VIII.

Cohomology of Sporadic Simple Groups

VIII.0 Introduction

In this chapter we will describe progress towards understanding the cohomology of the sporadic simple groups. Briefly we recall that from the classification of finite simple groups, [Gor], it was shown that there exist 26 simple groups not belonging to infinite families (i. e. not of alternating or Lie type) and we study ten of these groups here: four of the five Mathieu groups; the Janko groups J_1, J_2, J_3; the O'Nan group $O'N$; the McLaughlin group McL; and finally the Lyons group Ly.

Here are some of the reasons, (aside from pure curiosity), for understanding the cohomology of these groups. As we will see in Chap. IX, we can add two and three dimensional cells to the classifying space of a perfect group to obtain a simply connected topological space, B_G^+, with the same homology as B_G, and the homotopy groups of these spaces are of basic importance in homotopy theory. The symmetric groups build up $Q(S^0)$, a process discussed in the introduction to Chap. VI and proved in IX.3. The general linear groups over a finite field similarly build the classifying spaces B_O and B_U as well as certain fibers of "Adams operations", $\Psi^k - 1$, known as $Im(J)$-spaces (IX.3 again). Indeed, in IX.3.2 we point out that these lead to a product splitting $Q(S^0) = Im(J) \times Coker(J)$ with the $Im(J)$ space completely understood.

It is natural to expect that the sporadic groups should play a role in the structure of the $Coker(J)$ space, though we are only beginning to understand some of the smaller sporadic groups in this framework. The group M_{11} which has 2-rank two has been studied by F. Cohen and the three 2-rank three sporadic groups M_{12}, J_1 and $O'N$ are currently being analyzed from this point of view. We only have partial information about $O'N$, but both J_1 and M_{12} are closely tied to the exceptional Lie group G_2, facts which are still slightly mysterious, but which we will explain in Chap. IX. Among the rank four sporadic groups we have determined the cohomology of M_{22} and M_{23} and the cohomology rings are discussed in §5. Neither one is Cohen–Macaulay so the rings are quite complex, but M_{23} turns out to be the first example of a finite group for which $H^i(G; \mathbb{Z}) = 0$ for $i \leq 4$. Indeed, it had been conjectured by C. Giffen [Gi] that the only finite group for which this is possible is the trivial group. Also, the $Coker(J)$ space is 5-connected with $H_6(Coker(J); \mathbb{Z}) = \mathbb{Z}/2$, and the natural inclusions $M_{23} \subset \mathcal{S}_{23} \subset \mathcal{S}_\infty$ induce an isomorphism of $H_6(M_{23}; \mathbb{Z})$ with the $\mathbb{Z}/2$ which generates $H_6(Coker(J); \mathbb{Z})$ in this dimension. Again, we expand on these remarks in §5 and Chap. IX.

Likewise, from the point of view of modular representations, the sporadic groups are key examples and it is apparent that cohomological invariants play a fundamental part in recent and ongoing research in this field. As was mentioned in our introduction to this book the ideas here are discussed in the books of Benson and Evens.

What we shall concentrate on is furnishing detailed calculations of the cohomology of a few of these groups. We will apply all the different techniques described in the previous chapters, hoping perhaps that this will serve as an additional justification and description of our methods. In particular, our calculation of $H^*(M_{11}; \mathbb{F}_2)$ in §1 completely replaces the much more complex original calculation by Benson and Carlson. Also, our calculation of $H^*(M_{12}, \mathbb{F}_2)$, where M_{12} is the Mathieu group of order 95,040, replaces the less natural and longer original calculation in [AMM2].

We use $\mathbb{F}_2$ coefficients throughout, so they are often suppressed. At some points we use the ATLAS notation for groups, extensions, etc. In most cases it is self–explanatory.

VIII.1 The Cohomology of M_{11}

Let $G = M_{11}$, the first Mathieu group having order $7920 = 2^4\ 3^2\ 5\ 11$. M_{11} has 2-rank two with one conjugacy class of groups $(\mathbb{Z}/2)^2$ and one conjugacy class of involutions. From the Atlas, [Co], $N(2A) = 2 \cdot \Sigma_4 = \mathrm{GL}_2(\mathbb{F}_3)$, (that is to say, the normalizer of an involution is the non-split extension of the form

$$\mathbb{Z}/2 \triangleleft N(2A) \to \mathcal{S}_4$$

isomorphic to $GL_3(\mathbb{F}_3)$), and we can also check that $N((\mathbb{Z}/2)^2) = \Sigma_4$. Thus the quotient $|A_2(M_{11})|/M_{11}$ has the form

$$\Sigma_4 \bullet \underset{D_8}{\overline{\qquad\qquad}} \bullet\ \mathrm{GL}_2(\mathbb{F}_3)$$

Apply V.3.3 to obtain the formula

$$H^*(M_{11}) \oplus H^*(D_8) = H^*(\mathrm{GL}_2(3)) \oplus H^*(\Sigma_4)$$

We already know from IV.2.7 that $H^*(D_8; \mathbb{F}_2) = \mathbb{F}_2[\bar{x}_1, \bar{y}_1, w_2]/(\bar{x}\bar{y} = 0)$ which has Poincaré series

$$P_{D_8}(t) = \frac{2}{(1-x)(1-x^2)} - \frac{1}{1-x^2} = \frac{1}{(1-x)^2}.$$

Similarly, using VI.1.13, the Poincaré series for Σ_4 is $\frac{1+t+t^2+t^3}{(1-t^2)(1-t^3)}$. From VII.4.5 and III.4.2 the Poincaré series for $H^*(\mathrm{GL}_2(\mathbb{F}_3))$ is

$$\frac{(1+t)(1+t^3)}{(1-t^2)(1-t^4)} = \frac{1+t+t^2+t^3+t^4+t^5}{(1-t^3)(1-t^4)}$$

and so the Poincaré series for M_{11} is (as first computed in [We], compare [BC3])

$$\frac{1+t^5}{(1-t^3)(1-t^4)} .$$

The group $GL_2(\mathbb{F}_3)$ contains a Sylow 2-subgroup of M_{11} and has its mod(2) cohomology detected on its elementary 2-subgroups. Consequently the same is true for M_{11}. More restrictively it follows from VII.4.4 that

$$H^*(M_{11};\mathbb{F}_2) \subseteq \mathbb{F}_2[x_1,x_2]^{GL_2(\mathbb{F}_2)} = \mathbb{F}_2[d_2,d_3] ,$$

(the Dickson algebra described in III.2.3) and since there is only one element in each of the dimensions 3, 4, and 5 in this ring we see that $H^*(M_{11}) \cong \mathbb{F}_2[d_3, d_2^2](1, d_2d_3)$. Hence we have

Theorem 1.2.

$$H^*(M_{11}) \cong \mathbb{F}_2[v_3,v_4,v_5]/v_3^2v_4+v_5^2=0$$

with Poincaré series $p(t) = \frac{1+t^5}{(1-t^3)(1-t^4)}$.

Remark 1.3. From the action of the Steenrod Algebra on the Dickson algebra, we have the following table giving the action of the Steenrod algebra on $H^*(M_{11})$,

gen.\ Sq	Sq^1	Sq^2	Sq^3	Sq^4
v_3	0	v_5	v_3^2	
v_4	0	v_3^2	0	v_4^2
v_5	v_3^2	0	0	$v_3^3+v_4v_5$

VIII.2 The Cohomology of J_1

J_1 is the first Janko group, of order 175,560. It has a 2-Sylow subgroup isomorphic to $(\mathbb{Z}/2)^3$. Using this, we have already calculated its cohomology in II.6.9 using the invariant ring determined in III.1.9. For further remarks on $H^*(J_1)$ using a poset decomposition see V.2.11 and V.3. For convenience, we recall the result from II.6.9,

Theorem 2.1.

$$H^*(J_1) \cong \mathbb{F}_2[x_3,y_4,z_7](\gamma_5,\mu_6)\Big/ \begin{array}{r} \gamma^2+y\mu+xz=0 \\ \mu^2+x^4+x^2\mu+y^3+\gamma z=0. \end{array}$$

with Poincaré series

$$p(t) = \frac{(1+t^5)(1+t^6)}{(1-t^3)(1-t^4)(1-t^7)} = \frac{(1+x^3)(1+x^5)(1+x^6)}{(1-x^4)(1-x^6)(1-x^7)} .$$

The action of $\mathcal{A}(2)$ can be computed directly from the explicit form of the generators given in II.1.9, particularly x given in (II.1.10). We make the following remarks anticipating our discussion of the relation between J_1 and the exceptional group G_2 which we will give in Chap. IX. First $H^*(J_1; \mathbb{F}_2)$ is Cohen–Macaulay over the Dickson algebra $\mathbb{F}_2[d_4, d_6, d_7]$ and can be rewritten in the form

$$\mathbb{F}_2[d_4, d_6, d_7](1, x, Sq^2(x), x^2, xSq^2(x), x^3, x^2Sq^2(x), x^3Sq^2(x))$$

In particular the quotient algebra by the ideal generated by the Dickson elements is

$$\begin{aligned} &H^*(J_1; \mathbb{F}_2)/(d_4, d_6, d_7) \\ &\qquad \cong \mathbb{F}_2[x, Sq^2(x)]/(x^4 = (Sq^2(x))^2 = 0, x^2 = Sq^1(Sq^2(x))) \end{aligned} \tag{$*$}$$

and this is exactly $H^*(G_2; \mathbb{F}_2)$, a "coincidence" which we will try to explain in Chap. IX. (We are indebted to F. Cohen for this description of $H^*(J_1; \mathbb{F}_2)$ and the geometric explanation which we give in Chap. IX.)

VIII.3 The Cohomology of M_{12}

In this section we study the Mathieu group M_{12}. It is considerably more complex than M_{11} and J_1 and we begin with a detailed analysis of the structure of its elementary 2-groups. We explicitly construct subgroups of M_{12} with respect to which each conjugacy class of maximal elementary 2-groups is weakly closed. Then we prove that $H^*(Syl_2(M_{12}); \mathbb{F}_2)$ is detected by restriction to its elementary 2-subgroups and from that the determination of its cohomology is fairly direct. The discussion here is a modification, based on ideas in [FM], [M2], of the work of [AMM2] where $H^*(M_{12}; \mathbb{F}_2)$ was first determined.

