1 Introduction to Uncertain Systems

1.1 Uncertainty and Uncertain Systems

Uncertainty is one of the main features of complex and intelligent decision making systems. Various approaches, methods and techniques in this field have been developed for several decades, starting with such concepts and tools as adaptation, stochastic optimization and statistical decision theory (see e.g. [2, 3, 68, 79, 80]). The first period of this development was devoted to systems described by traditional mathematical models with unknown parameters. In the past two decades new ideas (such as learning, soft computing, linguistic descriptions and many others) have been developed as a part of modern foundations of knowledge-based Decision Support Systems (DSS) in which the decisions are based on *uncertain knowledge*. Methods and algorithms of decision making under uncertainty are especially important for design of computer control and management systems based on incomplete or imperfect knowledge of a decision plant. Consequently, problems of analysis and decision making in uncertain systems are related to the following fields:

1. General systems theory and engineering.
2. Control and management systems.
3. Information technology (knowledge-based expert systems).

There exists a great variety of definitions and formal models of uncertainties and uncertain systems. The most popular non-probabilistic approaches are based on fuzzy sets theory and related formalisms such as evidence and possibility theory, rough sets theory and fuzzy measures, including a probability measure as a special case (e.g. [4, 7, 9, 64, 65, 67, 69, 71, 74, 75, 78, 81, 83, 84, 96–100, 103, 104]). The different formulations of decision making problems and various proposals for reasoning under uncertainty are adequate for the different formal models of uncertainty. On the other hand, new forms of uncertain knowledge representations require new concepts and methods of information processing: from computing with numbers to granular computing [5, 72] and computing with words [101].

Special approaches have been presented for multiobjective programming and scheduling under uncertainty [91, 92], for uncertain object-oriented databases [63], and for uncertainty in expert systems [89]. A lot of works have been concerned with specific problems of uncertain control systems, including problems of stability and stabilization of uncertain systems and an idea of robust control (e.g. [31, 61, 62, 77, 87, 88]).

In recent years a concept of so-called *uncertain variables* and their applications to analysis and decision problems for a wide class of uncertain systems has been developed [25, 30, 35, 40, 42, 43, 44, 46, 50, 53, 54, 55]. The main aim of this book is to present a unified, comprehensive and compact description of analysis and decision problems in a class of uncertain systems described by traditional mathematical models and by relational knowledge representations. An attempt at a uniform theory of uncertain systems including problems and methods based on different mathematical formalisms may be useful for further research in this large area and for practical applications to the design of knowledge-based decision support systems. The book may be characterized by the following features:

1. The problems and methods are concerned with systems described by traditional mathematical models (with number variables) and by knowledge representations which are treated as an extension of classical functional models. The considerations are then directly related to respective problems and methods in traditional system and control theory.
2. The problems under consideration are formulated for systems with unknown parameters in the known form of the description (*parametric problems*) and for the direct non-deterministic input–output description (*non-parametric problems*). In the first case the unknown parameters are assumed to be values of random or uncertain variables. In the second case the values of input and output variables are assumed to be values of random, uncertain or fuzzy variables.
3. The book presents three new concepts introduced and developed by the author for a wide class of uncertain systems:
   a. Logic-algebraic method for systems with a logical knowledge representation [9 – 14].
   b. Learning process in systems with a relational knowledge representation, consisting in *step by step* knowledge validation and updating (e.g. [18, 22, 25]).
   c. Uncertain variables based on uncertain logics.
4. Special emphasis is placed on uncertain variables as a convenient tool for handling the uncertain systems under consideration. The main part of the book is devoted to the basic theory of uncertain variables and their application in different cases of uncertain systems. One of the main purposes of the book is to present recent developments in this area, a comparison with random and fuzzy variables and the generalization in the form of so-called *soft variables*.
5. Special problems such as pattern recognition and control of a complex of operations under uncertainty are included. Examples concerning the control of manufacturing systems, assembly processes and task distributions in computer systems indicate the possibilities of practical applications of uncertain variables and other approaches to decision making in uncertain systems.

The analysis and decision problems are formulated for input–output plants and two kinds of uncertainty:

1. The plant is non-deterministic, i.e. the output is not determined by the input.

2. The plant is deterministic, but its description (the input–output relationship) is not exactly known.

The different forms of the uncertainty may be used in the description of one plant. For example, the non-deterministic plant may be described by a relation such that the output is not determined by the input (i.e. is not a function of the input). This relation may be considered as a *basic description* of the uncertainty. If the relation contains unknown parameters, their description, e.g. in the form of probability distributions, may be defined as an *additional description* of the uncertainty or the *second-order uncertainty*.

In the wide sense of the word an uncertain system is understood in the book as a system containing any kind and any form of uncertainty in its description. In a narrow sense, an uncertain system is understood as a system with the description based on uncertain variables. In this sense, such names as “random, uncertain and fuzzy knowledge” or “random, uncertain and fuzzy controllers” will be used. Additional remarks will be introduced, if necessary, to avoid misunderstandings. Quite often the name “control” is used in the text instead of decision making for a particular plant. Consequently, the names “control plant, control system, control algorithm, controller” are used instead of “decision plant, decision system, decision algorithm, decision maker”, respectively.

1.2 Uncertain Variables

In the traditional case, for a static (memoryless) system described by a function $y = \Phi(u, x)$ where u, y, x are input, output and parameter vectors, respectively, the decision problem may be formulated as follows: to find the decision u^* such that $y = y^*$ (the desirable output value). The decision u^* may be obtained for the known function Φ and the value x. Let us now assume that x is unknown. In the probabilistic approach x is assumed to be a value of a random variable $\tilde{x}$ described by the probability distribution. In the approach based on uncertain variables the unknown parameter x is a value of an uncertain variable $\bar{x}$ for which an expert gives the certainty distribution $h(x) = v(\bar{x} \,\tilde{=}\, x)$ where v denotes a certainty index of the soft property: “$\bar{x}$ is approximately equal to x” or “x is the approximate value of $\bar{x}$”. The certainty distribution evaluates the expert’s opinion on approximate values of the uncertain variable. The uncertain variables, related to random variables and fuzzy numbers, are described by the set of values X and their certainty distributions which correspond to probability distributions for the random variables and to membership functions for the fuzzy numbers. To define the uncertain variable, it is necessary to give $h(x)$ and to determine the certainty indexes of the following soft properties:

1. “$\bar{x} \,\tilde{\in}\, D_x$” for $D_x \subset X$, which means “the approximate value of $\bar{x}$ belongs to D_x” or “$\bar{x}$ belongs approximately to D_x”.

