters μ, ranging on a prescribed set $\mathcal{P}$. In this way, plant (1.1) can be rewritten, taking into account (1.3) and dropping the superscript "a", as

$$\begin{array}{rcl}\dot x &=& A(\mu)x + B_1(\mu)u + B_2(\mu)w\\ e &=& C_1(\mu)x + D_{11}(\mu)u + D_{12}(\mu)w\\ y &=& C_2(\mu)x + D_{21}(\mu)u + D_{22}(\mu)w\,.\end{array} \tag{1.5}$$

Of course, μ can be regarded as an exogenous input as well, obeying the trivial dynamics

$$\dot\mu = 0\,,$$

and thus aggregated to w, but this would destroy the linearity of the model. For this reason, in dealing with linear systems, this kind of representation is more convenient (it will be dropped, though, in dealing with nonlinear systems, when there is no longer any special reason to keep the roles of μ and w separate, other than for expository purposes).

Again, in these more general terms, the problem in question is that of finding a feedback controller, *independent of* μ, such that, for all initial conditions, in the state spaces of the plant and of the controller, for all initial conditions in a prescribed subset $\mathcal{W}$ of the state space of the exosystem and for all values of μ in a prescribed subset $\mathcal{P}$, the trajectories of the resulting closed-loop system are bounded, if also are those of (1.4), and $e(t)$ converges to 0 as $t \to \infty$.

In the next few sections, we discuss some general results and some constructive procedures for the design of a controller which solves this kind of problem. Then, in the second half of the chapter, we turn our attention to the case of systems which are modeled by possibly *nonlinear* differential equations, i.e., systems which, instead of (1.5), are modeled by equations of the form

$$\begin{array}{rcl}\dot x &=& f(x,u,w,\mu)\\ e &=& h(x,u,w,\mu)\\ y &=& k(x,u,w,\mu)\,,\end{array} \tag{1.6}$$

in which $f(x,u,w,\mu)$, $h(x,u,w,\mu)$ and $k(x,u,w,\mu)$ are nonlinear functions of their arguments, and address similar design problems.

1.3 The Case of Linear Systems

Consider a linear time-invariant system described by equations of the form

$$\begin{aligned}\dot{x} &= Ax + Bu + Pw \\ y &= Cx + Qw \\ e &= C_e x + Q_e w \;.\end{aligned} \tag{1.7}$$

In these equations, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^r$ is a disturbance input, $y \in \mathbb{R}^p$ is the measured output, and $e \in \mathbb{R}^q$ is a tracking error, i.e., the regulated output. The disturbance input w affecting this system is generated by an autonomous linear time-invariant system

$$\dot{w} = Sw \;, \tag{1.8}$$

which, following the terminology introduced in Section 1.2, will be referred to as the *exosystem*.

The control of (1.7) is achieved by means of a dynamic feedback controller, which processes the measured output y and generates the control input u. This controller is itself a linear time-invariant system, modeled by equations of the form

$$\begin{aligned}\dot{\xi} &= F\xi + Gy \\ u &= H\xi + Ky\end{aligned} \tag{1.9}$$

with state $\xi \in \mathbb{R}^\nu$.

The interconnection of (1.7), (1.8) and (1.9), which is an autonomous linear time-invariant system with output e, modeled by equations of the form

$$\begin{aligned}\dot{w} &= Sw \\ \dot{x} &= (A + BKC)x + BH\xi + (P + BKQ)w \\ \dot{\xi} &= F\xi + GCx + GQw \\ e &= C_e x + Q_e w \,,\end{aligned} \tag{1.10}$$

will be in what follows referred to as the *forced closed-loop system.* The special subsystem obtained when the exosystem is disconnected and the output e is ignored, namely the system

$$\begin{aligned}\dot{x} &= (A + BKC)x + BH\xi \\ \dot{\xi} &= F\xi + GCx \,,\end{aligned} \tag{1.11}$$

will be referred to as the *unforced closed-loop system.*

In this and in the following sections, we will discuss some general aspects of the following design problem, which we will refer to as the *generalized tracking problem.* Given system (1.7) with exosystem (1.8) find, if possible, a controller of the form (1.9) such that:

(a) the equilibrium $(x, \xi) = (0, 0)$ of the unforced closed-loop system (1.11) is asymptotically stable,

(b) in the forced closed-loop system (1.10),

$$\lim_{t \to \infty} e(t) = 0$$

for every initial condition $(w(0), x(0), \xi(0))$.

In order to render the discussion as streamlined as possible, it is convenient to introduce from the very beginning a number of *standing assumptions*, some of which are trivially necessary, some of which can be proven to be necessary if certain additional design goals are to be obtained, and some of which carry with them the advantage of a sensibly simpler analysis without excessive compromise in terms of generality.

Assumption 1. The pair (A, B) is stabilizable and the pair (C, A) is detectable. This is a well-known necessary and sufficient condition for the existence of matrices F, G, H, K such that the matrix

$$J = \begin{pmatrix} (A + BKC) & BH \\ GC & F \end{pmatrix} \tag{1.12}$$

has all eigenvalues with negative real part. Thus, this is a trivial necessary condition for the fulfilment of requirement (a) of the problem and need not be discussed further.

Assumption 2. The exosystem (1.8) is stable, in the sense of Lyapunov, forward and backward in time, i.e., both (1.8) and

$$\dot{w} = -Sw$$

are stable in the sense of Lyapunov. The property in question will be referred to as *neutral stability.* This assumption holds if and only if all eigenvalues of S have zero real part and multiplicity one in the minimal polynomial. Thus, in suitable coordinates, S can always be expressed as a skew-symmetric matrix. If this assumption holds, all trajectories of the exosystem (1.8) are bounded in forward time and none of them decays to zero as $t \to \infty$. Boundedness in forward time guarantees that, if requirement (a) of the design problem is fulfilled, then for any $x(0), \xi(0), w(0)$ the trajectory of the forced closed-loop system (1.10) is bounded. In fact, $x(t), \xi(t)$ can be viewed as the response of an asymptotically stable linear system to a bounded input. The non-existence of trajectories of (1.8) which decay to zero as $t \to \infty$ on the other hand, singles-out non-interesting trajectories $w(t)$ for which the fulfilment of requirement (b) would be trivially implied by the fulfilment of requirement (a).

Assumption 3. There exists a $q \times p$ matrix E such that $e = Ey$. This property is usually referred to as the property that "*e is readable from y*" (see [16]). Note that, if this is the case, there is no loss of generality in considering, possibly after a change of coordinates in the output space, a regulated output e of the form

$$e = C_1 x + Q_1 w\,,$$

in which C_1 and Q_1 are obtained from a partition of C and Q as

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \qquad Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}.$$

Consistently, we will set $y_1 = e$ and

$$y_2 = C_2 x + Q_2 w\,.$$

This assumption essentially says that the set of variables which are accessible for measurement includes *all* the components of the regulated output e plus, possibly, an extra set of variables consisting of the components of the vector y_2. This assumption can be proven to be necessary, if certain robustness properties are required to hold (see [15]).

Assumption 4. The number q of components of e is equal to the number m of components of u. This is a very reasonable assumption to consider, if the components of e are viewed as components of a tracking error, in a control problem in which q variables are required to track an equal number of independent reference trajectories. In this case, in fact, the number of control inputs should at least be equal to the number of independent variables to be controlled. This assumption is not indispensable in general, but substantially simplifies the analysis. Note also that this, in the light of Assumption 3, trivially implies $p \geq q$.

We proceed now with the derivation of some general conditions for the solution of the generalized tracking problem, which will be used later in Section 1.5 for the specific design of appropriate control laws. The point of departure is the following result, which establishes a straightforward consequence of the existence of a controller fulfilling requirement (b).

Lemma 1.3.1. *Consider the closed-loop system (1.10) and suppose all the eigenvalues of the matrix (1.12) have negative real part. Then*

$$\lim_{t\to\infty} e(t) = 0$$

for each initial condition $(x(0), \xi(0), w(0))$ if and only if the unique solution pair (Π, Σ) of

$$\begin{aligned} \Pi S &= (A + BKC)\Pi + BH\Sigma + P + BKQ \\ \Sigma S &= F\Sigma + GC\Pi + GQ \end{aligned} \tag{1.13}$$

is such that

$$0 = C_1 \Pi + Q_1 \,. \tag{1.14}$$

Proof. Equation (1.13), written in matrix notation, is an equation of the form

$$\begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} S = \begin{pmatrix} (A+BKC) & BH \\ GC & F \end{pmatrix} \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} + \begin{pmatrix} P+BKQ \\ GQ \end{pmatrix},$$

i.e., is a Sylvester equation. By Assumption 2 all the eigenvalues of S have zero real part, while by the hypothesis of the lemma all the eigenvalues of the matrix (1.12) have negative real part; therefore Equation (1.13) does have a solution pair (Π, Σ), which is unique.

Consider now the coordinate transformation

$$\begin{pmatrix} w \\ \tilde{x} \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ -\Pi & I & 0 \\ -\Sigma & 0 & I \end{pmatrix} \begin{pmatrix} w \\ x \\ \xi \end{pmatrix} = \begin{pmatrix} w \\ x - \Pi w \\ \xi - \Sigma w \end{pmatrix}$$

and note that, in the new coordinates thus defined, the equations of the closed-loop system (1.10) assume the form

$$\begin{aligned} \dot{w} &= Sw \\ \dot{\tilde{x}} &= (A+BKC)(\tilde{x} + \Pi w) + BH(\tilde{\xi} + \Sigma w) + (P+BKQ)w - \Pi S w \\ \dot{\tilde{\xi}} &= F(\tilde{\xi} + \Sigma w) + GC(\tilde{x} + \Pi w) + GQw - \Sigma S w \,, \end{aligned}$$

which in view of (1.13) become

$$\begin{aligned} \dot{w} &= Sw \\ \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\xi}} \end{pmatrix} &= \begin{pmatrix} (A+BKC) & BH \\ GC & F \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\xi} \end{pmatrix} = J \begin{pmatrix} \tilde{x} \\ \tilde{\xi} \end{pmatrix}. \end{aligned}$$

In the new coordinates, the regulated variable e reads as

$$e = C_1 \tilde{x} + (C_1 \Pi + Q_1) w.$$

Integrating system (1.10) in the new coordinates yields

$$\begin{pmatrix} \tilde{x}(t) \\ \tilde{\xi}(t) \end{pmatrix} = e^{Jt} \begin{pmatrix} \tilde{x}(0) \\ \tilde{\xi}(0) \end{pmatrix}, \qquad w(t) = e^{St} w(0)$$

and therefore

$$e(t) = \begin{pmatrix} C_1 & 0 \end{pmatrix} e^{Jt} \begin{pmatrix} \tilde{x}(0) \\ \tilde{\xi}(0) \end{pmatrix} + (C_1 \Pi + Q_1) e^{St} w(0).$$

Since J has all eigenvalues with negative real part, the condition $\lim_{t\to\infty} e(t) = 0$ holds, for every $(w(0), \tilde{x}(0), \tilde{\xi}(0),)$, if and only if

$$\lim_{t\to\infty}(C_1\Pi + Q_1)e^{St} = 0$$

and this in turn occurs if and only if $C_1\Pi + Q_1 = 0$, because by Assumption 2 all the eigenvalues of S have nonnegative real part. This proves the lemma. $\lhd$

Remark 1.3.1. The construction described in the proof of this lemma can be given a simple and expressive geometric interpretation. Rewrite the equations of the closed-loop system (1.10) in the form

$$\dot{x}_{cl} = A_{cl}x_{cl}$$

with

$$x_{cl} = \begin{pmatrix} w \\ x \\ \xi \end{pmatrix}, \qquad A_{cl} = \begin{pmatrix} S & 0 \\ \Theta & J \end{pmatrix},$$

where

$$\Theta = \begin{pmatrix} P + BKQ \\ GQ \end{pmatrix}.$$

The $(r+n+\nu)\times(r+n+\nu)$ matrix A_{cl} has r eigenvalues with zero real part (those of S) and $n+\nu$ eigenvalues with negative real part (those of J). Let $\mathcal{V}^0$ denote the invariant subspace of A_{cl} associated with the eigenvalues with zero real part, and $\mathcal{V}^{\mathrm{s}}$ denote the invariant subspace of A_{cl} associated with the eigenvalues with negative real part.

It is immediate to realize that $\mathcal{V}^{\mathrm{s}}$ is spanned by the columns of the matrix

$$T^{\mathrm{s}} = \begin{pmatrix} 0 \\ I_{n+\nu} \end{pmatrix}.$$

In fact, the subspace spanned by the columns of T^{s} is invariant under A_{cl} and the restriction of A_{cl} to this subspace is precisely J.

The subspace $\mathcal{V}^0$, being complementary to $\mathcal{V}^{\mathrm{s}}$ in $\mathbb{R}^{r+n+\nu}$, will be spanned by the columns of a matrix T^0 of the form

$$T^0 = \begin{pmatrix} I_r \\ X \end{pmatrix},$$

in which X is a suitable matrix. It is immediate to realize that

$$X = \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix}$$

with (Π, Σ) the unique solution of (1.13). In fact, to impose the condition that the subspace spanned by the columns of T^0 is invariant under A_{cl} is to impose that for each $w \in \mathbb{R}^r$ there exists $\widetilde{w} \in \mathbb{R}^r$ satisfying

$$\begin{pmatrix} S & 0 \\ \Theta & J \end{pmatrix}\begin{pmatrix} I_r \\ X \end{pmatrix} w = \begin{pmatrix} I_r \\ X \end{pmatrix} \widetilde{w}.$$

Expanding the calculations, one obtains that necessarily

$$\widetilde{w} = Sw$$

and

$$\Theta + JX = XS\,.$$

The latter is an equation for X which coincides with (1.13). Thus the unique solution (Π, Σ) of (1.13) is such that the subspace spanned by the columns of

$$T^0 = \begin{pmatrix} I_r \\ \Pi \\ \Sigma \end{pmatrix},$$

is invariant under A_{cl}. Moreover, the previous calculation also shows that the restriction of A_{cl} to this invariant subspace is precisely S. Therefore, the columns of this matrix span the subspace $\mathcal{V}^0$.

In other words, condition (1.13) expresses the existence of an invariant subspace, for the closed-loop system (1.10), which has the form

$$\mathcal{V}^0 = \{(w, x, \xi) : x = \Pi w, \xi = \Sigma w\}$$

and on which the restriction of (1.10) reduces to

$$\dot{w} = Sw.$$

Condition (1.14), on the other hand, expresses the fact that the regulated output $e = C_1 x + Q_1 w$ is zero at each point of this invariant subspace.

From this interpretation it is rather easy to deduce why $e(t)$ converges to 0 as $t \to \infty$ if and only if the unique solution of (1.13) satisfies (1.14). The necessity derives from the fact that if the initial condition $(w(0), x(0), \xi(0))$ of (1.10) is in $\mathcal{V}^0$, the corresponding trajectory, which remains in $\mathcal{V}^0$ for all future times and is a copy of a trajectory of the exosystem, cannot converge to $(0, 0, 0)$ because the exosystem is neutrally stable. Thus, the only possibility of having $e(t) \to 0$ is that e, as a function of x and w, is zero at any point of $\mathcal{V}^0$. The sufficiency stems from the fact that all trajectories of (1.10) converge, as $t \to \infty$, to $\mathcal{V}^0$ and hence produce an error which asymptotically decays to zero. $\triangleleft$

We give now the condition established in this lemma a slightly different version, which is more useful in the sequel. First of all, split G and K (consistently with the partition of y) as

$$G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}, \qquad K = \begin{pmatrix} K_1 & K_2 \end{pmatrix},$$

in which case the controller (1.9) is rewritten as

$$\begin{aligned} \dot{\xi} &= F\xi + G_1 e + G_2 y_2 \\ u &= H\xi + K_1 e + K_2 y_2\,. \end{aligned} \tag{1.15}$$

Then, we have the following result.

Proposition 1.3.2. *Suppose the controller (1.15) stabilizes (1.7). Then, if*

$$\lim_{t\to\infty} e(t) = 0$$

there exist matrices Π, Σ, R such that

$$\begin{aligned} \Pi S &= A\Pi + BR + P \\ 0 &= C_1\Pi + Q_1 \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} \Sigma S &= F\Sigma + G_2(C_2\Pi + Q_2) \\ R &= H\Sigma + K_2(C_2\Pi + Q_2)\,. \end{aligned} \tag{1.17}$$

Proof. By Lemma 1.3.1, if $\lim_{t\to\infty} e(t) = 0$, the unique solution (Π, Σ) of (1.13) satisfies (1.14). The two equations of (1.13), rewritten as

$$\begin{aligned} \Pi S &= A\Pi + BK_1C_1\Pi + BK_2C_2\Pi + BH\Sigma + P + BK_1Q_1 + BK_2Q_2 \\ \Sigma S &= F\Sigma + G_1C_1\Pi + G_2C_2\Pi + G_1Q_1 + G_2Q_2\,, \end{aligned}$$

in the light of (1.14) reduce to

$$\begin{aligned} \Pi S &= A\Pi + BK_2C_2\Pi + BH\Sigma + P + BK_2Q_2 \\ \Sigma S &= F\Sigma + G_2C_2\Pi + G_2Q_2\,. \end{aligned}$$

Setting

$$R = H\Sigma + K_2(C_2\Pi + Q_2)$$

proves that (1.16) and (1.17) necessarily hold. ◁

Remark 1.3.2. Note that the parameters of the controller do not enter into equations (1.16). Thus, the existence of some solution pair Π, R can be seen as a necessary condition that a given plant must obey for the existence of *any* controller that solves the generalized tracking problem. On the other hand, Equations (1.17) express the ability, of a specific controller, to secure asymptotic decay to zero of the error for a given plant. ◁

Equations (1.16) are often referred to as the *regulator equations.* The first one of these expresses the property that the subspace

$$\mathcal{S} = \{(x, w) : x = \Pi w\}$$

is a *controlled invariant subspace* for the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u\,. \tag{1.18}$$

In fact, $\mathcal{S}$ is by construction invariant for

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \left[\begin{pmatrix} A & P \\ 0 & S \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \begin{pmatrix} 0 & R \end{pmatrix} \right] \begin{pmatrix} x \\ w \end{pmatrix} \tag{1.19}$$

which is the autonomous linear system obtained controlling (1.18) with the "feedback" law $u = Rw$. The second one, on the other hand, expresses the property that the regulated output is zero at each point of the subspace $\mathcal{S}$.