The Structure of the Mathieu Group M_{12}

The Mathieu group M_{12} is the subgroup of $\mathcal{S}_{12}$ generated by the following 6 elements as given for example in [Ha, pp. 79–80]:

$$\begin{aligned} u &= (1, 2, 3)(4, 5, 6)(7, 6, 9) \\ a &= (2, 4, 3, 7)(5, 6, 9, 8) \\ b &= (2, 5, 3, 9)(4, 8, 7, 6) \\ x &= (1, 10)(4, 5)(6, 8)(7, 9) \\ y &= (1, 11)(4, 6)(5, 9)(7, 8) \\ z &= (1, 12)(4, 7)(5, 6)(8, 9). \end{aligned}$$

It is 5-fold transitive and has order $2^6\ 3^3\ 5\ 11 = 95,040$. Note that $\langle a, b\rangle \cong \mathcal{Q}_8$ while $\langle x, y, z\rangle \cong \mathcal{S}_4$. When we conjugate a, b by x, y, z we obtain $xax = b$, $xbx = a$, $yay = ba$, $yby = b^{-1}$, $zaz = a^{-1}$, and $zbz = ba$, so $\mathcal{Q}_8$ is normal in the subgroup $W = \langle a, b, x, y, z\rangle$ which consequently has order $2^6\ 3 = 192$ and contains a 2-Sylow subgroup of M_{12}. In particular

$$H = Syl_2(M_{12}) = \langle a, b, d, k\rangle$$

where

$$\begin{aligned} d &= xyz = (1, 10, 11, 12)(4, 8, 7, 6) \\ k &= xyxz = (1, 12)(5, 9)(6, 8)(10, 11). \end{aligned}$$

Note that $bd = db$ so $\langle b, d\rangle = (\mathbb{Z}/4)^2$, and $k\theta k = \theta^{-1}$ for each $\theta \in \langle b, d\rangle$. Likewise, $ka = ak$, $ada^{-1} = b^{-1}d$, so setting $s = ad^2$ we have $s^2 = 1$, $\langle s, k\rangle = (\mathbb{Z}/2)^2$ and we can write H as a split extension $(\mathbb{Z}/4)^2 \times_T (\mathbb{Z}/2)^2$ where the twisting by s is given by $sds = b^{-1}d$, $sbs = b^{-1}$.

There are seven conjugacy classes of elements of order two in H, the central element $a^2 = (2, 3)(4, 7)(5, 9)(6, 8)$, one having two elements and representative

$$d^2 = (1, 11)(4, 7)(6, 8)(10, 12),$$

four with four elements in each conjugacy class having representatives

$$\begin{aligned} k &= (1, 10)(4, 7)(5, 9)(11, 12), \\ &\quad (1, 12)(2, 6)(3, 8)(4, 5)(7, 9)(10, 11), \\ &\quad (1, 10)(2, 5)(3, 9)(4, 6)(7, 8)(11, 12), \\ &\quad (1, 11)(2, 6)(3, 8)(4, 9)(5, 7)(10, 12), \end{aligned}$$

and finally one class with eight elements and representative z. The centralizers of the elements in the last 5 classes are given as follows, four copies of $D_8 \times \mathbb{Z}/2$ for the four classes with four elements and $(\mathbb{Z}/2)^3$ for the class with 8 elements. In $\mathcal{S}_{12}$ the involutions contained in M_{12} lie in two conjugacy classes, the class $\{4\}$ consisting of elements which are products of 4 disjoint involutions and $\{6\}$ consisting of those elements which are products of six disjoint involutions, so there are at least two distinct conjugacy classes of involutions in M_{12}. In fact, it is well known, [Co], that there are exactly two.

From the structure of the centralizers in H it is easy to see that every elementary 2-subgroup of H is contained in a $(\mathbb{Z}/2)^3$ and that there are exactly 8 distinct $(\mathbb{Z}/2)^3$'s, three of the form 4^7 (by which we mean that each non-identity element is in the class $\{4\}$), three of the form $4^3 6^4$, and finally two of the form $4^1 6^6$. These last two are conjugate in H and consequently weakly closed in $H \subset M_{12}$. In the other two classes two of the groups are conjugate and the third is normal in H.

Remark 3.1. These $(\mathbb{Z}/2)^3$'s are given as follows. The normal subgroup of type 4^7 is $V_1 = \langle b^2, d^2, k\rangle$ and the two other subgroups of this type are $V_2 = \langle b^2, d^2, bd^{-1}k\rangle$,

$\langle b^2, d^2, d^{-1}k\rangle$. The normal subgroup of type $4^3 6^4$ is $V_3 = \langle b^2, d^2, bk\rangle$, and the two non-normal subgroups of this type are $V_4 = \langle b^2, k, s\rangle$, $\langle b^2, d^2k, bs\rangle$, while the two subgroups of type $4^1 6^6$ are $V_5 = \langle b^2, bd^2k, s\rangle$ and $\langle b^2, bk, ks\rangle$.

Set $H_{21} = \langle a, b, k, dkd^{-1}\rangle = \mathcal{Q}_8 \times_T K$ where $K \subset \mathcal{S}_4$ is the Klein group. Since k and dkd^{-1} act on $\mathcal{Q}_8$ by conjugation with a, ab respectively, it follows that $\langle ak, abdkd^{-1}\rangle \cong \mathcal{Q}_8$ commutes with $\langle a, b\rangle$ and H_{21} is the central product, $\mathcal{Q}_8 * \mathcal{Q}_8 \cong D_8 * D_8$, an extra special 2-group. H_{21} is also the subgroup of H which is spanned by the three groups of the form $4^3 6^4$, and the element of order three,

$$xy = (1, 10, 11)(4, 9, 8)(5, 6, 7) \tag{3.2}$$

which normalizes H_{21} permutes the three subgroups cyclically. Thus these groups form a single conjugacy class in M_{12} and are weakly closed in $W \subset M_{12}$. H_{21} also contains the normal 4^7 subgroup, and xy normalizes this group as well. Hence, W can be rewritten $(\mathbb{Z}/2)^3 \times_T \mathcal{S}_4$. Finally, H_{21} contains both of the $4^1 6^6$ subgroups, and each is normal in $\langle H_{21}, xy\rangle$.

The subgroup, $H_{22} \subset H$ which is spanned by the three subgroups of the form 4^7 also has order 32. It is $\langle b, d, k\rangle$ so $H_{22} \cong (\mathbb{Z}/4)^2 \times_T \mathbb{Z}/2$ with the element of order two acting to invert the elements in $(\mathbb{Z}/4)^2$. Consequently there are exactly four subgroups of the form $(\mathbb{Z}/2)^3 \subset H_{22}$, $\langle b^2, d^2, k\rangle$, $\langle b^2, d^2, bk\rangle$, $\langle b^2, d^2, dk\rangle$ and $\langle b^2, d^2, bdk\rangle$. The first, third, and fourth each have the form 4^7 and the second has the from $4^3 6^4$. Also, the element

$$T = (1, 4, 2)(3, 11, 7)(5, 10, 6)(8, 9, 12) \in M_{12} \tag{3.3}$$

normalizes H_{22} and cyclically permutes the three 4^7 subgroups while normalizing the $4^3 6^4$. Finally, the element $d^2a = b^2s \in H$ satisfies $d^2aTd^2a = T^{-1}$, so

$$W' = \langle H, T\rangle = H_{22} \times_T \mathcal{S}_3.$$

and V_1 is weakly closed in $W' \subset M_{12}$. Summarizing the discussion above we have

Theorem 3.4.

a. *There are precisely three conjugacy classes of groups $(\mathbb{Z}/2)^3$ contained in M_{12}. Under the inclusion $M_{12} \subset \mathcal{S}_{12}$ they remain non-conjugate. The first, I, has the form 4^7, the second II, has the form $4^3 6^4$, and the third, III, has the form $4^1 6^6$.*
b. *There are subgroups W and W' of M_{12} described above so that $W \cap W' = H$ and $I \subset W'$, $II \subset W$ and $III \subset H$ are all weakly closed in M_{12}.*
c. *The Weyl group of I in M_{12} and the Weyl group of II in M_{12} are copies of $\mathcal{S}_4$.*
d. *The Weyl group of III in M_{12} is $\mathcal{A}_4$, the Weyl group of III in W.*

(There isn't much to add to the discussion above to complete the proof. In the Atlas, [Co], it is noted that the centralizer of the involution of type $\{4\}$ is W, and the centralizer of an involution of type $\{6\}$ is $\mathbb{Z}/2 \times \mathcal{S}_5$ and from this the result is direct.)

For later reference we give the explicit actions of xy and T on the eight $(\mathbb{Z}/2)^3$'s in H now. The notation is that the element in the n^{th} position in the image group is the image of the n^{th} element in the domain group. First the normalizing actions.

$$T\colon \langle b^2, d^2, bk\rangle \longrightarrow \langle d^2, b^2d^2, d^2bk\rangle, \qquad xy\colon \langle b^2, d^2, k\rangle \longrightarrow \langle b^2, k, kd^2\rangle \tag{3.5}$$

The action of d^2xy on a representative group V_5 is

$$\langle b^2, d^2bk, s\rangle \longrightarrow \langle b^2, b^2d^2bks, d^2bk\rangle. \tag{3.6}$$

Next we give the action of xy on the three subgroups of type $4^3 6^4$

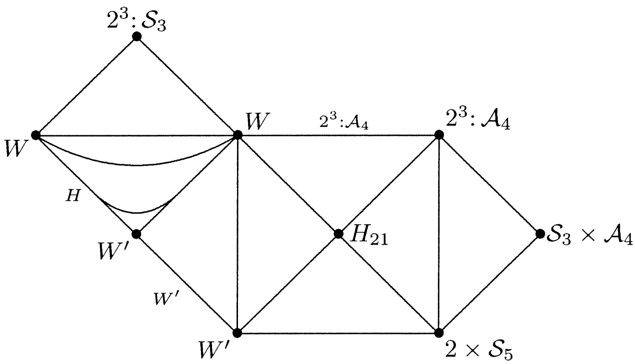
$$\langle b^2, d^2, bk\rangle \longrightarrow \langle b^2, k, b^2sk\rangle \longrightarrow \langle b^2, d^2k, bs\rangle, \tag{3.7}$$

and the acton of T on the three subgroups of type 4^7,

$$\langle b^2, d^2, k\rangle \longrightarrow \langle d^2, b^2d^2, bd^{-1}k\rangle \longrightarrow \langle b^2d^2, b^2, d^{-1}k\rangle. \tag{3.8}$$

Using the results above, and, for example, the detailed subgroup results in [AMM2] or [BR] we have the following picture of the poset space

3.9 $|A_2(M_{12})|/M_{12}$



There are 9 vertices, 17 edges and 9 triangles in this orbit complex. We have only shown the isotropy information for the vertices and a few edges. The full details are given in [AMM2, p. 106]. In Webb's formula, V.3.3, most of the groups cancel out and we are left with only

$$H^*(M_{12}) \oplus H^*(H) \cong H^*(W) \oplus H^*(W'). \tag{3.10}$$

Additionally, a close analysis of the structure of the finite Chevellay groups $G_2(q)$ shows that for $q \equiv 3, 5 \bmod (8)$, $Syl_2(G_2(q)) \cong Syl_2(M_{12})$, and the configuration $W \cup_H W'$ is also contained in $G_2(q)$. See, e. g. [M2].

This completes our discussion of the subgroup structure of M_{12}. We now turn to the cohomology ring.

