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Introduction

In Part I of this monograph, we develop, via the method of matched asymptotic expansions (MAE), a rational approach to obtaining the complete large-t (dimensionless time) structure of the solution to initial-boundary value problems (IBVPs) and initial value problems (IVPs) for reaction-diffusion equations of the Fisher-Kolmogorov type, which exhibit the formation of a permanent form travelling wave (PTW) structure. In particular, this approach allows the wave speed for the large-t PTW, the correction to the wave speed and the rate of convergence of the solution of the IBVP or IVP onto the PTW to be determined. This large-t structure is obtained by careful consideration of the asymptotic structures as $t \to 0$ ($0 \leq x < \infty$) (where x is the dimensionless distance) and as $x \to \infty$ ($t \geq O(1)$).

We exemplify this approach by considering in detail two classes of scalar reaction-diffusion equations, namely,

$$u_t = u_{xx} + F(u), \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

where the reaction function, $F(u)$, is given either by:

(A) The generalized Fisher nonlinearity. Where the reaction function, $F(u)$, satisfies the normalized conditions $(F1)-(F5)$, as described in Section 1.1.
(B) The mth-order ($m > 1$) Fisher nonlinearity. In this case the reaction function, $F(u)$, is given by $F(u) = u^m(1-u)$ (known as the Zeldovich nonlinearity when $m = 2$).

Equation (1.1) (with either nonlinearity (A) or (B)) is to be solved subject to the initial condition

$$u(x, 0) = u_0(x), \quad -\infty < x < \infty,$$

and the boundary condition

$$u(x, t) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Sketches of the reaction function, $F(u)$, when $F(u)$ is given by (A) and (B) are given in Figures 1.1 and 1.2 respectively. We note that these nonlinearities have similar qualitative behaviour near $u = 1$ but differ significantly near $u = 0$. In particular, the mth-order ($m > 1$) Fisher nonlinearity has $F(u) \sim u^m$ as $u \to 0^+$ with $F'(0) = 0$ (and zero derivatives up to order $m-1$ for $m \geq 2$), whereas the generalized Fisher nonlinearity has $F(u) \sim u$ as $u \to 0^+$ with $F'(0) = 1$.

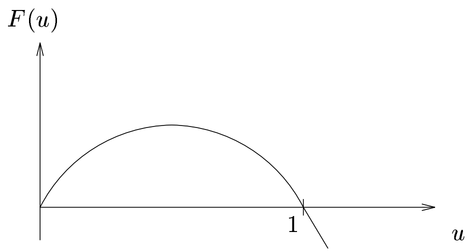

Fig. 1.1. The generalized Fisher nonlinearity.

Reaction-diffusion equations of the form (1.1) with associated nonlinearities (A) or (B) arise in many diverse scientific areas, for example chemistry (e.g. chemical kinetics) and biology (e.g. population dynamics and genetics). The dependant variable $u(x,t)$ may accordingly represent, for example the concentration of a chemical reactant or the population density of a biological species. A comprehensive review of the literature regarding mathematical models, based on (1.1), which arise from chemical and biological systems is given in Section 1 of McCabe, Leach and Needham [38], with a review of the basic properties of equation (1.1) being found in Section 2 of Xin [74]. For a general introduction to the mathematical modelling of chemical and biological systems see Gray and Scott [21], Murray [52] and Winfree [73].

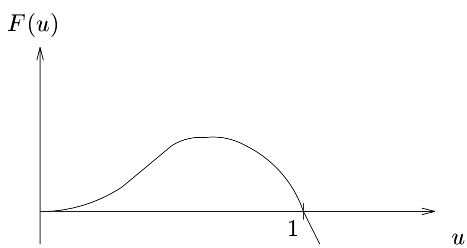

Fig. 1.2. The mth-order ($m > 1$) Fisher nonlinearity.

We note throughout that the initial data, $u_0(x)$, is assumed to be continuous everywhere and analytic within the closure of its support. However, the more general case when $u_0(x)$ is simply piecewise differentiable can be treated in entirely the same manner but may require the inclusion of additional passive asymptotic regions in the structure of the solution as $t \to 0$. These additional regions have no influence on the final large-t structure and for simplicity in what follows we restrict our attention to the former case.