This simple geometric interpretation highlights the roles that Π and R have in the solution of the generalized problem of tracking. In fact, consider again system (1.7), pick an initial condition $w(0)$ for the exosystem (1.8), and suppose the following:

- the initial condition $x(0)$ of (1.7) is equal to $x(0) = \Pi w(0)$,
- the control input $u(t)$ of (1.7) is equal to $u(t) = Rw(t)$.

Then, it is immediate to conclude that $e(t) = 0$ for all $t \in \mathbb{R}$. To see that this is the case, it suffices to look at system (1.19) and pick any initial condition on $\mathcal{S}$, i.e., a condition satisfying $x(0) = \Pi w(0)$. Since the latter is invariant, we have $x(t) = \Pi w(t)$ for all $t \in \mathbb{R}$. By construction, the trajectory thus found (which is a trajectory of a *closed-loop system*, namely system (1.19)) is such that $x(t)$ can be interpreted as response of an *open-loop system*, namely system (1.7) itself, to the control input $u(t) = Rw(t)$ and to the disturbance input $w(t)$. Since the regulated output is zero at any point of $\mathcal{S}$ and $(x(t), w(t))$ remains in $\mathcal{S}$ for all $t \in \mathbb{R}$, it is concluded that $e(t) = 0$ for all $t \in \mathbb{R}$. In other words, $u(t) = Rw(t)$ can be seen as a *feedforward input* capable of keeping $e(t)$ identically at zero if the initial condition of (1.7) is appropriately set. In essence, all methods for solving the generalized problem of tracking, which will be discussed in the following, stem from this simple interpretation. As a matter of fact, the focus of the various design procedures is on how to asymptotically reproduce the control input $Rw(t)$ (as the latter *is not available* in real time) and to asymptotically approach the subspace $\mathcal{S}$.

Equations (1.17), which we like to refer to as the *internal model property*, express the fact that the control input $Rw(t)$ in question can be viewed as generated by the autonomous finite-dimensional linear dynamical system

$$\begin{aligned} \begin{pmatrix} \dot{\xi} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} F & G_2(C_2\Pi + Q_2) \\ 0 & S \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix} \\ u &= \begin{pmatrix} H & K_2(C_2\Pi + Q_2) \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix}. \end{aligned} \tag{1.20}$$

In fact, the first equation of (1.17) expresses nothing else than the property that the subspace

$$\mathcal{R} = \{(\xi, w) : \xi = \Sigma w\}$$

is an invariant subspace for (1.20). Thus, if $\xi(0) = \Sigma w(0)$, then $\xi(t) = \Sigma w(t)$ for all $t \in \mathbb{R}$. As a consequence, for those initial conditions, the output $u(t)$ of (1.20) becomes

$$u(t) = H\Sigma w(t) + K_2(C_2\Pi + Q_2)w(t) = Rw(t)$$

as the second equation of (1.17) holds.

Equations (1.17) essentially express the property that, embedded in the controller (1.15), there is a generator (an *internal model*) for those control inputs $Rw(t)$ which are capable of keeping the state of (1.7) – (1.8) evolving on the subspace $\mathcal{S}$ and, accordingly, keeping the regulated output $e(t)$ identically at zero. Indeed, since the y_1 component of the measured output vanishes when e is zero, system (1.20) only reflects the behavior of the controller when the latter is driven by the restriction of y_2 to $\mathcal{S}$.

Proposition 1.3.2 has a natural converse.

Proposition 1.3.3. *Suppose a controller of the form (1.15) stabilizes (1.7) and, for some triplet of matrices Π, Σ, R, conditions (1.16) and (1.17) hold. Then this controller solves the generalized tracking problem.*

Proof. Consider the closed-loop system, which has the form

$$\begin{aligned}
\dot{w} &= Sw \\
\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= \begin{pmatrix} A + BK_1C_1 + BK_2C_2 & BH \\ G_1C_1 + G_2C_2 & F \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \\
&\qquad + \begin{pmatrix} P + BK_1Q_1 + BK_2Q_2 \\ G_1Q_1 + G_2Q_2 \end{pmatrix} w \,.
\end{aligned}$$

By hypothesis, the matrix

$$J = \begin{pmatrix} A + BK_1C_1 + BK_2C_2 & BH \\ G_1C_1 + G_2C_2 & F \end{pmatrix}$$

has all eigenvalues with negative real part. As a consequence, the Sylvester equation

$$\begin{pmatrix} \hat{\Pi} \\ \hat{\Sigma} \end{pmatrix} S = \begin{pmatrix} A + BK_1C_1 + BK_2C_2 & BH \\ G_1C_1 + G_2C_2 & F \end{pmatrix} \begin{pmatrix} \hat{\Pi} \\ \hat{\Sigma} \end{pmatrix} + \begin{pmatrix} P + BK_1Q_1 + BK_2Q_2 \\ G_1Q_1 + G_2Q_2 \end{pmatrix} \tag{1.21}$$

has a unique solution $\hat{\Pi}, \hat{\Sigma}$. Using (1.16) and (1.17), it is trivial to check that Π and Σ do provide a solution of this equation. Thus, necessarily

$$\hat{\Pi} = \Pi \,, \qquad \hat{\Sigma} = \Sigma \,.$$

Using again (1.16), we deduce that the unique solution of (1.21) is such that

$$C_1\hat{\Pi} + Q_1 = 0$$

and this, in view of Lemma 1.3.1 in its sufficient part, proves that $\lim_{t\to\infty} e(t) = 0$, i.e., proves the proposition. ◁

Propositions 1.3.2 and 1.3.3 together provide a necessary and sufficient condition for the existence of a controller which solves the generalized tracking problem. They are not yet usable for design, though, because they do not describe how a controller meeting all these conditions can be constructed. This issue will be dealt with in the following sections. For the time being, we discuss the particular form to which these conditions reduce in the special case in which $e = y$ and we discuss the important issue of the robustness.

In the special case in which $e = y$, the controller (1.15) reduces to a controller of the form

$$\begin{aligned} \dot{\xi} &= F\xi + G_1 e \\ u &= H\xi + K_1 e\,, \end{aligned} \tag{1.22}$$

and the two Propositions 1.3.2 and 1.3.3 together yield the following.

Corollary 1.3.4. *Suppose a controller of the form (1.22) stabilizes (1.7). This controller solves the generalized tracking problem if and only if there exist matrices Π, Σ, R such that*

$$\begin{aligned} \Pi S &= A\Pi + BR + P \\ 0 &= C_1\Pi + Q_1 \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} \Sigma S &= F\Sigma \\ R &= H\Sigma\,. \end{aligned} \tag{1.24}$$

Remark 1.3.3. It is worth stressing that the simpler structure assumed in this case by the internal model property (1.17), namely form (1.24), may occur also in cases in which the measured output y does not consist only of e, but contains a nontrivial set of extra variables y_2. For the special form (1.24) to hold, in fact, it suffices that the solution Π of the regulator equations (1.23) is such that $C_2\Pi + Q_2 = 0$. $\triangleleft$

1.4 The Issue of Robustness

As observed at the end of Section 1.2, a good design should also take into account the fact that the model of the controlled plant is subject to (at least parametric) uncertainties. That is, the design has to take into account the fact that the quintuplet of matrices A, B, P, C, Q which characterize the model (1.7) (matrices C_e and Q_e are submatrices of C and Q, in the light of Assumption 3) depend on a vector μ of uncertain parameters, ranging on a given set $\mathcal{P}$. On the other hand, the controller has to be independent of μ, whose actual value is unknown, and this entails some relevant consequences on the analysis presented in the previous section.

In what follows, we consider a μ-dependent plant

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C_1(\mu)x + Q_1(\mu)w \\ y_2 &= C_2(\mu)x + Q_2(\mu)w\,. \end{aligned} \tag{1.25}$$

and say that the (μ-independent) controller (1.9) is *robust* if it is able to solve the generalized tracking problem for any value of $\mu \in \mathcal{P}$. Note that in this formulation we have implicitly assumed that the exosystem (1.8) is not subject to parameter uncertainties. The presence of parametric uncertainties in the exosystem is a delicate problem, which requires a separate analysis and will be dealt with in Section 1.6.

Since a controller that solves a generalized tracking problem in particular asymptotically stabilizes the plant, a trivial necessary condition for the existence of a robust controller is the existence of a robust stabilizer. We postulate this as an additional standing assumption, which strengthens Assumption 1 in a obvious way.

Assumption 1′. There exist matrices F, G, H, K such that, for all $\mu \in \mathcal{P}$, all the eigenvalues of the matrix

$$J = \begin{pmatrix} (A(\mu) + B(\mu)KC(\mu)) & B(\mu)H \\ GC(\mu) & F \end{pmatrix} \tag{1.26}$$

have negative real part.

We do not pursue here the subject of determining the existence and providing the construction of a robust stabilizer, which are topics addressed in many other works, and extend beyond the scope of this book. We limit ourselves to record the necessity of the existence of one such robust stabilizer, and proceed with the analysis of the consequence of robustness of the various developments presented in Section 1.3. Indeed, the results of Propositions 1.3.2 and 1.3.3 provide the following immediate consequence.

Proposition 1.4.1. *Suppose a controller of the form (1.15) robustly stabilizes (1.25). This controller is robust if and only if, for every value $\mu \in \mathcal{P}$, there exist matrices Π, Σ, R such that*

$$\begin{aligned} \Pi S &= A(\mu)\Pi + B(\mu)R + P(\mu) \\ 0 &= C_1(\mu)\Pi + Q_1(\mu) \end{aligned} \tag{1.27}$$

and

$$\begin{aligned} \Sigma S &= F\Sigma + G_2(C_2(\mu)\Pi + Q_2(\mu)) \\ R &= H\Sigma + K_2(C_2(\mu)\Pi + Q_2(\mu))\,. \end{aligned} \tag{1.28}$$

It is worth observing that the three matrices Π, Σ, R, which are requested to solve Equations (1.27) and (1.28), need not be independent of μ. To stress this we may, occasionally, use the more explicit (but heavier) notation $\Pi(\mu), \Sigma(\mu), R(\mu)$.

The existence, for each μ, of a solution pair Π, R of the first set of Equations (1.27), namely the regulator equations, can be given a very simple, and expressive, characterization if it is assumed that *all entries* of the various coefficient matrices in this equation are susceptible to *independent variations*, i.e., if the uncertain parameter μ is identified with the quintuplet

$$\{A, B, P, C_1, Q_1\} \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times r} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times r},$$

and $\mathcal{P}$ is a set having nonempty interior (in the topology of the vector space where this quintuplet is defined). In this case, in fact, to say that Equations (1.27) have solution for all $\mu \in \mathcal{P}$ is to say that equations of the form

$$\begin{aligned} \Pi S &= A\Pi + BR + P \\ 0 &= C_1\Pi + Q_1 \end{aligned} \tag{1.29}$$

have solutions for all $\{A, B, P, C_1, Q_1\}$ in a set having nonempty interior. For this to be the case, there is a simple condition available.

Lemma 1.4.2. *Equations (1.29) have solution for all* $\{A, B, P, C_1, Q_1\}$ *in an open set if and only if*

$$\det\begin{pmatrix} A-\lambda I & B \\ C_1 & 0 \end{pmatrix} \neq 0$$

for all λ *which are eigenvalues of* S*. If this is the case, the solution is unique.*

Proof. Let A, B, C_1, S be fixed matrices and consider the map

$$\mathcal{R} \;:\; \mathbb{R}^{n\times r} \times \mathbb{R}^{m\times r} \quad\to\quad \mathbb{R}^{n\times r} \times \mathbb{R}^{m\times r}$$

defined as follows:

$$\mathcal{R}(\Pi, R) = (\Pi S - A\Pi - BR, -C\Pi)\,.$$

To say that Π, R is a solution of (1.29) is to say that

$$\mathcal{R}(\Pi, R) = (P, Q_1)\,. \tag{1.30}$$

The map $\mathcal{R}$ is a linear map and it is known that this map is invertible if and only if the condition of the lemma holds (see [21] and [39]). Invertibility means that for any pair (P, Q_1) there is one and only one pair (Π, R) such that (1.30) holds and this indeed proves that the condition in question is sufficient. To show that the condition is necessary, pick any quintuplet $\{A, B, P, C_1, Q_1\}$ in the open set $\mathcal{Q}$ where these quintuplets are suppose to range. Given any pair $(\bar{P}, \bar{Q}_1) \in \mathbb{R}^{n\times r} \times \mathbb{R}^{m\times r}$, if ε is sufficiently small,

$$\{A, B, P + \varepsilon\bar{P}, C_1, Q_1 + \varepsilon\bar{Q}_1\} \in \mathcal{Q}\,,$$

because $\mathcal{Q}$ is open. Equations (1.29) must be solved for all those ε and, as a simple calculation shows, the corresponding solution can be put in the form

$$\begin{aligned} \Pi &= \Pi_0 + \varepsilon\bar{\Pi} \\ R &= R_0 + \varepsilon\bar{R} \end{aligned}$$

in which $\bar{\Pi}, \bar{R}$ are such that

$$\mathcal{R}(\bar{\Pi}, \bar{R}) = (\bar{P}, \bar{Q}_1)\,. \tag{1.31}$$

The pair $(\bar{P}, \bar{Q}_1)$ being arbitrary, this proves that the linear map $\mathcal{R}$ is onto its range and, since domain and range of the map have the same dimension, the map is invertible. Hence, the condition in the lemma, a necessary condition for $\mathcal{R}$ to be invertible, follows. ◁

Motivated by this, it is natural to add the following new assumption.

Assumption 5. For any $\mu \in \mathcal{P}$, the matrix

$$\begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C_1(\mu) & 0 \end{pmatrix} \tag{1.32}$$

is nonsingular for all λ which are eigenvalues of S. This assumption will be referred to as *nonresonance condition*.

It follows trivially from the previous discussion that if this assumption holds then Equations (1.27) do have a solution for every $\mu \in \mathcal{P}$, as required. However, it should be also clear that the assumption in question, strictly speaking, is not necessary, unless the entries of the various coefficient matrices are free to independently vary, which may not be the case in a specific situation. It does, indeed, simplify the discussion and it is for this reason that it is convenient to take it as an explicit standing assumption.

Remark 1.4.1. It is also important to stress that the roots of the determinant of (1.32) coincide with the so-called *transmission zeros* of the system

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u \\ e &= C_1(\mu)x\,, \end{aligned} \tag{1.33}$$

if the latter is controllable and observable. Thus, Assumption 5 is simply the assumption that, for any $\mu \in \mathcal{P}$, none of the eigenvalues of the exosystem (1.8) is a transmission zero of (1.33). ◁

If the regulator Equations (1.27) are known to have a solution for every μ (which is indeed the case, as shown above, if Assumption 5 holds), it is not difficult to construct a controller such that the second set of equations in Proposition 1.4.1 is also solved, i.e., such that the internal model property holds. To this end, let

$$m(\lambda) = \lambda^s + a_{s-1}\lambda^{s-1} + \cdots + a_1\lambda + a_0$$

denote the minimal polynomial of the matrix S. It is well known that

$$S^s + a_{s-1}S^{s-1} + \cdots + a_1 S + a_0 I = 0$$

and, hence, for any $m \times s$ matrix V,

$$VS^s = -(a_1 VS^{s-1} + \cdots + a_1 VS + a_0 V)\,. \tag{1.34}$$

Consider now the pair of matrices

$$\Phi = \begin{pmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & I \\ -a_0 I & -a_1 I & -a_2 I & \cdots & -a_{s-2} I & -a_{s-1} I \end{pmatrix} \tag{1.35}$$

$$\Gamma = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Using (1.34), it is immediate to check that the matrix

$$T_S(V) = \begin{pmatrix} V \\ VS \\ \cdots \\ VS^{s-2} \\ VS^{s-1} \end{pmatrix} \tag{1.36}$$

satisfies

$$\begin{aligned} T_S(V)S &= \Phi T_S(V) \\ V &= \Gamma T_S(V)\,. \end{aligned}$$

Note that Φ, Γ depend only on the coefficients of the minimal polynomial of S and not on the parameter μ, while $T_S(V)$ may depend on this parameter if V does. Using these matrices, it is not difficult to construct a controller such that the internal model property (1.28) holds.

Proposition 1.4.3. *Let $\Pi(\mu), R(\mu)$ be a solution of (1.27). Suppose the matrices F, G_1, G_2, H in (1.15) have the form*

$$F = \begin{pmatrix} \Phi & 0 \\ 0 & L \end{pmatrix}, \qquad G_1 = \begin{pmatrix} \Theta_1 \\ M_1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} \Theta_2 \\ M_2 \end{pmatrix}, \qquad H = \begin{pmatrix} \Gamma & N \end{pmatrix}$$

with Φ, Γ as in (1.35). Suppose also that none of the eigenvalues of L is an eigenvalue of S and that

$$\Theta_2(C_2(\mu)\Pi(\mu) + Q_2(\mu)) = 0 \tag{1.37}$$

for all $\mu \in \mathcal{P}$. Then, there exists a matrix $\Sigma(\mu)$ such that $\Pi(\mu), \Sigma(\mu), R(\mu)$ is a solution of (1.28).