Write $D_8 = \{x, y \mid x^2 = y^2 = (xy)^4 = 1\}$. Then $H_{22} \subset D_8 \times D_8$ with embedding given by $b \mapsto ((xy)^{-1}, xy)$, $d \mapsto (1, xy)$, and $y \mapsto (x, x)$, and this extends to a map $\pi\colon H \to D_8 \wr \mathbb{Z}/2$ by $\pi(s) = E$, the element which exchanges the two copies of D_8.

Theorem 3.11. *$H^*(H_{22}; \mathbb{F}_2)$ is detected by restriction to its four maximal elementary $(\mathbb{Z}/2)^3$ subgroups. In particular*

$$H^*(H_{22}; \mathbb{F}_2) = \mathbb{F}_2[w_1, w_2, c](1, x_1, x_2, x_1x_2)$$

with relations $x_1^2 = x_1c$, $x_2^2 = x_2c$ where the w's are two dimensional and the remaining generators are one dimensional.

Proof. Recall from IV.2.7 or IV.1.10 that $H^*(D_8; \mathbb{F}_2) = \mathbb{F}_2[\bar{x}, \bar{y}, w_2]/(\bar{x}\bar{y} = 0)$, where

$$H^1(D_8; \mathbb{F}_2) = Hom(D_8, \mathbb{F}_2^+)$$

with generators $\bar{x}(x) = 1$, $\bar{x}(y) = 0$, $\bar{y}(x) = 0$, $\bar{y}(y) = 1$. In $H^*(D_8 \times D_8; \mathbb{F}_2)$ write $x_1 = \bar{x} \otimes 1$, $x_2 = 1 \otimes \bar{x}$, $y_1 = \bar{y} \otimes 1$,$y_2 = 1 \otimes \bar{y}$, $w_1 = w \otimes 1$ and $w_2 = 1 \otimes w$. Then in the Gysin sequence for the inclusion $H_{22} \subset D_8 \times D_8$ we have that the map $\cup\chi\colon H^*(D_8 \times D_8; \mathbb{F}_2) \to H^{*+1}(D_8 \times D_8; \mathbb{F}_2)$ is just $\cup(x_1 + x_1 + y_1 + y_2)$. We have

$$H^*(D_8 \times D_8; \mathbb{F}_2) = \mathbb{F}_2[x_1, y_1, x_2, y_2, w_1, w_2]/(x_1y_1, x_2y_2)$$

and this can be rewritten as

$$\mathbb{F}_2[\chi, x_1 + y_1, x_1, x_2, w_1, w_2]/(\chi x_2 = x_2^2 + x_2(x_1 + y_1), x_1(x_1 + y_1) = x_1^2).$$

In particular, multiplication by χ is injective, and from this the result is direct where c_1 is the image of $x_1 + y_1$. □

In IV.7.3 a special class, $\Gamma(x) \in H^{2i}(G \wr \mathbb{Z}/2; \mathbb{F}_2)$ is constructed for each $x \in H^i(G; \mathbb{F}_2)$. Moreover, these classes, their cup products with e^i where $e \in H^1(G \wr \mathbb{Z}/2; \mathbb{F}_2)$ is dual to E, and classes of the form $tr\ (y)$, $y \in H^*(G \times G; \mathbb{F}_2)$ generate $H^*(G \wr \mathbb{Z}/2; \mathbb{F}_2)$. Also, in the Gysin sequence for the inclusion

$$\pi\colon H \hookrightarrow D_8 \wr \mathbb{Z}/2 \qquad (*)$$

the map $H^*(D_8 \wr \mathbb{Z}/2; \mathbb{F}_2) \xrightarrow{\chi} H^{*+1}(D_8 \wr \mathbb{Z}/2; \mathbb{F}_2)$ is multiplication by $tr\ (x_1 + y_1)$ which restricts to $H^*(D_8 \times D_8; \mathbb{F}_2)$ as $x_1 + x_2 + y_1 + y_2$. We can now state

Corollary 3.12.

a. *$H^*(H; \mathbb{F}_2)$ is detected by restriction to the cohomology of its 5 conjugacy classes of maximal elementary two groups.*

b. Up to extensions,

$$H^*(H;\mathbb{F}_2) = \mathbb{F}_2[c,\pi^*\,tr\,(w_1),\pi^*\Gamma(w)](1,\pi^*\,tr\,(x_1),\pi^*\Gamma(\bar{x}),\pi^*\,tr\,(x_1w_2)) \oplus e_1\mathbb{F}_2[e_1,c,\pi^*\Gamma(w)](1,\pi^*\Gamma(\bar{x}))$$

where e_1 is one dimensional and dual to E.

Proof. We begin by consdering the Gysin sequence for the inclusion (∗). The kernel of $\cup\chi$ is $(e) = e\mathbb{F}_2[\Gamma(\bar{x}),\Gamma(\bar{y}),\Gamma(w),e]/(\Gamma(\bar{x})\Gamma(\bar{y}))$, but note that the square of any element in (e) is again a non-zero element of (e). Now, the transfer, $tr : H^*(H;\mathbb{F}_2)\to H^*(D_8\wr\mathbb{Z}/2;\mathbb{F}_2)$, while it does not commute with cup products, does commute with squares (since they are stable cohomology operations), so there are no possible nilpotents in the cokernel of the restriction map $H^*(D_8\wr\mathbb{Z}/2;\mathbb{F}_2)\to H^*(H;\mathbb{F}_2)$.

Next we need $H^*(D_8\wr\mathbb{Z}/2)/(\cup\chi)$. We have an exact sequence

$$0\longrightarrow(e)\longrightarrow H^*(D_8\wr\mathbb{Z}/2;\mathbb{F}_2)\longrightarrow H^*(D_8\times D_8;\mathbb{F}_2)^{\mathbb{Z}/2}\longrightarrow 0.$$

When we cup this sequence with χ we obtain an exact sequence

$$0\longrightarrow(e)\longrightarrow H^*(D_8\wr\mathbb{Z}/2;\mathbb{F}_2)/(\chi) \longrightarrow H^*(D_8\times D_8)^{\mathbb{Z}/2}/(\chi H^*(D_8\times D_8)^{\mathbb{Z}/2})\longrightarrow 0,$$

and since the right hand quotient is easily seen to have no nilpotent elements the first statement follows.

We now use the spectral sequence of the extension $H_{22}\lhd H\xrightarrow{\pi}\mathbb{Z}/2$ with E_2-term $H^*_T(\mathbb{Z}/2;\,H^*(H_{22};\mathbb{F}_2))$. Here the action of $\mathbb{Z}/2$ on $H^*(H_{22};\mathbb{F}_2)$ is given by $T(c)=c$ since $c=x_1+y_1\sim x_2+y_2$, $T(x_1)=x_2$, $T(w_1)=w_2$. Consequently we can calculate the E_2-term explicitly as

$$E_2 = \mathbb{F}_2[c,\,w_1+w_2,w_1w_2](1,x_1+x_2,x_1x_2,x_1w_2+x_2w_1) \oplus e\mathbb{F}_2[e,c,w_1w_2](1,x_1x_2).$$

But each of the generators above, except c is in the image of π^* and hence is an infinite cycle, while c is also an infinite cycle, since $H^1(H;\mathbb{F}_2) = Hom(H,\mathbb{F}_2^+) = (\mathbb{Z}/2)^3$, and the spectral sequence collapses. This proves the second statement. □

From 3.12 it follows that the restriction images of all the generators to the five conjugacy classes of $(\mathbb{Z}/2)^3$'s are explicit. They can be determined as follows. From IV.7.1 and IV.7.2 we obtain the restrictions of $\Gamma(\theta)$ to the basic subgroups $K\times K$, $S\times K$, $K\times S$, $S\times S$, $\langle\Delta(K),E\rangle$ and $\langle\Delta(S),E\rangle$, and the image of tr is zero when restricted to these last two groups. But also, under π we obtain the following inclusions where $A=(xy)^2$:

$$\begin{aligned} V_1 &= \langle b^2,d^2,k\rangle &&\mapsto \langle A_1+A_2,A_2,x_1+x_2\rangle\subset K\times K\\ V_2 &= \langle b^2,d^2,bd^{-1}k\rangle &&\mapsto \langle A_1+A_2,A_2,y_1+x_2\rangle\subset S\times K\\ V_3 &= \langle b^2,d^2,bk\rangle &&\mapsto \langle A_1+A_2,A_2,y_1+y_2+A_2\rangle\subset S\times S\\ V_4 &= \langle b^2,k,s\rangle &&\mapsto \langle A_1+A_2,x_1+x_2,E\rangle = \langle\Delta(K),E\rangle\\ V_5 &= \langle b^2,bd^2k,s\rangle &&\mapsto \langle A_1+A_2,y_1+y_2,E\rangle = \langle\Delta(S),E\rangle. \end{aligned} \tag{3.13}$$

For definiteness, assume that w is given so that $res^*(w) = a^2 + ah$ in both $H^*(K)$ and $H^*(S)$ where a is dual to A and h is dual to x in K, y in S. In the each of the groups V_i write λ as the dual to b^2, τ is dual to the second generator, and h is dual to the third. Also, write $n = \tau^2 + \tau h$, $v = \lambda^4 + \lambda^2(\tau^2 + h^2 + \tau h) + \lambda(\tau^2 h + \tau h^2)$. Then we have the following table for the restrictions to the five $(\mathbb{Z}/2)^3$'s in 3.13.

gen.\ group	V_1	V_2	V_3	V_4	V_5
c	h	h	h	τ	τ
$\pi^* tr(x_1)$	0	h	0	0	0
e	0	0	0	h	h
$\pi^*\Gamma(\bar{x})$	h^2	0	0	$\tau^2 + \tau h$	0
$\pi^* tr(w_1)$	n	n	n	0	0
$\pi^* tr(x_1 w_2)$	$\tau^2 h + \tau h^2$	$\lambda^2 h + \lambda h^2$	0	0	0
$\pi^*\Gamma(w)$	v	v	v	v	v

(3.14)

From this the detailed structure of $H^*(H; \mathbb{F}_2)$ can be easily obtained.

However, we now have a simple criterion for determining the cohomology of M_{12}, W, and W' directly from the table above by using the actions of T and (xy) detailed in 3.5–3.8, which lead to the following maps in cohomology:

$$\begin{array}{cc} \text{Map} & \text{On Elements} \\ (xy)^*\colon H^*(V_1) \to H^*(V_1) & h \mapsto h + \tau, \tau \mapsto h, \lambda \mapsto \lambda \\ T^*\colon H^*(V_2) \to H^*(V_1) & h \mapsto h, \tau \mapsto \lambda + \tau, \lambda \mapsto \tau \\ T^*\colon H^*(V_3) \to H^*(V_3) & h \mapsto h, \tau \mapsto h + \tau + \lambda, \lambda \mapsto \tau \\ (xy)^*\colon H^*(V_4) \to H^*(V_3) & h \mapsto h, \tau \mapsto \tau + h, \lambda \mapsto \lambda + h \\ (d^2xy)^*\colon H^*(V_5) \to H^*(V_5) & h \mapsto \tau, \tau \mapsto h + \tau, \lambda \mapsto \lambda + \tau \end{array} \tag{3.15}$$

Now the stability conditions for elements to be in $H^*(W)$, $H^*(W')$ and $H^*(M_{12})$ are easily written down.