2. "$\bar{x}\,\tilde{\notin}\,D_x$" = "$\neg(\bar{x}\,\tilde{\in}\,D_x)$", which means "$\bar{x}$ does not belong approximately to D_x".

To determine the certainty indexes for the properties: $\neg(\bar{x}\,\tilde{\in}\,D_x)$, $(\bar{x}\,\tilde{\in}\,D_1)\vee(\bar{x}\,\tilde{\in}\,D_2)$ and $(\bar{x}\,\tilde{\in}\,D_1)\wedge(\bar{x}\,\tilde{\in}\,D_2)$ where $D_1, D_2 \subseteq X$, it is necessary to introduce an *uncertain logic*, which deals with the soft predicates of the type "$\bar{x}\,\tilde{\in}\,D_x$". In Chapter 4 four versions of the uncertain logic have been defined and used for the formulation of the respective versions of the uncertain variable.

For the proper interpretation (semantics) of these formalisms it is convenient to consider $\bar{x} = g(\omega)$ as a value assigned to an element $\omega \in \Omega$ (a universal set). For fixed ω its value $\bar{x}$ is determined and $\bar{x} \in D_x$ is a crisp property. The property $\bar{x}\,\tilde{\in}\,D_x$ = $x \in D_x$ = "the approximate value of $\bar{x}$ belongs to D_x" is a soft property because $\bar{x}$ is unknown and the evaluation of "$\bar{x}\,\tilde{\in}\,D_x$" is based on the evaluation of $\bar{x}\,\tilde{=}\,x$ for the different $x \in X$ given by an expert. In the first version of the uncertain variable, $v(\bar{x}\,\tilde{\in}\,D_x) \neq v(\bar{x}\,\tilde{\notin}\,\overline{D}_x)$ where $\overline{D}_x = X - D_x$ is the complement of D_x. In the version called the *C*-uncertain variable, $v_c(\bar{x}\,\tilde{\notin}\,D_x) = v_c(\bar{x}\,\tilde{\in}\,\overline{D}_x)$ where v_c is the certainty index in this version

$$v_c(\bar{x}\,\tilde{\in}\,D_x) = \frac{1}{2}[v(\bar{x}\,\tilde{\in}\,D_x) + v(\bar{x}\,\tilde{\notin}\,\overline{D}_x)]\,.$$

The uncertain variable in the first version may be considered as a special case of the possibilistic number with a specific interpretation of $h(x)$ described above. In our approach we use soft properties of the type "P is approximately satisfied" where P is a crisp property, in particular $P =$ "$\bar{x} \in D_x$". It allows us to accept the difference between $\bar{x}\,\tilde{\in}\,D_x$ and $\bar{x}\,\tilde{\notin}\,\overline{D}_x$ in the first version. More details concerning the relations to random variables and fuzzy numbers are given in Chapter 6. Now let us pay attention to the following aspects which will be more clear after the presentation of the formalisms and semantics in Chapter 4:

1. To compare the meanings and practical utilities of different formalisms, it is necessary to take into account their semantics. It is specially important in our approach. The definitions of the uncertain logics and consequently the uncertain variables contain not only the formal description but also their interpretation. In particular, the uncertain logics may be considered as special cases of multi-valued predicate logic with a specific semantics of the predicates. It is worth noting that from the formal point of view the probabilistic measure is a special case of the fuzzy measure and the probability distribution is a special case of the membership function in the formal definition of the fuzzy number when the meaning of the membership function is not described.
2. Even if the uncertain variable in the first version may be formally considered as a very special case of the fuzzy number, for simplicity and unification it is better to introduce it independently (as has been done in the book) and not as a special case

of the much more complicated formalism with different semantics and applications.
3. *Uncertainty* is understood here in the narrow sense of the word, and concerns an incomplete or imperfect knowledge of something which is necessary to solve the problem. In our considerations, it is the knowledge of the parameters in the mathematical description of the system or the knowledge of a form of the input–output relationships, and is related to a fixed expert who gives the description of the uncertainty.
4. In the majority of interpretations the value of the membership function means a *degree of truth* of a soft property determining the fuzzy set. In our approach, "$\overline{x} \in D_x$" and "$x \in D_x$" are crisp properties, the soft property "$\overline{x} \,\tilde{\in}\, D_x$" is introduced because the value of $\overline{x}$ is unknown and $h(x)$ is a *degree of certainty* (or $1 - h(x)$ is a degree of uncertainty).

1.3 Basic Deterministic Problems

The problems of analysis and decision making under uncertainty described in the book correspond to the respective problems for deterministic (functional) plants with the known mathematical models. Let us consider a static plant described by a function $y = \Phi(u)$ where $u \in U = R^p$ is the input vector, $y \in Y = R^l$ is the output vector, U and Y are p-dimensional and l-dimensional real number vector spaces, respectively. The function Φ may be presented as a set of functions

$$y^{(i)} = \Phi_i(u^{(1)}, u^{(2)}, ..., u^{(p)}); \quad i = 1, 2, ..., l$$

where $y^{(i)}$ is the i-th component of y and $u^{(j)}$ is the j-th component of u.

Analysis problem: Given the function Φ and the value $u = u^*$, find the corresponding output $y^* = \Phi(u^*)$.

Decision problem: For the given function Φ and the value y^* required by a user, find the decision u^* such that $y = y^*$.

The solution of the problem is reduced to solving the equation $y^* = \Phi(u)$ with respect to u. In general, we may obtain a set of decisions

$$D_u = \{ u \in U : \quad \Phi(u) = y^* \} .$$

In particular $D_u = \varnothing$ (an empty set), which means that the solution does not exist. For the plant described by a function $y = \Phi(u, z)$ where z is a vector of external disturbances, the set of solutions $D_u(z)$ depends on z. In the case of a unique solution we obtain $u^* \stackrel{\Delta}{=} \Psi(z)$, i.e. the deterministic decision (control) algorithm in an open-loop decision system when z is measured (Fig. 1.1). For the plant described by the function $y = \Phi(u)$, on the assumption that the equation $\Phi(u) = y^*$ has a unique solution, the decision u^* may be determined by the following recursive algorithm:

$$u_{n+1} = u_n - K[y^* - \Phi(u_n)]; \quad n = 0, 1, \ldots \tag{1.1}$$

where u_n denotes the n-th approximation of u^* and K is a matrix of coefficients. Under some conditions concerning Φ and K, the sequence u_n converges to u^* for any u_0. The algorithm (1.1) may be executed in a closed-loop decision system (Fig. 1.2) where the output $y_n = \Phi(u_n)$ is measured. It is worth noting that to assure the convergence, it is not necessary to know exactly the function Φ. Then feedback is a way to achieve the proper decision u^* for the uncertain plant, i.e. it is one of the possible approaches to decision making under uncertainty.