Proof. Set

$$\Sigma = \begin{pmatrix} T_S(V) \\ W \end{pmatrix}.$$

Then, it is easily seen that for a controller of the above structure, equations (1.28) reduce to (we drop, for convenience, the dependence on μ)

$$\begin{aligned} T_S(V)S &= \Phi T_S(V) + \Theta_2(C_2\Pi + Q_2) \\ WS &= LW + M_2(C_2\Pi + Q_2) \\ R &= \Gamma T_S(V) + NW + K_2(C_2\Pi + Q_2)\,. \end{aligned}$$

The second of these is a Sylvester equation in W which, as none of the eigenvalues of S is an eigenvalue of L, has indeed a unique solution W (which may depend on μ, as $C_2\Pi + Q_2$ does). Define now the (possibly μ-dependent) matrix

$$V = R - NW - K_2(C_2\Pi + Q_2)\;.$$

In this case, the last equation becomes

$$V = \Gamma T_S(V)\,,$$

while the first one, since by hypothesis $\Theta_2(C_2\Pi + Q_2) = 0$, reduces to

$$T_S(V)S = \Phi T_S(V)\,.$$

Thus, the matrix Σ solves (1.28). ◁

This result essentially states that, if the regulator Equations (1.27) have a solution for all μ (which is anyway a necessary condition the plant must fulfil for the existence of a controller that solves the generalized problem of tracking), any controller in which F, G_1, G_2, H have the form indicated, under the only hypotheses that none of the eigenvalues of S is an eigenvalue of L and that condition (1.37) holds, is such that both sets of Equations (1.27) and (1.28) of Proposition 1.4.1 have a solution. Thus, as this proposition says, if the degrees of freedom left in the design, namely the matrices

$$L, M_1, M_2, N, K_1, K_2, \Theta_1, \Theta_2$$

can be chosen so as to robustly stabilize the plant (1.25), this controller is a robust controller, i.e., it solves the generalized tracking problem for every value of $\mu \in \mathcal{P}$. Various options for the choice of these matrices will be discussed in the next section. For the time being, we give a result which shows that the structure of the controller suggested in Proposition 1.4.1 is, to some extent, a mandated choice for fulfilment of the conditions of Proposition 1.4.3.

As a matter of fact, consider the same scenario in which Assumption 5 is necessary for the existence of a robust controller, namely the case in which the uncertain parameter μ can be identified with the quintuplet $\{A, B, P, C_1, Q_1\}$ and $\mathcal{P}$ is a set having nonempty interior. Then, it is possible to prove that the second set of equations in Proposition 1.4.1, namely the internal model property, entails an important consequence on the structure of any robust controller.

Proposition 1.4.4. *A controller is robust only if, for every $m \times r$ matrix V, there exists a $\nu \times r$ matrix T such that*

$$\begin{aligned} TS &= FT \\ V &= HT\,. \end{aligned} \tag{1.38}$$

Proof. Pick, as in the proof of Lemma 1.4.2, a quintuplet of the form

$$\{A, B, P + \varepsilon\bar{P}, C_1, Q_1 + \varepsilon\bar{Q}_1\}\,.$$

with $\{A, B, P, C_1, Q_1\}$ in the interior of $\mathcal{P}$. Equations (1.16) and (1.17) must have a solution Π, Σ, Γ for all small ε and a simple calculation shows that these solutions can be put in the form

$$\begin{aligned} \Pi &= \Pi_0 + \varepsilon\bar{\Pi} \\ \Sigma &= \Sigma_0 + \varepsilon\bar{\Sigma} \\ R &= R_0 + \varepsilon\bar{R} \end{aligned}$$

in which $\bar{\Pi}, \bar{R}$ are such that (1.31) holds and

$$\begin{aligned} \bar{\Sigma}S &= F\bar{\Sigma} + G_2C_2\bar{\Pi} \\ \bar{R} &= H\bar{\Sigma} + K_2C_2\bar{\Pi}\,. \end{aligned} \tag{1.39}$$

Since the map $\mathcal{R}$ is invertible, as shown in the proof of Lemma 1.4.2, it is always possible to find $(\bar{P}, \bar{Q}_1)$ such that the unique pair $(\bar{\Pi}, \bar{R})$ which solves (1.31) is such that $\bar{\Pi}$ satisfies $C_2\bar{\Pi} = 0$, while $\bar{R}$ coincides with any arbitrary matrix V. In this case, Equations (1.39) reduce to

$$\bar{\Sigma}S = F\bar{\Sigma}\,, \qquad V = H\bar{\Sigma}\,,$$

and this proves the proposition. ◁

We conclude the section with an obvious corollary of Proposition 1.4.3, which considers the case in which the matrix F in (1.15) has the form

$$F = \begin{pmatrix} \Phi & \Delta \\ 0 & L \end{pmatrix}\,.$$

In this case, if none of the eigenvalues of L is an eigenvalue of S, the Sylvester equation

$$ZL = \Phi Z + \Delta \tag{1.40}$$

has a (unique) solution Z. In fact, by construction, the characteristic polynomial of Φ is m-times the minimal polynomial of S, hence none of the eigenvalues of L is an eigenvalue of Φ. The matrix Z can be used to construct a change of coordinates, in the state space of (1.15), that brings F to the form considered in Proposition 1.4.3.

Corollary 1.4.5. *Let $\Pi(\mu), R(\mu)$ be a solution of (1.27). Suppose the matrices F, G_1, G_2, H in (1.15) have the form*

$$F = \begin{pmatrix} \Phi & \Delta \\ 0 & L \end{pmatrix}, \qquad G_1 = \begin{pmatrix} \Theta_1 \\ M_1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} \Theta_2 \\ M_2 \end{pmatrix}, \qquad H = \begin{pmatrix} \Gamma & N \end{pmatrix}$$

with Φ, Γ as in (1.35). Suppose also that none of the eigenvalues of L is an eigenvalue of S and that

$$(\Theta_2 - ZM_2)\,(C_2(\mu)\Pi(\mu) + Q_2(\mu)) = 0 \tag{1.41}$$

for all $\mu \in \mathcal{P}$. Then, there exists a matrix $\Sigma(\mu)$ such that $\Pi(\mu), \Sigma(\mu), R(\mu)$ is a solution of (1.28).

1.5 Design Methods for Linear Systems

The discussion in the previous section has identified a potential structure of a controller that solves the generalized tracking problem, and appropriate conditions under which this controller is robust. To summarize this discussion, suppose (see Assumption 5) that for every $\mu \in \mathcal{P}$ the matrix (1.32) is nonsingular for all λ which are eigenvalues of S and observe that, if this is the case, for every $\mu \in \mathcal{P}$ the regulator Equations (1.27) have a unique solution $\Pi(\mu), R(\mu)$. Then, the results of the previous section yield the following conclusion.

Proposition 1.5.1. *Consider system (1.25). Suppose that, for all $\mu \in \mathcal{P}$,*

$$\det \begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C_1(\mu) & 0 \end{pmatrix} \neq 0 \tag{1.42}$$

for all λ which are eigenvalues of S. Let $\Pi(\mu), R(\mu)$ be the unique solution of (1.27) and suppose Θ is a matrix satisfying

$$\Theta(C(\mu)\Pi(\mu) + Q(\mu)) = 0\,, \tag{1.43}$$

for all $\mu \in \mathcal{P}$. Suppose L, M, N, K is a quadruplet of matrices, in which none of the eigenvalues of L is an eigenvalue of S, such that, for all $\mu \in \mathcal{P}$, all the eigenvalues of the matrix

$$\begin{pmatrix} A(\mu) + B(\mu)KC(\mu) & B(\mu)\Gamma & B(\mu)N \\ \Theta C(\mu) & \Phi & 0 \\ MC(\mu) & 0 & L \end{pmatrix} \tag{1.44}$$

have negative real part. Then, the controller

$$\begin{aligned} \dot\xi' &= \Phi\xi' + \Theta y \\ \dot\xi'' &= L\xi'' + My \\ u &= \Gamma\xi' + N\xi'' + Ky\,. \end{aligned} \tag{1.45}$$

is a robust controller.

Proof. By hypothesis, controller (1.45) stabilizes (1.25) for all $\mu \in \mathcal{P}$. Since the hypotheses of Proposition 1.4.3 hold, there is a matrix $\Sigma(\mu)$ solving also the set of Equations (1.28) for all $\mu \in \mathcal{P}$. Thus, in view of Proposition 1.4.1, this controller solves the generalized tracking problem for all $\mu \in \mathcal{P}$. ◁

Remark 1.5.1. Of course, in view of Corollary 1.4.5, a similar result holds if, instead of (1.45), a controller of the form

$$\begin{aligned} \dot{\xi}' &= \Phi\xi' + \Delta\xi'' + \Theta y \\ \dot{\xi}'' &= L\xi'' + My \\ u &= \Gamma\xi' + N\xi'' + Ky\,, \end{aligned} \tag{1.46}$$

is considered. In fact, as the matrices Φ and L have no eigenvalues in common, the latter is equivalent to a controller of the form (1.45). To express the conditions of Proposition 1.5.1 directly in terms of the parameters of (1.46), the hypothesis (1.43) must be replaced by the hypothesis

$$(\Theta - ZM)(C(\mu)\Pi(\mu) + Q(\mu)) = 0\,, \tag{1.47}$$

with Z unique solution of (1.40). The resulting controller is robust if L, M, N, K are such that all eigenvalues of

$$\begin{pmatrix} A(\mu) + B(\mu)KC(\mu) & B(\mu)\Gamma & B(\mu)N \\ \Theta C(\mu) & \Phi & \Delta \\ MC(\mu) & 0 & L \end{pmatrix} \tag{1.48}$$

have negative real part for all $\mu \in \mathcal{P}$. ◁

This results essentially shows that the design of a robust controller can be achieved, using a structure of the form (1.45), if a robust stabilizer can be found for a system of the form

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi}' \end{pmatrix} &= \begin{pmatrix} A(\mu) & B(\mu)\Gamma \\ \Theta C(\mu) & \Phi \end{pmatrix} \begin{pmatrix} x \\ \xi' \end{pmatrix} + \begin{pmatrix} B(\mu) \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} C(\mu) & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi' \end{pmatrix}. \end{aligned} \tag{1.49}$$

Controller (1.45) has a simple structure. In fact, it consists of the *parallel interconnection* (see Figure 1.1) of two subsystems, one of which is modeled by equations of the form

$$\begin{aligned} \dot{\xi}' &= \Phi\xi' + \Theta_1 e + \Theta_2 y_2 \\ u' &= \Gamma\xi'\,, \end{aligned} \tag{1.50}$$

while the other one is modeled by equations of the form

$$\begin{aligned} \dot{\xi}'' &= L\xi'' + M_1 e + M_2 y_2 \\ u'' &= N\xi'' + K_1 e + K_2 y_2\,. \end{aligned} \tag{1.51}$$

In what follows, we will sometime refer to (1.50) as the *internal model* unit, or *servocompensator*, while we will refer to (1.51) as the *stabilizer*. We show in this section how the degrees of freedom still left in the design of these two subsystems can be chosen in such a way as to meet the conditions of Proposition 1.5.1 and to obtain robust stability.

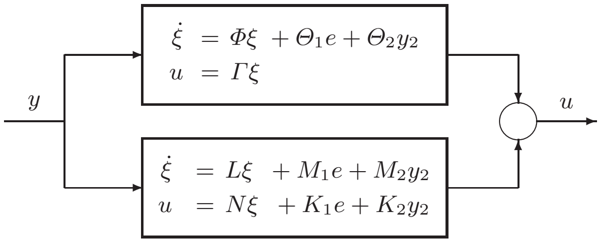

Figure 1.1. Controller (1.45).

We describe, to begin with, the design of a controller able to achieve the desired goal for all values of the parameter μ in *some* open neighborhood of given (and known) nominal value $\bar\mu$. Let, for convenience,

$$\begin{aligned} \dot x &= Ax + Bu + Pw \\ e &= C_1 x + Q_1 w \\ y_2 &= C_2 x + Q_2 w \end{aligned} \tag{1.52}$$

describe the controlled plant when $\mu = \bar\mu$. Since by definition $C_1\Pi + Q_1 = 0$, a trivial way to meet the condition $\Theta_2(C_2\Pi + Q_2) = 0$ of Proposition 1.5.1 is to set $\Theta_2 = 0$. Setting also, for simplicity, $K_1 = 0$ and $K_2 = 0$ yields a controller modeled by equations of the form

$$\begin{aligned} \dot\xi' &= \Phi\xi' + \Theta_1 e \\ \dot\xi'' &= L\xi'' + M_1 e + M_2 y_2 \\ u &= \Gamma\xi' + N\xi''\,. \end{aligned} \tag{1.53}$$

The corresponding unforced closed-loop system is

$$\begin{pmatrix} \dot x \\ \dot\xi' \\ \dot\xi'' \end{pmatrix} = \begin{pmatrix} A & B\Gamma & BN \\ \Theta_1 C_1 & \Phi & 0 \\ MC & 0 & L \end{pmatrix} \begin{pmatrix} x \\ \xi' \\ \xi'' \end{pmatrix}, \tag{1.54}$$

in which, for convenience, we have set

$$M = (\, M_1 \quad M_2\,)\,, \qquad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Observe now that the pair (Γ, Φ) in (1.35) is by construction observable, and that it is always possible to find Θ_1 such that the pair (Φ, Θ_1) is controllable. For this choice of Θ_1, the following holds.

Lemma 1.5.2. *Suppose the pair (A, B) is stabilizable and the pair (C, A) is detectable. Suppose*

$$\det \begin{pmatrix} A - \lambda I & B \\ C_1 & 0 \end{pmatrix} \neq 0 \tag{1.55}$$

for all λ which are eigenvalues of S. Let Φ and Γ be as in (1.35), and Θ_1 such that the pair (Φ, Θ_1) is controllable. Then, the pair

$$\begin{pmatrix} C & 0 \end{pmatrix}, \qquad \begin{pmatrix} A & B\Gamma \\ 0 & \Phi \end{pmatrix} \tag{1.56}$$

is detectable and the pair

$$\begin{pmatrix} A & 0 \\ \Theta_1 C_1 & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix} \tag{1.57}$$

is stabilizable.

Proof. Suppose, without loss of generality, that the pair (C_1, A) is decomposed as

$$\begin{pmatrix} 0 & C_{1d} \end{pmatrix}, \qquad \begin{pmatrix} A_{uu} & A_{ud} \\ 0 & A_{dd} \end{pmatrix}$$

with (C_{1d}, A_{dd}) a detectable pair. Split B and C_2 accordingly, as

$$B = \begin{pmatrix} B_u \\ B_d \end{pmatrix}, \qquad C_2 = \begin{pmatrix} C_{2u} & C_{2d} \end{pmatrix}.$$

Suppose that the pair (1.56) is not detectable. Then, there exist a number λ with nonnegative real part and a nonzero vector $\mathrm{col}(x_u, x_d, \xi)$ such that

$$\begin{aligned} (A_{uu} - \lambda I)x_u + A_{ud}x_d + B_u\Gamma\xi &= 0 \\ (A_{dd} - \lambda I)x_d + B_d\Gamma\xi &= 0 \\ (\Phi - \lambda I)\xi &= 0 \\ C_{1d}x_d &= 0 \\ C_{2u}x_u + C_{2d}x_d &= 0\,. \end{aligned}$$

The vector ξ cannot be zero. Otherwise, $(x_u, x_d) \neq (0, 0)$ and the remaining equations would contradict the detectability of the pair (C, A). Therefore, from the third equation, it is seen that λ is an eigenvalue of Φ and hence an eigenvalue of S. Note now that also $\Gamma\xi$ cannot be zero because, otherwise, the identity

$$\begin{pmatrix} (\Phi - \lambda I) \\ \Gamma \end{pmatrix} \xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

would contradict the observability of (Γ, Φ). We have found in this way a nonzero vector $\mathrm{col}(x_u, x_d, \Gamma\xi)$ satisfying

$$\begin{pmatrix} (A_{uu} - \lambda I) & A_{ud} & B_u \\ 0 & (A_{dd} - \lambda I) & B_d \\ 0 & C_{1d} & 0 \end{pmatrix} \begin{pmatrix} x_u \\ x_d \\ \Gamma\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with λ an eigenvalue of S, and this contradicts the hypothesis (1.55). Thus, the pair (1.56) is detectable. The proof that the pair (1.57) is stabilizable is identical. ◁

Corollary 1.5.3. *Suppose the hypotheses of Lemma 1.5.2 hold. Then, there exists a triplet L, M, N, in which none of the eigenvalues of L is an eigenvalue of S, such that system (1.54) is asymptotically stable.*

Proof. Since detectability and stabilizability are invariant under output injection and, respectively, state feedback, it is immediate to see that the pair

$$\begin{pmatrix} C & 0 \end{pmatrix}, \qquad \begin{pmatrix} A & B\Gamma \\ \Theta_1 C_1 & \Phi \end{pmatrix}$$

is detectable and the pair

$$\begin{pmatrix} A & B\Gamma \\ \Theta_1 C_1 & \Phi \end{pmatrix}, \qquad \begin{pmatrix} B \\ 0 \end{pmatrix}$$

is stabilizable. Thus, looking at the structure of (1.54), the existence of a triplet L, M, N which makes this system asymptotically stable follows from standard results. Small variations on L do not destroy asymptotic stability and therefore L can always be chosen so that none of its eigenvalues is an eigenvalue of S. ◁

It is seen from this result that if the pair (A, B) is stabilizable, if the pair (C, A) is detectable and condition (1.55) holds, it is always possible to choose Θ_1 and L, M, N, K such that controller (1.45) meets all the requirements indicated in Proposition 1.5.1. Therefore, this controller solves the generalized tracking problem for (1.52). This controller is also *robust*, by construction, in the presence of plant parameter variations, so long as these variations are such that condition (1.42) continues to hold and all the eigenvalues of (1.44) remain with negative real part. In this respect, it may be worth observing that if those conditions hold for one value of $\bar{\mu}$, they continue to hold for all μ in *some* open neighborhood of $\bar{\mu}$. Thus, the controller in question solves the generalized tracking problem for all values of μ in some open neighborhood of $\bar{\mu}$. However, it must be stressed that, in the previous design procedure, there is no *a priori* guarantee that the second condition (the stability of the closed-loop system) continues to hold if μ is free to range on a *given* set $\mathcal{P}$. For this to be the case, a more refined design of L, M, N, K, and perhaps Θ, is required.