Theorem 3.16.

a. *$\alpha \in H^*(H; \mathbb{F}_2)$ is contained in the image of $res^*\colon H^*(W; \mathbb{F}_2) \to H^*(H; \mathbb{F}_2)$ if and only if $res^*(\alpha) \in H^*(V_1)^{\mathbb{Z}/3}$, and also in $H^*(V_5)^{\mathbb{Z}/3}$, while the map above from $H^*(V_4)$ to $H^*(V_3)$ stabilizes $res^*(\alpha)$.*
b. *$\alpha \in res^*(H^*(W'; \mathbb{F}_2))$ if and only if $res^*(\alpha) \in H^*(V_3)^{\mathbb{Z}/3}$ and the map from $H^*(V_2)$ to $H^*(V_1)$ stabilizes α.*
c. *$\alpha \in res^*(H^*(M_{12}))$ if and only if the conditions in both (a.) and (b.) are satisfied, i. e., if and only if $\alpha \in H^*(W) \cap H^*(W') \subset H^*(H)$.*

Remark 3.17. $\pi^*\Gamma(w) + c^4 + e^4 + (ce)^2 + \pi^* tr(w_1)^2$ restricts to the Dickson element d_4 at each of the V_i. From this and the fact that $d_6 = Sq^2(d_4)$, $d_7 = Sq^1(d_6)$ it follows that $H^*(M_{12}; \mathbb{F}_2)$ contains a copy of $\mathbb{F}_2[d_4, d_6, d_7]$. In fact it turns out that $H^*(M_{12}; \mathbb{F}_2)$ is actually Cohen–Macaulay, that is to say, free and finitely generated, over this subalgebra.

Remark 3.18. V_5 is weakly closed in $H \subset M_{12}$ and, from 3.14, the image of

$$res^*\colon\ H^*(H;\mathbb{F}_2)\longrightarrow H^*(V_5;\mathbb{F}_2)$$

is $\mathbb{F}_2[h,\tau,v_4]$ in $H^*(V_5)$. From 3.15 the action of $(d^2xy)^*$ fixes v_4 and acts on $\mathbb{F}_2[h,\tau]$ in the same way $\mathbb{Z}/3$ acts in III.1.3. Consequently, $H^*(V_5)^{\mathbb{Z}/3} = \mathbb{F}_2[h^2 + h\tau+\tau^2, h^2\tau+h\tau^2, d_4](1, h^3+h^2\tau+\tau^3)$ is the image of restriction from $H^*(M_{12};\mathbb{F}_2)$. Thus, besides the copy of the Dickson algebra there is one two dimensional generator α and there are two three dimensional generators, $Sq^1(\alpha)$ and l_3 in $H^*(M_{12};\mathbb{F}_2)$. They are constructed as follows: $\alpha = (\pi^*tr(x_1)^2 + \pi^*\Gamma(\bar{x}) + e^2 + c^2 + ec$ which restricts to $(0, 0, h^2, h^2, h^2 + \tau h + \tau^2)$ and $l_3 = c^3 + e^3 + \pi^*tr(x_1)^3 + (c + e)\pi^*\Gamma(\bar{x}) + e^2c$ which restricts to $(0, 0, h^3, h^3, h^3 + h^2\tau + \tau^3)$.

Remark 3.19. The map $T^{-*}(xy)^*T^*\colon\ H^*(V_2)\to H^*(V_2)$ is given on elements by $h\mapsto h+\lambda$, $\tau\mapsto h+\lambda+\tau$, $\lambda\mapsto h$, so $\pi^*tr(x_1w_2)$ is stable for T^* and is also $\mathbb{Z}/3$ invariant in both $H^*(V_2)$, $H^*(V_1)$. Consequently, since it restricts to 0 in the remaining groups it is in the image from $H^*(M_{12})$. This gives us a third independent generator $m_3 \in H^3(M_{12})$, and a generator $Sq^2(m_3) \in H^5(M_{12})$.

The remaining details of the determination of $H^*(M_{12};\mathbb{F}_2)$ are direct and simplified considerably by the weak closure conditions of 3.4 as 3.18 shows. We leave them to the reader and content ourselves with quoting the result from [AMM2].

Theorem 3.20. *$H^*(M_{12};\mathbb{F}_2)$ has the form $\mathbb{F}_2[\alpha_2, x_3, y_3, z_3, d_4, \gamma_5, d_6, d_7]/\mathcal{R}$ where the d_i are described above and $\mathcal{R}$ is the relation set*

$$\begin{array}{ll}
\alpha(x+y+z)=0 & x^3=\alpha^3x+\alpha d_4x+xd_6\\
xy=\alpha^3+x^2+y^2 & xz=\alpha^3+y^2\\
x^2y=\alpha^3z+\alpha d_4z+yd_6+\alpha d_7 & yz=\alpha^3+x^2\\
d_7x=d_4x^2+\alpha^2x^2 & \alpha\gamma=\alpha^2y\\
d_7y=\alpha^2d_6+\alpha^2y^2+d_4x^2+d_4y^2 & y\gamma=\alpha y^2\\
d_7z=\gamma^2+\alpha^2d_6+\alpha^2x^2+d_4x^2+d_4z^2 & x\gamma=\alpha^4+\alpha z^2\\
 & d_7^2=z^3\gamma+\alpha^2d_4d_6+\alpha^5d_4+\\
z^4=\gamma d_7+x^4+\alpha^4d_4+z^2d_6 & \quad zd_4d_7+zd_6(\gamma+\alpha z)\\
 & \quad +d_4^2(\alpha^3+xz+yz).
\end{array}$$

The Poincaré series for $H^*(M_{12};\mathbb{F}_2)$ is

$$\frac{1+t^2+3t^3+t^4+3t^5+4t^6+2t^7+4t^8+3t^9+t^{10}+3t^{11}+t^{12}+t^{14}}{(1-t^4)(1-t^6)(1-t^7)}$$

and $H^*(M_{12};\mathbb{F}_2)$ is Cohen–Macaulay over $\mathbb{F}_2[d_4, d_6, d_7]$.

(Note that all the generators have been constructed in 3.17-3.19 but x, y and z are linear combinations of $Sq^1(\alpha)$, l, and m and not these generators themselves.)

As a test the reader should calculate the Poincaré series for $H^*(W;\mathbb{F}_2)$ and $H^*(W';\mathbb{F}_2)$. Applying the result of Webb's formula, 3.10, then gives the Poincaré series in 3.20. This was a critical step in [AMM2], but here, using the weak closure conditions, it only serves the role of assuring us that we have made no numerical errors.

VIII.4 Discussion of $H^*(M_{12}; \mathbb{F}_2)$

Given a situation such as that of H, W, and W', we can find a universal completion $\Gamma = W *_H W'$ which makes the diagram below commute,

$$\begin{array}{ccc} H & \hookrightarrow & W' \\ \downarrow & & \downarrow \phi_2 \\ W & \xrightarrow{\phi_1} & \Gamma \end{array}$$

and such that any Γ' (generated by W and W') that occurs in such a push–out diagram is a quotient of Γ. Γ is called the amalgamated product of W and W' over H. It is well known, (see [Se3]), that an amalgamated product as above will act on a tree with finite isotropy, and orbit space of the form

$$W \bullet \overset{H}{\longrightarrow} \bullet W'$$

In [Go], Goldschmidt analyzed the situation for actions on the cubic tree (the tree of valence 3) and obtained a classification of finite primitive amalgams of index (3, 3) (this refers to the indexes $[W\colon H]$, $[W'\colon H]$). He shows that M_{12} is one of 15 such amalgams, necessarily a quotient of the universal one Γ.

From this we deduce the existence of an extension

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow M_{12} \longrightarrow 1 \tag{4.1}$$

where Γ' is a free group (it has cohomological dimension 1). Using the formula for Euler characteristics in [Brown], we have, on the one hand

$$\chi(\Gamma) \;=\; \frac{1}{192} + \frac{1}{192} - \frac{1}{64} = -\frac{1}{192}$$

(amalgamated product), and also

$$\chi(\Gamma) \;=\; \frac{\chi(\Gamma')}{|M_{12}|}.$$

Hence $\chi(\Gamma') = 95,040(-\frac{1}{192}) = -495$ and it follows that $\Gamma' \cong *_1^{496}\mathbb{Z}$, the free group on 496 generators.

We can now state

Theorem 4.2. *The natural map $\Gamma \to M_{12}$ induces an isomorphism*

$$H^*(M_{12}; \mathbb{F}_2) \longrightarrow H^*(\Gamma; \mathbb{F}_2).$$

Proof. Consider the map

$$res\,^W_H \oplus\, res\,^{W'}_H : H^*(W) \oplus H^*(W') \longrightarrow H^*(H)\;.$$

Its kernel is clearly $\mathrm{im}(res\,{}^W_H)\cap\mathrm{im}(res\,{}^{W'}_H)\cong H^*(M_{12})$. On the other hand, (3.10) gives that $H^*(W)\oplus H^*(W')\cong H^*(M_{12})\oplus H^*(H)$. Hence $res\,{}^W_H\oplus res\,{}^{W'}_H$ is onto. On the other hand, from the structure of the orbit space of the tree described at the beginning of this section there is a classifying space for $W *_H W'$ of the form $B_W\cup_{B_H} B_{w'}$, and, applying the Mayer-Vietoris sequence we have a long exact sequence

$$\cdots\longrightarrow H^i(\Gamma)\longrightarrow H^i(W)\oplus H^i(W')\longrightarrow H^i(H)\longrightarrow H^{i+1}(\Gamma)\longrightarrow\cdots$$

As it comes from a Mayer-Vietoris sequence the same map as before arises, hence the sequence splits and

$$H^*(W)\oplus H^*(W')\cong H^*(\Gamma)\oplus H^*(H).$$

Consequently, by rank considerations and the fact that the finite subgroups in Γ are mapped isomorphically into M_{12} under the projections the proof is complete. □

Corollary 4.3. $H^1(\Gamma';\mathbb{F}_2)$ *is an* M_{12}*-acyclic* $\mathbb{F}_2(M_{12})$*-module of rank 496 which is not projective.*

Proof. The proof follows from considering the spectral sequence over $\mathbb{F}_2$ associated to (4.4) below and the observation that 64 does not divide 496. □

This representation has radically different cohomological behavior at distinct primes dividing $|M_{12}|$. For example, at $p=3$ we have a sequence

$$H^{p-2}(M_{12};H^1(\Gamma';\mathbb{F}_3))\longrightarrow H^p(M_{12};\mathbb{F}_3)\longrightarrow H^p(W;\mathbb{F}_3)\oplus H^p(W';\mathbb{F}_3)$$

and clearly the term on the left must be non-trivial. It appears, however, that this module restricted to $M_{11}\subseteq M_{12}$ is the same one associated to the poset space for M_{11}.

