Exercise 4.8.4. Interpret the constant $c_{-1;2}$ and compute it.
[**Answer:** $\frac{1}{72}\,\mathrm{tr}(\Lambda^{-1})^6$].

In the examples above, the contribution $c_{g;n}$ of all genus g gluings of $2n$ copies of 3-stars is a monomial in $\mathrm{tr}(\Lambda^{-(2k+1)})$ with rational coefficients. In the next section, we will show that this is the case for all g and n.

4.9 A Sketch of Kontsevich's Proof of Witten's Conjecture

In this section we discuss very briefly exciting connections of the Kontsevich model with the one-matrix model on the one hand, and with the intersection theory model on the other hand.

4.9.1 The Generating Function for the Kontsevich Model

The sample calculations in the previous section show that the contribution $c_{g;n}$ of all genus g gluings of $2n$ copies of 3-stars is a monomial in $\mathrm{tr}(\Lambda^{-1})$, $\mathrm{tr}(\Lambda^{-3})$, $\dots$. It is more convenient, however, to use a slightly different normalized infinite sequence of independent variables: $t_0 = -\,\mathrm{tr}(\Lambda^{-1})$, $t_1 = -1!!\,\mathrm{tr}(\Lambda^{-3})$, $t_2 = -3!!\,\mathrm{tr}(\Lambda^{-5})$, $\dots$, $t_i = -(2i-1)!!\,\mathrm{tr}(\Lambda^{-2i-1})$, $\dots$.

Theorem 4.9.1 ([178]). *The integral* (4.6) *is a formal power series in the variables $t_0, t_1, \dots$ with rational coefficients.*

Let us denote this series by $K(t_0, t_1, \dots)$. We have already computed the first few terms:

$$K(t_0, t_1, \dots) = \log\left(1 + \frac{1}{3!}t_0^3 + \frac{1}{24}t_1 + \frac{25}{144}t_0^3 t_1 + \frac{1}{72}t_0^6 + \dots\right) . \qquad (4.10)$$

Remark 4.9.2. The integral (4.7) can be easily interpreted as a formal power series since each monomial in $\mathrm{tr}(\Lambda^{-1})$, $\mathrm{tr}(\Lambda^{-3}), \dots$ appears in the integral evaluation only a finite number of times. Equation (4.10) is valid for arbitrary value of the dimension N. However, if we want to compute a specific coefficient in this expansion, the value of N must be chosen sufficiently large.

Kontsevich's proof of the Witten conjecture consists of two parts. First, he shows that the coefficient of $t_0^{l_0} \dots t_s^{l_s}/(l_0! \dots l_s!)$ in the expansion of his integral $K(t_0, t_1, \dots)$ in the variables t_i coincides with the intersection number $\langle \tau_0^{l_0} \dots \tau_s^{l_s} \rangle$. This part of the proof is based on the study of the combinatorial model for the moduli space of curves. The second part consists in verification that the integral is a τ-function for the KdV hierarchy. This means, essentially, that the second derivative $\partial^2 K/\partial t_0^2$ is a solution to the KdV equation. The proof of this statement is achieved by treating the function K as a matrix Airy function.

4.9.2 The Kontsevich Model and Intersection Theory

A formal justification of the argument in this section requires the construction of a “minimal compactification” of the moduli space of smooth marked curves (elaborated by Looijenga in [203]) and an analysis of circle bundles over this compactification. The latter part is accurately written in the Ph.D. thesis of D. Zvonkine [313]. Below, we simply outline the original Kontsevich’s argument.