Motivated by this observation, we describe now a design method by means of which, under appropriate hypotheses, robust stability for all values of μ within a prescribed set $\mathcal{P}$ can be achieved. For the sake of simplicity, the analysis will be limited to the case of a system for which $m = 1$, as the discussion of this case is sufficient to present the basic design ideas. The extension to systems having $m > 1$ is not terribly more difficult, but requires a nonnegligible additional notational burden. Consider a system of the form

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C_1(\mu)x + Q_1(\mu)w \end{aligned} \tag{1.58}$$

in which $A(\mu), B(\mu), C_1(\mu)$ are continuous functions of μ, a vector of uncertain parameters ranging over a compact set $\mathcal{P}$, and suppose that, for all values of $\mu \in \mathcal{P}$, this system has the same *relative degree* between the input u and the output e. This is to say that for some integer $r \geq 1$

$$C_1(\mu)B(\mu) = C_1(\mu)A(\mu)B(\mu) = \cdots = C_1(\mu)A^{r-2}(\mu)B(\mu) = 0$$
$$C_1(\mu)A^{r-1}(\mu)B(\mu) \neq 0\,.$$

It is well known that, by means of a suitable μ-dependent change of coordinates, this system can be written in the form

$$\begin{aligned} \dot{x}_1 &= A_{11}(\mu)x_1 + A_{12}(\mu)x_2 + P_1(\mu)w \\ \dot{x}_2 &= \bar{A}x_2 + \bar{B}\,(A_{21}(\mu)x_1 + A_{22}(\mu)x_2 + b(\mu)u) + P_2(\mu)w \\ e &= \bar{C}x_2 + Q_1(\mu)w \end{aligned}$$

in which $\dim(x_1) = n - r$, $\dim(x_2) = r$,

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad \bar{C} = (\,1 \quad 0 \quad 0 \quad \cdots \quad 0\,)\,,$$

and

$$b(\mu) = C_1(\mu)A^{r-1}(\mu)B(\mu)\,.$$

An additional change of coordinates

$$\tilde{x}_2 = x_2 + Z(\mu)w\,,$$

in which

$$Z(\mu) = \begin{pmatrix} Q_1(\mu) \\ \bar{C}P_2(\mu) + Q_1(\mu)S \\ \cdots \\ \bar{C}\bar{A}^{r-2}P_2(\mu) + \bar{C}\bar{A}^{r-3}P_2(\mu)S + \ldots + Q_1(\mu)S^{r-1} \end{pmatrix},$$

yields a system of the form

$$\begin{array}{rcl} \dot{x}_1 & = & A_{11}(\mu)x_1 + A_{12}(\mu)\tilde{x}_2 + \bar{P}_1(\mu)w \\ \dot{\tilde{x}}_2 & = & \bar{A}\tilde{x}_2 + \bar{B}\left(A_{21}(\mu)x_1 + A_{22}(\mu)\tilde{x}_2 + b(\mu)u + \bar{p}_2(\mu)w\right) \\ e & = & \bar{C}\tilde{x}_2 \end{array} \tag{1.59}$$

whose peculiar structure will be exploited in the sequel.

Remark 1.5.2. Note that the ith element of the vector $\tilde{x}_2(t)$ coincides with the $(i-1)$th derivative of the function $e(t)$ with respect to time. This property will be used later, for the construction of a robust stabilizer driven by e. ◁

A simple calculation, which is left to the reader, shows that the matrix in (1.42) is singular if and only if λ is an eigenvalue of $A_{11}(\mu)$. Thus, the basic condition (1.42) holds if and only if none of the eigenvalues of S is an eigenvalue of $A_{11}(\mu)$. In particular, this is the case if all eigenvalues of $A_{11}(\mu)$ have negative real part. Alternatively, one may wish to check that the regulator Equations (1.27) have solutions for every μ. To this end, split $\Pi(\mu)$ in two blocks, as

$$\Pi(\mu) = \begin{pmatrix} \Pi_1(\mu) \\ \Pi_2(\mu) \end{pmatrix},$$

consistently with the partition on the state vector in (1.59). If none of the eigenvalues of S is an eigenvalue of $A_{11}(\mu)$, the Sylvester equation

$$\Pi_1(\mu)S = A_{11}(\mu)\Pi_1(\mu) + \bar{P}_1(\mu) \tag{1.60}$$

has a unique solution $\Pi_1(\mu)$. Now, set $\Pi_2(\mu) = 0$ and

$$R(\mu) = \frac{1}{b(\mu)}\left[-A_{21}(\mu)\Pi_1(\mu) - \bar{p}_2(\mu)\right]. \tag{1.61}$$

A straightforward check shows that the pair $\Pi(\mu), R(\mu)$ thus constructed is a solution of (1.27). Thus, if none of the eigenvalues of S is an eigenvalue of $A_{11}(\mu)$, one of the basic conditions indicated in Proposition 1.5.1 is fulfilled.

Even though not strictly necessary, it may be convenient to use the solution $\Pi_1(\mu)$ of (1.60) to change the coordinate x_1 as

$$\tilde{x}_1 = x_1 - \Pi_1(\mu)w,$$

and this, in view of (1.61), transforms system (1.59) into

$$\begin{array}{rcl} \dot{\tilde{x}}_1 & = & A_{11}(\mu)\tilde{x}_1 + A_{12}(\mu)\tilde{x}_2 \\ \dot{\tilde{x}}_2 & = & \bar{A}\tilde{x}_2 + \bar{B}(A_{21}(\mu)\tilde{x}_1 + A_{22}(\mu)\tilde{x}_2) + \bar{B}b(\mu)(u - R(\mu)w) \\ e & = & \bar{C}\tilde{x}_2\,. \end{array} \tag{1.62}$$

The fulfilment of the other basic condition of Proposition 1.5.1, namely robust stabilization of system (1.49), can be achieved as follows. We begin by observing that a system of the form (1.59) can be robustly stabilized if the matrix $A_{11}(\mu)$ and the coefficient $b(\mu)$ meet certain conditions. As a matter of fact, the following well-known result holds.

Lemma 1.5.4. *Let $\mathcal{P}$ be a compact set. Suppose there is a number $\bar{b} > 0$ such that $b(\mu) \geq \bar{b}$ and suppose that the eigenvalues of $A_{11}(\mu)$ have negative real part, for all $\mu \in \mathcal{P}$. Then, there is an r-dimensional row vector N, independent of μ, such that, for all $\mu \in \mathcal{P}$, the eigenvalues of*

$$\begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) \\ \bar{B}A_{21}(\mu) & \bar{A} + \bar{B}A_{22}(\mu) \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{B}b(\mu) \end{pmatrix} \begin{pmatrix} 0 & N \end{pmatrix} \tag{1.63}$$

have negative real part.

Proof. Consider system (1.59) and set $w = 0$ (in fact, the role of w is irrelevant in the stability analysis). Let

$$d(\lambda) = \lambda^{r-1} + d_{r-2}\lambda^{r-2} + \cdots + d_1\lambda + d_0$$

be a polynomial having all roots with negative real part and set

$$D = \begin{pmatrix} d_0 & d_1 & \cdots & d_{r-2} \end{pmatrix}, \qquad \bar{N} = \begin{pmatrix} D & 1 \end{pmatrix}.$$

Let $\tilde{x}_{22}$ denote the last entry of $\tilde{x}_2$, split $\tilde{x}_2$ as

$$\tilde{x}_2 = \begin{pmatrix} \tilde{x}_{21} \\ \tilde{x}_{22} \end{pmatrix},$$

and define a set of new coordinates as

$$\theta_1 = \begin{pmatrix} x_1 \\ \tilde{x}_{21} \end{pmatrix}, \qquad \theta_2 = \tilde{x}_{22} + D\tilde{x}_{21} = \bar{N}\tilde{x}_2\,.$$

This yields a system of the form

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} F_{11}(\mu) & F_{12}(\mu) \\ F_{21}(\mu) & F_{22}(\mu) \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b(\mu) \end{pmatrix} u \tag{1.64}$$

in which $F_{11}(\mu)$ is a matrix having the following structure

$$F_{11}(\mu) = \begin{pmatrix} A_{11}(\mu) & * \\ 0 & \bar{F} \end{pmatrix}$$

with

$$\bar{F} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{r-2} \end{pmatrix}.$$

By assumption (and by definition of D) this matrix $F_{11}(\mu)$ has eigenvalues with negative real part for all $\mu \in \mathcal{P}$. Thus, given any number $\alpha > 0$, there exists a positive definite symmetric matrix $Z(\mu)$ such that

$$Z(\mu)F_{11}(\mu) + F_{11}^{\mathrm{T}}(\mu)Z(\mu) = -\alpha I\,.$$

Since the entries of $F_{11}(\mu)$ are continuous functions of μ, so are the entries of $Z(\mu)$. Consider now the positive definite quadratic form

$$V(\theta_1, \theta_2) = \theta_1^{\mathrm{T}} Z(\mu)\theta_1 + \theta_2^{\mathrm{T}}\theta_2\,. \tag{1.65}$$

The derivative of this function along the trajectories of (1.64) with control $u = -k\theta_2$ is the quadratic form

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -\alpha I & Z(\mu)F_{12}(\mu) + F_{21}^{\mathrm{T}}(\mu) \\ F_{12}^{\mathrm{T}}(\mu)Z(\mu) + F_{21}(\mu) & 2F_{22}(\mu) - 2b(\mu)k \end{pmatrix} \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}.$$

As μ ranges on a compact set, the off-diagonal blocks are continuous functions of μ and $b(\mu) \geq \bar{b} > 0$, standard arguments show that there is a number k^* such that, if $k \geq k^*$, this form is negative definite and hence the (1.64) with control $u = -k\theta_2$ is (robustly) asymptotically stable. Reverting back to the coordinates of (1.59) yields a control

$$u = -k\bar{N}\tilde{x}_2$$

and this proves the lemma. ◁

Remark 1.5.3. Note that, to say that $b(\mu) \geq \bar{b}$ is to say that the so-called *high-frequency gain* of the system is positive, while to say that all eigenvalues of $A_{11}(\mu)$ have negative real part is to say that all zeros of the transfer function of (1.59) between the input u and output e have negative real part. In the lemma above, these properties are required to hold for all $\mu \in \mathcal{P}$. Note also that the matrix N determined in the proof of the lemma is a matrix of the form $N = -k\bar{N}$ in which $\bar{N}$ is fixed and k is a number required to satisfy $k \geq k^*$ for some $k^* > 0$. ◁

In other words, this lemma says that if the two hypotheses on $A_{11}(\mu)$ and $b(\mu)$ hold, there is a memoryless state-feedback law

$$u = N\tilde{x}_2$$

that stabilizes system (1.59). It is important to stress that the feedback law thus defined, expressed in the original coordinates in which system (1.58) is given, is not μ-independent and not even w-independent. In fact, the changes of variables needed to obtain $\tilde{x}_2$ from x depend on μ and on w. There might be cases, though, in which the vector $\tilde{x}_2$ can be regarded as part of a vector y of measured variables, and in these cases, indeed, the feedback law thus found does provide robust stability. If the vector $\tilde{x}_2$ is *not* directly accessible for measurement, a dynamic state feedback, driven by the measured variable e, can be constructed instead. This is made possible by the fact that the various components of the vector $\tilde{x}_2(t)$ coincide with the derivatives of $e(t)$ with respect to time and that these derivatives can be approximated by means of a suitable estimator driven by $e(t)$. To this end, consider a system modeled by

$$\dot{\xi}'' = L\xi'' + M_1 e$$

in which $\xi'' \in \mathbb{R}^r$,

$$L = \begin{pmatrix} -gc_{r-1} & 1 & 0 & \cdots & 0 \\ -g^2 c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -g^{r-1}c_1 & 0 & 0 & \cdots & 1 \\ -g^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} gc_{r-1} \\ g^2 c_{r-2} \\ \cdot \\ g^{r-1}c_1 \\ g^r c_0 \end{pmatrix}, \tag{1.66}$$

where $c_0, c_1, \ldots, c_{r-1}$ are the coefficients of a fixed polynomial

$$c(\lambda) = \lambda^r + c_{r-1}\lambda^{r-1} + \cdots + c_1\lambda + c_0$$

having all roots with negative real part, and $g > 0$ is a parameter to be determined. It is well known that, if g is large enough, the dynamic feedback law

$$\begin{aligned} \dot{\xi}'' &= L\xi'' + M_1 e \\ u &= N\xi'' \end{aligned} \tag{1.67}$$

robustly stabilizes (1.59).

Lemma 1.5.5. *Let $\mathcal{P}$ be a compact set. Suppose there is a number $\bar{b} > 0$ such that $b(\mu) \geq \bar{b}$ and suppose that the eigenvalues of $A_{11}(\mu)$ have negative real part, for all $\mu \in \mathcal{P}$. Let N be as in Lemma 1.5.4. Then, there is a number g^* such that, if $g > g^*$, the eigenvalues of*

$$\begin{pmatrix} \begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) \\ \bar{B}A_{21}(\mu) & \bar{A} + \bar{B}A_{22}(\mu) \end{pmatrix} & \begin{pmatrix} 0 \\ \bar{B}b(\mu) \end{pmatrix} N \\ M_1 \begin{pmatrix} 0 & \bar{C} \end{pmatrix} & L \end{pmatrix} \tag{1.68}$$

have negative real part for all $\mu \in \mathcal{P}$.

Proof. Change the matrix (1.68), by similarity, using a transformation

$$T = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & D_g & -D_g \end{pmatrix}$$

in which D_g is the matrix

$$D_g = \begin{pmatrix} g^{r-1} & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & g & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} .$$

In this way, matrix (1.68) becomes

$$\begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) & 0 \\ \bar{B}A_{21}(\mu) & \bar{A} + \bar{B}[A_{22}(\mu) + b(\mu)N] & -\bar{B}b(\mu)ND_g^{-1} \\ \bar{B}A_{21}(\mu) & \bar{B}[A_{22}(\mu) + b(\mu)N] & -\bar{B}b(\mu)ND_g^{-1} + g\bar{L} \end{pmatrix} \tag{1.69}$$

with

$$\bar{L} = \begin{pmatrix} -c_{r-1} & 1 & 0 & \cdots & 0 \\ -c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -c_1 & 0 & 0 & \cdots & 1 \\ -c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

a matrix having all eigenvalues with negative real part.

The submatrix consisting of the four blocks in the upper-left corner of (1.69) is precisely the matrix (1.63) which, if N is chosen as in Lemma 1.5.4, has all eigenvalues with negative real part, for all $\mu \in \mathcal{P}$. Moreover, if $g \geq 1$, the matrix D_g^{-1} is bounded, in norm, by 1. Thus, arguments identical to those used in the proof of Lemma 1.5.4 prove the claim. $\triangleleft$

With this result in mind, we revert now to the problem of designing a robust regulator for (1.49). The idea is to choose the matrix Θ in such a way that system (1.49) becomes a system which fulfils the hypotheses of Lemma 1.5.5, so that the robust stabilizer described in the previous lemma can be used. This is actually possible, in the light of the following simple but very important result.

Lemma 1.5.6. *Let F_0 be any $s \times s$ Hurwitz matrix and let G_0 be any $s \times 1$ vector such that the pair (F_0, G_0) is controllable. Let Φ be any $s \times s$ matrix whose eigenvalues have nonnegative real part, and let Γ be any $1 \times s$ vector such that the pair (Γ, Φ) is observable. Then, there exist a $1 \times s$ vector Ψ and a nonsingular $s \times s$ matrix T such that*

$$\begin{aligned} (F_0 + G_0\Psi)T &= T\Phi \\ \Psi T &= \Gamma . \end{aligned}$$

Proof. Observe, first of all, that the Sylvester equation

$$T\Phi = F_0 T + G_0 \Gamma$$

has a unique solution T, because Φ and F_0 have no eigenvalues in common. We prove that T is nonsingular. Suppose, by contradiction, that the kernel of T is nonzero. Let $\{v_1, \ldots, v_k\}$ be a basis for $\ker(T)$. Then

$$T\Phi v_j = G_0 \Gamma v_j\,, \qquad \text{for} \qquad j = 1, \ldots, k. \tag{1.70}$$

As T is square, there exists also a set of independent row vectors $\{w_1, \ldots, w_k\}$ such that $w_i T = 0$ for $i = 1, \ldots, k$. Then

$$w_i G_0 \Gamma v_j = 0\,, \qquad \text{for} \qquad i, j = 1, \ldots, k.$$

Suppose $\Gamma v_j = 0$ for all j. In this case, (1.70) yields $T\Phi v_j = 0$, i.e.,

$$\Phi v_j \in \ker(T)\,, \qquad \text{for} \qquad j = 1, \ldots, k. \tag{1.71}$$

We find, in this way, that $\ker(T)$ is invariant under Φ and contained in $\ker(\Gamma)$, and this contradicts observability of (Γ, Φ). If $\Gamma v_j \neq 0$ for at least one value of j, then $w_i G_0$ must be zero for all i and, with a dual argument, we can prove that this contradicts controllability of (F_0, G_0). Having shown that T is nonsingular, to complete the proof it suffices to set $\Psi = \Gamma T^{-1}$. ◁

Remark 1.5.4. Note that the matrix Ψ is the unique row vector which assigns to $F_0 + G_0\Psi$ a set of eigenvalues which coincide with the eigenvalues of Φ. ◁

Remark 1.5.5. Note that, if Φ and Γ are matrices of the form (1.35), written for $m = 1$, the matrix

$$T_S(R(\mu)) = \begin{pmatrix} R(\mu) \\ R(\mu)S \\ \cdots \\ R(\mu)S^{s-1} \end{pmatrix}$$

satisfies

$$\begin{aligned} T_S(R(\mu))S &= \Phi T_S(R(\mu)) \\ R(\mu) &= \Gamma T_S(R(\mu))\,. \end{aligned}$$

Hence, the matrix $T(\mu) = T T_S(R(\mu))$ satisfies

$$\begin{aligned} T(\mu)S &= (F_0 + G_0\Psi)T(\mu) \\ R(\mu) &= \Psi T(\mu)\,. \quad \triangleleft \end{aligned}$$

With this result in mind, consider now for (1.62) a controller of the form

$$\begin{aligned} \dot{\xi}' &= F_0\xi' + G_0 u \\ u &= \Psi\xi' + v\,, \end{aligned} \tag{1.72}$$

in which v is viewed as an additional input, to be used later for robust stabilization. This controller will be referred to as the *canonical internal model*, in view of some relevant features that will be highlighted in the sequel.