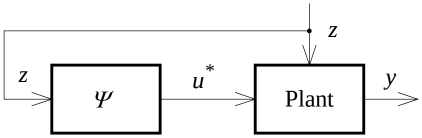

Figure 1.1. Open-loop decision system

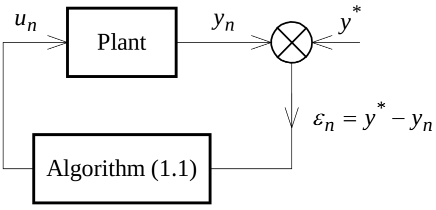

Figure 1.2. Closed-loop decision system

If there are additional constraints and/or the solution of the equation $\Phi(u) = y^*$ does not exist, the decision problem may be formulated as an optimization problem consisting in finding u^* minimizing a quality index $\varphi(y, y^*)$, e.g.

$$\varphi(y, y^*) = (y - y^*)^{\mathrm{T}}(y - y^*)$$

where vectors are presented as one-column matrices and T denotes transposition of a matrix.

The formulations of basic analysis and decision problems may be extended to deterministic dynamical plants. Let us consider a plant described by the equation

$$s_{n+1} = f(s_n, u_n)\,; \quad n = 0, 1, \ldots$$

where s_n is a state vector.

Analysis problem: For the given function f, initial state s_0 and the sequence $u_0, u_1, \ldots, u_{N-1}$ one should find the sequence $s_1, s_2, \ldots, s_N$.

One of the possible formulations of a decision problem is the following: for the given function f, s_0 and $s_N = s^*$ required by a user, one should determine the sequence of decisions $u_0, u_1, ..., u_{N-1}$ such that $s_N = s^*$. The solution exists for sufficiently large N if the plant is controllable. The optimization problem corresponding to the minimization of $\varphi(y, y^*)$ for a static plant may be formulated as follows.

Optimal decision problem: For the given function f, state s_0 and a quality index $\varphi(s, s^*)$, one should determine the sequence $u_0, u_1, ..., u_{N-1}$ minimizing the global performance index

$$Q_N = \sum_{n=1}^{N} \varphi(s_n, s^*) = \sum_{n=0}^{N-1} \varphi[\, f(s_n, u_n), s^*\,]\,.$$

1.4 Structure of the Book

The book consists of two informal parts. The first part containing Chapters 2–7 presents basic analysis and decision problems for static plants. The second part containing Chapters 8–14 concerns dynamical systems and special problems connected with learning and complex systems, pattern recognition and operation systems. The parts are organized as follows.

Chapter 2 presents basic analysis and decision problems for static plants described by relations. A general concept of so-called *determinization*, consisting in replacing an uncertain description by its deterministic representation, is introduced. Two kinds of relational knowledge representation are considered: the knowledge of the plant and the knowledge of the decision making.

Chapter 3 deals with the application of random variables to the description of the uncertainty. In the first part of the chapter, analysis and decision problems are considered for the functional and relational plant with random parameters. The second part is devoted to the respective problems with a non-parametric description of the uncertainty. In this case the knowledge of the plant has a form of conditional probability distribution. In both cases it is shown how the probabilistic knowledge of the decision making (i.e. the random decision algorithm in an open-loop decision system) may be obtained from the probabilistic knowledge of the plant, and how to obtain the deterministic decision algorithm as a result of determinization.

Chapters 4 and 5 are devoted to uncertain variables and their applications to uncertain systems. The basic definitions and properties of the uncertain logics and variables are given in Chapter 4. We consider four versions of the uncertain variables with different definitions of the certainty distributions and operations. The application of the uncertain variables to the formulation and solving of the analysis and decision problems for the functional and relational static plant is the topic of Chapter 5. The chapter is completed with considerations for the non-parametric case in which the knowledge of the plant has the form of conditional certainty distributions. The uncertain decision algorithm is obtained from the uncertain knowledge of the plant.

In the first part of Chapter 6 the applications of fuzzy numbers (fuzzy variables) to non-parametric analysis and decision problems for the static plant are presented. In the second part of the chapter the comparison of uncertain variables with random and fuzzy variables and analogies between the non-parametric problem statements and solutions for the descriptions based on random, uncertain and fuzzy variables are discussed. These analogies lead to a generalization in the form of so-called *soft variables* and their application in analysis and decision problems for the static plant.

Chapter 7 is concerned with relational static plants described by a logical knowledge representation, which may be treated as a special form of the relational knowledge representation that consists of relations in the form of logical formulas concerning input, output and additional variables. Consequently, to formulate and solve the analysis and decision problems, one may apply the so-called *logic-algebraic method*. The modification of this method may be applied to a plant with random and uncertain parameters.

The purpose of Chapter 8 is to show how the approaches and methods presented in the first part of the book for static plants (in particular, the considerations based on the relational knowledge representation and uncertain variables) may be applied to dynamical plants. The application of the presented approach to knowledge-based control of an assembly process is described.

Chapter 9 has a special character, and is devoted to the general idea of parametric optimization and its application to uncertain, random and fuzzy controllers in closed-loop decision systems with dynamical plants. The chapter is completed with remarks concerning so-called descriptive and prescriptive approaches, and the quality of the decisions based on different forms of the knowledge given by an expert.

The idea and algorithms of learning based on *step by step* knowledge validation and updating are presented in Chapter 11. Two cases are considered. In the first case the validation and updating concerns the knowledge of the plant, and in the second case – the knowledge of the decision making. The idea of learning is illustrated by an example of the application to the assembly system considered in Chapter 8.

Chapters 12, 13 and 14 deal with specific problems and systems: the decision problems for plants with three-level uncertainty, complex relational systems (with an application to a complex manufacturing system), control of a complex of operations (with an application to task allocation in a group of parallel processors), and knowledge-based pattern recognition under uncertainty.

2 Relational Systems

This chapter is concerned with analysis and decision making problems for a static input–output plant described by a relation which is not reduced to the function Φ considered in Sect. 1.3. Consequently, for the given relation, the output is not determined by the input. The analysis problem consists in finding the *output property* (or the set of possible outputs) for the given *input property* (or the given set of inputs), and the decision problem consists in finding the *input property* (or the set of possible inputs) for the given *output property* (or the set of acceptable outputs, required by a user). For the functional plant presented in Sect. 1.3, the input and output properties have the form "$u = u^*$" and "$y = y^*$", respectively. For the relational plant the respective properties have the form "$u \in D_u$" and "$y \in D_y$" where D_u and D_y are subsets of U and Y, respectively.