To complete our discussion on M_{12}, we will explain the nature of its Poincaré series. Recall from 4.1 that $H^*(M_{12};\mathbb{F}_2)$ is Cohen–Macaulay over the Dickson algebra $\mathbb{F}_2[d_4,d_6,d_7]$. For any finite group G with Cohen–Macaulay cohomology Carlson and Benson, [BC2], have shown that the Poincaré series must satisfy a functional equation which in our case is

$$p_{M_{12}}(t)=(-t)^{rk\,M_{12}}p_{M_{12}}(t). \tag{$*$}$$

The method they use is to construct a projective $\mathbb{Z}G$-complex P^* of dimension $\sum_{i=1}^{rk\,G}(n_i-1)$ (where the $\{n_i\}$ are the dimensions of the generators of a polynomial subalgebra over which the cohomology is free and finitely generated), having the chain homotopy type of $C^*\left(\prod_{i=1}^{rk\,G}S^{n_i-1}\right)$. This is done by using cohomological varieties [BC1].

Then they consider the spectral sequence

$$E_2^{p,q} = H^p(G, H^q(P^*)) \Longrightarrow H^{p+q}((P^*)^G)$$

Let $v_i \in H^{n_i-1}(P^*)$ be the cohomology generators; by construction they transgress to $\rho_i \in H^{n_i}(G)$ and we have

$$E_\infty^{*,0} \cong H^*((P^*)^G) \cong H^*(G)/(\rho_i)$$

and so if $q(t) = P.S.\, H^*(P^{*G})$, then

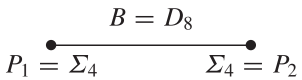
$$p_G(t) = q(t) \Big/ \prod_{i=1}^{rkG} (1 - t^{n_i})$$

$(P^*)^G$ is the algebraic orbit cochain complex, and hence will also satisfy Poincaré Duality, from wich $p_G(t)$ satisfies $(*)$. The construction of a geometric complex X satisfying this is more delicate, and obstructions certainly exist in the general case.

For M_{12} the existence of such a complex can be proved, [M2], by considering, besides the map $\Gamma \to M_{12}$ of 4.2, also a map $\Gamma \to G_2(\mathbb{F}_{3^\infty})$ constructed in [M2] as a consequence of the remark following (3.10). Taking plus constructions as described in Chap. IX, we obtain a fibering $B_\Gamma^+ \to B^+_{G_2(\mathbb{F}_{3^\infty})}$ and the fiber is a (2-local) finite complex with the correct Poincaré series. On the other hand the (2-local) homotopy equivalence $B_\Gamma^+ \to B^+_{M_{12}}$ gives the desired map on the fiber complex. We do not know if this fiber is a manifold or not though it is a (2-local) finite dimensional Poincaré duality complex.

On the other hand, there is a closely related complex which is a manifold. We now elaborate on this.

Let Y be the graph associated to the Tits Building of $L_3(\mathbb{F}_2)$ first described in Chap. V. We recall that one associates a vertex to each proper subgroup in $(\mathbb{F}_2)^3$ and an edge to any proper flag. This is a trivalent graph with a transitive $L_3(\mathbb{F}_2)$-action, having as orbit space the edge



The Tits Building has the equivariant homotopy type of $A_2(L_3(\mathbb{F}_2))$, and we have

$$\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}[G/D_8] \cong \mathbb{Z}_{(2)}[G/\Sigma_4] \oplus \mathbb{Z}_{(2)}[G/\Sigma_4] \oplus P \tag{4.4}$$

where $P = H_1(Y, \mathbb{Z}_{(2)})$ is an 8-dimensional projective module, the so-called Steinberg representation.

The above also arises by considering the amalgamated product $\Gamma = \Sigma_4 \underset{D_8}{*} \Sigma_4$. The graph Y is a quotient of the cubic tree under a free normal group $\Gamma' \subseteq \Gamma$, with quotient $L_3(\mathbb{F}_2)$. As for M_{12}, $H^*(\Gamma) \cong H^*(L_3(\mathbb{F}_2))$.

We consider the non-split extension E:

$$1 \to (\mathbb{Z}/2)^3 \to E \to L_3(\mathbb{F}_2) \to 1 \tag{4.5}$$

E is a group of order 1344 and it contains the subgroups W, W' which appear in M_{12}, realized as

$$\begin{aligned} 1 \to (\mathbb{Z}/2)^3 \to W \to P_1 \to 1 \\ 1 \to (\mathbb{Z}/2)^3 \to W' \to P_2 \to 1 \end{aligned} \tag{4.6}$$

and $\mathrm{Syl}_2(E) = \mathrm{Syl}_2(M_{12})$, realized as

$$1 \to (\mathbb{Z}/2)^3 \to H \to D_8 \to 1$$

Denote $Q = (\mathbb{Z}/2)^3$, $G = L_3(\mathbb{F}_2)$ as before. Then

$$\begin{aligned} H^*(E) &\cong H^*(\mathrm{Hom}_E(F_*, \mathbb{F}_2)) \cong H^*(\mathrm{Hom}_Q(F_*, \mathbb{F}_2)^G) \\ &\cong H^*(G, \mathrm{Hom}_Q(F_*, \mathbb{F}_2)) \end{aligned}$$

where F_* is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}E$. From this and Shapiro's formula, we deduce

$$\begin{aligned} H^*(E) \oplus H^*(H) \cong H^*(W) \oplus H^*(W') \\ \oplus H^*(G, \mathrm{Hom}_Q(F_*, \mathbb{F}_2) \otimes St) \end{aligned}$$

Rearranging, we obtain

Theorem 4.7.

$$H^*(E) \cong H^*(M_{12}) \oplus (H^*(Q) \otimes St)^{L_3(\mathbb{F}_2)}$$

The group E is the normalizer of a $(\mathbb{Z}/2)^3$ in the compact Lie group G_2, which is a 14-dimensional manifold, with

$$H^*(B_{G_2}) \cong \mathbb{F}_2[d_4, d_6, d_7] \qquad \text{([Bo2])}$$

Now E acts freely on G_2 and one can in fact show

Theorem 4.8.

$$P.S.(H^*(G_2/E)) = P_E(t) \cdot (1 - t^4)(1 - t^6)(1 - t^7).$$

In [M2], $P_E(t)$ was determined to be

$$P_E(t) = \frac{1+t^2+3t^3+2t^4+4t^5+5t^6+4t^7+5t^8+4t^9+2t^{10}+3t^{11}+t^{12}+t^{14}}{(1-t^4)(1-t^6)(1-t^7)}.$$

The numerator represents the Poincaré series of the manifold G_2/E, and it clearly dominates our answer for $P_{M_{12}}(t)$, explaining the leading terms. As a corollary we obtain that the Poincaré series for $(H^*(Q) \otimes St)^{L_3(\mathbb{F}_2)}$ is

$$z(t) = \frac{t^4 + t^5 + t^6 + 2t^7 + t^8 + t^9 + t^{10}}{(1 - t^4)(1 - t^6)(1 - t^7)}.$$

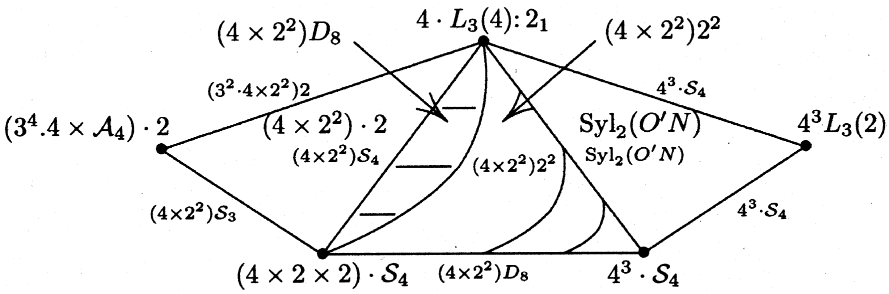
Algebraically, the denominator is explained by the action of the Dickson algebra

$$\mathbb{F}_2[d_4, d_6, d_7] \cong H^*(Q)^{L_3(\mathbb{F}_2)}.$$

VIII.5 The Cohomology of Other Sporadic Simple Groups

The O'Nan Group $O'N$

The O'Nan group $O'N$ has order $460,815,505,920 = 2^9\ 3^4\ 5\ 7^3\ 11\ 19\ 31$, and in [AM3] we determine the poset space $|A_2(O'N)|/O'N$, obtaining the following picture:



From this picture some easy cancellations give

$$\begin{aligned} &H^*(O'N)\oplus H^*((\mathbb{Z}/4)^3\cdot \Sigma_4)\\ &\qquad\cong H^*((\mathbb{Z}/4)^3\cdot \mathrm{GL}_3(\mathbb{F}_2))\oplus H^*(\mathbb{Z}/4\cdot \mathrm{SL}_3(\mathbb{F}_4)\times_T \mathbb{Z}/2). \end{aligned}$$

Our calculations show that the cohomology will be Cohen–Macaulay. Indeed, in this case the cohomology of $\mathrm{Syl}_2(O'N)$ is already Cohen–Macaulay, but is not detected by restriction to elementary 2-groups. We refer to [AM3] for complete details. It is worth noting that a key part of the cohomology of $O'N$ is detected by restriction to $H^*((\mathbb{Z}/4)^3)^{GL_3(\mathbb{F}_2)}$, which has been analyzed in Chap. III. However, this calculation is somewhat delicate since the image is a proper subset of the invariant ring, though the Dickson elements are all present.