The composition of (1.62) with the controller (1.72) yields the system

$$\begin{aligned} \dot{\xi}' &= (F_0 + G_0\Psi)\xi' + G_0 v \\ \dot{\tilde{x}}_1 &= A_{11}(\mu)\tilde{x}_1 + A_{12}(\mu)\tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}(A_{21}(\mu)\tilde{x}_1 + A_{22}(\mu)\tilde{x}_2) + \bar{B}b(\mu)(\Psi\xi' + v - R(\mu)w)\,. \end{aligned}$$

The latter, changing ξ' into

$$\tilde{\xi}' = \xi' - T(\mu)w$$

and using the properties of $T(\mu)$ indicated in Remark 1.5.5, becomes

$$\begin{aligned}\dot{\tilde{\xi}}' &= (F_0 + G_0\Psi)\tilde{\xi}' + G_0 v\\ \dot{\tilde{x}}_1 &= A_{11}(\mu)\tilde{x}_1 + A_{12}(\mu)\tilde{x}_2\\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}(A_{21}(\mu)\tilde{x}_1 + A_{22}(\mu)\tilde{x}_2) + \bar{B}b(\mu)(\Psi\tilde{\xi}' + v)\,.\end{aligned} \tag{1.73}$$

Remark 1.5.6. Note that, if v is set equal to the output of a system of the form

$$\begin{aligned}\dot{\xi}'' &= L\xi'' + My\\ v &= N\xi'' + Ky\,,\end{aligned}$$

the resulting controller is a controller of the form

$$\begin{aligned}\dot{\xi}' &= (F_0 + G_0\Psi)\xi' + G_0 N\xi'' + G_0 K y\\ \dot{\xi}'' &= L\xi'' + My\\ u &= \Psi\xi' + N\xi'' + Ky\,,\end{aligned}$$

i.e., a special case of the controller considered in Remark 1.5.1. $\triangleleft$

The advantage in using the canonical internal model (1.72) is that the composite system (1.73), with v viewed as input and e viewed as output, still meets the conditions, indicated in Lemma 1.5.5, for the existence of a robust stabilizer. To see that this is the case, consider the change of state variables

$$\chi = \tilde{\xi}' - \frac{1}{b(\mu)} G_0\bar{C}\bar{A}^{r-1}\tilde{x}_2\,,$$

which yields (recall that $\bar{C}\bar{A}^{r-1}\bar{B} = 1$ and that $\bar{A}^r = 0$)

$$\begin{aligned}\dot{\chi} &= (F_0 + G_0\Psi)\tilde{\xi}' + G_0 v\\ &\qquad -\frac{1}{b(\mu)}\,G_0\Big(A_{21}(\mu)\tilde{x}_1 + A_{22}(\mu)\tilde{x}_2 + b(\mu)(\Psi\tilde{\xi}' + v)\Big)\,,\end{aligned}$$

in which the terms $G_0(\Psi\tilde{\xi}' + v)$ and $-G_0(\Psi\tilde{\xi}' + v)$ cancel out. In the new variables, system (1.73) reduces to a system of the form

$$\begin{aligned}\begin{pmatrix}\dot{\chi}\\ \dot{\tilde{x}}_1\end{pmatrix} &= \begin{pmatrix}F_0 & F_{01}(\mu)\\ 0 & A_{11}(\mu)\end{pmatrix}\begin{pmatrix}\chi\\ \tilde{x}_1\end{pmatrix} + \begin{pmatrix}F_{02}(\mu)\\ A_{12}(\mu)\end{pmatrix}\tilde{x}_2\\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\Big(\begin{pmatrix}\Psi b(\mu) & A_{21}(\mu)\end{pmatrix}\begin{pmatrix}\chi\\ \tilde{x}_1\end{pmatrix} + F_{22}(\mu)\tilde{x}_2 + b(\mu)v\Big)\end{aligned} \tag{1.74}$$

in which $F_{01}(\mu), F_{02}(\mu), F_{22}(\mu)$ are suitable matrices. This system has exactly the same structure as system (1.59), the place of the matrix $A_{11}(\mu)$ being taken by the matrix

$$\begin{pmatrix} F_0 & F_{01}(\mu) \\ 0 & A_{11}(\mu) \end{pmatrix} . \tag{1.75}$$

As F_0 is by hypothesis a Hurwitz matrix, if the eigenvalues of $A_{11}(\mu)$ have negative real part for all $\mu \in \mathcal{P}$, so have the eigenvalues of (1.75). Hence, the results of Lemma 1.5.4 and 1.5.5 can be invoked. In particular, it is possible to claim the existence of matrices L, M_1, N such that the eigenvalues of the matrix

$$\begin{pmatrix} F_0 & F_{01}(\mu) & F_{02}(\mu) \\ 0 & A_{11}(\mu) & A_{12}(\mu) \\ \bar{B}\Psi b(\mu) & \bar{B}A_{21}(\mu) & \bar{A}+\bar{B}F_{22}(\mu) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \bar{B}b(\mu) \end{pmatrix} \begin{pmatrix} 0 & 0 & N \end{pmatrix} \tag{1.76}$$

and those of the matrix

$$\begin{pmatrix} \begin{pmatrix} F_0 & F_{01}(\mu) & F_{02}(\mu) \\ 0 & A_{11}(\mu) & A_{12}(\mu) \\ \bar{B}\Psi b(\mu) & \bar{B}A_{21}(\mu) & \bar{A}+\bar{B}F_{22}(\mu) \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \bar{B}b(\mu) \end{pmatrix} N \\ M_1 \begin{pmatrix} 0 & 0 & \bar{C} \end{pmatrix} & L \end{pmatrix} \tag{1.77}$$

have negative real part for all $\mu \in \mathcal{P}$.

These matrices L, M_1, N can be directly used for the design of a robust controller. For the convenience of the reader, we summarize the conclusion in a formal statement, where all relevant assumptions are indicated.

Proposition 1.5.7. *Consider system (1.59). Suppose there is a number $\bar{b} > 0$ such that $b(\mu) \geq \bar{b}$ and suppose that the eigenvalues of $A_{11}(\mu)$ have negative real part, for all μ in a compact set $\mathcal{P}$. Let s denote the dimension of the minimal polynomial of S. Let F_0 be any $s \times s$ Hurwitz matrix and let G_0 be any $s \times 1$ vector such that the pair (F_0, G_0) is controllable. Let Ψ be the unique row vector which assigns to $F_0 + G_0\Psi$ a set of eigenvalues which coincide with the eigenvalues of Φ. Let L, M_1, N be such that all eigenvalues of (1.77) have negative real part for all $\mu \in \mathcal{P}$. The controller*

$$\begin{aligned} \dot{\xi}' &= (F_0 + G_0\Psi)\xi' + G_0 N \xi'' \\ \dot{\xi}'' &= L\xi'' + M_1 e \\ u &= \Psi\xi' + N\xi'' \end{aligned} \tag{1.78}$$

is a robust controller for (1.59).

Proof. Controlling (1.59) by means of (1.78) yields a system which, in suitable coordinates, appears as (1.74) controlled by

$$\begin{aligned} \dot{\xi}'' &= L\xi'' + M_1 e \\ v &= N\xi''. \end{aligned}$$

For all $\mu \in \mathcal{P}$, the system obtained in this way is asymptotically stable and, in particular, $\lim_{t\to\infty} e(t) = 0$. Thus, the controller is a robust controller. ◁

Remark 1.5.7. There might be cases in which not only the first component of $\tilde{x}_2$, which is equal to e, but also all other components of this vector are available for measurement. In this case, the controller described in the previous proposition can be simplified. In fact, set $y = \tilde{x}_2$ and consider the controller

$$\begin{aligned} \dot{\xi}' &= (F_0 + G_0\Psi)\xi' + G_0 N y \\ u &= \Psi\xi' + Ny. \end{aligned} \tag{1.79}$$

If N is such that the eigenvalues of matrix (1.76) have negative real part for all $\mu \in \mathcal{P}$, this controller is a robust controller. ◁

Remark 1.5.8. Controller (1.78) and controller (1.79) have by construction the internal model property (1.28), because they are special cases of the controller (1.46) and, respectively, (1.45). It may be convenient, for the sake of completeness, to determine the explicit expression of the matrix $\Sigma(\mu)$ which renders condition (1.28) fulfilled. Consider, for instance, the case of controller (1.79). In this case, $y = \tilde{x}_2$ and we have shown above that $\Pi_2(\mu) = 0$. Thus, (1.28) reduce to

$$\begin{aligned} \Sigma(\mu)S &= (F_0 + G_0\Psi)\Sigma(\mu) \\ R(\mu) &= \Psi\Sigma(\mu). \end{aligned}$$

Comparing with the observations in Remark 1.5.5, it is concluded that $\Sigma(\mu) = T(\mu)$. In the case of controller (1.78), the internal model property (1.28) reduces to

$$\begin{aligned} \Sigma(\mu)S &= \begin{pmatrix} (F_0 + G_0\Psi) & G_0 N \\ 0 & L \end{pmatrix} \Sigma(\mu) \\ R(\mu) &= \begin{pmatrix} \Psi & N \end{pmatrix} \Sigma(\mu). \end{aligned}$$

Thus, again comparing with the observations in Remark 1.5.5, it is concluded that

$$\Sigma(\mu) = \begin{pmatrix} T(\mu) \\ 0 \end{pmatrix}. \quad \triangleleft$$

1.6 Internal Model Adaptation

The remarkable feature of a robust controller is the ability to secure asymptotic decay of the regulated output $e(t)$ in spite of parameter uncertainties. As a matter of fact, so long as the controller is such that the internal model property (1.28) holds, the subspace

$$\mathcal{V}_0(\mu) = \{(x, \xi, w) : x = \Pi(\mu)w, \xi = \Sigma(\mu)w\}$$

is *invariant* in the closed-loop system

$$\begin{aligned}
\dot{w} &= Sw \\
\dot{x} &= [A(\mu) + B(\mu)KC(\mu)]x + B(\mu)H\xi + [P(\mu) + B(\mu)KQ(\mu)]w \\
\dot{\xi} &= F\xi + GC(\mu)x + GQ(\mu)w\,,
\end{aligned}$$

and *attractive* if the unforced closed-loop system is robustly stable. Thus, all trajectories of the forced closed-loop system asymptotically converge to $\mathcal{V}_0(\mu)$ and, since the map

$$e = C_1(\mu)x + Q_1(\mu)w$$

is zero on $\mathcal{V}_0(\mu)$, the regulation goal $\lim_{t\to\infty} e(t) = 0$ is achieved.

Of course, the subspace $\mathcal{V}_0(\mu)$ depends on the uncertain parameter μ, but the remarkable feature of the special structure of the controller considered in Proposition 1.4.3, or in the subsequent Corollary 1.4.5, is to have the internal model property secured by means of a controller which is *independent of* μ. As a matter of fact, the controller in question is able to generate, regardless of the specific value of μ, the feedforward control input $R(\mu)w(t)$ which would force the state $x(t)$ of the controlled plant to remain on the subspace

$$\mathcal{S}(\mu) = \{(x, w) : x = \Pi(\mu)w\}\,.$$

In fact, for every $w(0)$ there is a initial state $\xi(0)$ (that is, $\xi(0) = \Sigma(\mu)w(0)$) from which the controller, driven by the measured output

$$y(t) = [C(\mu)\Pi(\mu) + Q(\mu)]w(t)\,,$$

which is precisely the measured output occurring when $x(t) = \Pi(\mu)w(t)$, produces a control input which coincides with $R(\mu)w(t)$.

Thus, control schemes incorporating a robust controller efficiently address the problem of rejecting all disturbance inputs generated by the exosystem (1.8). In this sense, they generalize the classical way in which integral-control-based schemes cope with constant but unknown disturbances. There still is a limitation, though, in these schemes: the necessity for a precise model of the exosystem. As a matter of fact, the controller considered in Proposition 1.4.3 (or in Corollary 1.4.5) contains a pair of matrices (Φ, Γ) whose construction (see (1.35)) requires the knowledge of the precise values of the coefficients of the minimal polynomial of S. The reader will have no difficulty in checkings that, in general, the internal model property will be lost if inaccurate values of these coefficients are used to construct the matrix Φ. This limitation is not sensed in a problem of set point control, where the uncertain exogenous input is constant and thus obeys a trivial, parameter independent, differential equation, but becomes immediately evident in the problem of rejecting, for example, a *sinusoidal* disturbance of unknown amplitude and phase. A robust controller is able to cope with uncertainties on amplitude and phase of the exogenous sinusoidal signal, but the frequency at which the internal model oscillates must exactly match the frequency of the exogenous signal: any mismatch in such frequencies results in a nonzero steady-state error.

In what follows we show how this limitation can be removed, by automatically tuning the "natural frequencies" of the robust controller. For the sake of simplicity, we limit ourselves to sketch here the main philosophy of the design method; further details and proofs will be provided later in specific applications.

Consider again system (1.58), for which we have learned how to design a robust controller but suppose, now, that the model of exosystem which generates the disturbance w depends on a vector ϱ of uncertain parameters, ranging on a prescribed set $\mathcal{Q}$, as in

$$\dot{w} = S(\varrho)w\,. \tag{1.80}$$

We retain the assumption that the exosystem is neutrally stable, in which case $S(\varrho)$ can only have eigenvalues on the imaginary axis (with simple multiplicity in the minimal polynomial). Therefore, uncertainty in the value of ϱ is reflected in uncertainty in the value of the imaginary part of these eigenvalues.

Let

$$m_\varrho(\lambda) = \lambda^s + a_{s-1}(\varrho)\lambda^{s-1} + \cdots + a_1(\varrho)\lambda + a_0(\varrho)$$

denote the minimal polynomial of $S(\varrho)$ and assume that the coefficients $a_{s-1}(\varrho), \ldots, a_1(\varrho), a_0(\varrho)$ are continuous functions of ϱ. Define a pair of matrices Φ_ϱ, Γ as (1.35), the former of which is a continuous function of ϱ. Appealing to Lemma 1.5.6, it can be asserted that, if (F_0, G_0) is a controllable pair in which F_0 is a Hurwitz matrix, there exists a vector Ψ_ϱ and a nonsingular matrix T_ϱ such that

$$\begin{aligned} (F_0 + G_0\Psi_\varrho)T_\varrho &= T_\varrho\Phi_\varrho \\ \Psi_\varrho T_\varrho &= \Gamma\,. \end{aligned}$$

In particular, Ψ_ϱ and T_ϱ depend continuously on ϱ, as Φ_ϱ does.

If ϱ were known, the controller considered in Proposition 1.5.7, with $\Psi = \Psi_\varrho$ would be a robust controller (having assumed, of course, that the assumptions of the proposition are fulfilled). In case ϱ is not known, one may wish to replace the vector Ψ in (1.78) with an *estimate* $\hat{\Psi}$ of Ψ_ϱ, *to be tuned* by means of an appropriate adaptation law. We illustrate how this works in the simpler situation in which the entire vector $\tilde{x}_2$ is available for measurement, in which case the simpler control law (1.79) can be taken as a paradigm.

Consider a control law of the form

$$\begin{aligned} \dot{\xi}' &= (F_0 + G_0\hat{\Psi})\xi' + G_0 N y \\ u &= \hat{\Psi}\xi' + Ny\,, \end{aligned} \tag{1.81}$$

in which $\hat{\Psi}$ is a $1 \times s$ vector to be tuned, and choose for $\hat{\Psi}$ an adaptation law of the form

$$\dot{\hat{\Psi}} = -\gamma(\bar{N}y)(\xi')^{\mathrm{T}} \tag{1.82}$$

in which $\gamma > 0$ is an arbitrary design parameter. The row vector $\bar{N}$ is a vector of the form

$$\bar{N} = \begin{pmatrix} d_0 & d_1 & \cdots & d_{r-2} & 1 \end{pmatrix}$$

in which $d_0, d_1, \ldots, d_{r-2}$ are coefficients of a Hurwitz polynomial

$$d(\lambda) = \lambda^{r-1} + d_{r-2}\lambda^{r-2} + \cdots + d_1\lambda + d_0$$

and $N = -k\bar{N}$. Then, the following result holds.

Proposition 1.6.1. *Consider system (1.59), with exosystem (1.80), in which ϱ is a vector of uncertain parameters ranging on a compact set $\mathcal{Q}$. Suppose there is a number $\bar{b} > 0$ such that $b(\mu) \geq \bar{b}$ and suppose that the eigenvalues of $A_{11}(\mu)$ have negative real part, for all μ in a compact set $\mathcal{P}$. Let s denote the dimension of the minimal polynomial of S. Let F_0 be any $s \times s$ Hurwitz matrix, and let G_0 be any $s \times 1$ vector such that the pair (F_0, G_0) is controllable. Then, there is a number k^* such that, for all $k \geq k^*$, the control law (1.81) with adaptation law (1.82), driven by the measured output $y = \tilde{x}_2$, is such that, in the corresponding closed-loop system, all trajectories are bounded and* $\lim_{t\to\infty} e(t) = 0$.