Further, we assume in general that the initial data is symmetric about $x = 0$ and impose a symmetry condition, via the Neumann boundary condition

$$u_x(0,t) = 0, \quad t > 0,$$

and restrict attention to solving (1.1) in $x, t > 0$. In a chemical context this is appropriate to model the situation where the reaction proceeds on the domain $x \geq 0$ with an impermeable wall positioned at $x = 0$. This is not a technical restriction and is adopted merely for the convenience of presentation.

The methodology developed is applicable to a wide range of problems of the Fisher-Kolmogorov type provided:

(i) (1.1) is parabolic.
(ii) (1.1) is semilinear.
(iii) The equilibrium state $u(x,t) \equiv 0$ of (1.1) is temporally unstable.
(iv) A detailed knowledge of the PTW theory for (1.1) is available.

Thus we expect this method to be applicable to a wide variety of problems of the Fisher-Kolmogorov type, but not to excitable (bistable) problems where the equilibrium state $u(x,t) \equiv 0$ is temporally stable. We further note that the method is readily adaptable to parabolic systems of Fisher-Kolmogorov type and to problems in higher spatial dimensions. We conclude Part I by considering the extension of the presented method to a system of Fisher-Kolmogorov equations which arise as a simple model for an ionic autocatalytic system.

1.1 Generalized Fisher Nonlinearity

In this section we introduce the following initial-boundary-value problem for a scalar reaction-diffusion equation,

$$u_t = Du_{xx} + R(u), \quad x, t > 0, \tag{1.2}$$

$$u(x,0) = \begin{cases} u_0 g(x), & 0 \leq x \leq \sigma, \\ 0, & x > \sigma, \end{cases} \tag{1.3}$$

$$u_x(0,t) = 0, \quad t > 0, \tag{1.4}$$

$$u(x,t) \to 0 \quad \text{as} \quad x \to \infty, \quad t \geq 0. \tag{1.5}$$

Here $g : [0,\sigma] \to \mathbb{R}$ is positive, analytic, has $\max_{x\in[0,\sigma]} g(x) = 1$ and $g(x) \sim g_\sigma(\sigma - x)^r$ as $x \to \sigma^-$ (with $g_\sigma > 0$ and $r \in \mathbb{N}$), with the parameters $\sigma, D, u_0 > 0$. The function $R : (-\infty,\infty) \to \mathbb{R}$ has the following properties:

(R1) $R(u)$ is continuous and differentiable for all $u \in (-\infty,\infty)$,
(R2) $R(0) = R(u_s) = 0 \quad (u_s > 0)$,
(R3) $R'(0) > 0, R'(u_s) < 0$,
(R4) $R(u) > 0$ for all $u \in (0, u_s)$,
(R5) $R(u) < 0$ for all $u \in (u_s, \infty)$.

The problem (1.2)-(1.5) can be simplified by introducing the scaled variables

$$t' = R'(0)t, \quad x' = \left[\frac{R'(0)}{D}\right]^{1/2} x, \quad u' = \frac{u}{u_s}. \tag{1.6}$$

In terms of the above scaled variables (1.2)-(1.5) may be rewritten as (dropping primes for convenience)

$$u_t = u_{xx} + F(u), \quad x, t > 0, \tag{1.7}$$

$$u(x,0) = \begin{cases} u_0 g(x), & 0 \le x \le \sigma, \\ 0, & x > \sigma, \end{cases} \tag{1.8}$$

$$u_x(0,t) = 0, \quad t > 0, \tag{1.9}$$

$$u(x,t) \to 0, \quad \text{as} \quad x \to \infty, \quad t \ge 0, \tag{1.10}$$

where now $F : (-\infty,\infty) \to \mathbb{R}$ satisfies the normalized conditions,

(F1) $F(u)$ is continuous and differentiable for $u \in (-\infty,\infty)$,
(F2) $F(0) = 0, \quad F(1) = 0$,
(F3) $F'(0) = 1, \quad F'(1) < 0$,
(F4) $F(u) > 0$ for all $u \in (0,1)$,
(F5) $F(u) < 0$ for all $u \in (1,\infty)$.