The Rank Four Sporadic Groups

Perhaps one of the most interesting things about M_{12} is that

$$Syl_2(M_{12})\cong Syl_2(G_2(q))\cong Syl_2({}^3D_4(q))$$

for $q\equiv 3,5 \bmod (8)$. A second group which is the Sylow 2-subgroup of an entire series of interesting groups is

$$Syl_2(\tilde{\mathcal{A}}_8)=Syl_2(M_{22})=Syl_2(M_{23})=Syl_2(PSU_4(3))=Syl_2(McL). \qquad (5.1)$$

These are all rank four groups and three of them are sporadics. In fact, there are eight sporadic groups of rank four at the prime 2:

Group	Name	Order	Basic 2-locals, etc.
M_{22}	3^{rd} Mathieu	443,520	$2^4\colon \mathcal{A}_6,\ 2^4\colon \mathcal{S}_5,\ 2^3\colon L_3(2)$
J_2	Janko-Hall	604,800	$(D_8 * Q_8)\colon \mathcal{A}_5,\ 2^4\colon (2^2\colon 3^2)$
M_{23}	4^{th} Mathieu	10,200,960	$2^4\colon \mathcal{A}_7,\ 2^4\colon (GL_2(4)\colon 2)$
HS	Higman–Sims	44,352,000	$4^3\colon L_3(2),\ (4 * Q_8 * Q_8)\colon \mathcal{S}_5,\ \mathcal{S}_8$
J_3	Janko	50,232,960	$(Q_8 * D_8)\colon \mathcal{A}_5,\ 2^4\colon SL_2(4)$
McL	McLaughlin	898,128,000	$2^4\colon \mathcal{A}_7,\ L_3(4)\colon 2_2,\ \tilde{\mathcal{A}}_8$
Co_3	3^{rd} Conway	495,766,656,000	$McL\colon 2,\ HS,\ M_{23}$
Ly	Lyons	51,765,179,004,000,000	$G_2(5),\ 3\cdot McL\colon 2,\ \tilde{\mathcal{A}}_{11}$

In this section we will describe the calculations of the mod 2 cohomology for the groups M_{22}, M_{23}, McL, J_2, J_3 and Ly. We will present "second generation" versions of some of these computations. As a clearer picture has increasingly emerged. we feel that they are more enlightening. The cohomology of M_{22} was determined by Adem–Milgram [AM4], the cohomology of M_{23} by Milgram [M5], that of McL by Adem–Milgram [AM5], that of Ly by Adem–Karagueuzian–Milgram–Umland, [AKMU], and that of J_2, J_3 by Carlson–Maginnis–Milgram. For the Higman–Sims group HS, the cohomology of the 2–Sylow subgroup was calculated in [ACKM], and the full mod 2 cohomology of HS is now available, albeit not that easy to describe. From there an obvious immediate goal is to understand the cohomology of Co_3, which as we have seen, seems to have an intriguing role to play in homotopy theory. Here we should note that the size of these groups and the technical advantages now available in computer algebra, via the MAGMA program have led to the development of interesting new hybrid techniques, which promise to lead to substantial further progress. In particular the cohomology of the final Mathieu group M_{24} can now be determined, as can the cohomology of He (the Held group), which shares the same 2–Sylow subgroup. The mod 2 cohomology ring of $UT_5(2)$, the Sylow 2–subgroup of both M_{24} and He has recently been determined. The biggest hurdle that remains in determining $H^*(M_{24}, \mathbb{F}_2)$ and $H^*(He, \mathbb{F}_2)$ is the calculation and close study of the various invariant subrings in $H^*((\mathbb{Z}/2)^6, \mathbb{F}_2)$, that occur for the different normalizers of the two non–conjugate $(\mathbb{Z}/2)^6$'s in each of these groups. However, these groups represent a new level of complexity and progress along these lines is not expected to be rapid.

The group $L_3(4) = PSL_3(\mathbb{F}_4) = SL_4(\mathbb{F}_4)/3$ and its Sylow 2-subgroup play a critical role in studying many of these groups. We have a natural $2^2 \subset Out(L_3(4)) = 2 \times \mathcal{S}_3$ generated by the element 2_2 given by acting on the coefficients of the matrices with the (non-trivial) Galois automorphism of $\mathbb{F}_4$ over $\mathbb{F}_2$, $x \mapsto x^2$. Similarly, there is the standard automorphism, $2_3\colon A \mapsto A^{-t}$, taking A to its transpose-inverse. We write 2_1 for the composite of 2_2, 2_3 and note that the subgroup $\langle 2_1, 2_2\rangle = 2^2 \subset Aut(L_3(4))$. We have

$$\begin{aligned}
Syl_2(M_{22}) = Syl_2(M_{23}) = Syl_2(McL) &= Syl_2(L_3(4))\colon \langle 2_2 \rangle \\
Syl_2(J_2) = Syl_2(J_3) &= Syl_2(L_3(4))\colon \langle 2_1 \rangle \\
Syl_2(Ly) &= Syl_2(L_3(4))\colon 2^2 \\
&= Syl_2(L_3(4))\colon \langle 2_1, 2_2 \rangle \\
Syl_2(HS) &= 4^3\colon D_8 \\
Syl_2(Co_3) &= Syl_2(HS)\colon 2
\end{aligned} \tag{5.2}$$

We also have inclusions of index four $Syl_2(M_{22}) \subset Syl_2(HS)$, $Syl_2(J_2) \subset Syl_2(HS)$. The importance of $L_3(4)$ can be explained by the fact that it is really the Mathieu group M_{21}. Because it has Lie type it has a simple poset-geometry:

$$2^4 : \mathcal{A}_5 \bullet\!\!\xrightarrow[]{2^{2+4}:3^2}\!\!\bullet\ 2^4 : \mathcal{A}_5$$

where we have abbreviated $2^{2+4} = Syl_2(L_3(4))$ following ATLAS notation. Here note that $SL_2(4) = \mathcal{A}_5$, and the action of $\mathcal{S}_5$ on 2^4 is, in both cases given by regarding 2^4 as the 2-dimensional vector space over $\mathbb{F}_4$, $(\mathbb{F}_4)^2$, while $2^{2+4}\colon 3^2$ is just the subgroup of upper-triangular matrices in $SL_3(\mathbb{F}_4)$ quotiented out by the central $\mathbb{Z}/3$.

We will tie these subgroups together by studying the subgroups of the 3-fold wreath product $2 \wr 2 \wr 2$.

The Lattice of Subgroups of $2 \wr 2 \wr 2$

Write $2 \wr 2 = D_8 = \{x, y \mid x^2 = y^2 = (xy)^4 = 1\}$ and $2 \wr 2 \wr 2 = (D_8)^2\colon 2$ with the new generator acting to interchange the two copies of D_8. Thus $2 \wr 2 \wr 2$ is generated by x, y, s with $x' = sxs$, $y' = sys$ generating a second copy of D_8 which commutes with the first. In particular the quotient of $2 \wr 2 \wr 2$ by the Frattini subgroup is $2^3 = \langle x, y, s \rangle$ and the commutator subgroup which in this case equals the Frattini subgroup is given as

$$(2 \wr 2 \wr 2)' = \langle xx', yy', (xy)^2 \rangle = D_8 \times 2. \tag{5.3}$$

Note that there is an outer automorphism of D_8 which exchanges x, y that extends to an outer automorphism of $2 \wr 2 \wr 2$ exchanging x, y, then exchanging x', y', but fixing s. We now have:

Lemma 5.4. *There are seven index two subgroups of* $2 \wr 2 \wr 2$*:*

homomorphism	*kernel*	*name*
(1, 0, 0)	$\langle y, y', (xy)^2, (x'y')^2, s, xx'\rangle$	$UT_4(2)$
(0, 1, 0)	$\langle x, x', (xy)^2, (x'y')^2, s, yy'\rangle$	$UT_4(2)$
(0, 0, 1)	$\langle x, x', y, y'\rangle$	$D_8 \times D_8$
(1, 1, 0)	$\langle xy, x'y', xx', s\rangle = 4^2 : 2^2$	H
(1, 0, 1)	$\langle y, y', (xy)^2, (x'y')^2, sx\rangle = 2^4 : 4$	S
(0, 1, 1)	$\langle x, x', (xy)^2, (x'y')^2, ys\rangle = 2^4 : 4$	S
(1, 1, 1)	$\langle xy, x'y', xs\rangle = 4^2 : 4$	T

where the group in question is given as the kernel of a homomorphism $\phi_{a,b,c} : 2 \wr 2 \wr 2 \longrightarrow 2$ *and* $\phi_{a,b,c}(x) = a$, $\phi_{a,b,c}(y) = b$, $\phi_{a,b,c}(s) = c$ *with* $a, b, c \in (0, 1) = \mathbb{Z}/2$.

Here $H = Syl_2(M_{12})$ and the copies of $UT_4(2)$ are each isomorphic to the Sylow 2-subgroup of $\mathcal{A}_8 \cong L_4(2)$. There is a single copy of $Q_8 * Q_8 = D_8 * D_8$ in $2 \wr 2 \wr 2$,

$$Q_8 * Q_8 \ = \ \{xx', (xy)^2, yy', s\}. \tag{5.5}$$

which is the intersection

$$Q_8 * Q_8 = H \cap UT_4(2)_1 = H \cap UT_4(2)_2 = UT_4(2)_1 \cap UT_4(2)_2. \tag{5.6}$$

We now wish to go a little deeper into the structure of the Sylow 2-subgroup of the central extension $2\mathcal{A}_{10} = \tilde{\mathcal{A}}_{10}$. For this we need the relatively well known result below.

Lemma 5.7. *The wreath product* $2 \wr 2 \wr 2$ *has three conjugacy classes of* 2^4*'s.*

Proof. Since D_8 has two subgroups of the form 2^2,

$$K = \langle x, (xy)^2\rangle, \ J = \langle y, (xy)^2\rangle$$

there are at least three conjugacy classes of $2^4 \subset 2 \wr 2 \wr 2$ given as $K \times K$, $J \times J$, and $K \times J$, all contained in $D_8 \times D_8$ with the first two normal. To see that there are no more than three one can look at the decomposition

$$\langle (xy)^2, xx', yy'\rangle = 2 \times D_8 \lhd 2 \wr 2 \wr 2 \longrightarrow \langle x, y', s\rangle = 2^3$$

and verify that a 2^4 will either have image 2 or $2^2 = \langle x, y'\rangle$ in the quotient.

Now, consider the alternating group $\mathcal{A}_{10} \supset \mathcal{S}_8$. Note that

$$Syl_2(\mathcal{A}_{10}) = Syl_2(\mathcal{S}_8) = 2 \wr 2 \wr 2. \tag{5.8}$$

Also, $\mathcal{A}_{10}$ has a unique 2-fold cover $\tilde{\mathcal{A}}_{10}$ with extension data described as follows: an involution $v \in \mathcal{A}_{10}$ has $v^2 = z$, the new central element, if and only if $v = (a, b)(c, d)$ in cycle notation. Thus, for two involutions α, β, we have $\alpha\beta = \beta\alpha$ if and only if

$$\alpha\beta = (a, b)(c, d)(e, f)(g, h) \in \mathcal{A}_{10}$$

for eight distinct elements $a, b, \dots, h$. In particular, the three conjugacy classes of 2^4's in $2 \wr 2 \wr 2$ lift as follows when we set $K = \langle (1,3)(2,4),\ (1,2)(3,4) \rangle$, $J = \langle (1,2),\ (3,4) \rangle$ so their representatives in $\mathcal{S}_{10}$ are given as

$$\begin{aligned} K \times K &\cong \langle (1,2)(3,4),\ (1,3)(2,4),\ (5,6)(7,8),\ (5,7)(6,8) \rangle \\ J \times J &\cong \langle (1,2)(9,10),\ (3,4)(9,10),\ (5,6)(9,10),\ (7,8)(9,10) \rangle \\ K \times J &\cong \langle (1,2)(3,4),\ (1,3)(2,4),\ (5,6)(9,10),\ (7,8)(9,10) \rangle \end{aligned}$$

and we see that $\widetilde{K \times K} \cong \widetilde{K \times J} \cong Q_8 * Q_8$ while $\widetilde{J \times J} \cong D_8 * Q_8$ since

$$\langle (1,2)(3,4)(5,6)(7,8),\ (5,6)(7,8) \rangle \subset \widetilde{J \times J}$$

is a D_8 while $\langle (1,2)(9,10),\ (3,4)(9,10) \rangle \cong Q_8$ and the two subgroups together span the entire lift and are directly seen to commute.