2.1 Relational Knowledge Representation

Let us consider a static plant with input vector $u \in U$ and output vector $y \in Y$, where U and Y are real number vector spaces. The plant is described by a relation

$$u \ \rho \ y \overset{\Delta}{=} R(u, y) \subset U \times Y \tag{2.1}$$

which may be called a *relational knowledge representation* of the plant. It is an extension of the traditional functional model $y = \Phi(u)$ considered in Sect. 1.3. The relation $R(u, y)$ denotes a set of all possible pairs (u, y) in the Cartesian product $U \times Y$, which may appear in the plant. In other words, the plant is described by a property (a predicate) concerning (u, y), and $R(u, y)$ denotes the set of all pairs (u, y) for which this property is satisfied, i.e.

$$R(u, y) = \{(u, y) \in U \times Y : w[\varphi(u, y)] = 1\} \overset{\Delta}{=} \{(u, y) \in U \times Y : \varphi(u, y)\}$$

where $w[\varphi(u, y)] \in [0, 1]$ for the fixed values (u, y) denotes a logic value of the property $\varphi(u, y)$. When the relation $R(u, y)$ is not a function, the description (2.1) given by an expert may have two practical interpretations:

1. The plant is deterministic, i.e. at every moment n

$$y_n = \Phi(u_n),$$

but the expert has no full knowledge of the plant and for the given u he/she can determine only the set of possible outputs:

$$D_y(u) \subset Y : \{y \in Y : (u,y) \in R(u,y)\}.$$

For example, in a one-dimensional case $y = cu$, the expert knows that $c_1 \le c \le c_2$; $c_1, c_2 > 0$. Then as a description of the plant he/she gives a relation presented in the following form:

$$\left.\begin{array}{l} c_1 u \le y \le c_2 u \quad \text{for} \quad u \ge 0 \\ c_2 u \le y \le c_1 u \quad \text{for} \quad u \le 0 \end{array}\right\}. \tag{2.2}$$

The situation is illustrated in Fig. 2.1, in which the set of points (u_n, y_n) is marked.

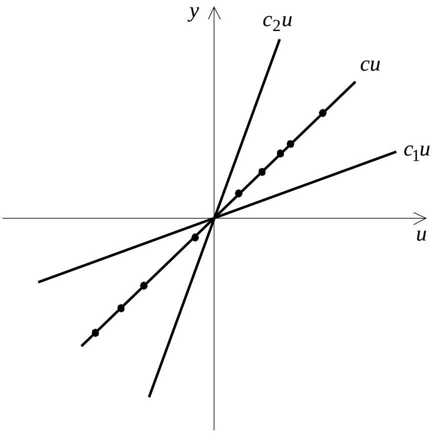

Figure 2.1. Illustration of a relation – the first case

2. The plant is not deterministic, which means that at different n we may observe different values y_n for the same values u_n. Then $R(u, y)$ is a set of all possible points (u_n, y_n), marked for the example (2.2) in Fig. 2.2, and $D_y(u)$ is a set of all possible values which may be observed at the output for the fixed value u.
In the first case the relation (which is not a function) is a result of the expert's uncertainty and in the second case – a result of uncertainty in the plant. For simplicity, in both cases we shall talk about an *uncertain plant*, and the plant described by a relational knowledge representation will be shortly called a *relational plant*.
In more complicated cases the relational knowledge representation given by an expert may have the form of a set of relations:

$$R_i(u, \overline{w}, y) \subset U \times W \times Y, \quad i = 1, 2, ..., k \tag{2.3}$$

where $\overline{w} \in W$ is a vector of additional auxiliary variables used in the description of the knowledge. The set of relations (2.3) may be called a *basic knowledge representation*. It may be reduced to a *resulting knowledge representation* $R(u, y)$:

$$R(u, y) = \{(u, y) \in U \times Y : \bigvee_{\overline{w} \in W} (u, \overline{w}, y) \subset \overline{R}(u, w, y)\}$$

where

$$\overline{R}(u, \overline{w}, y) = \bigcap_{i=1}^{k} R_i(u, \overline{w}, y) .$$

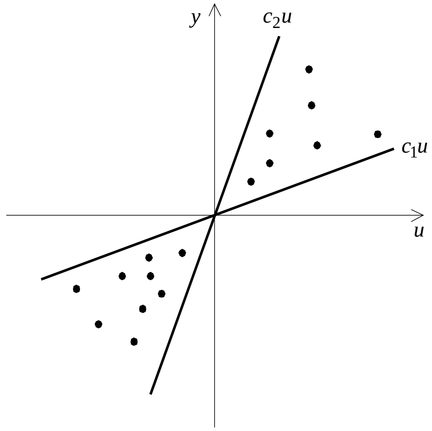

Figure 2.2. Illustration of a relation – the second case

The relations $R_i(u, w, y)$ may have the form of a set of inequalities and/or equalities concerning the components of the vectors *u*, *w*, *y*.

In Chapter 7 we shall consider a special form of the relational knowledge representation, in which the relations (2.3) are expressed by logical formulas concerning $(u, \overline{w}, y)$, and in Chapter 8 the extension of the relational knowledge representation to a dynamical plant will be presented.

The relational knowledge representation has a specific form in a discrete case when *U* and *Y* are finite sets of vectors. Assume that *U* is a finite discrete set

$$U = \{\overline{u}_1, \overline{u}_2, ..., \overline{u}_\alpha\} .$$

Then the relation $R(u, y)$ is reduced to the family of sets

$$D_y(\overline{u}_j) = \{ y \in Y : (\overline{u}_j, y) \in R(u, y)\} , \quad j = 1,2,..., \alpha ,$$

i.e. the sets of possible outputs for all inputs $\overline{u}_j$. If $Y = \{\overline{y}_1, \overline{y}_2, ..., \overline{y}_\beta\}$ then $R(u, y)$ is a set of pairs $(\overline{u}_j, \overline{y}_i)$ selected from $U \times Y$ and $D_y(\overline{u}_j)$ is a corresponding finite set of the points $\bar{y}_i$ (a subset of Y).

For the plant with external disturbances, the relational knowledge representation has the form of a relation

$$R(u, y, z) \subset U \times Y \times Z$$

where $z \in Z$ is a vector of the disturbances.