Henceforth, we will refer to (1.7)-(1.10) as IBVP (which will be discussed in detail in Chapter 2). It is readily established that IBVP has a unique, global, solution (see, for example, Smoller [66], Chapter 14) with

$$0 < u(x,t) < \max[1, u_0]$$

for all $x, t > 0$.

The particular case of IBVP which arises when

$$F(u) = u(1-u) \tag{1.11}$$

has been studied extensively (see, for example, Fisher [16], Kolmogorov *et al* [29], McKean [43], Bramson [9], Larson [31], and Merkin and Needham [44]), when equation (1.7) is referred to as the Fisher-Kolmogorov equation. The starting point in analyzing IBVP with (1.11), is to examine the existence of propagating, permanent form travelling waves (PTW) which may be supported by equation (1.7). Any such PTW should have a constant propagation speed $v > 0$, have u non-negative throughout the wave profile, whilst achieving the unreacted state $u = 0$ ahead of the wave front and the fully reacted state $u = 1$ to the rear of the wave front. Introducing a travelling wave coordinate $z = x - v\,t$, the existence of a PTW requires the existence of a solution to the following nonlinear boundary value problem

$$\left.\begin{array}{l} u_{zz} + v\,u_z + u(1-u) = 0, \quad -\infty < z < \infty, \\ u(z) \to \begin{cases} 1 & \text{as} \quad z \to -\infty, \\ 0 & \text{as} \quad z \to +\infty, \end{cases} \\ u(z) \geq 0 \quad \text{for all} \quad -\infty < z < \infty. \end{array}\right\} \quad \text{BVP1}$$

This (BVP1) may be thought of as a nonlinear eigenvalue problem for the propagation speed $v > 0$. BVP1 has been studied extensively, with a review of the main results being given by Fife [14] (Chapter 4). In the present context, we recall the main result, that BVP1 has a unique (up to translation) solution if and only if

$$v \geq 2 \tag{1.12}$$

(which, in the original variables, requires $v_d \geq 2\sqrt{R'(0)D}$, with v_d being the dimensional propagation speed).

In relation to IBVP, we may now enquire as to whether or not the structure of the solution to IBVP for $t \gg 1$ involves the formation of a PTW, and if so, what is the propagation speed $v \geq 2$ of this evolving PTW. For the case when $g(x) \equiv 1$ and $u_0 = 1$ this has been analyzed rigorously by Kolmogorov *et al* [29] and McKean [43]. It was established that a PTW does evolve in the solution of IBVP as $t \to \infty$, and this PTW is the one with minimum propagation speed, that is, the PTW with $v = 2$. For this specific case, the analysis was extended by Bramson [9], who obtained the following asymptotic estimate of the propagation speed as $t \to \infty$, namely,

$$\dot{s}(t) = 2 - \frac{3}{2}t^{-1} + o\left(t^{-1}\right), \tag{1.13}$$

where $s(t)$ is a measure of the location of the PTW wave front at time t. For more general initial data, when $u_0 \ll 1$, a formal theory for IBVP has been developed by Needham [54] which is primarily based on linearization of IBVP for $t \ll 1, x = O(1)$ followed by $t \geq O(1), x \gg O(t)$. This theory reproduces (1.13) for the case when $g(x)$ has finite support (In addition it enables the cases when the initial data has exponential and algebraic tails to be analyzed, which can lead to the propagation of a PTW which has $v > 2$ when $t \gg 1$). For the purpose of this chapter and that of Chapter 2 it is useful to highlight this theory for the case when $g(x)$ has finite support. The linearized version of IBVP, with $F(u)$ given by (1.11) and $u_0 \ll 1$, is

$$u_t = u_{xx} + u, \quad x, t > 0, \tag{1.14}$$

together with conditions (1.8)-(1.10). The solution to the linearized problem may be written as