The group 2^{2+4} has a presentation as follows:

$$2^{2+4} = \left\{ x, y, e, f, z, t \;\middle|\; \begin{array}{l} \langle x, y, z, t \rangle \cong \langle e, f, z, t \rangle \cong 2^4, \\ [x, e] = z, [x, f] = t, [y, e] = t, [y, f] = tz \end{array} \right\} \quad (5.9)$$

and contains exactly two copies of 2^4, $\langle x, y, z, t \rangle$, $\langle e, f, z, t \rangle$, and three copies of the group $Q_8 \times 2$:

$$\begin{aligned} (Q_8 \times 2)_z &= \{xe, yef, t\}, \\ (Q_8 \times 2)_t &= \{xf, ye, z\}, \\ (Q_8 \times 2)_{tz} &= \{xef, yf, t\}. \end{aligned} \quad (5.10)$$

Lemma 5.11. *The lift of* $Q_8 * Q_8 = UT_4(2)_1 \cap UT_4(2)_2$ *is just* 2^{2+4}. *Moreover,* $Syl_2(Ly)$ *contains a unique copy of* 2^{2+4}.

Proof. One checks easily that the quotient of 2^{2+4} by a single central element is $Q_8 * Q_8$. Also, the table of index two subgroups of $2 \wr 2 \wr 2$ above shows that there is no copy of $2^{2+4} \subset 2 \wr 2 \wr 2$. Consequently there is at most one copy $2^{2+4} \subset Syl_2(Ly)$ but we already know there is at least one since $Syl_2(McL) \subset Syl_2(Ly)$. □

Remark 5.12. The lift of

$$2^3_I = \langle (1,2)(3,4)(5,6)(7,8),\ (1,3)(2,4)(5,7)(6,8),\ (1,5)(2,6)(3,7)(4,8) \rangle$$

is the first $2^4 \subset 2^{2+4}$ and the lift of its conjugate,

$$2^3_{II} = \langle (1,2)(3,4)(5,6)(7,8),\ (1,4)(2,3)(5,7)(6,8),\ (1,6)(2,5)(3,7)(4,8) \rangle$$

gives the second $2^4 \subset 2^{2+4}$.

Now, notice that $\langle Q_8 * Q_8, J \times J\rangle$ is one of the $UT_4(2)$'s while $\langle Q_8 * Q_8, K \times K\rangle$ is the other. The lift of the first is $Syl_2(J_2) = 2^{2+4}\colon 2_1$ while the lift of the second is $Syl_2(M_{22}) = 2^{2+4}\colon 2_2$, and the entire group is $Syl_2(Ly)$. Finally, the lift of H is $2^{2+4}\colon 2_3$.

We will now look at the subgroup structure for the Lyons group Ly; the following table summarizes the information about maximal subgroups.

The Maximal Subgroups of the Lyons Group Ly

Order	Group
5, 895, 000, 000	$G_2(5)$
5, 388, 768, 000	$3 \cdot McL\colon 2$
46, 500, 000	$5^3 \cdot L_4(5)$
29, 916, 800	$\tilde{\mathcal{A}}_{11}$
9, 000, 000	$5_+^{1+4}\colon 4\mathcal{S}_6$
3, 849, 120	$3^5\colon (2 \times M_{11})$
699, 840	$3^{2+4}\colon (\tilde{\mathcal{A}}_5).D_8$
1474	$67\colon 22$
666	$37\colon 18$

Consequently, Ly contains the two subgroups $3 \cdot McL$ and a three extension of $L_3(4)\colon 2_1 \subset L_3(4)\colon \langle 2_1, 2_2\rangle$, which is a subgroup of $3 \cdot McL\colon 2$, but not of $3 \cdot McL$. However, from our analysis above there are only two possible candidates for the intersections of these groups with $Syl_2(Ly)$, so $3\colon McL$ intersects in the lift of the $UT_4(2)$ which contains $K \times K$, while $L_3(4)\colon 2_1$ intersects in the lift of the $UT_4(2)$ which contains $J \times J$.

Remark 5.13. Note that the ATLAS table of maximal subgroups of Ly shows that $G_2(5) \subset Ly$ is a maximal subgroup, [Co] p. 174. Consequently, $Syl_2(Ly)$ must contain a copy of $Syl_2(G_2(5)) \cong Syl_2(M_{12})$. In fact one can show that $Syl_2(Ly)$ contains a *unique* copy of $Syl_2(M_{12})$ and identify it.

The Cohomology Structure of 2^{2+4}

We review the description of $H^*(2^{2+4})$ given in [AM1] (see also [Mag]):

$$\mathbb{F}_2[v_4, w_4] \otimes \left\{ \begin{array}{r} (\mathbb{F}_2[x, y] \oplus \mathbb{F}_2[e, f])(1, L_3, M_3, LM) \\ \oplus\langle xf, ye, x^2 f, xf^2, xfL, R_6\rangle \end{array} \right\}. \tag{5.14}$$

The radical is the piece

$$\mathbb{F}_2[v_4, w_4](xf, ye, x^2 f, xf^2, xfL)$$

and the restriction to each $H^*(2^4)$ is the entire invariant subring

$$H^*(2^4)^{2^2} = \mathbb{F}_2[x, y, v_4, w_4](1, L, M, LM).$$

Also, R_6 restricts to $(LM, 0)$ in the cohomology of the two copies of 2^4 while

$$
\begin{aligned}
x &\mapsto (x,0),\\
y &\mapsto (y,0),\\
e &\mapsto (0,x),\\
f &\mapsto (0,y),\\
L &\mapsto (L,L),\\
M &\mapsto (M,M).
\end{aligned}
$$

describes the rest of the restriction to the two 2^4's.

An essential step in the work of [CMM] was the following sharpening of (5.14). It helps clarify the existing results on $H^*(M_{22})$, $H^*(M_{23})$ and makes it possible to determine $H^*(J_2)$, $H^*(J_3)$ as well.

Lemma 5.15. *The two copies of* 2^4 *and the three copies of* $Q_8 \times 2$ *contained in* 2^{2+4} *detect* $H^*(2^{2+4})$ *under restriction.*

Proof. A computer calculation using MAGMA results in the following table giving the restrictions of the generators above to the cohomology of the three $Q_8 \times 2 \subset 2^{2+4}$. We have $H^*(Q_8 \times 2) = \mathbb{F}_2[\gamma_4, t_1](1, a, b, a^2, b^2, a^2b)$ where a is dual to the first generator in the corresponding group of (2.3) while b is dual to the second and t is dual to the third:

element	$(Q_8 \times 2)_z$	$(Q_8 \times 2)_t$	$(Q_8 \times 2)_{zt}$
x	$a+b$	a	b
y	b	b	$a+b$
e	a	b	b
f	b	a	a
L	b^2t+bt^2	$(a+b)^2t+(a+b)t^2$	b^2t+bt^2
v_4	$t^4+a^2t^2$	$\gamma_4+a^2t^2$	$t^4+\gamma_4$
w_4	γ_4	γ_4+t^4	γ_4

Consequently we have the following restriction images for the generators of the radical:

element	$(Q_8 \times 2)_z$	$(Q_8 \times 2)_t$	$(Q_8 \times 2)_{zt}$
xf	a^2	a^2	ab
ye	ab	b^2	a^2
x^2f	a^2b	0	a^2b
xye	0	a^2b	a^2b
xfL	a^2bt^2	a^2bt^2	a^2bt^2
v_4	$t^4+a^2t^2$	$\gamma_4+a^2t^2$	$t^4+\gamma_4$
w_4	γ_4	γ_4+t^4	γ_4

and the image of the radical clearly has the form

$$\mathbb{F}_2[v_4, w_4](xf, ye, x^2 f, xye, xfL)$$

as required. □

Detection and the Cohomology of J_2, J_3

The spectral sequence for the group extension, converging to $H^*(2^{2+4}: 2_2)$ collapses at E_2, as does the spectral sequence converging to $H^*(2^{2+4}: 2_1)$. Moreover, one again gets detection theorems for the cohomology of these groups. We obtain the following theorem (see [Mag])

Theorem 5.16. *There is a copy of the group $Q_8 * D_8 \subset 2^{2+4}: 2_1$ and $H^*(Q_8 * D_8)$ $H^*(2^{2+4})$ detect $H^*(2^{2+4}: 2_1)$.*

Using 5.15, this can be immediately sharpened to

Theorem 5.17. *There are three conjugacy classes of subgroups isomorphic to $Q_8 \times 2$ and one conjugacy class of 2^4's in $2^{2+4}: 2_1$ and restriction to these four subgroups detects $H^*(2^{2+4}: 2_1)$.*

In J_2 and J_3 the three conjugacy classes of $Q_8 \times 2$'s in $2^{2+4}: 2_1$ all become conjugate and we have, with only a little further work (see [CMM])

Theorem 5.18. *In both $H^*(J_2)$ and $H^*(J_3)$ the radical has the form*

$$\mathbb{F}_2[d_8, d_{12}](k_5, a_7, a_{11})$$

while the restriction of the image of $H^*(J_2)$ to $H^*(2^4)$ is the inverse image in $H^*(2^4)^{2^2:3^2}$ of the subalgebra $H^*(2^2)^{\mathcal{S}_3}$ under the inclusion of the center of 2^{2+4} in 2^4_I. Similarly, the image of $H^*(J_3)$ in $H^*(2^4)$ is the inverse image in $H^*(2^4)^{GL_2(4)}$ of $H^*(2^2)^{\mathcal{S}_3}$.

The Cohomology of the Groups M_{22}, M_{23}, $SU_4(3)$, McL, and Ly

We view $Syl_2(M_{22})$ as the split extension $2^{2+4}: 2_2$ given explicitly by adjoining an element a to the presentation of 2^{2+4} above, where $a^2 = 1$ and the action of a on 2^{2+4} is given by setting

$$\begin{array}{ll} x^a = x & y^a = xy \\ e^a = e & f^a = ef \\ z^a = z & t^a = zt \end{array}$$

The group $(Q_8 \times 2)_z$ is normalized by a while a exchanges the other two copies of $(Q_8 \times 2)$. Likewise, a normalizes both copies of 2^4 in 2^{2+4}. Besides the two 2^4's there are now two other conjugacy classes of extremal elementary two groups in $Syl_2(M_{22})$ with representatives given as follows:

$$\begin{aligned} 2_I^3 &= \langle a, x, z\rangle \\ 2_{II}^3 &= \langle a, e, z\rangle \end{aligned}$$

and we have the following detection result which sharpens the results of [AM4]:

Theorem 5.19.

1. *Restriction to* 2_I^3, 2_{II}^3 *and* 2^{2+4} *detects* $H^*(2^{2+4}: 2_2)$.
2. *Restriction to the subgroups* 2_I^4, 2_{II}^4, 2_I^3, 2_{II}^3, $(Q_8 \times 2)_z$ *and* $(Q_8 \times 2)_t$ *detects* $H^*(2^{2+4}: 2_2)$.