2.2 Analysis and Decision Making for Relational Plants

The formulations of the analysis and decision making problems for a relational plant analogous to those for a functional plant described by a function $y = \Phi(u)$ are adequate for the knowledge of the plant [24].

Analysis problem: For the given $R(u, y)$ and $D_u \subset U$ find the smallest set $D_y \subset Y$ such that the implication

$$u \in D_u \rightarrow y \in D_y \tag{2.4}$$

is satisfied.

The information that $u \in D_u$ may be considered as a result of observation. For the given D_u one should determine the best estimation of y in the form of the set of possible outputs D_y . It is easy to note that

$$D_y = \{ y \in Y : \bigvee_{u \in D_u} (u, y) \in R(u, y)\} . \tag{2.5}$$

This is then a set of all such values of y for which there exists $u \in D_u$ such that (u, y) belongs to R. In particular, if the value u is known, i.e. $D_u = \{u\}$ (a singleton), then

$$D_y(u) = \{ y \in Y : (u, y) \in R(u, y)\} \tag{2.6}$$

where $D_y(u)$ is a set of all possible y for the given value u. The analysis problem is illustrated in Fig. 2.3 where the shaded area illustrates the relation $R(u, y)$ and the interval D_y denotes the solution for the given interval D_u .

Example 2.1.

Let us consider the plant with two inputs $u^{(1)}$ and $u^{(2)}$, described by the inequality

$$c_1 u^{(1)} + d_1 u^{(2)} \leq y \leq c_2 u^{(1)} + d_2 u^{(2)} ,$$

and the set D_u is determined by

$$au^{(1)} + bu^{(2)} \leq \alpha \tag{2.7}$$

$$u^{(1)} \geq u^{(1)}_{\min}, \quad u^{(2)} \geq u^{(2)}_{\min}. \tag{2.8}$$

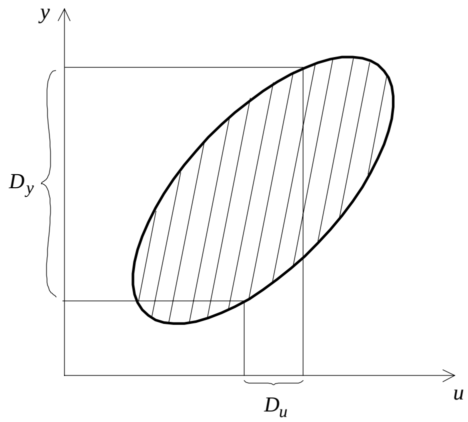

Figure 2.3. Illustration of analysis problem

For example, y may denote the amount of a product in a production process, $u^{(1)}$ and $u^{(2)}$ – amounts of two kinds of raw material, and the value $au^{(1)} + bu^{(2)}$ – the cost of the raw material. Assume that c_1, c_2, d_1, d_2, a, b, α are positive numbers and $c_1 < c_2$, $d_1 < d_2$. It is easy to see that the set (2.5) is described by the inequality

$$c_1 u^{(1)}_{\min} + d_1 u^{(2)}_{\min} \leq y \leq y_{\max} \tag{2.9}$$

where

$$y_{\max} = \max_{u^{(1)}, u^{(2)}} (c_2 u^{(1)} + d_2 u^{(2)}) \tag{2.10}$$

subject to constraints (2.7) and (2.8).

The maximization in (2.10) leads to the following results:
If

$$\frac{c_2}{d_2} \leq \frac{a}{b}$$

then

$$y_{\max} = c_2 u^{(1)}_{\min} + \frac{d_2}{b}(\alpha - au^{(1)}_{\min}). \tag{2.11}$$

If

$$\frac{c_2}{d_2} \geq \frac{a}{b}$$

then

$$y_{\max} = \frac{c_2}{a}(\alpha - bu_{\min}^{(2)}) + d_2 u_{\min}^{(2)} . \qquad (2.12)$$

For the numerical data $c_1 = 1$, $c_2 = 2$, $d_1 = 2$, $d_2 = 4$, $a = 1$, $b = 4$, $\alpha = 3$, $u_{\min}^{(1)} = 1$, $u_{\min}^{(2)} = 0.5$

$$\frac{c_2}{d_2} = \frac{1}{2}, \qquad \frac{a}{b} = \frac{1}{4} < \frac{c_2}{d_2} .$$

From (2.12) we obtain $y_{\max} = 4$ and according to (2.9) $y_{\min} = c_1 u_{\min}^{(1)} + d_1 u_{\min}^{(2)} = 2$. The set D_y is then determined by inequality $2 \leq y \leq 4$. □

Now let us consider a decision making problem for the relational plant described by $R(u, y)$ which is not a function. In this case the requirement $y = y^*$ (where y^* is a given value) cannot be satisfied and should be replaced by a weaker requirement $y \in D_y$ for a given set D_y. As a result we may obtain not one particular decision u^*, but a set of possible decisions D_u.

Decision problem: For the given $R(u, y)$ and $D_y \subset Y$ find the largest set $D_u \subset U$ such that the implication (2.4) is satisfied.

The set D_y is given by a user, the property $y \in D_y$ denotes the user's requirement and D_u denotes the set of all possible decisions for which the requirement concerning the output y is satisfied. It is easy to note that

$$D_u = \{u \in U : \quad D_y(u) \subseteq D_y\} \qquad (2.13)$$

where $D_y(u)$ is the set of all possible y for the fixed value u, determined by (2.6), or

$$D_u = \{u \in U : (u, y) \in R(u, y) \rightarrow y \in D_y\} .$$

The solution may not exist, i.e. $D_u = \varnothing$ (empty set). Such a case is illustrated in Fig.2.4: for the given interval D_y, a set $D_u \neq \varnothing$ satisfying the implication (2.4) does not exist. This means that the requirement is too strong, i.e. the interval D_y is too small. The requirement may be satisfied for a larger interval D_y (see Fig. 2.3). In the example illustrated in Fig. 2.2, if $D_y = [y_{\min}, y_{\max}]$ and $c_1, c_2 > 0$ then