Consider the projection

$$\pi : \mathcal{M}_{g;n}^{\mathrm{comb}} \cong \mathcal{M}_{g;n} \times \mathbb{R}_+^n \to \mathbb{R}_+^n$$

of the combinatorial model to the second factor. This projection takes a marked graph with a metric to the n-tuple of the lengths of the perimeters of the marked points. Introduce the real 2-forms ω_i defined only on open strata of $\mathcal{M}_{g;n}^{\mathrm{comb}}$ by the following formulas:

$$\omega_i = \sum d(l_{e'}/p_i) \wedge d(l_{e''}/p_i),$$

where p_i is the perimeter of the ith face, and e', e'' run over all pairs of distinct edges of the ith face, e' preceding e'' in some fixed order with a chosen starting vertex. The 2-form ω_i represents the class ψ_i. Indeed, fix a smooth curve $(X; x_1, \ldots, x_n)$ and take the canonical Jenkins–Strebel quadratic differential associated to the n-tuple $p_1, \ldots, p_n$. Then vertical trajectories of this quadratic differential through x_i identify the perimeter of the ith face of the corresponding embedded graph with the “spherized” cotangent line L_i considered as a real plane (that is, the fiber punctured at the origin is projected to the unit circle along the half-lines passing through the origin) at the ith point. Now it is possible to represent the intersection numbers $\langle \tau_{m_1} \ldots \tau_{m_n} \rangle$ in terms of integrals of very explicit differential forms:

$$\langle \tau_{m_1} \ldots \tau_{m_n} \rangle = \int_{\pi^{-1}(\bar{p})} \prod_{i=1}^{n} \omega_i^{m_i}$$

over any generic point $\bar{p} \in \mathbb{R}_+^n$.

From now on we use the notation d for the complex dimension of $\mathcal{M}_{g;n}$, $d = 3g - 3 + n$. Introduce the volume form on (the open strata of) $\mathcal{M}_{g;n}^{\mathrm{comb}}$:

$$\mathrm{Vol}(\lambda_1, \ldots, \lambda_n) = \frac{1}{d!} \Omega^d \times \prod_{i=1}^{n} e^{-\lambda_i p_i} dp_i,$$

where $\Omega = p_1^2 \omega_1 + \cdots + p_n^2 \omega_n$ and λ_i are real positive parameters.

Then the volume of $\mathcal{M}_{g;n}^{\mathrm{comb}}$ with respect to this volume form can be computed in two ways: directly, under the projection to $\mathbb{R}_+^n$, and summing the volumes of all open cells. The first computation gives

$$\begin{aligned}
\int_{\mathcal{M}_{g;n}^{\mathrm{comb}}} \mathrm{Vol}(\lambda_1,\dots,\lambda_n) &= \frac{1}{d!}\int_{\mathbb{R}_+^n}\left(\int_{\pi^{-1}(\bar{p})}\Omega^d\right) e^{-\sum\lambda_i p_i}\,dp_1\wedge\dots\wedge dp_n \\
&= \sum_{m_1+\dots+m_n=d}\frac{\langle\tau_{m_1}\dots\tau_{m_n}\rangle}{m_1!\dots m_n!}\prod_i\int_0^\infty p_i^{2m_i}e^{-\lambda_i p_i}\,dp_i \\
&= \sum_{m_1+\dots+m_n=d}\langle\tau_{m_1}\dots\tau_{m_n}\rangle\prod_{i=1}^n\frac{(2m_i)!}{m_i!}\lambda_i^{-(2m_i+1)} \\
&= 2^d\sum_{m_1+\dots+m_n=d}\langle\tau_{m_1}\dots\tau_{m_n}\rangle\prod_{i=1}^n\frac{(2m_i-1)!!}{\lambda_i^{(2m_i+1)}}.
\end{aligned}$$

The first computation is completed, and we start the second one. Consider the open cell in $\mathcal{M}_{g;n}^{\mathrm{comb}}$ corresponding to a 3-valent embedded graph Γ. The lengths $l_1,\dots,l_{|E(\Gamma)|}$ of the edges of Γ form a set of coordinates on this cell. In these coordinates, the volume form $\mathrm{Vol}(\lambda_1,\dots,\lambda_n)$ can be rewritten as

$$\mathrm{Vol}_\Gamma(\lambda_1,\dots,\lambda_n) = 2^{d+|E(\Gamma)|-|V(\Gamma)|}e^{-\sum_j l_j\tilde{\lambda}_j}\,dl_1\wedge\dots\wedge dl_{|E(\Gamma)|}.$$