Proof. Here, 5.19.1 is contained in [AM4] while 5.19.2 follows directly from 5.19.1 and 5.15.

This result allows a direct understanding of the cohomology of M_{22}, M_{23}, $PSU_4(3)$ and McL. The two 2^4's remain non-conjugate in all four groups, and consequently the two pairs $2_I^4 \subset 2^{2+4}: 2_2$, $2_{II}^4 \subset 2^{2+4}: 2_2$ are weakly closed in each. The Weyl groups are given as follows:

$$\begin{array}{lcc} \text{Group} & V_4 & W_4 \\ M_{22} & \mathcal{A}_6 & \mathcal{S}_5 \\ M_{23} & \mathcal{A}_7 & GL_2(4): 2 \\ PSU_4(3) & \mathcal{A}_6 & \mathcal{A}_6 \\ McL & \mathcal{A}_7 & \mathcal{A}_7 \end{array}$$

It follows from our determination of $H^*(2^{2+4})$ that the intersection of the image of $H^*(G)$ with $H^*(2_j^4)$ is the entire invariant subring under the action of the Weyl group. Moreover, the invariants for each of the groups above are known, as we mentioned in Chap. III. From [AM2] we have that

$$\begin{aligned} \mathbb{F}_2[x_1, x_2, x_3, x_4]^{\mathcal{A}_6} &= \mathbb{F}_2[w_3, \gamma_5, d_8, d_{12}](1, \gamma_9, b_{15}, \gamma_9 b_{15}) \\ \mathbb{F}_2[x_1, x_2, x_3, x_4]^{\mathcal{A}_7} &= D_4(1, x_{18}, x_{20}, x_{21}, x_{24}, x_{25}, x_{27}, x_{45}) \end{aligned}$$

where $D_4 = \mathbb{F}_2[d_8, d_{12}, d_{14}, d_{15}]$ is the rank 4 Dickson algebra. Similarly, the $\mathcal{S}_5$ invariant subring can be described as

$$\begin{aligned} &\mathbb{F}_2[a, b, c, d]^{\mathcal{S}_5} = \\ &\qquad \mathbb{F}_2[\bar{w}_3, \gamma_5, d_8, d_{12}](1, n_6, n_8, \gamma_9, n_{10}, n_{12}, x_{12}, x_{14}, x_{15}, x_{16}, x_{18}, x_{24}) \end{aligned}$$

where $Sq^2(n_6) = n_8$, $Sq^4(n_6) = n_{10}$, $n_{12} = n_6^2$, $x_{12} = Sq^4(n_8)$ and $x_{14} = n_6 n_8$.

The invariant subring for $GL_2(4): 2$ is more involved and was determined in [M5]. The Poincaré series has the form $p(x)/q(x)$ where $q(x)$ is the polynomial

$$q(x) = (1 - x^{10})(1 - x^{12})(1 - x^{15})(1 - x^{24}),$$

and $p(x)$ is

$$1 + x^6 + 2x^8 + x^9 + x^{11} + 2x^{12} + x^{13} + 3x^{14} + 2x^{15} + 3x^{16} + 3z^{17} + 2x^{18} + 2x^{19} + 4x^{20} + 3x^{21} + 4x^{22} + 4x^{23} + 4x^{24} + 4x^{25} + 4x^{26} + 4x^{27} + 5x^{28} + 5x^{29} + 4x^{30} + 4x^{31} + 4x^{32} + 4x^{33} + 4x^{34} + 4x^{35} + 3x^{36} + 4x^{37} + 2x^{38} + 2x^{39} + 3x^{40} + 3x^{41} + 2x^{42} + 3x^{43} + x^{44} + 2x^{45} + x^{46} + x^{48} + 2x^{49} + x^{51} + x^{57}$$

Expanding out into a Taylor series we obtain

Corollary 5.20. *The Poincaré series for the invariants*

$$\mathbb{F}_2[x_1, x_2, x_3, x_4]^{GL_2(4):2}$$

has Taylor series of the form

$$1 + x^6 + 2x^8 + x^9 + x^{10} + x^{11} + 3x^{12} + x^{13} + 3x^{14} + 3x^{15} + 4x^{16} + 3x^{17} + 5x^{18} + \cdots .$$

It remains to discuss the radicals and the 2^3's. In M_{22} and M_{23} one of the two 2^3's becomes conjugate to a subgroup of one of the 2^4's but the other remains extremal. Consequently, it is also weakly closed in M_{22}, M_{23}, and has Weyl group $L_3(2)$ in both M_{22}, M_{23}. However, the intersection is not the entire invariant subring, $\mathbb{F}_2[d_4, d_6, d_7]$, but $\mathbb{F}_2[d_4^2, d_6, d_7](1, d_4d_6, d_4d_7)$ so this is the restriction image from both $H^*(M_{22})$, $H^*(M_{23})$.

For M_{22} the radical is

$$\mathbb{F}_2[d_8, d_{12}](a_2, a_7, a_{11}, a_{14})$$

while M_{23} has the smaller radical

$$\mathbb{F}_2[d_8, d_{12}](a_7, a_{11}).$$

The image of restriction in each of the $H^*(2^4; \mathbb{F}_2)$'s is the entire invariant subring. Thus, to describe the image of $H^*(M_{22}; \mathbb{F}_2)$ in the direct sum $H^*(V_4; \mathbb{F}_2) \oplus H^*(W_4; \mathbb{F}_2) \oplus H^*(V_3; \mathbb{F}_2)$ we need to describe the multiple image classes, i. e. those classes which have non-trivial image in more than one of the three rings. It turns out that they are generated by $(\bar{w}_3, \bar{w}_3, 0)$, $(0, n_6, d_6)$, $(0, n_{10}, d_4d_6)$ together with the polynomial ring $\mathbb{F}_2[d_8, d_{12}]$, where $d_8 \mapsto (d_8, d_8, d_4^2)$, $d_{12} \mapsto (d_{12}, d_{12}, d_6^2)$.

The non-nilpotent part of $H^*(M_{22}; \mathbb{F}_2)$ is given in [AM4] as the direct sum

$$H^*(V_4; \mathbb{F}_2)^{\mathcal{A}_6} \oplus H^*(W_4; \mathbb{F}_2)^{\mathcal{S}_5} \oplus d_7\mathbb{F}_2[d_4, d_6, d_7]$$

where the two copies of $\mathbb{F}_2[d_8, d_{12}](1, \bar{w}_3)$ in the first two rings are identified.

The result for M_{23} is similar.

Theorem 5.21. *For M_{23} there is a long exact sequence*

$$0 \longrightarrow \mathbb{F}_2[d_8, d_{12}](a_7, a_{11}) \longrightarrow H^*(M_{23}) \longrightarrow H^*(V)^{\mathcal{A}_7} \oplus H^*(W)^{GL_2(4):2} \oplus \mathbb{F}_2[d_4, d_6, d_7]d_7 \longrightarrow \mathbb{F}_2[d_8, d_{12}] \longrightarrow 0$$

Finally, we have the groups $PSU_4(3)$ and McL. In both of these groups the remaining 2^3 becomes conjugate to a subgroup of the other 2^4 and so $H^*(PSU_4(3))$, $H^*(McL)$ are completely detected by restriction to the two $H^*(2^4)^{W_G(2^4)}$ together with the determination of the radicals.

Theorem 5.22. *There is a long exact sequence*

$$0 \to \mathbb{F}_2[d_8, d_{12}](a_2, a_7, a_{11}, a_{14}) \to H^*(PSU_4(3)) \to H^*(2^4)^{\mathcal{A}_6} \oplus H^*(2^4)^{\mathcal{A}_6} \to \mathbb{F}_2[d_8, d_{12}](1, \bar{w}_3, b_{15}, \bar{w}_3 b_{15}) \to 0.$$

Here, it appears that the class corresponding to the Lie group $PSU_4(\mathbb{C})$ is the (double image) b_{15}.

In the case of McL the result takes the form below, but note that the class b_{15} is no longer present.

Theorem 5.23. *There is a long exact sequence*

$$0 \longrightarrow \mathbb{F}_2[d_8, d_{12}](a_7, a_{11}) \longrightarrow H^*(McL) \longrightarrow H^*(2^4)^{\mathcal{A}_7} \oplus H^*(2^4)^{\mathcal{A}_7} \longrightarrow \mathbb{F}_2[d_8, d_{12}](1, x_{18}) \longrightarrow 0.$$

In [AKMU] a calculation was given for the ring of invariants,

$$H^*(2^4)^{L_3(2)} = \mathbb{F}_2[d_2, d_3, d_4, d_8](1, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{21})$$

where the action of $L_3(2)$ is the twisted action of the Weyl group of either of the 2^4's in $\tilde{\mathcal{A}}_8$. Using this, the cohomology rings of $\tilde{\mathcal{A}}_8$, $\tilde{\mathcal{S}}_8$ and $\tilde{\mathcal{A}}_{10}$ can be quickly determined. It turns out that the rings for both $\tilde{\mathcal{S}}_8$ and $\tilde{\mathcal{A}}_{10}$ are detected by restriction to the two maximal elementary 2-subgroups: a 2^4 and a 2^3. From this it follows that the same is true for $H^*(Ly)$ and we have a complete determination of the cohomology ring of Ly.

Theorem 5.24. *There is a short exact sequence*

$$0 \longrightarrow H^*(Ly) \longrightarrow H^*(2^4)^{\mathcal{A}_7} \oplus \mathbb{F}_2[d_4^2, d_6^2, d_7](1, d_4 d_7, d_6 d_7, d_4 d_6 d_7) \longrightarrow \mathbb{F}_2[d_8, d_{12}] \longrightarrow 0.$$

Here the elements d_8 and d_{12} in $\mathbb{F}_2[d_8, d_{12}]$ are the images of (d_8, d_4^2) and (d_{12}, d_6^2) in the direct sum above.

Remark on the Cohomology of M_{23}

Some time ago it was conjectured by J.L. Loday and later by C. Giffen (see [Gi]) that if a finite group G satisfies $H_i(G, \mathbb{Z}) = 0$ for $i = 1, 2, 3$, then $G = \{1\}$. The sporadic group M_{23} is the first known counterexample to this conjecture. Indeed, combining the arguments given here together with easier computations at odd primes, Milgram proved [M]:

Theorem 5.25. $H^*(M_{23}, \mathbb{Z}) = 0$ *for* $0 < i < 5$

Note that M_{23} is somewhat unusual among the sporadic groups in that $Out(M_{23}) = Mult(M_{23}) = 1$. We also have that $H_6(M_{23}; \mathbb{Z}) = \mathbb{Z}/2$ is the first non-zero homology group. In particular, when we look at the usual inclusion $M_{23} \subset \mathcal{S}_{23}$ we can ask about the image of this first non-trivial class. This will be discussed in Chap. IX, where we will consider homotopy theoretic aspects of these calculations.