$$D_u = [\frac{y_{\min}}{c_1}, \frac{y_{\max}}{c_2}]$$

and the solution exists on the condition

$$\frac{y_{\min}}{c_1} \le \frac{y_{\max}}{c_2}.$$

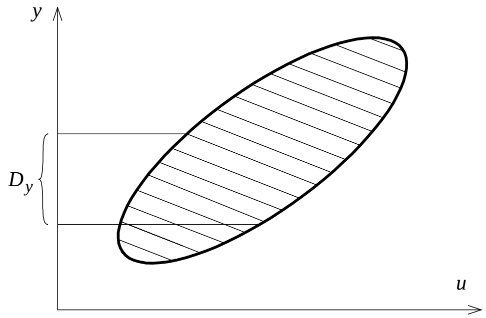

Figure 2.4. Illustration of the case where the solution does not exist

The analysis and decision problems for the relational plant are the extensions of the respective problems for the functional plant, presented in Sect. 1.3. The properties "$u \in D_u$" and "$y \in D_y$" may be called *input and output properties*, respectively. For the functional plant we considered the input and output properties in the form: "$u = u^*$" and "$y = y^*$" where u^*, y^* denote fixed values. For the relational plant the analysis problem consists in finding the best output property (the smallest set D_y) for the given input property, and the decision problem consists in finding the best input property (the largest set D_u) for the given output property required. The procedure for determining the effective solution D_u or D_y based on the general formulas (2.5) or (2.13) depends on the form of $R(u, y)$ and may be very complicated. If $R(u, y)$ and the given property (i.e. the given set D_u or D_y) are described by a set of equalities and/or inequalities concerning the components of the vector u and y, then the procedure is reduced to "solving" this set of equalities and/or inequalities.

Example 2.2.

Consider a plant with a single output, described by a relation

$$G_1(u) \le y \le G_2(u) \tag{2.14}$$

where G_1 and G_2 are the functions

$$G_1 : U \to R^+, \quad G_2 : U \to R^+; \quad R^+ = [0, \infty),$$

and

$$\bigwedge_{u \in U} [G_1(u) \le G_2(u)] .$$

For example, y is the amount of a product as in Example 2.1, and the components of the vector u are features of the raw materials. For a user's requirement

$$y_{\min} \le y \le y_{\max} , \tag{2.15}$$

i.e. $D_y = [y_{\min}, y_{\max}]$, we obtain

$$D_u = \{u \in U : [G_1(u) \ge y_{\min}] \wedge [G_2(u) \le y_{\max}]\} .$$

In particular, if the relation (2.14) has the form

$$c_1 u^{\mathrm{T}} u \le y \le c_2 u^{\mathrm{T}} u , \quad c_1 > 0 , \; c_2 > c_1$$

where $u \in R^k$ and

$$u^{\mathrm{T}} u = (u^{(1)})^2 + (u^{(2)})^2 + ... + (u^{(k)})^2$$

then D_u is described by the inequality

$$\frac{y_{\min}}{c_1} \le u^{\mathrm{T}} u \le \frac{y_{\max}}{c_2}$$

and the decision u satisfying the requirement (2.15) exists if

$$\frac{y_{\max}}{c_2} \ge \frac{y_{\min}}{c_1} . \qquad \square$$

2.3 Relational Plant with External Disturbances

The considerations may by extended to a plant with external disturbances, described by a relation $R(u, y, z) \subset U \times Y \times Z$ where $z \in Z$ is a vector of the disturbances which may be observed. The property $z \in D_z$ for the given $D_z \subset Z$ may be considered as a result of observations. Our plant has two inputs (u, z) and the analysis problem is formulated in the same way as for the relation $R(u, y)$, with $(u, z) \in D_u \times D_z$ in place of $u \in D_u$.

Analysis problem: For the given $R(u,y,z)$, D_z and D_u find the smallest set $D_y \subset Y$ such that the implication

$$(u \in D_u) \wedge (z \in D_z) \rightarrow y \in D_y \qquad (2.16)$$

is satisfied.

The result analogous to (2.5) is

$$D_y = \{y \in Y : \bigvee_{u \in D_u} \bigvee_{z \in D_z} (u, y, z) \in R(u, y, z)\}.$$

The decision making is an inverse problem consisting in the determination of the set of all decisions *u* such that for every decision from this set and for every $z \in D_z$ the required property $y \in D_y$ is satisfied.

Decision problem: For the given $R(u,y,z)$, D_y (the requirement) and D_z (the result of observations), find the largest set D_u such that the implication (2.16) is satisfied. The general form of the solution is as follows:

$$D_u = \{u \in U : \bigwedge_{z \in D_z} [D_y(u,z) \subseteq D_y]\} \qquad (2.17)$$

where

$$D_y(u,z) = \{y \in Y : (u,y,z) \in R(u,y,z)\}. \qquad (2.18)$$

It is then the set of all such decisions u that for every $z \in D_z$ the set of possible outputs y belongs to D_y. For the fixed z (the result of measurement) the set D_u is determined by (2.17) with the relation

$$R(u,y,z) \overset{\Delta}{=} R(u, y\,; z) \subset U \times Y.$$

In this notation z is the parameter in the relation $R(u, y\,; z)$. Then

$$D_u(z) = \{u \in U : D_y(u,z) \subseteq D_y\} \overset{\Delta}{=} \overline{R}(z,u) \qquad (2.19)$$

where $D_y(u,z)$ is defined by (2.18). The formula (2.19) defines a relation between z and u denoted by $\overline{R}(z,u)$. The relation $\overline{R}(z,u)$ may be called a knowledge representation for the decision making (a description of the knowledge of the decision making) or a *relational decision algorithm*. The block scheme of the open-loop decision system (Fig. 2.5) is analogous to that of Fig. 1.1 for a functional plant. The *knowledge of the decision making*

$$< \overline{R}(z,u) > \stackrel{\Delta}{=} \text{KD}$$

has been obtained for the given *knowledge of the plant*

$$< R(u,y,z) > \stackrel{\Delta}{=} \text{KP}$$

and the given requirement $y \in D_y$.

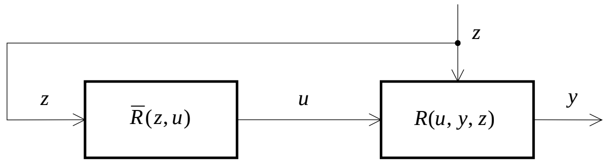

Figure 2.5. Open-loop decision system

Example 2.3.
Consider a plant with one output, described by a relation

$$G_1(u,z) \le y \le G_2(u,z) \tag{2.20}$$

where G_1 and G_2 are the functions

$$G_1 : U \times Z \to \text{R}^+, \quad G_2 : U \times Z \to \text{R}^+; \quad \text{R}^+ = [0, \infty),$$

and

$$\bigwedge_{u \in U} \bigwedge_{z \in Z} [G_1(u,z) \le G_2(u,z)].$$

For example, y is the amount of a product (see Example 2.2), the components of the vector u are the features of the raw material which may be chosen by a decision maker, and the components of the vector z are the features of the raw material which may be observed. For a user's requirement

$$y_{\min} \le y \le y_{\max} , \tag{2.21}$$

i.e. $D_y = [y_{\min}, y_{\max}]$, we obtain

$$D_u : \{u \in U : [\overline{G}_1(u) \ge y_{\min}] \wedge [\overline{G}_2(u) \le y_{\max}]\}$$

where

$$\overline{G}_1(u) = \min_{z \in D_z} G_1(u,z), \qquad \overline{G}_2(u) = \max_{z \in D_z} G_2(u,z).$$

