10 The Finite Element Method for a Parabolic Problem

In this chapter we consider the approximation of solutions of the model heat equation in two space dimensions by means of Galerkin's method, using piecewise linear trial functions. In Sect. 10.1 we consider the discretization with respect to the space variables only, and in the following Sect. 10.2 we study some completely discrete schemes.

10.1 The Semidiscrete Galerkin Finite Element Method

Let $\Omega \subset \mathbf{R}^2$ be a bounded convex domain with smooth boundary Γ, and consider the initial-boundary value problem,

$$\begin{aligned} &u_t - \Delta u = f, && \text{in } \Omega \times \mathbf{R}_+, \\ &u = 0, && \text{on } \Gamma \times \mathbf{R}_+, \\ &u(\cdot, 0) = v, && \text{in } \Omega, \end{aligned} \tag{10.1}$$

where u_t denotes $\partial u/\partial t$ and Δ the Laplacian $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. In the first step we shall approximate the solution $u(x,t)$ by means of a function $u_h(x,t)$ which, for each fixed t, is a piecewise linear function of x over a triangulation $\mathcal{T}_h$ of Ω, thus depending on a finite number of parameters.

Thus, let $\mathcal{T}_h = \{K\}$ denote a triangulation of Ω of the type considered in Sect. 5.2 and let $\{P_j\}_{j=1}^{M_h}$ be the interior nodes of $\mathcal{T}_h$. Further, let S_h denote the continuous piecewise linear functions on $\mathcal{T}_h$ which vanish on $\partial\Omega$ and let $\{\Phi_j\}_{j=1}^{M_h}$ be the standard basis of S_h corresponding to the nodes $\{P_j\}_{j=1}^{M_h}$. Recall the definition (5.28) of the interpolant $I_h : \mathcal{C}_0(\bar{\Omega}) \to S_h$, and the error bounds (5.34) with $r = 2$.

For the purpose of defining thus an approximate solution to the initial boundary value problem (10.1) we first write this in weak form as in Sect. 8.3, i.e., with the definitions there,

$$(u_t, \varphi) + a(u, \varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1,\ t > 0. \tag{10.2}$$

We then pose the approximate problem to find $u_h(t) = u_h(\cdot, t)$, belonging to S_h for each t, such that

$$(10.3) \qquad \begin{aligned} (u_{h,t}, \chi) + a(u_h, \chi) &= (f, \chi), \quad \forall \chi \in S_h,\ t > 0, \\ u_h(0) &= v_h, \end{aligned}$$

where $v_h \in S_h$ is some approximation of v. Since we have discretized only in the space variables, this is referred to as a *spatially semidiscrete* problem. In the next section, we shall discretize also in the time variable to produce completely discrete schemes.

In terms of the basis $\{\Phi_j\}_{j=1}^{M_h}$ our semidiscrete problem may be stated: Find the coefficients $\alpha_j(t)$ in

$$u_h(x,t) = \sum_{j=1}^{M_h} \alpha_j(t)\Phi_j(x),$$

such that

$$\sum_{j=1}^{M_h} \alpha_j'(t)(\Phi_j, \Phi_k) + \sum_{j=1}^{M_h} \alpha_j(t) a(\Phi_j, \Phi_k) = (f(t), \Phi_k), \quad k = 1, \dots, M_h,$$

and, with γ_j denoting the nodal values of the given initial approximation v_h,

$$\alpha_j(0) = \gamma_j, \quad j = 1, \dots, M_h.$$

In matrix notation this may be expressed as

$$(10.4) \qquad B\alpha'(t) + A\alpha(t) = b(t), \quad \text{for } t > 0, \quad \text{with } \alpha(0) = \gamma,$$

where $B = (b_{kj})$ is the mass matrix with elements $b_{kj} = (\Phi_j, \Phi_k)$, $A = (a_{kj})$ the stiffness matrix with $a_{kj} = a(\Phi_j, \Phi_k)$, $b = (b_k)$ the vector with entries $b_k = (f, \Phi_k)$, $\alpha(t)$ the vector of unknowns $\alpha_j(t)$, and $\gamma = (\gamma_j)$. The dimension of all these items equals M_h, the number of interior nodes of $\mathcal{T}_h$.