$$u(x,t) = e^t D(x,t), \quad x,t \geq 0, \tag{1.15}$$

where $D(x,t)$ is the solution of the corresponding pure diffusion problem. Due to the temporal, exponential growth in (1.15), the linearized theory fails when $t \gg 1$ and $x = O(1)$, when $u = O(1)$. However, we expect (1.15) to remain a valid approximation for $t \gg 1$ when $x \gg 1$ and $u \ll 1$ (for details see Needham [54]). Approximating $D(x,t)$ for $t \gg 1$ and $x \gg O(t)$ (via steepest descents) we obtain, via (1.15),

$$u(x,t) \sim u_0 t^{-1/2} \exp\left[-t\left(\frac{y^2}{4} - 1\right)\right] \tag{1.16}$$

for $t \gg 1$ where $y = xt^{-1} = O(1)$. The approximation (1.15) followed by (1.16) will only remain accurate when $u(x,t)$ remains small, $u \leq O(u_0)$. An examination of (1.16) then shows that, when $t \gg 1$, (1.16) may be expected to remain valid for $y > 2$ but will fail for fixed $y < 2$. The overall conclusion is that for small initial data, with finite support, the solution to IBVP will be well approximated by (1.16) for fixed $y > 2$, with u decaying exponentially to zero as $t \to \infty$. However, the exponential growth in (1.16) when $y < 2$ indicates that the linearized approximation fails when $y < 2$ and that $u = O(1)$ as $t \to \infty$ when $y < 2$. Therefore we may expect that a transition occurs in the solution to IBVP when $t \gg 1$ and $y \sim 2 + o(1)$; that is, when $t \gg 1$ and $x \sim 2t + o(t)$. This transition, from $u = O(1)$ to $u \ll 1$, when $x \sim 2t + o(t)$ and $t \gg 1$, is interpreted as the large-t development of the PTW with minimum speed $v = 2$ in IBVP. This argument is in agreement with the rigorous results discussed earlier, and indicates that the mechanism which leads to the development of a PTW in the solution to IBVP when $t \gg 1$, and in particular the mechanism which selects the propagation speed of the emerging PTW, is based on the linearized approximation (1.16); that is, the selection of propagation speed from those available ($v \geq 2$) is determined via the evolution when $x \gg 1$, $t \geq 0$ (in the "far field").

We now move onto the more general case of IBVP, when the only restrictions on $F(u)$ are those given by (F1)-(F5). A PTW in this case requires the existence of a solution to the nonlinear boundary value problem

$$\left.\begin{array}{l} u_{zz} + v\,u_z + F(u) = 0, \quad -\infty < z < \infty, \\ u(z) \to \begin{cases} 1 & \text{as} \quad z \to -\infty, \\ 0 & \text{as} \quad z \to +\infty, \end{cases} \\ u(z) \geq 0 \quad \text{for all} \quad -\infty < z < \infty, \end{array}\right\} \quad \text{BVP2}$$

with the same notation as in BVP1. A general theory for BVP2 has been developed (and will be discussed further in Chapter 2). For the present it is sufficient to note that again there exists a value $v^* > 0$ such that BVP2 has a unique solution if and only if

$$v \geq v^*. \tag{1.17}$$

In particular, in the present context, we note that when $F(u)$ satisfies the additional condition

$$F(u) \leq F'(0)u = u \quad \text{for all} \quad u \in [0,1] \tag{1.18}$$

then

$$v^* = 2, \tag{1.19}$$

as in the case of the Fisher-Kolmogorov equation, when $F(u)$ is given by (1.11). However, when condition (1.18) is not satisfied by $F(u)$ then it is possible that