Assume that $z^{\mathrm{T}} = [z^{(1)} \;\; z^{(2)}]$, the relation (2.20) has the form

$$c_1 z^{(1)} u^{\mathrm{T}} u \le y \le c_2 z^{(2)} u^{\mathrm{T}} u,$$

$$c_1 > 0, \;\; c_2 > c_1, \quad z_1 > 0, \;\; z_2 > z_1$$

and D_z is described by the inequality

$$r_{\min}^2 \le (z^{(1)})^2 + (z^{(2)})^2 \le r_{\max}^2.$$

Then

$$\overline{G}_1(u) = c_1 z_{\min}^{(1)} u^{\mathrm{T}} u, \quad \overline{G}_2(u) = c_2 z_{\max}^{(1)} u^{\mathrm{T}} u$$

where

$$z_{\min}^{(1)} = \frac{r_{\min}}{\sqrt{2}}, \quad z_{\max}^{(2)} = \frac{r_{\max}}{\sqrt{2}}.$$

Consequently, the set D_u is described by the inequality

$$\frac{\sqrt{2}\, y_{\min}}{c_1 r_{\min}} \le u^{\mathrm{T}} u \le \frac{\sqrt{2}\, y_{\max}}{c_2 r_{\max}}$$

and the decision u satisfying the requirement (2.21) exists if

$$\frac{y_{\max}}{c_2 r_{\max}} \ge \frac{y_{\min}}{c_1 r_{\min}}. \qquad \square$$

Example 2.4.

A plant with $u, y, z \in R^2$ (two-dimensional vectors) is described by the inequalities

$$z^{(1)} u^{(1)} \le y^{(1)} \le 2 z^{(1)} u^{(1)}$$

$$z^{(2)} u^{(2)} \le y^{(2)} \le 2 z^{(2)} u^{(2)},$$

$z^{(1)}, \; z^{(2)}, \; u^{(1)}, \; u^{(2)} > 0$. The requirement concerning the output is the following

$$\alpha \le (y^{(1)})^2 + (y^{(2)})^2 \le \beta$$

for the given $\alpha, \; \beta > 0$. From the description of the plant we have

$$(z^{(1)})^2 (u^{(1)})^2 \le (y^{(1)})^2 \le 4(z^{(1)})^2 (u^{(1)})^2$$

$$(z^{(2)})^2 (u^{(2)})^2 \le (y^{(2)})^2 \le 4(z^{(2)})^2 (u^{(2)})^2 .$$

If $z^{(1)} \in [z_{\min}^{(1)}, z_{\max}^{(1)}]$ and $z^{(2)} \in [z_{\min}^{(2)}, z_{\max}^{(2)}]$, then the set D_u is determined by the inequalities

$$4[(z_{\max}^{(1)})^2 (u^{(1)})^2 + (z_{\max}^{(2)})^2 (u^{(2)})^2] \le \beta$$

$$(z_{\min}^{(1)})^2 (u^{(1)})^2 + (z_{\min}^{(2)})^2 (u^{(2)})^2 \ge \alpha .$$

2.4 Determinization

The deterministic decision algorithm based on the knowledge KD may be obtained as a result of *determinization* (see Sect. 1.4) of the relational decision algorithm $\overline{R}(z,u)$ by using the mean value

$$\tilde{u}(z) = \int\limits_{D_u(z)} u\,du \cdot [\int\limits_{D_u(z)} du]^{-1} \triangleq \tilde{\Psi}(z) .$$

In such a way the relational decision algorithm $\overline{R}(z,u)$ is replaced by the deterministic decision algorithm $\tilde{\Psi}(z)$.

For the given desirable value y^* we can consider two cases: in the first case the deterministic decision algorithm $\Psi(z)$ is obtained via determinization of the knowledge of the plant KP, and in the second case the deterministic decision algorithm $\Psi_d(z)$ is based on the determinization of the knowledge of the decision making KD obtained from KP for the given y^*. In the first case we determine the mean value

$$\tilde{y}(z) = \int\limits_{D_y(u,z)} y\,dy \cdot [\int\limits_{D_y(u,z)} dy]^{-1} \triangleq \Phi(u,z) \tag{2.22}$$

where $D_y(u,z)$ is described by formula (2.18). Then, by solving the equation

$$\Phi(u,z) = y^* \tag{2.23}$$

with respect to u, we obtain the deterministic decision algorithm $u = \Psi(z)$, on the assumption that Equation (2.23) has a unique solution.

In the second case we use

$$R(u, y^*, z) \triangleq R_d(z,u), \tag{2.24}$$

i.e. the set of all pairs (u,z) for which it is possible that $y = y^*$. The relation $R_d(z,u) \subset Z \times U$ may be considered as the knowledge of the decision making KD, i.e. the relational decision algorithm obtained for the given KP and the value y^*. The determinization of the relational decision algorithm R_d gives the deterministic decision algorithm

$$u_d(z) = \int_{D_{ud}(z)} u\, du \cdot [\int_{D_{ud}(z)} du\,]^{-1} \triangleq \Psi_d(z) \tag{2.25}$$

where

$$D_{ud}(z) = \{u \in U : \ (u,z) \in R_d(z,u)\} .$$

Two cases of the determination of the deterministic decision algorithm are illustrated in Figs. 2.6 and 2.7. The results of these two approaches may be different, i.e. in general $\Psi(z) \neq \Psi_d(z)$ (see Example 2.5).

Example 2.5.