Proof. Define an estimation error $\tilde{\Psi} = \hat{\Psi} - \Psi_\varrho$, and consider the aggregate of (1.59) and (1.81), with $\hat{\Psi}$ replaced by $\Psi_\varrho + \tilde{\Psi}$ and $y = \tilde{x}_2$,

$$\begin{aligned}
\dot{\xi}' &= (F_0 + G_0\Psi_\varrho)\xi' + G_0 N\tilde{x}_2 + G_0\tilde{\Psi}\xi' \\
\dot{x}_1 &= A_{11}(\mu)\tilde{x}_1 + A_{12}(\mu)\tilde{x}_2 + \bar{P}_1(\mu)w \\
\dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\left(A_{21}(\mu)x_1 + A_{22}(\mu)\tilde{x}_2 + b(\mu)\Psi_\varrho\xi' + b(\mu)N\tilde{x}_2 + \bar{p}_2(\mu)w\right) \\
&\qquad + \bar{B}b(\mu)\tilde{\Psi}\xi' \,.
\end{aligned}$$

Recall (see Remark 1.5.5) that the matrix

$$\Sigma_\varrho(\mu) = T_\varrho T_S(R(\mu))$$

satisfies

$$\Sigma_\varrho(\mu)S(\varrho) = (F_0 + G_0\Psi_\varrho)\Sigma_\varrho(\mu), \qquad R(\mu) = \Psi_\varrho\Sigma_\varrho(\mu)\,.$$

Hence, changing variables as

$$\begin{aligned}
\tilde{x}_1 &= x_1 - \Pi_1(\mu)w \\
\tilde{\xi}' &= \xi' - \Sigma_\varrho(\mu)w
\end{aligned}$$

yields, in view of (1.60) and (1.61),

$$\begin{aligned}
\dot{\tilde{\xi}}' &= (F_0 + G_0\Psi_\varrho)\tilde{\xi}' + G_0 N\tilde{x}_2 + G_0\tilde{\Psi}\xi' \\
\dot{\tilde{x}}_1 &= A_{11}(\mu)\tilde{x}_1 + A_{12}(\mu)\tilde{x}_2 \\
\dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\left(A_{21}(\mu)\tilde{x}_1 + A_{22}(\mu)\tilde{x}_2 + b(\mu)\Psi_\varrho\tilde{\xi}' + b(\mu)N\tilde{x}_2\right) + \bar{B}b(\mu)\tilde{\Psi}\xi'
\end{aligned}$$

(we have not changed the coordinate ξ' in the term multiplied by $\tilde{\Psi}$ for reasons that will become clear in a moment). The additional change of variables

$$\chi = \tilde{\xi}' - \frac{1}{b(\mu)} G_0 \bar{C}\bar{A}^{r-1}\tilde{x}_2\,,$$

(already used in the discussion preceding Proposition 1.5.7) yields a system of the form

$$\begin{aligned}
\dot{\chi} &= F_0\chi + F_{01}(\mu)\tilde{x}_1 + F_{02}(\mu)\tilde{x}_2 \\
\dot{\tilde{x}}_1 &= A_{11}(\mu)\tilde{x}_1 + A_{12}(\mu)\tilde{x}_2 \\
\dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\left(A_{21}(\mu)\tilde{x}_1 + [F_{22}(\mu) + b(\mu)N]\tilde{x}_2 + b(\mu)\Psi_\varrho\chi\right) \\
&\qquad + \bar{B}b(\mu)\tilde{\Psi}\xi'
\end{aligned}$$

in which the matrices $F_{01}(\mu), F_{02}(\mu), F_{22}(\mu)$ are precisely the same matrices found in (1.76). Setting

$$\mathbf{x} = \begin{pmatrix} \chi \\ \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ \bar{B} \end{pmatrix}$$

the system thus obtained can be simply written as

$$\dot{\mathbf{x}} = \mathbf{A}(\mu,\varrho)\mathbf{x} + \mathbf{b}\,b(\mu)\tilde{\Psi}\xi'$$

in which $\mathbf{A}(\mu,\varrho)$ is precisely the matrix (1.76), written for $\Psi = \Psi_\varrho$.

From the proof of Lemma 1.5.4 it is known that there exists $\mathbf{Z}(\mu)$ such that, if k is large enough,

$$\mathbf{Z}(\mu)\mathbf{A}(\mu,\varrho) + \mathbf{A}^{\mathrm{T}}(\mu,\varrho)\mathbf{Z}(\mu) < 0\,.$$

Moreover, an easy calculation shows that this matrix $\mathbf{Z}(\mu)$ is such that

$$\mathbf{b}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{x} = \bar{N}\tilde{x}_2\,. \tag{1.83}$$

Observing that Ψ_ϱ is constant and using (1.83), it is seen that

$$\dot{\tilde{\Psi}}^{\mathrm{T}} = \dot{\hat{\Psi}}^{\mathrm{T}} = -\gamma\xi'\bar{N}\tilde{x}_2 = -\gamma\xi'\mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{b}\,.$$

Compute now the derivative along the trajectories of the closed-loop system of the positive definite quadratic form

$$U(\mathbf{x},\tilde{\Psi}) = \mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{x} + \frac{b(\mu)}{\gamma}\tilde{\Psi}\tilde{\Psi}^{\mathrm{T}}\,.$$

This yields

$$\begin{aligned}\dot{U} &= \mathbf{x}^{\mathrm{T}}[\mathbf{Z}(\mu)\mathbf{A}(\mu,\varrho) + \mathbf{A}^{\mathrm{T}}(\mu,\varrho)\mathbf{Z}(\mu)]\mathbf{x} + 2\mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{b}\,b(\mu)\tilde{\Psi}\xi' + 2\frac{b(\mu)}{\gamma}\tilde{\Psi}\dot{\tilde{\Psi}}^{\mathrm{T}}\\ &\leq 2\mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{b}\,b(\mu)\tilde{\Psi}\xi' - 2b(\mu)\tilde{\Psi}\xi'\mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{b} = 0\,.\end{aligned}$$

Thus, since $U(\mathbf{x},\tilde{\Psi})$ is positive definite, the trajectories of the closed-loop system are bounded. Moreover, the classical arguments of La Salle's invariance principle show that $\lim_{t\to\infty}\mathbf{x}(t) = 0$. In particular, this implies that $\lim_{t\to\infty} e(t) = 0$, proving the Proposition. ◁

1.7 The Case of Nonlinear Systems

We proceed now with the description of how the methods presented in the previous section can be extended to nonlinear systems. Of course, we cannot expect to obtain results as sharp and complete as those obtained for linear systems, essentially for two main reasons: (i) the derivation of necessary conditions cannot count on a nonlinear equivalent of the Sylvester equation (1.13) on which the various necessary conditions presented in Section 1.3 were based, (ii) the design of robust stabilizers cannot count on methods as general as those presented in the first part of Section 1.5. If the problem is to be solved *locally* about a prescribed equilibrium, the available theory is pretty satisfactory because the Sylvester equation (1.13) can be replaced by a nonlinear analogue which characterizes the existence of a *center manifold*, while local stability can be guaranteed by linear methods exactly as in the first part of Section 1.5. The interested reader is referred to [29, Chapter 8]. The scope of the applications presented later in this book, though, is that of seeking solutions with global or arbitrarily fixed domain of validity, and in this case the analysis based on a local theory is insufficient. Because of this limitation, the analysis which follows will be essentially focused on the presentation of certain nonlinear versions of conditions and constructions derived in the previous sections, rather than on a systematic motivation of the necessity of certain hypotheses. This is actually possible mainly because the geometric interpretation of the two key conditions (1.16) and (1.17) has an appealing nonlinear counterpart.

To begin with, we consider, as a nonlinear correspondent of system (1.7), a system modeled by equations of the form

$$\begin{aligned}\dot{x} &= f(x,u,w)\\ e &= h(x,w)\\ y &= k(x,w)\,,\end{aligned} \tag{1.84}$$

in which the various variables x, u, w, y, e have the same meaning (and dimensions) as in (1.7). In particular it is assumed that $f(x, u, w)$, $h(x, w)$ and $k(x, w)$ are smooth functions of their arguments, and that $f(0, 0, 0) = 0$, $h(0, 0) = 0$ and $k(0, 0) = 0$. The disturbance w affecting the system is generated by a possibly nonlinear autonomous system

$$\dot{w} = s(w)\,, \tag{1.85}$$

even though, in essentially all known design methods, the case of a linear exosystem of the form (1.8) is considered. Again, here it is assumed that $s(w)$ is a smooth function and that $s(0) = 0$. There is an important novelty, though, in dealing with nonlinear models: the role of the exogenous input w and that of a possibly unknown parameter μ in the model (1.84) no longer need to be kept separate. The reason why they have been kept separate in the case of a linear model was to take advantage of the linearity in w. However, so long as the right-hand sides of the Equations (1.84) are nonlinear functions of w, it is natural – and convenient – to regard the various components of μ as components of w, obeying the trivial autonomous differential equation $\dot{\mu} = 0$.

In general, one may expect that system (1.84) is controlled by a fully nonlinear version of (1.9), such as a system modeled by equations of the form

$$\begin{array}{rcl} \dot{\xi} & = & \phi(\xi, y) \\ u & = & \theta(\xi, y)\,, \end{array} \tag{1.86}$$

in which $\phi(\xi, y)$ and $\theta(\xi, y)$ are smooth functions of their arguments, satisfying $\phi(0, 0) = 0$ and $\theta(0, 0) = 0$. However, as will be seen soon, in most cases in which a design is successful, sensibly simpler structures are used. As in Section 1.3, we consider the forced closed-loop system

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{x} & = & f(x, \theta(\xi, k(x, w)), w) \\ \dot{\xi} & = & \phi(\xi, k(x, w)) \\ e & = & h(x, w) \end{array} \tag{1.87}$$

and we define the *nonlinear* generalized tracking problem as follows. Given system (1.84) with exosystem (1.85), and two sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^r$, find, if possible, a controller of the form (1.86) and a set $\Xi \subset \mathbb{R}^\nu$, such that, in the closed-loop system:

(a) the trajectory $(x(t), \xi(t), w(t))$ is bounded,

(b) $\lim_{t\to\infty} e(t) = 0$,

for every initial condition $(x(0), \xi(0), w(0)) \in \mathcal{X} \times \Xi \times \mathcal{W}$.

Note that, since $\mathcal{X}$ and $\mathcal{W}$ are *a priori* fixed sets, a local analysis is in general insufficient to provide a meaningful insight. Note also that the requirement of asymptotic stability of the unforced closed-loop system, which in the case of linear systems was instrumental in determining certain sharp necessary conditions such as those of Lemma 1.3.1, has been traded with the weaker, but still very reasonable, requirement of boundedness of all trajectories. We also assume that all trajectories of the exosystem (1.85) are bounded forward and backward in time, and that the equilibrium $w = 0$ of (1.85) is stable in the sense of Lyapunov, again forward and backward in time.

The point of departure of the analysis is a nonlinear analogue of the geometric interpretation, given in Remark 1.3.1, of the conditions in Lemma 1.3.1. Let

$$\begin{aligned} \pi &: \mathbb{R}^r \to \mathbb{R}^n \\ \sigma &: \mathbb{R}^r \to \mathbb{R}^\nu \end{aligned}$$

be two smooth mappings, and suppose that the smooth manifold

$$\mathcal{M}_0 = \{(x, \xi, w) : x = \pi(w), \xi = \sigma(w)\}$$

is invariant for the forced closed-loop system (1.87). To say that $\mathcal{M}_0$ is invariant for (1.87) is to say that $\pi(w)$ and $\sigma(w)$ are solutions of the pair of partial differential equations

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), \theta(\sigma(w), k(\pi(w), w)), w) \\ \frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), k(\pi(w), w))\,. \end{aligned} \tag{1.88}$$

These equations are the nonlinear counterparts of the linear Equations (1.13). In fact, if the mappings $\pi(w)$ and $\sigma(w)$ were linear functions, like $x = \Pi w$ and $\xi = \Sigma w$, and if (1.87) were a linear system like (1.10), these equations would reduce exactly to Equations (1.13), and the manifold $\mathcal{M}_0$ would reduce to the subspace $\mathcal{V}_0$.

Suppose now that in system (1.87) all trajectories with initial conditions in a set $\mathcal{X} \times \Xi \times \mathcal{W}$ are *bounded* and *attracted* by the manifold $\mathcal{M}_0$ (as happens in the case of linear systems, if the unforced closed-loop system is asymptotically stable) and that the regulated output e is zero at each point of $\mathcal{M}_0$, i.e., that

$$0 = h(\pi(w), w)\,. \tag{1.89}$$

Then, obviously, $\lim_{t\to\infty} e(t) = 0$. Indeed, condition (1.89) is the nonlinear counterpart of condition (1.14).

We can in this way assert that, if there are mappings $\pi(w)$ and $\sigma(w)$ such that (1.88) and (1.89) hold and, in the forced closed-loop system (1.87), all trajectories with initial conditions in a set $\mathcal{X} \times \Xi \times \mathcal{W}$ are bounded and attracted by the manifold $\mathcal{M}_0$, the controller (1.86) solves the generalized tracking problem.

As in Section 1.3, the equations which $\pi(w)$ and $\sigma(w)$ are expected to solve can be rewritten as two separate sets of equations, the nonlinear counterparts of (1.16) and (1.17), which can be given similar interpretations. As a matter of fact, it is clear that conditions (1.88) and (1.89) hold if and only if there exists a triplet of mappings $\pi(w), \sigma(w), c(w)$ such that

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), c(w), w) \\ 0 &= h(\pi(w), w) \end{aligned} \tag{1.90}$$

and

$$\begin{aligned} \frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), k(\pi(w), w)) \\ c(w) &= \theta(\sigma(w), k(\pi(w), w))\,. \end{aligned} \tag{1.91}$$

Equations (1.90) are referred to as the *nonlinear regulator equations*. The first one of these expresses the property that the submanifold

$$\mathcal{S} = \{(x, w) : x = \pi(w)\}$$

is a *controlled invariant submanifold* for the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ \dot{w} &= s(w)\,. \end{aligned} \tag{1.92}$$

In fact, $\mathcal{S}$ is by construction invariant for

$$\begin{aligned} \dot{x} &= f(x, c(w), w) \\ \dot{w} &= s(w) \end{aligned} \tag{1.93}$$

which is the autonomous nonlinear system obtained controlling (1.92) with "feedback" law $u = c(w)$. The second one, on the other hand, expresses the property that the regulated output is zero at each point of the submanifold $\mathcal{S}$.

As in Section 1.3, it is clear that, for any initial condition $w(0)$ of the exosystem (1.85), if:

- the initial condition $x(0)$ of (1.84) is equal to $x(0) = \pi(w(0))$,
- the control input $u(t)$ of (1.84) is equal to $u(t) = c(w(t))$,

then $e(t) = 0$ for all $t \in \mathbb{R}$. In fact, any trajectory of system (1.93) with $x(0) = \pi(w(0))$ satisfies $x(t) = \pi(w(t))$ for all $t \in \mathbb{R}$, because $\mathcal{S}$ is invariant. But this trajectory is such that $x(t)$ can be interpreted as the response of the open-loop system (1.84) to the control input $u(t) = c(w(t))$ and to the disturbance input $w(t)$. Since the regulated output is zero at any point of $\mathcal{S}$ and $(x(t), w(t))$ remains in $\mathcal{S}$ for all $t \in \mathbb{R}$, it is concluded that $e(t) = 0$ for all $t \in \mathbb{R}$. In other words, $u(t) = c(w(t))$ is a *feedforward input* capable of keeping $e(t)$ identically at zero, if the initial condition of (1.84) is appropriately set.

Equations (1.91), which will be referred to as the *nonlinear internal model property*, express the fact the control input $c(w(t))$ in question can be viewed to be generated by the autonomous finite-dimensional nonlinear dynamical system

$$\begin{aligned} \dot{\xi} &= \phi(\xi, k(\pi(w), w)) \\ \dot{w} &= s(w) \\ u &= \theta(\xi, k(\pi(w), w)) \,. \end{aligned} \tag{1.94}$$

In fact, the first one expresses the property that the submanifold

$$\mathcal{R} = \{(\xi, w) : \xi = \sigma(w)\}$$

is invariant for (1.94). Thus, if $\xi(0) = \sigma(w(0))$, then $\xi(t) = \sigma(w(t))$ for all $t \in \mathbb{R}$. As a consequence, for those initial conditions, the output $u(t)$ of (1.94) becomes

$$u(t) = \theta(\sigma(w(t)), k(\pi(w(t)), w(t))) = c(w(t))$$

since the second one of (1.91) holds.

As for Equations (1.17) in the case of linear systems, Equations (1.91) essentially express the property that, embedded in the controller, there is a generator for those control inputs $c(w(t))$ which are capable of keeping the regulated output $e(t)$ identically at zero.

We summarize this discussion in a form which provides a nonlinear analogue of Proposition 1.3.3.

Proposition 1.7.1. *Suppose a controller of the form (1.86) is such that conditions (1.90) and (1.91) hold, for some triplet of mappings* $\pi(w), \sigma(w), c(w)$. *Suppose that all trajectories of the forced closed-loop system, with initial conditions in a set* $\mathcal{X} \times \Xi \times \mathcal{W}$, *are bounded and attracted by the manifold* $\mathcal{M}_0$. *Then, the controller solves the generalized tracking problem.*

Note that if the measured output y and regulated output e coincide, condition (1.91) simplifies. In fact, if $k(x, w) = h(x, w)$, since the mapping $\pi(w)$ by hypothesis satisfies $h(\pi(w), w) = 0$, condition (1.91) reduces to

$$\begin{aligned} \frac{\partial \sigma}{\partial w} s(w) &= \phi(\sigma(w), 0) \\ c(w) &= \theta(\sigma(w), 0) \,. \end{aligned} \tag{1.95}$$

The same kind of simplification occurs, of course, if $k(x, w)$ is zero at any point of the manifold $\mathcal{S}$.

1.8 Design Methods for Nonlinear Systems

The existence of a solution pair $\pi(w), c(w)$ for the regulator Equations (1.90) is a condition that does not depend on the specific controller used. On the other hand, the existence of a mapping $\sigma(w)$ which, along with that particular pair $\pi(w), c(w)$, satisfies (1.91) is a property of the specific controller. As seen in Section 1.4, it is important to look at controllers whose structure automatically guarantees the existence of such $\sigma(w)$. To simplify matters, we *assume*, in what follows, that the map $k(x,w)$ which defines the measured output satisfies

$$k(\pi(w),w) = 0\,, \tag{1.96}$$

so that the simplified version (1.95) of the internal model property applies. With a view to the design procedure illustrated in Section 1.5, which we plan to extend to the case of nonlinear systems, we consider controllers of the form

$$\begin{aligned} \dot\xi' &= \varphi(\xi') + \Delta(\xi'',y) \\ \dot\xi'' &= L(\xi'',y) \\ u &= \gamma(\xi') + N(\xi'',y)\,, \end{aligned} \tag{1.97}$$

in which $\varphi(\xi'), \gamma(\xi')$ and, respectively, $\Delta(\xi'',y), L(\xi'',y), N(\xi'',y)$ are smooth functions vanishing at $(\xi',\xi'',y) = (0,0,0)$. This being the case, it is immediately realized that the internal model property (1.95) holds if there exists a mapping $\sigma'(w)$ such that

$$\begin{aligned} \frac{\partial\sigma'}{\partial w}s(w) &= \varphi(\sigma'(w)) \\ c(w) &= \gamma(\sigma'(w))\,. \end{aligned} \tag{1.98}$$

In fact, it is trivial to check that, if $\sigma'(w)$ satisfies (1.98), the mapping

$$\sigma(w) = \begin{pmatrix} \sigma'(w) \\ \sigma''(w) \end{pmatrix} = \begin{pmatrix} \sigma'(w) \\ 0 \end{pmatrix}$$

satisfies (1.95).