$$v^* = 2 \quad \text{or} \quad v^* > 2. \tag{1.20}$$

As an example, we exhibit in Chapter 2 that when

$$F(u) = u(1-u)(1+\hat{\sigma}u) \tag{1.21}$$

with the parameter $\hat{\sigma} \geq 0$, then (1.20(a)) holds for $0 \leq \hat{\sigma} \leq 2$, whilst (1.20(b)) holds for $2 < \hat{\sigma} < \infty$. With reference to IBVP, we may again enquire as to whether or not a PTW emerges for $t \gg 1$, and if so, what is the propagation speed $v \geq v^*$ of this emerging PTW. If we again follow the linearized theory, we obtain from IBVP, together with condition (F3), the same linearized equation as for the Fisher-Kolmogorov equation, and so we may conclude again that the solution to IBVP has the structure (1.16) when $y = O(1)$ and $t \gg 1$. This again *suggests* that a wave front emerges in the solution to IBVP when $t \gg 1$, and the wave front location, $x \sim s(t)$, has speed $\dot{s}(t) \sim 2 + o(1)$ as $t \to \infty$. This is consistent with the PTW theory when $v^* = 2$ and indicates that the PTW of minimum speed emerges in the solution to IBVP when $t \gg 1$, and this selection is dictated by the linear mechanisms when $t \gg 1$ and $x \gg 1$. However, a "paradox" emerges when we consider the situation when $v^* > 2$: the linearized theory indicates that the solution to IBVP when $x \gg 1$ and $t \gg 1$ will develop, via linear mechanisms, an emerging PTW which has location $x \sim s(t)$ and speed $\dot{s}(t) \sim 2 + o(1)$; but this is not possible, for in this case no PTW exists with speed $v = 2$ (as the minimum propagation speed $v^* > 2$). If a PTW emerges at all then its propagation speed must have $v \geq v^* > 2$. Thus, in this case the linearized mechanism fails to determine the large-t propagation speed of the emerging PTW. This "paradox" has important consequences. For many reaction-diffusion problems of the Fisher-Kolmogorov type, when rigorous results are not available, linearized arguments have been extensively used to predict asymptotic wave speeds (see, for example, Murray [52], Sherratt [63], Snita *et al* [67], Gray *et al* [22]). The above argument demonstrates that this approach must be treated with caution: for IBVP the linearized approach can only be guaranteed to be accurate when $F(u)$ satisfies the additional condition (1.18).

In Chapter 2 we present a rational approach to establishing the large-t structure of IBVP via a full analysis using the theory of matched asymptotic expansions. The results of the linearized theory are reproduced and extended when IBVP is such that $v^* = 2$. The situation when $v^* > 2$ is fully developed, and it is established that in this case a travelling wave of speed $v = v^*$ emerges in IBVP for $t \gg 1$, and the apparent paradox between this situation and that as presented by the linearized theory is resolved. In each case correction terms to the asymptotic PTW wave speed are obtained, which agree with the correction of Bramson when $F(u)$ has the form (1.11), and show that this correction remains the same whenever $F(u)$ satisfies (1.18). However, the correction term changes when $F(u) > u$ for some $u \in [0, 1]$, and the correction terms in this case when $v^* = 2$ and $v^* > 2$ are presented. It is particularly important to note that the theory developed in Chapter 2 is flexible, and may be applied to coupled systems of Fisher-Kolmogorov type equations, and to problems in higher spatial dimensions. At the end of Chapter 2, we illustrate the theory when applied to IBVP with $F(u)$ given by (1.21).

1.2 mth-Order ($m > 1$) Fisher Nonlinearity

In this section we introduce the following initial-boundary value problem for a scalar reaction-diffusion equation (which will be discussed in detail in Chapters 3 and 4),

$$\left.\begin{array}{lll} u_t = u_{xx} + u^m(1-u), & x, t > 0, & \text{(P1)} \\ u(x,0) = u_0(x), & x \geq 0, & \text{(P2)} \\ u_x(0,t) = 0, & t > 0, & \text{(P3)} \\ u(x,t) \to 0 \ \text{ as } \ x \to \infty, & t \geq 0, & \text{(P4)} \end{array}\right\} \quad [\mathbf{P}, \mathbf{m}]$$

where the reaction order $m > 1$, and $u_0(x)$ is a continuous, piecewise analytic, non-negative and monotone decreasing function in $x \geq 0$, with $u_0(x) \to 0$ as $x \to \infty$. In particular, we consider the following classes of initial data:

(i) $u_0(x)$ is positive, analytic and has exponential decay rate as $x \to \infty$, with

$$u_0(x) \sim \begin{cases} u_\infty e^{-\sigma x} + O[e^{-f(x)}] \ \text{ as } \ x \to \infty, & \text{(g1)} \\ \tilde{u}_0 + \sum_{l=1}^{\infty} \tilde{u}_l x^l \ \text{ as } \ x \to 0^+, & \text{(g2)} \end{cases}$$

for some $f(x) > O(x)$ as $x \to \infty$, where $u_\infty, \sigma, \tilde{u}_o > 0$ and $\tilde{u}_l$ are constants.

(ii) $u_0(x)$ has compact support. In this case

$$u_0(x) = \begin{cases} u_0 g(x), & 0 \leq x \leq \sigma, \\ 0, & x > \sigma, \end{cases} \quad \text{(d1)}$$

where g:[0,σ] $\to \mathbb{R}$ is positive in [0,σ), non-negative and analytic in [0,σ], has $g(0) = 1$ and

$$g(x) \sim \begin{cases} g_\sigma(\sigma - x)^r, & \text{as} \quad x \to \sigma^-, \\ 1 + g_{\tilde{m}} x^{\tilde{m}}, & \text{as} \quad x \to 0^+. \end{cases}$$

Here r, $\tilde{m} \in \mathbb{N}$ with constants $g_{\tilde{m}} \neq 0$ and $g_\sigma, g_0 > 0$. The parameters σ, u_0 are positive.

(iii) $u_0(x)$ is positive, analytic and has algebraic decay rate as $x \to \infty$, with

$$u_0(x) \sim \begin{cases} u_\infty x^{-\alpha} + \mathrm{EST}(x) & \text{as} \quad x \to \infty, \\ \tilde{u}_0 + \sum_{l=1}^{\infty} \tilde{u}_l x^l & \text{as} \quad x \to 0^+, \end{cases} \quad \text{(d2)}$$

for some $\alpha \geq \frac{1}{(m-1)}$, where $u_\infty > 0$ and $\mathrm{EST}(x)$ denotes exponentially small terms in x as $x \to \infty$.

In all cases the global existence and uniqueness of a solution to **[P,m]** follows directly via the comparison theorem for parabolic operators (see, for example, Smoller Chapter 14 [66]) with

$$0 < u(x,t) < \max[1, u_0] \tag{1.22}$$

for all $x, t > 0$.

The case when m=1, when (P1) is the Fisher-Kolmogorov equation, has been studied extensively (see Fisher [16], Kolmogorov *et al* [29], McKean [43], Bramson [9], Larson [31], Merkin and Needham [44]). On considering **[P,1]** it has been shown that travelling waves of permanent form (PTWs), which connect the equilibrium state $u=0$ (ahead) to the equilibrium state $u=1$ (at the rear), develop for initial data $u_0(x)$ such that

$$u_0(x) \leq O\left(e^{-\lambda x}\right) \quad \text{as} \quad x \to \infty, \quad (\lambda > 0), \tag{1.23}$$

with these PTWs having speed

$$v = \begin{cases} 2, & \lambda \geq 1, \\ \lambda + \frac{1}{\lambda}, & 0 < \lambda < 1, \end{cases} \tag{1.24}$$

(see Larson [31], McKean [43], Needham [54], Billingham and Needham [8]). However, when $u_0(x)$ is such that

$$u_0(x) e^{\lambda x} \to \infty \quad \text{as} \quad x \to \infty, \tag{1.25}$$