Here j runs over the set of all edges of Γ, and $\tilde{\lambda}_j$ is the sum

$$\tilde{\lambda}_j = \lambda_- + \lambda_+$$

of the two $\lambda's$ corresponding to the two faces of Γ adjacent to the jth edge. Note that the two faces neighboring to an edge may coincide, and in this case $\lambda_- = \lambda_+$. Obtaining the correct power of 2 in the last formula (and hence showing that it is independent of the chosen cell) is a rather cumbersome task, and we refer the reader to [178] for details. An immediate calculation gives

$$\mathrm{Vol}_\Gamma(\lambda_1,\dots,\lambda_n) = \prod_{j=1}^{|E(\Gamma)|}\frac{1}{\tilde{\lambda}_j}.$$

The contribution of a marked embedded graph to the total volume is proportional to the inverse cardinality of the automorphism group of the graph, whence summing over all 3-valent marked genus g embedded graphs with n marked faces and multiplying by 2^{-d} we obtain the *main combinatorial identity*

$$\sum_{m_1+\dots+m_n=d}\langle\tau_{m_1}\dots\tau_{m_n}\rangle\prod_{i=1}^n\frac{(2m_i-1)!!}{\lambda_i^{2m_i+1}} = \sum_\Gamma\frac{2^{-|V(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|}\prod_{j=1}^{|E(\Gamma)|}\frac{2}{\tilde{\lambda}_j}. \tag{4.11}$$

The main combinatorial identity is an identity between two rational functions in variables λ_i. Making an arbitrary substitution of the form $\lambda_i = \Lambda_{k_i}$, $1\le k_i\le N$ and summing the resulting identities over all such substitutions one gets

$$\sum_{m_1+\cdots+m_n=d} \langle \tau_{m_1} \dots \tau_{m_n} \rangle \prod_{i=1}^{n} (2m_i - 1)!! \operatorname{tr}(\Lambda^{-(2m_i-1)})$$
$$= \sum_{\Gamma} \frac{2^{-|V(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|} \prod_{j=1}^{|E(\Gamma)|} \frac{2}{\widetilde{\Lambda}_j}. \tag{4.12}$$

Here $\widetilde{\Lambda}_j = \Lambda_- + \Lambda_+$ and the sum on the right-hand side is taken over all possible ways to color the faces of the graph Γ in N colors $\Lambda_1, \dots, \Lambda_N$. Recall that Λ denotes the diagonal $N \times N$ matrix with positive entries $\Lambda_1, \dots, \Lambda_N$.

The right-hand side of the last equation coincides with the matrix integral expansion in the Kontsevich model, and we obtain the first part of the Kontsevich theorem: the generating function K of the Kontsevich model coincides with the generating function F of the intersection model.

4.9.3 The Kontsevich Model and the KdV Equation

The second part of the proof consists in showing that the integral of the Kontsevich model is a τ-function for the KdV-hierarchy, in other words, that it obeys the Korteweg–de Vries equation.

Let

$$a(y) = \int_{-\infty}^{\infty} e^{i(\frac{1}{3}x^3 - yx)} dx$$

be the classical Airy function, i.e., the unique (up to a scalar factor) bounded solution to the linear differential equation

$$a''(y) + ya(y) = 0.$$

We are interested in the "asymptotic behavior" of this function as $y \to \infty$. An application of the stationary phase method (which must be justified in this case) gives

$$a(y) \sim e^{-\frac{2i}{3}y^{3/2}} \int_{U(y^{1/2})} e^{i(\frac{1}{3}x^3 + y^{1/2}x^2)} dx + e^{\frac{2i}{3}y^{3/2}} \int_{U(-y^{1/2})} e^{i(\frac{1}{3}x^3 - y^{1/2}x^2)} dx,$$

where the integration is carried out over arbitrary neighborhoods of the points $\pm y^{1/2}$.