Consider a plant with $u, z, y \in \mathrm{R}^1$ (one-dimensional variables), described by the inequality

$$cu + z \le y \le 2cu + z , \quad c > 0 . \tag{2.26}$$

For $D_y = [y_{\min}, y_{\max}]$ and the given z, the set (2.19) is determined by the inequality

$$\frac{y_{\min} - z}{c} \le u \le \frac{y_{\max} - z}{2c} .$$

The determinization of the knowledge KP according to (2.22) gives

$$\tilde{y} = \frac{3}{2} cu + z = \Phi(u,z) .$$

From the equation $\Phi(u,z) = y^*$ we obtain the decision algorithm

$$u = \Psi(z) = \frac{2(y^* - c)}{3z} .$$

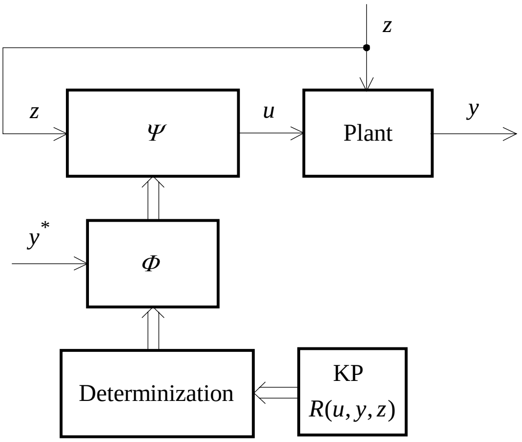

Figure 2.6. Decision system with determinization – the first case

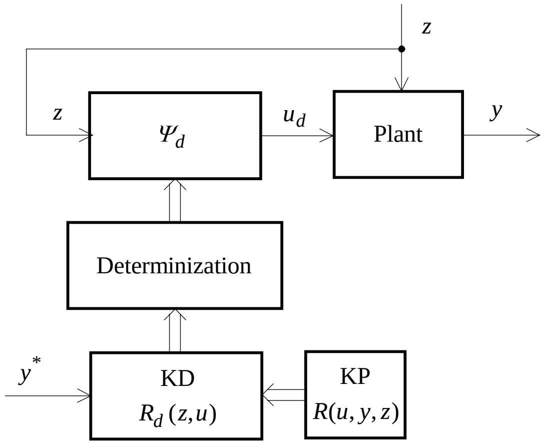

Figure 2.7. Decision system with determinization – the second case

Substituting y^* into (2.26) we obtain the relational decision algorithm $R_d(z,u)$ in the form

$$\frac{y^* - z}{2c} \le u \le \frac{y^* - z}{c}$$

and after the determinization

$$u_d = \Psi_d(z) = \frac{3(y^* - z)}{4c} \ne \Psi(z). \qquad \square$$

2.5 Discrete Case

Assume that

$$U = \{\bar{u}_1, \bar{u}_2, ..., \bar{u}_\alpha\}, \quad Y = \{\bar{y}_1, \bar{y}_2, ..., \bar{y}_\beta\}.$$

Now the relation $R(u, y)$ is a set of pairs $(\bar{u}_j, \bar{y}_i)$ selected from $U \times Y$, and may be described by the zero-one matrix

$$\chi_{ji} = \begin{cases} 1 & \text{if} \quad (\bar{u}_j, \bar{y}_i) \in R \\ 0 & \text{if} \quad (\bar{u}_j, \bar{y}_i) \notin R, \quad j = 1, ..., \alpha, \quad i = 1, ..., \beta. \end{cases}$$

The sets $D_u \subset U$ and $D_y \subset Y$ may be determined by the sets of the respective indexes

$$J \subset \{1, 2, ..., \alpha\} \triangleq S_u, \quad I \subset \{1, 2, ..., \beta\} \triangleq S_y,$$

i.e.

$$\bar{u}_j \in D_u \leftrightarrow j \in J, \quad \bar{y}_j \in D_y \leftrightarrow i \in I.$$

Analysis problem: For the given matrix $[\chi_{ij}]$ and the set J find the smallest set I such that

$$j \in J \rightarrow i \in I. \tag{2.27}$$

According to (2.5)

$$I = \{i \in S_y : \bigvee_{j \in S_u} (\chi_{ji} = 1)\} \tag{2.28}$$

Decision problem: For the given matrix $[\chi_{ij}]$ and the set I required by a user, find the largest set J such that the implication (2.27) is satisfied.

According to (2.13)

$$J = \{j \in S_u : S_y(j) \subseteq I\}$$

where

$$S_y(j) = \{i \in S_y : \chi_{ji} = 1\}, \tag{2.29}$$

or

$$J = \{j \in S_u : \chi_{ji} = 1 \rightarrow i \in I\}. \tag{2.30}$$

It is worth noting that the sets (2.28) and (2.29) may be easily generated by a computer containing the matrix $[\chi_{ij}]$ as a knowledge base. For the plant with external disturbances

$$z \in Z = \{\bar{z}_1, \bar{z}_2, ..., \bar{z}_\gamma\},$$

the relation $R(u, y, z)$ may be described by the three-dimensional zero-one matrix

$$\chi_{jik} = \begin{cases} 1 & \text{if} \quad (\bar{u}_j, \bar{y}_i, \bar{z}_k) \in R \\ 0 & \quad\quad \text{otherwise,} \end{cases}$$

$j = 1, ..., \alpha$, $i = 1, ..., \beta$, $k = 1, ..., \gamma$. The set D_z may be determined by the set of respective indexes $K \subset \{1, 2, ..., \gamma\}$, i.e. $\bar{z}_k \in D_z \leftrightarrow k \in K$.

The decision problem consists in finding the largest set J such that the implication

$$(j \in J) \wedge (k \in K) \rightarrow i \in I$$

is satisfied.

The solution is analogous to (2.17) and (2.18):

$$J = \{ j \in S_u : \bigwedge_{k \in K} S_y(j, k) \subseteq I \}$$

where

$$S_y(j, k) = \{i \in S_y : \chi_{jik} = 1\}.$$

The form corresponding to (2.30) is as follows:

$$J = \{ j \in S_u : \bigwedge_{k \in K} (\chi_{jik} = 1 \rightarrow i \in I)\}.$$

Remark 2.1. Note that in the discrete case it may be possible to satisfy the requirement $y = y^* \in Y$, i.e. $i = i^*$ for R which is not a function. The solution has the form

$$J = \{ j \in S_u : S_y(j, k) = i^*\}. \qquad \square$$

Example 2.6.

Let $\alpha = 5$, $\beta = 6$,

$$\chi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

and the requirement is determined by $I = \{3, 4, 5\}$, which means that $D_y = \{\bar{y}_3, \bar{y}_4, \bar{y}_5\}$. According to (2.29)

$$S_y(1) = \{1, 6\}, \quad S_y(2) = \{4, 5\}, \quad S_y(3) = \{3, 4, 6\},$$

$$S_y(4) = \{4\}, \quad S_y(5) = \{2, 3, 4, 6\}.$$

Then $J = \{2, 4\}$, which means that the requirement is satisfied for the decisions $\bar{u}_2$ and $\bar{u}_4$. It is easy to see that for

$$\chi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

the solution does not exist. □