Remark 1.8.1. Equations (1.98) can be interpreted in these terms. Consider the pair of autonomous systems with output

$$\dot w = s(w)\,, \qquad u = c(w) \tag{1.99}$$

and

$$\dot\xi' = \varphi(\xi')\,, \qquad u = \gamma(\xi')\,. \tag{1.100}$$

These systems are defined on two different state spaces, as $w \in \mathbb{R}^r$ and $\xi' \in \mathbb{R}^{\nu'}$, but have a common output space, as $u \in \mathbb{R}^m$. Suppose a mapping $\sigma'(w)$ fulfilling (1.98) exists, pick any initial condition $w(0) \in \mathbb{R}^r$, and choose, as initial condition $\xi'(0) \in \mathbb{R}^{\nu'}$ the value $\xi'(0) = \sigma'(w(0))$. Then, it is easily checked that, for all $t \in \mathbb{R}$,

$$\xi'(t) = \sigma'(w(t))\,.$$

In fact, fulfilment of the first condition (1.98) guarantees that $\sigma'(w(t))$ is a solution of the differential equation $\dot\xi' = \varphi(\xi')$, the unique solution satisfying $\xi'(0) = \sigma'(w(0))$. If the second condition in (1.98) also holds, the trajectory $w(t)$ of (1.99) and the trajectory $\xi'(t)$ of (1.100) satisfy

$$c(w(t)) = \gamma(\xi'(t))\,.$$

Thus, for any $w(0) \in \mathbb{R}^r$ there is a state $\xi'(0) \in \mathbb{R}^{\nu'}$ such that the output generated by (1.100) from $\xi'(0) \in \mathbb{R}^{\nu'}$ reproduces exactly the output generated by (1.99) from $w(0) \in \mathbb{R}^r$. In other words, system (1.100) is a system able to generate all possible outputs generated by system (1.99) and it is for this reason that it is customary to say that, if condition (1.98) holds, system (1.99) is *immersed* into system (1.100). ◁

Thus, a structure of the form (1.97) guarantees the fulfilment of the internal model property so long as the functions $\varphi(\xi')$ and $\gamma(\xi')$ are such that (1.98) holds for some $\sigma'(w)$. Since the condition in question is indeed trivially satisfied if (1.100) coincides with (1.99), being $\sigma'(w)$ the identity map, one may be tempted to use a controller of the form

$$\begin{aligned}\dot{\xi}' &= s(\xi') + \Delta(\xi'', y)\\ \dot{\xi}'' &= L(\xi'', y)\\ u &= c(\xi') + N(\xi'', y)\,.\end{aligned}$$

However, this solution usually doesn't work, because it might result in a closed-loop system for which it is impossible to obtain the desired asymptotic properties. For instance, it may be impossible to choose the remaining functions $\Delta(\xi'', y), L(\xi'', y), N(\xi'', y)$ so as to make the manifold $\mathcal{M}_0$ attractive. To better understand this point a comparison with the (robust) linear case is helpful. In the case of a μ-dependent linear system (having assumed that the simplifying assumption (1.96) continues to hold) system (1.99) is a system of the form

$$\begin{pmatrix}\dot{w}\\ \dot{\mu}\end{pmatrix} = \begin{pmatrix}Sw\\ 0\end{pmatrix}, \qquad u = R(\mu)w\,.$$

Embedding an identical copy of this system in (1.97), and choosing linear functions $\Delta(\xi'', y), L(\xi'', y), N(\xi'', y)$, would yield a controller of the form

$$\begin{aligned}\dot{\xi}'_1 &= S\xi'_1 + \Delta_1\xi'' + \Theta_1 y\\ \dot{\xi}'_2 &= \Delta_2\xi'' + \Theta_2 y\\ \dot{\xi}'' &= L\xi'' + My\\ u &= R(\xi'_2)\xi'_1 + N\xi'' + Ky\,.\end{aligned}$$

The controller thus obtained, which – as expected – is nonlinear, has a linear approximation which is *not detectable*, as a simple verification shows. Thus, if this controller is used, obtaining even local stability for the forced closed-loop system would be a difficult, if not impossible, task. This obstruction did not appear in the analysis described before because, in (1.53), the observable pair

$$\varphi(\xi') = \Phi\xi'\,, \qquad \gamma(\xi') = \Gamma\xi'\,. \tag{1.101}$$

was used instead.

This observation suggests the opportunity of looking for a pair $\varphi(\xi'), \gamma(\xi')$ having similar properties. As a matter of fact, such a pair exists, under appropriate – but not terribly restrictive – hypotheses. One of these hypotheses is that the *exosystem* is a *linear* system. Since in this case the exogenous input w may contain constant uncertain parameters, it is convenient – without loss of generality – to change coordinates in the state space of the exosystem so as to separate the constant modes from the oscillatory ones and obtain

$$\begin{aligned} \dot{w}_1 &= S w_1 \\ \dot{w}_2 &= 0\,, \end{aligned} \tag{1.102}$$

in which S has only nonzero purely imaginary eigenvalues, with simple multiplicity in the minimal polynomial (recall that the system in question was assumed to be neutrally stable). The other hypothesis is that the map $c(w)$ is a *polynomial*, in the various components of w_1, with coefficients which are arbitrary smooth functions of w_2. In this case, the condition (1.98) can be fulfilled by means of a pair of functions $\varphi(\xi'), \gamma(\xi')$ of the form (1.101).

To show that this is the case, it is convenient to recall a standard differential operation which, given a map $c : \mathbb{R}^r \to \mathbb{R}^m$, defines a new map $L_s c : \mathbb{R}^r \to \mathbb{R}^m$ as

$$L_{s(w)} c(w) = \frac{\partial c}{\partial w} s(w)\,.$$

This operation will be used repeatedly, with

$$L^k_{s(w)} c(w) = L_{s(w)} L^{k-1}_{s(w)} c(w)\,, \qquad L^0_{s(w)} c(w) = c(w)\,.$$

Then, we have the following interesting result.

Lemma 1.8.1. *Suppose the exosystem (1.85) is a linear system. Suppose that the map $c(w)$ is a polynomial in the components of w_1, with w_2-dependent coefficients. Then, there exist an integer s and real numbers $a_0, a_1, \ldots, a_{s-1}$ such that*

$$L^s_{s(w)} c(w) + a_{s-1} L^{s-1}_{s(w)} c(w) + \cdots + a_1 L_{s(w)} c(w) + a_0 c(w) = 0\,. \tag{1.103}$$

As a consequence, the map

$$\sigma'(w) = \begin{pmatrix} c(w) \\ L_{s(w)} c(w) \\ \cdots \\ L^{s-1}_{s(w)} c(w) \end{pmatrix} \tag{1.104}$$

is such that

$$\begin{aligned} \frac{\partial \sigma'}{\partial w} s(w) &= \Phi \sigma'(w) \\ c(w) &= \Gamma \sigma'(w)\,, \end{aligned} \tag{1.105}$$

with Φ and Γ as in (1.35).

Proof. Consider, for simplicity, the case $m = 1$, and note that, if the exosystem is written in the form (1.102), we have

$$L_{s(w)}c(w) = \frac{\partial c}{\partial w_1} S w_1 \,. \tag{1.106}$$

Now, if $c(w)$ is a polynomial in the components of w_1, of degree less than or equal to k, then also the right-hand side of (1.106) is a polynomial in w_1 of degree less than or equal to k. In other words, the set $\mathcal{P}_k$ of all polynomials in w_1 of degree less than or equal to k with real coefficients, is a finite-dimensional vector space which is closed under the action of the mapping

$$\begin{array}{rcl} L_{Sw_1} : \mathcal{P}_k & \to & \mathcal{P}_k \\ p(w_1) & \mapsto & \dfrac{\partial p}{\partial w_1} S w_1 \,. \end{array} \tag{1.107}$$

Since L_{Sw_1} is a linear mapping of the finite-dimensional vector space $\mathcal{P}_k$ into itself, its minimal polynomial

$$m(\lambda) = \lambda^s + a_{s-1}\lambda^{s-1} + \cdots + a_1\lambda + a_0$$

is such that

$$L^s_{Sw_1} p(w_1) + a_{s-1} L^{s-1}_{Sw_1} p(w_1) + \cdots + a_1 L_{Sw_1} p(w_1) + a_0 c(w) = 0 \,,$$

for any polynomial $p(w_1)$ in $\mathcal{P}_k$. This, in view of (1.106), proves (1.103). The map $\sigma'(w)$ indicated in the lemma indeed satisfies

$$\frac{\partial \sigma'}{\partial w} s(w) = \begin{pmatrix} L_{s(w)}c(w) \\ L^2_{s(w)}c(w) \\ \cdots \\ L^s_{s(w)}c(w) \end{pmatrix}$$

and this, in view of (1.103) and of the definition (1.35) of Φ and Γ, proves the lemma. The proof in the case $m > 1$ is a trivial extension. ◁

The hypotheses that the exosystem is a linear system and that $c(w)$ is a polynomial in the (nontrivial) components of w make it possible to have the internal model property fulfilled by means of a linear, and observable, pair of functions $\varphi(\xi'), \gamma(\xi')$. This fact indeed simplifies the subsequent stage of the design, in which stabilization has to be achieved. For convenience, we summarize the discussion up to this point in a way that, to some extent, provides a nonlinear counterpart of Proposition 1.5.1.

Proposition 1.8.2. *Consider system (1.84). Suppose the exosystem is linear, as in (1.102). Suppose there exist mappings $\pi(w)$ and $c(w)$ satisfying (1.90) for all $w \in \mathbb{R}^r$. Suppose, in particular, that $c(w)$ is a polynomial in the components of w_1, with w_2-dependent coefficients. Let $\sigma'(w)$ be the map* *defined in (1.104) and Φ and Γ the matrices defined in (1.35). Consider a controller of the form*

$$\begin{aligned} \dot{\xi}' &= \Phi\xi' + \Delta(\xi'', y) \\ \dot{\xi}'' &= L(\xi'', y) \\ u &= \Gamma\xi' + N(\xi'', y) \,. \end{aligned} \tag{1.108}$$

Suppose $\Delta(\xi'', y), L(\xi'', y), N(\xi'', y)$ are such that, in the corresponding forced closed-loop system, all trajectories with initial conditions in a set $\mathcal{X} \times \Xi \times \mathcal{W}$ are bounded and attracted by the manifold

$$\mathcal{M}_0 = \{(x, \xi', \xi'', w) : x = \pi(w), \xi' = \sigma'(w), \xi'' = 0\} \,.$$

Then, this controller solves the generalized tracking problem.

As in the case of linear systems, this kind of result reduces a generalized problem of tracking to a stabilization problem. Note, in this respect, that if in the closed-loop system one changes coordinates as

$$\begin{aligned} \tilde{x} &= x - \pi(w) \\ \tilde{\xi}' &= \xi' - \sigma'(w) \,, \end{aligned} \tag{1.109}$$

the manifold $\mathcal{M}_0$ becomes the set

$$\mathcal{M}_0 = \{(\tilde{x}, \tilde{\xi}', \xi'', w) : \tilde{x} = 0, \tilde{\xi}' = 0, \xi'' = 0\} \,.$$

We describe in what follows two examples of how the design of the controller can be completed. The first of these illustrates a controller in which convergence of the trajectories to $\mathcal{M}_0$ is achieved only for initial conditions in *some* neighborhood of the equilibrium point $(w, x, \xi) = (0, 0, 0)$. Suppose the various assumptions of Proposition 1.8.2 are fulfilled, i.e., that the exosystem is linear and that there exist mappings $\pi(w)$ and $c(w)$ satisfying (1.90), with $c(w)$ a polynomial in the components of w_1, with w_2-dependent coefficients. Choose a controller of the form

$$\begin{aligned} \dot{\xi}' &= \Phi\xi' + \Theta_1 e \\ \dot{\xi}'' &= L\xi'' + My \\ u &= \Gamma\xi' + N\xi'' \,, \end{aligned} \tag{1.110}$$

that is, a controller identical to (1.53). The corresponding forced closed-loop system has the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x, \Gamma\xi' + N\xi'', w) \\ \dot{\xi}' &= \Phi\xi' + \Theta_1 h(x, w) \\ \dot{\xi}'' &= L\xi'' + Mk(x, w) \,. \end{aligned} \tag{1.111}$$

Changing variables as in (1.109) and using (1.90) and (1.105) yields

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\tilde{x}} &= f(\tilde{x}+\pi(w), \Gamma\tilde{\xi}' + N\xi'' + c(w), w) - f(\pi(w), c(w), w) \\ \dot{\tilde{\xi}}' &= \Phi\tilde{\xi}' + \Theta_1 h(\tilde{x}+\pi(w), w) \\ \dot{\xi}'' &= L\xi'' + Mk(\tilde{x}+\pi(w), w)\,. \end{aligned} \tag{1.112}$$

This system will be referred to as the *error system*. The subsystem consisting of the three bottom equations has by construction an equilibrium at $(\tilde{x}, \tilde{\xi}', \xi'') = (0,0,0)$ which corresponds, as observed before, to the manifold $\mathcal{M}_0$. Thus, if this equilibrium can be rendered asymptotically stable, the tracking problem is solved.

Now, consider the *linear approximation* of the three bottom equations of (1.112) at the point

$$(\tilde{x}, \tilde{\xi}', \xi'', w) = (0,0,0,0)\,. \tag{1.113}$$

Elementary manipulations show that this approximation is a system of the form (for convenience we retain, for the state variables, the same notation as in (1.112))

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\Gamma\tilde{\xi}' + BN\xi'' \\ \dot{\tilde{\xi}}' &= \Phi\tilde{\xi}' + \Theta_1 C_1\tilde{x} \\ \dot{\xi}'' &= L\xi'' + MC\tilde{x}\,, \end{aligned} \tag{1.114}$$

in which

$$A = \left[\frac{\partial f}{\partial x}\right]_{(0,0,0)} \qquad B = \left[\frac{\partial f}{\partial u}\right]_{(0,0,0)}$$

$$C_1 = \left[\frac{\partial h}{\partial x}\right]_{(0,0)} \qquad C = \left[\frac{\partial k}{\partial x}\right]_{(0,0)}.$$

The linear system (1.114) is identical to system (1.54) and therefore the result of Corollary 1.5.3 applies. In other words, if A, B, C, C_1 are such that the hypotheses of Lemma 1.5.2 hold, there exist L, M, N such that this system is asymptotically stable. By the principle of stability of the first approximation, the equilibrium $(\tilde{x}, \tilde{\xi}', \xi'') = (0,0,0)$ of the three bottom equations of (1.112) is *locally* asymptotically stable. Hence, there exist open neighborhoods $\mathcal{X}$, $\Xi' \times \Xi''$ and $\mathcal{W}$ of the point (1.113) such that, for all initial conditions in these sets, trajectories are bounded and converge to the set $\mathcal{M}_0$. The controller in question solves the generalized tracking problem for these sets of initial data.

The second example of design considers a nonlinear system whose properties are similar to those assumed for system (1.58) in Section 1.5. In this case, it is possible to design a controller that solves the generalized tracking problem for any *arbitrarily large*, but compact, set of initial data. For convenience, we limit the analysis to the case of a system in which $m = 1$, and assume first of all that the right-hand side of (1.84) is an affine function of u, i.e., that the system in question is modeled by equations of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x,0,w) + g(x,w)u \\ e &= h(x,w) \\ y &= k(x,w)\,. \end{aligned}$$

Under appropriate hypotheses, which are discussed in detail, for example in [29, Chapter 9], there exists a change of coordinates that transforms the last three equations of this system into equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \bar{C}\tilde{x}_2, w) \\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}(f_2(x_1,\tilde{x}_2,w) + b(x_1,\tilde{x}_2,w)u) \\ e &= \bar{C}\tilde{x}_2 \\ y &= \tilde{x}_2 \end{aligned} \tag{1.115}$$

in which $\dim(x_1) = n - r$, $\dim(\tilde{x}_2) = r$, $b(x_1, \tilde{x}_2, w) \neq 0$, and $\bar{A}, \bar{B}$ are the matrices introduced in Section 1.5. The first three equations in (1.115) are the nonlinear counterpart of Equations (1.59). The last equation, on the other hand, reflects the simplifying assumption that all components of the vector $\tilde{x}_2$ are available for measurement (see Remark 1.5.7).

Remark 1.8.2. Note that while in (1.59) the full vector $\tilde{x}_2$ is allowed in the right-hand side of $\dot{x}_1$, in (1.115) only $\bar{C}\tilde{x}_2$, i.e. only the first component of $\tilde{x}_2$ (which coincides with e), is present. For a linear system, this does not make any special difference, and in fact it is possible to show that Equations (1.59) can always be transformed into equations in which only $\bar{C}\tilde{x}_2$ is present in the right-hand side of $\dot{x}_1$. For a nonlinear system, though, the existence of the "normal" form (1.115) is something more restrictive than the existence of a form in which full vector $\tilde{x}_2$ is allowed in the right-hand side of $\dot{x}_1$. The special structure of (1.115), on the other hand, substantially eases the design of stabilizing control. $\triangleleft$

System (1.115) is supposed to satisfy a number of additional assumptions, which in the case of a linear system are either automatically satisfied or consequences of assumptions considered in Proposition 1.5.7. These assumptions are the following ones:

(i) There exists a smooth mapping $\zeta(w)$ satisfying

$$\frac{\partial \zeta}{\partial w} s(w) = f_1(\zeta(w), 0, w)\,.$$

If this is the case, the regulator Equations (1.90) have a solution. In fact, splitting $\pi(w)$ in two blocks $\pi_1(w), \pi_2(w)$ consistently with the partition of the state vector in (1.115), it is easy to check that the mappings defined as

$$\pi(w) = \begin{pmatrix} \pi_1(w) \\ \pi_2(w) \end{pmatrix} = \begin{pmatrix} \zeta(w) \\ 0 \end{pmatrix}$$

and

$$c(w) = -\frac{f_2(\zeta(w), 0, w)}{b(\zeta(w), 0, w)}$$

solve the regulator equations.