for all $\lambda > 0$, there are no PTW solutions which may develop in **[P,1]** as $t \to \infty$. In particular, Bramson [9] considered **[P,1]** when the initial data has compact support with a step function initial profile as $u_0(x)$ and determined that the PTW propagation speed is given by $v(t) \sim 2 - \frac{3}{2}\frac{1}{t}$ as $t \to \infty$. This result was also obtained formally by Billingham and Needham [8] via the method of matched asymptotic expansions. Further, Billingham and Needham [8] obtained (via the method of matched asymptotic expansions) the large time solution to **[P,1]** for initial data with compact support and unbounded support with algebraic and exponential decay rates as $x \to \infty$. They established the wave speed of the PTW in the cases when the initial data has compact support and unbounded support with exponential decay rate as $x \to \infty$ and that the asymptotic correction to the wave speed is of $O\left(\frac{1}{t}\right)$ as $t \to \infty$ in these cases. Moreover, when the initial data has unbounded support with algebraic decay rate as $x \to \infty$ (when via (1.25) no PTW exists) the large time solution exhibits an accelerating phase wave **PHW** structure (Needham and Barnes [56]). An alternative approach for $m = 1$ has been presented by Ebert and Van Sarloos [13]. However this approach does not generalize to the degenerate case $m > 1$.

On considering **[P,m]** for $m > 1$ fixed, it has been shown that travelling waves of permanent form, travelling with constant speed $v \geq v^*(m)(> 0)$ exist (see Billingham and Needham [5], Merkin and Needham [45] and Barnes [4]) which connect the equilibrium state $u = 0$ (ahead) to the equilibrium state $u = 1$ (at the rear). The PTW with minimum propagation speed $v = v^*(m)$ has exponential decay ahead of the wave front. However, each PTW with speed $v > v^*(m)$ decays algebraically ahead of the wavefront, with degree $(m-1)^{-1}$.

The initial-boundary value problem **[P,m]** (with $m > 1$) has recently been considered by Needham and Barnes [56] in the complementary case to (iii) when $u_0(x)$ is positive, analytic and has algebraic decay rate as $x \to \infty$, given by (d2) when now $\alpha < \frac{1}{(m-1)}$ and $u_\infty > 0$. It was established that, when

$$u_0(x) x^{\frac{1}{(m-1)} - \delta} \to \infty \quad \text{as} \quad x \to \infty, \tag{1.26}$$

for some $\delta > 0$, then no PTW structure develops in the solution to **[P,m]** ($m > 1$) as $t \to \infty$ but an accelerating phase wave (**PHW**) structure develops as $t \to \infty$.

The asymptotic theory we develop for **[P,m]** with ($m > 1$), is similar in spirit to that developed by Billingham and Needham [8] for a system of reaction-diffusion equations which correspond to **[P,m]** with $m = 1$. For $m > 1$, this approach needs considerable adaptation due to the degenerate linearization of (P1) about $u = 0$, which leads to nonlinear effects being dominant for $x \gg 1, t \geq O(1)$ when $m > 1$, whereas linear effects are dominant when $m = 1$.

In Chapter 3 we obtain, using the method of matched asymptotic expansions, the full structure of the large-t solution to **[P,m]** (with $m > 1$) for the cases when the initial data $u_0(x)$ has unbounded support with exponential decay rate (given by (g1), (g2), case (i)) as $x \to \infty$ and when the initial data has compact support (given by (d1), case (ii)). We establish that in both cases a PTW develops as $t \to \infty$ in **[P,m]** ($m > 1$). Further, we establish in both cases the wave speed of this PTW (this being the minimum available speed, $v = v^*(m)$), its asymptotic correction as $t \to \infty$, together with the rate of convergence of the solution to **[P,m]** onto the PTW as $t \to \infty$.

In Chapter 4 we consider **[P,m]** with initial data of the form (iii) with $\alpha \geq \frac{1}{(m-1)}$. It is demonstrated, via the method of matched asymptotic expansions, that $\alpha = \frac{1}{(m-1)}$ is a bifurcation point between the development of a PTW or a **PHW** in the solution to [**P,m**] ($m > 1$) as $t \to \infty$, with the critical decay rate $\alpha = \frac{1}{(m-1)}$ falling into the PTW case. Moreover, we are able to determine the dependence of the propagation speed of the PTW upon the parameters u_∞, α and m, together with its asymptotic correction as $t \to \infty$. As may be expected in the critical case $\alpha = \frac{1}{(m-1)}$, the propagation speed of the PTW, and its correction depends sensitively on the parameters m and u_∞.