Similar constructions are valid for the case of the *matrix Airy function*

$$A(Y) = \int_{\mathcal{H}_N} e^{i(\frac{1}{3}\operatorname{tr} H^3 - HY)} d\mu(H),$$

for a positive diagonal matrix Y. This function obeys the *matrix Airy equation*

$$\Delta A(Y) + \operatorname{tr} Y \cdot A(Y) = 0,$$

where Δ denotes the Laplace operator. Similarly to the 1-dimensional Airy function, the matrix Airy function admits an asymptotic expansion as a sum of 2^N expressions of the form

$$e^{-i\frac{2}{3}\operatorname{tr} Y^{3/2}} \int e^{i\operatorname{tr}(\frac{1}{3}H^3 - H^2 Y^{1/2})} d\mu(H) = e^{-i\frac{2}{3}\operatorname{tr} Y^{3/2}} \int e^{i\operatorname{tr}\frac{1}{3}H^3} d\mu_{Y^{1/2}}(H).$$

The sum is taken over all 2^N quadratic roots $Y^{1/2}$ of the matrix Y, and the integral is taken over a neighborhood of the origin in $\mathcal{H}_N$. As $Y \to \infty$, the integral can be replaced with that over the entire space $\mathcal{H}_N$, i.e., it becomes the integral of the Kontsevich model for $\Lambda = Y^{1/2}$. The asymptotic expansion of the latter we already know.

Another way to compute the matrix Airy function consists in the application of formulas borrowed from [146] and [216]:

$$\begin{aligned} A(Y) &= c_N \Delta(Y_i)^{-1} \int_{\mathbb{R}^N} \prod_{i=1}^{n} \Delta(X_i) e^{i(\frac{1}{3}X_i^3 - X_i Y_i)} dX_i \\ &= c_N \frac{\det(a^{(j-1)}(Y_i))}{\det(Y_i^{j-1})}, \end{aligned}$$

where this time Δ denotes the Vandermonde determinant. Here we made use of the obvious identity

$$\int e^{i(x^3/3 - xy)} x^{j-1} dx = (ia(y))^{(j-1)}.$$

The derivatives of the Airy function admit natural asymptotic expansions

$$a^{(j-1)}(y) \sim \sum_{y^{1/2}} \text{const} \cdot y^{-3/4} e^{-\frac{2i}{3} y^{3/2}} \cdot f_j(y^{-1/2})$$

for some Laurent series $f_j(z) = z^{-j} + \cdots \in \mathbb{Q}((z))$. Substituting the last formula into the expression for the matrix Airy function we obtain

$$A(Y) = \sum_{Y^{1/2}} \text{const} \times e^{-\frac{2i}{3}\operatorname{tr} Y^{3/2}} \prod_{i=1}^{N} Y_i^{-3/4} \cdot \frac{\det(f_j(Y_i^{-1/2}))}{\det(Y_i^{j-1})}.$$

The last expression relates the matrix Airy function to the τ-function corresponding to the subspace $\langle f_1, f_2, \dots \rangle \subset \mathbb{C}((z^{-1}))$, see Sec. 3.6.4. The proposition and the argument in the end of that section complete the proof of Witten's conjecture.

* * *

The main theorem established in this chapter permits to compute the intersection indices for certain classes; but the structure of the cohomology ring of the moduli spaces remains unknown. There also remains one more Witten's conjecture (it is discussed, in particular, in [178] and [202]), and though it is not apparent from its formulation, it is also related to embedded graphs.

The general idea behind the notion of a moduli space is that of "the space of parameters". In this chapter we parametrized algebraic curves. It is no less interesting to parametrize the pairs (X, f) where X is a curve and f is a meromorphic function on X. The corresponding parameter spaces are called *Hurwitz spaces*. The reader will find an introduction to this theory – from the point of view of embedded graphs, to be sure – in the next chapter.