(ii) The exosystem is *linear*, as in (1.102), and the map $c(w)$ is a *polynomial*, in the various components of w_1, with coefficients which are arbitrary smooth functions of w_2.

(iii) The coefficient $b(x_1, \tilde{x}_2, w)$ only depends on the component w_2 of w, and is bounded from below by a positive number $\bar{b}$.

If these assumptions hold, the dynamics of (1.115) can be conveniently transformed, by means of the change of variable

$$\tilde{x}_1 = x_1 - \zeta(w)\,,$$

into a set of equations of the form (compare with (1.62))

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{f}_1(\tilde{x}_1, \bar{C}\tilde{x}_2, w) \\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\tilde{f}_2(\tilde{x}_1, \tilde{x}_2, w) + \bar{B}\tilde{b}(w_2)(u - c(w))\,, \end{aligned} \tag{1.116}$$

in which we have written $\tilde{b}(w_2)$ for $b(x_1, \tilde{x}_2, w)$, and in which

$$\begin{aligned} \tilde{f}_1(\tilde{x}_1, \bar{C}\tilde{x}_2, w) &= f_1(\tilde{x}_1 + \zeta(w), \bar{C}\tilde{x}_2, w) - f_1(\zeta(w), 0, w) \\ \tilde{f}_2(\tilde{x}_1, x_2, w) &= f_2(x_1 + \zeta(w), \tilde{x}_2, w) - f_2(\zeta(w), 0, w)\,. \end{aligned}$$

Note, in this respect, that

$$\begin{aligned} \tilde{f}_1(0, 0, w) &= 0 \\ \tilde{f}_2(0, 0, w) &= 0\,. \end{aligned}$$

Moreover, the result of Lemma 1.8.1 applies, i.e. there exists a mapping $\sigma'(w)$ such that (1.105) hold, for a pair of matrices Φ and Γ of the form (1.35) in which Φ has only eigenvalues on the imaginary axis and Γ is such that the pair (Γ, Φ) is observable.

Appealing to Lemma 1.5.6, let F_0 be an $s \times s$ Hurwitz matrix (with s equal to the dimension of Φ) and G_0 an $s \times 1$ vector such that the pair (F_0, G_0) is controllable, let T and Ψ be such that

$$T^{-1}(F_0 + G_0\Psi)T = \Phi\,, \qquad \Psi T = \Gamma\,, \tag{1.117}$$

and consider again the controller (1.72), namely the controller

$$\begin{aligned} \dot{\xi}' &= F_0\xi' + G_0 u \\ u &= \Psi\xi' + v\,, \end{aligned} \tag{1.118}$$

in which v is an additional input, to be used for stabilization. Note, in this respect, that a result similar to that indicated in Remark 1.5.5 holds. In fact, composing (1.105) with (1.117), it is seen that the mapping

$$\tau(w) = T\sigma'(w)$$

satisfies

$$\begin{aligned} \frac{\partial \tau}{\partial w} s(w) &= (F_0 + G_0\Psi)\tau(w) \\ c(w) &= \Psi\tau(w)\,. \end{aligned} \tag{1.119}$$

Controlling (1.116) by means of (1.118) yields the system

$$\begin{aligned} \dot{\xi}' &= (F_0 + G_0\Psi)\xi' + G_0 v \\ \dot{\tilde{x}}_1 &= \tilde{f}_1(\tilde{x}_1, \bar{C}\tilde{x}_2, w) \\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\tilde{f}_2(\tilde{x}_1, \tilde{x}_2, w) + \bar{B}\tilde{b}(w_2)(\Psi\xi' + v - c(w))\,. \end{aligned}$$

Changing the state variable ξ' in

$$\tilde{\xi}' = \xi' - \tau(w)\,,$$

using (1.119) and adding the dynamics of the exosystem, a system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\tilde{\xi}}' &= (F_0 + G_0\Psi)\tilde{\xi}' + G_0 v \\ \dot{\tilde{x}}_1 &= \tilde{f}_1(\tilde{x}_1, \bar{C}\tilde{x}_2, w) \\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\tilde{f}_2(\tilde{x}_1, \tilde{x}_2, w) + \bar{B}\tilde{b}(w_2)(\Psi\tilde{\xi}' + v) \end{aligned} \tag{1.120}$$

is obtained (compare with system (1.73), in which there was no need to add the exosystem, as the equations were independent of w and μ was a constant parameter).

This system plays a role identical to that of the *error system* (1.112) of the previous example. In fact, the point $(\tilde{\xi}', \tilde{x}_1, \tilde{x}_2) = (0, 0, 0)$, which is an equilibrium for $v = 0$, corresponds to the manifold

$$\mathcal{M}_0 = \{(\xi', x_1, \tilde{x}_2) : \xi' = \tau(w), x_1 = \zeta_1(w), \tilde{x}_2 = 0\}\,.$$

On this manifold, the regulated output e is zero. Thus, if a control

$$v = v(\tilde{x}_2)$$

can be found, with $v(0) = 0$, such that all trajectories are bounded and the manifold $\mathcal{M}_0$ is attractive, the controller (1.118) solves the generalized tracking problem.

As a matter of fact, under an assumption which corresponds to the hypothesis, in Proposition 1.5.7, that the eigenvalues of $A_{11}(\mu)$ have negative real part, such a control v exists, as shown in the following result.

Proposition 1.8.3. *Consider system (1.115). Suppose the assumptions* (i), (ii), (iii) *above hold. Suppose also that there exist a smooth function $V : \mathbb{R}^{n-r} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$ and real numbers $\delta > 0$, $a > 0$, $b > 0$ such that*

$$\underline{\alpha}(\|\tilde{x}_1\|) \leq V(\tilde{x}_1) \leq \overline{\alpha}(\|\tilde{x}_1\|)$$
$$\frac{\partial V}{\partial \tilde{x}_1} \tilde{f}_1(\tilde{x}_1, 0, w) \leq -\alpha(\|\tilde{x}_1\|)$$

for all $\tilde{x}_1 \in \mathbb{R}^{n-r}$ and for all w, and

$$\underline{\alpha}(s) = as^2, \qquad \alpha(s) = bs^2$$

for all $s \in [0, \delta]$. Then, for every choice of compact sets $\mathcal{X}$, Ξ, $\mathcal{W}$, there is an r-dimensional row vector N such that all trajectories of system (1.120), with control $v = N\tilde{x}_2$, for any initial condition in the set $\mathcal{X} \times \Xi \times \mathcal{W}$ are bounded and attracted by the set $\mathcal{M}_0$. As a consequence, the controller

$$\begin{aligned} \dot{\xi}' &= (F_0 + G_0\Psi)\xi' + G_0 N y \\ u &= \Psi\xi' + Ny\,, \end{aligned}$$

solves the generalized tracking problem for the given set of initial data.

Sketch of the proof. Changing, in (1.120), coordinates as

$$\chi = \tilde{\xi}' - \frac{1}{b(w_2)} G_0 \bar{C} \bar{A}^{r-1} \tilde{x}_2\,,$$

yields a system of the following structure (compare with (1.74))

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{\chi} &= F_0\chi + \varphi_0(\tilde{x}_1, \tilde{x}_2, w) \\ \dot{\tilde{x}}_1 &= \tilde{f}_1(\tilde{x}_1, \bar{C}\tilde{x}_2, w) \\ \dot{\tilde{x}}_2 &= \bar{A}\tilde{x}_2 + \bar{B}\varphi_2(\tilde{x}_1, \tilde{x}_2, \chi, w) + \bar{B}\tilde{b}(w_2)v\,, \end{aligned}$$

in which $\varphi_0(\tilde{x}_1, \tilde{x}_2, w)$ and $\varphi_2(\tilde{x}_1, \tilde{x}_2, \chi, w)$ are such that

$$\varphi_0(0, 0, w) = 0\,, \qquad \varphi_2(0, 0, 0, w) = 0\,.$$

From this, standard arguments prove that, since F_0 is a Hurwitz matrix and since the system

$$\dot{\tilde{x}}_1 = \tilde{f}_1(\tilde{x}_1, 0, w)$$

has the properties indicated in the proposition, a matrix N with the required the properties exists.[1] $\triangleleft$

We conclude the chapter with the discussion of an example of a class of nonlinear systems in which the conditions of this proposition are easily verifiable and the design procedure can be implemented.

Example 1.8.3. Consider a nonlinear system described by equations of the form

$$\begin{aligned}
\dot{z} &= A_{11}(w_2)z + p_0(\xi_1, w_1, w_2)\\
\dot{\xi}_1 &= \xi_2 + p_1(z, \xi_1, w_1, w_2)\\
&\cdots\\
\dot{\xi}_{r-1} &= \xi_r + p_{r-1}(z, \xi_1, \ldots, \xi_{r-1}, w_1, w_2)\\
\dot{\xi}_r &= p_r(z, \xi_1, \ldots, \xi_r, w_1, w_2) + b(w_2)u\\
e &= \xi_1 + q(w_1, w_2)\,,
\end{aligned} \tag{1.121}$$

in which z is a ℓ-dimensional vector and $\xi_1, \ldots, \xi_r, e, u$ are scalar variables. The exogenous inputs w_1 and w_2 are generated by a linear exosystem as in (1.102), with (w_1, w_2) ranging over a compact set $\mathcal{W}_1 \times \mathcal{W}_2$. If this is the case, in the equations above w_2 is a constant, possibly uncertain, parameter. Assume that:

(a) there is a positive definite symmetric matrix P and a number $a_0 > 0$ such that, for every $w_2 \in \mathcal{W}_2$,

$$PA_{11}(w_2) + A_{11}^{\mathrm{T}}(w_2)P \le -a_0 I\,,$$

(b) there is a number $\bar{b} > 0$ such that, for every $w_2 \in \mathcal{W}_2$, $b(w_2) \ge \bar{b}$,

(c) the functions $p_0(\xi_1, w_1, w_2), p_1(z, \xi_1, w_1, w_2), \ldots, p_r(z, \xi_1, \ldots, \xi_r, w_1, w_2)$ and $q(w_1, w_2)$ are polynomials in $z, \xi_1, \ldots, \xi_r, w_1$, with coefficients which are smooth functions of w_2.

This system can be transformed, by means of a simple recursive calculation, into a system of the form (1.115). In fact, set $\phi_1(w_1, w_2) = q(w_1, w_2)$, define

$$e_1 = \xi_1 + \phi_1(w_1, w_2)$$

and observe that

$$\dot{e}_1 = \dot{\xi}_1 + \frac{\partial \phi_1}{\partial w_1} S w_1 = \xi_2 + p_1(z, \xi_1, w_1, w_2) + \frac{\partial \phi_1}{\partial w_1} S w_1\,.$$

Thus, $\dot{e}_1$ can be expressed as

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[1] The details of how N is determined are not as simple as those indicated in the proof of Lemma 1.5.4. The interested reader can find a detailed presentation of this subject, for instance, in [30, Chapter 12].

$$\dot{e}_1 = \xi_2 + \phi_2(z, \xi_1, w_1, w_2)$$

in which $\phi_2(z, \xi_1, w_1, w_2)$, as a consequence of the hypotheses on the system, is a polynomial in z, ξ_1, w_1, with coefficients which are smooth functions of w_2. Set

$$e_2 = \xi_2 + \phi_2(z, \xi_1, w_1, w_2)$$

so that

$$\dot{e}_1 = e_2 \,.$$

In the same way we can give $\dot{e}_2$ a similar expression. In fact,

$$\dot{e}_2 = \dot{\xi}_2 + \frac{\partial \phi_2}{\partial z}\dot{z} + \frac{\partial \phi_2}{\partial \xi_1}\dot{\xi}_1 + \frac{\partial \phi_2}{\partial w_1} S w_1 \,,$$

and, using the expressions of $\dot{z}, \dot{\xi}_1, \dot{\xi}_2$, it is possible to write

$$\dot{e}_2 = \xi_3 + \phi_3(z, \xi_1, \xi_2, w_1, w_2)$$

in which $\phi_3(z, \xi_1, \xi_2, w_1, w_2)$, as a consequence of the hypotheses on the system, is a polynomial in z, ξ_1, ξ_2, w_1, with coefficients which are smooth functions of w_2. Set now

$$e_3 = \xi_3 + \phi_3(z, \xi_1, \xi_2, w_1, w_2)$$

and proceed recursively.

In this way one defines a (partial) set $\tilde{x}_2$ of new state variables as

$$\tilde{x}_2 = \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_r \end{pmatrix} = \begin{pmatrix} \xi_1 + \phi_1(w_1, w_2) \\ \xi_2 + \phi_2(z, \xi_1, w_1, w_2) \\ \cdots \\ \xi_r + \phi_r(z, \xi_1, \ldots, \xi_{r-1}, w_1, w_2) \end{pmatrix} . \tag{1.122}$$

Since the dependence of the various components of $\tilde{x}_2$ on the ξ_i is triangular, the map

$$(\xi_1, \ldots, \xi_r) \quad \rightarrow \quad \tilde{x}_2$$

defined by (1.122) is globally invertible and the ξ_i are polynomials in $z, \tilde{x}_2, w_1$, with coefficients which are smooth functions of w_2.

In the new variables, system (1.121) reads as

$$\begin{array}{rcl} \dot{z} & = & A_{11}(w_2)z + p_0(e_1 - q(w_1, w_2), w_1, w_2) \\ \dot{e}_1 & = & e_2 \\ & & \cdots \\ \dot{e}_{r-1} & = & e_r \\ \dot{e}_r & = & \psi(z, e_1, \ldots, e_r, w_1, w_2) + b(w_2)u \\ e & = & e_1 \end{array} \tag{1.123}$$

in which $p_0(e_1 - q(w_1, w_2), w_1, w_2)$ and $\psi(z, e_1, \ldots, e_r, w_1, w_2)$ are polynomials in $z, e_1, \ldots, e_r, w_1$, with coefficients which are smooth functions of w_2. This system is clearly a system of the form (1.115), with $x_1 = z$, $\tilde{x}_2 = \mathrm{col}(e_1, \ldots, e_r)$, and

$$\begin{aligned} f_1(x_1, \bar{C}\tilde{x}_2, w) &= A_{11}(w_2)z + p_0(e_1 - q(w_1, w_2), w_1, w_2) \\ f_2(x_1, \tilde{x}_2, w) &= \psi(z, e_1, \ldots, e_r, w_1, w_2) \\ b(x_1, \tilde{x}_2, w) &= b(w_2)\,. \end{aligned}$$

In particular, $\psi(z, e_1, \ldots, e_r, w_1, w_2)$ is a polynomial in $z, e_1, \ldots, e_r, w_1$.

We show now that hypotheses (i) and (ii) hold (hypothesis (iii) is identical to hypothesis (b) above). To this end, observe that $f_1(x_1, 0, w)$ is a function of the form

$$f_1(x_1, 0, w) = A_{11}(w_2)z + p(w_1, w_2)$$

in which $p(w_1, w_2)$ is a polynomial in w_1. To say that hypothesis (i) holds is to say that there exist a map $\zeta(w_1, w_2)$ such that

$$\frac{\partial \zeta}{\partial w_1} S w_1 = A_{11}(w_2)\zeta(w_1, w_2) + p(w_1, w_2)\,, \tag{1.124}$$

which, using the notation introduced in the proof of Lemma 1.8.1, can be written also as

$$L_{Sw_1}\zeta(w_1, w_2) = A_{11}(w_2)\zeta(w_1, w_2) + p(w_1, w_2)\,.$$

Now, let k be an integer such that, for every $w_2 \in \mathcal{W}_2$, the ℓ entries of $p(w_1, w_2)$ are polynomials of degree not exceeding k in w_1. If the ℓ entries of $\zeta(w_1, w_2)$ are polynomials of degree not exceeding k in w_1, the entries of $L_{Sw_1}\zeta(w_1, w_2)$ can be interpreted as values at $\zeta(w_1, w_2)$ of a linear mapping of the form (1.107). As a consequence, the equation above is an identity between the value of a linear map of a finite-dimensional vector space (the ℓ-fold Cartesian product of $\mathcal{P}_k$) into itself, and the value of an affine map (the one on the right-hand side) of this vector space into itself. In other words, the equation above can be seen as a Sylvester equation. Hypothesis (a) implies that the eigenvalues of $A_{11}(w_2)$ have negative real parts. Since the eigenvalues of S have zero real part, the eigenvalues of the map (1.107) have zero real part (see [39, Lemma 1.2]). Then, for every fixed w_2, the Sylvester equation in question does have a (unique) solution $\zeta(w_1, w_2)$, whose entries are polynomials in w_1. Moreover, since the equation is linear and the coefficients of $p(w_1, w_2)$, viewed as polynomial in w_1, are smooth functions of w_2, it follows that also the coefficients of $\zeta(w_1, w_2)$, viewed as polynomial in w_1, are smooth functions of w_2. We conclude in this way that hypothesis (i) holds.

To verify hypothesis (ii), it suffices to observe that in this case

$$c(w) = -\frac{\psi(\zeta(w_1, w_2), 0, \ldots, 0, w_1, w_2)}{b(w_2)}\,.$$

As $\psi(z, 0, \ldots, 0, w_1, w_2)$ is a polynomial in z, w_1, and $\zeta(w_1, w_2)$ is a polynomial in w_1, the hypothesis in question indeed holds.

Finally, we check that also the main hypothesis of Proposition 1.8.3 holds. To this end, note that the change of variables $\tilde{x}_1 = z - \zeta(w_1, w_2)$ yields

$$\tilde{f}_1(\tilde{x}_1, 0, w) = A_{11}(w_2)\tilde{x}_1 \, .$$

Hence, the function $V(\tilde{x}_1) = \tilde{x}_1^{\mathrm{T}} P \tilde{x}_1$ has the desired properties. ◁