Preface

The body of this text is written. It remains to find some words to explain what to expect in this book. A first attempt of characterizing the content could be:

MSC 2000: 05-01, 05C10, 05C62, 52-01, 52C10, 52C30, 52C42.

In words: The questions posed and partly answered in this book are from the intersection of graph theory and discrete geometry. The reader will meet some graph theory with a geometric flavor and some combinatorial geometry of the plane. Though, the investigations always start in the geometry of the plane it is sometimes appropriate to pass on to higher dimensions to get a more global understanding of the structures under investigation. This is the in Chapter 7, for example, when the study of triangulations of a point configuration leads to the definition of secondary polytopes.

I like to think of the book as a collection which makes up a kind of bouquet. A bouquet of problems, ideas and results, each of a special character and beauty, put together with the intention that they supplement each other to form an interesting and appealing whole.

The main mathematical part of the text contains only few citations and references to related material. These additional bits of information are provided in the last section of each chapter, 'Notes and References'. On average the bibliography of a chapter contains about thirty items. This is far from being a complete list of the relevant literature. The intention is to just indicate the most valuable literature so that these sections can serve as entry points for further studies. The text is supplemented by many figures to make the material more attractive and help the reader get a sensual impression of the objects. In some cases, I have confined the presentation to results which fall behind today's state of the art. I wanted to emphasize the main ideas and stop before technical complexity starts taking over. This strategy should make the mathematics accessibility to a relatively broad audience including students of computer science, students of mathematics, instructors and researchers.

The book can serve different purposes. It may be used as textbook for a course or as a collection of material for a seminar. It should also be helpful to people who want to learn something about specific themes. They may concentrate on single chapters because all the chapters are self-contained and can be read as stand alone surveys.

Topics

Chapter 1. We introduce basic notion graph theory and explain what geometric and topological graphs are. Planar graphs and some important theorems about them are reviewed. The main results of this chapter are bounds for some extremal problems for geometric graphs.

Chapter 2. We show that a 3-connected planar graph with f faces admits a convex drawing on the $(f-1) \times (f-1)$ grid. The result is based on Schnyder woods, a special cover of the edges of a 3-connected planar graph with three trees. Schnyder woods bring along connections to geodesic embeddings of planar graphs and to the order dimension of planar graphs and 3-polytopes.

Chapter 3. This is about non-planar graphs. How many crossing pairs of edges do we need in any drawing of a given graph in the plane? The Crossing Lemma provides a bound and has beautiful applications to deep extremal problems. We explain some of them.

Chapter 4. Let $\mathcal{P}$ be a configuration of n points in the plane. A k-set of $\mathcal{P}$ is a subset $\mathcal{S}$ of k points of $\mathcal{P}$ which can be separated from the complement $\bar{\mathcal{S}} = \mathcal{P} \setminus \mathcal{S}$ by a line. The notorious k-set problem of discrete geometry asks for asymptotic bounds of this number as a function of n. We present bounds, Welzl's generalization of the Lovász Lemma to higher dimensions and close with the surprisingly related problem of bounding the rectilinear crossing number of complete graphs from below.

Chapter 5. This chapter contains selected results from the extremal theory for configurations of points and arrangements of lines. The main results are bounds for the number of ordinary lines of a point configuration and for the number of triangles of an arrangement.

Chapter 6. Compared to arrangements of lines, arrangements of pseudolines have the advantage that they can be nicely encoded by combinatorial data. We introduce several combinatorial representations and prove relations between them. For each representation we give an applications which makes use of specific properties. The encoding by triangle orientations has a natural generalization which leads to higher Bruhat orders.

Chapter 7. In this chapter we study triangulations of a point configuration. The flip operation allows to move between different triangulations. The Delaunay triangulation is investigated as a special element in the graph of triangulations. This graph is shown to be related to the skeleton graph of the secondary polytope. In the special case of a point configuration in convex position they coincide. In this case, we make use of hyperbolic geometry to get a lower bound for the diameter of the graph of triangulations.

Chapter 8. Rigidity allows a different view to geometric graphs. We introduce rigidity theory and prove three characterizations of minimal generically rigid graphs in the plane. Pseudotriangulations are shown to be the planar minimal generically rigid graphs in the plane. The set of pseudotriangulations with vertices embedded in a fixed point configuration $\mathcal{P}$ has a nice structure. There is a notion of flip that allows to move between different pseudotriangulations. The flip-graph is a connected graph and it turns out that it is the skeleton graph of a polytope. The beautiful theory finds a surprising application in the Carpenter's Rule Problem.

The selection of topics is clearly governed by my personal taste. This is a drawback, because your taste is likely to differ from mine, at least in some details. The advantage is that though there are several books on related topics each of these books is clearly distinguishable by its style and content. In the spirit of 'Customers who bought this book also bought' I recommend the following four books:

- J. Matoušek
  Lectures on Discrete Geometry
  Graduate Texts in Math. 212
  Springer-Verlag, 2002.
- J. Pach and P. K. Agarwal
  Combinatorial Geometry
  John Wiley & Sons, 1995.
- G. M. Ziegler
  Lectures on Polytopes
  Graduate Texts in Math. 152
  Springer-Verlag, 1994.
- H. Edelsbrunner
  Geometry and Topology for Mesh Generation,
  Cambridge University Press, 2001.

Feedback

There will be something wrong. You may find errors of different nature. Inadvertently, I may not have given proper credit for certain contribution. You may know of relevant work that I have overlooked or you may have additional comments. In all these cases: Please let me know.

- `felsner@math.tu-berlin.de`

You can find a list of errata and a collection of comments and pointers related to the book at the following web-location:

- `http://www.math.tu-berlin.de/~felsner/gga-book.html`

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