We recall from Sect. 5.2 that the stiffness matrix A is symmetric positive definite, and this holds also for the mass matrix B since

$$\sum_{k,j=1}^{M_h} \xi_j \xi_k (\Phi_j, \Phi_k) = \Big\| \sum_{j=1}^{M_h} \xi_j \Phi_j \Big\|^2 \geq 0,$$

and since equality can only occur if the vector $\xi = 0$. In particular, B is invertible, and therefore the above system of ordinary differential equations may be written

$$\alpha'(t) + B^{-1}A\alpha(t) = B^{-1}b(t), \quad \text{for } t > 0, \quad \text{with } \alpha(0) = \gamma,$$

and hence obviously has a unique solution for t positive.

We begin our analysis by considering the stability of the semidiscrete method. Since $U^n \in S_h$ we may choose $\chi = u$ in (10.3) to obtain

$$(u_{h,t}, u_h) + a(u_h, u_h) = (f, u_h), \quad \text{for } t > 0,$$

or, since the first term equals $\frac{1}{2}\frac{d}{dt}\|u_h\|^2$ and the second is non-negative,

$$\tfrac{1}{2}\frac{d}{dt}\|u_h\|^2 = \|u_h\|\frac{d}{dt}\|u_h\| \le \|f\|\,\|u_h\|.$$

This yields

$$\frac{d}{dt}\|u_h\| \le \|f\|,$$

which after integration shows the stability estimate

$$\|u_h(t)\| \le \|v_h\| + \int_0^t \|f\|\,\mathrm{d}s. \tag{10.5}$$

For the purpose of writing equation in (10.3) in operator form, we introduce a *discrete Laplacian* Δ_h, which we think of as an operator from S_h into itself, defined by

$$(-\Delta_h\psi, \chi) = a(\psi, \chi), \quad \forall \psi, \chi \in S_h. \tag{10.6}$$

This discrete analogue of Green's formula clearly defines $\Delta_h\psi = \sum_{j=1}^{M_h} d_j\Phi_j$ from

$$\sum_{j=1}^{M_h} d_j(\Phi_j, \Phi_k) = -a(\psi, \Phi_k), \quad k = 1, \ldots, M_h,$$

since the matrix of this system is the positive definite mass matrix encountered above. The operator Δ_h is easily seen to be selfadjoint and $-\Delta_h$ is positive definite in S_h with respect to the L_2-inner product, see Problem 10.3. With P_h denoting the L_2-projection onto S_h, the equation in (10.3) may now be written

$$(u_{h,t} - \Delta_h u_h - P_h f, \chi) = 0, \quad \forall \chi \in S_h,$$

or, noting that the first factor is in S_h, so that χ may be chosen equal to it, it follows that

$$u_{h,t} - \Delta_h u_h = P_h f, \quad \text{for } t > 0, \quad \text{with } u_h(0) = v_h, \tag{10.7}$$

We denote by $E_h(t)$ the solution operator of the homogeneous case of the semidiscrete equation in (10.7), with $f = 0$. Hence $E_h(t)$ is the operator which takes the initial data $u_h(0) = v_h$ into the solution $u_h(t)$ at time t, so that $u_h(t) = E_h(t)v_h$. It is then easy to show (cf. Duhamel's principle (8.22)) that the solution of the initial value problem (10.7) is

$$u_h(t) = E_h(t)v_h + \int_0^t E_h(t-s)P_h f(s)\,\mathrm{d}s. \tag{10.8}$$

We now note that it follows from (10.5) that $E_h(t)$ is stable in L_2, or

$$\|E_h(t)v_h\| \le \|v_h\|, \quad \forall v_h \in S_h. \tag{10.9}$$

Since also P_h has unit norm in L_2 this, together with (10.8), re-establishes the stability estimate (10.5) for the inhomogeneous equation, so that, in fact, it suffices to show stability for the homogeneous equation.

We shall prove the following estimate for the error between the solutions of the semidiscrete and continuous problems.

Theorem 10.1. *Let u_h and u be the solutions of* (10.3) *and* (10.1)*. Then*

$$\|u_h(t) - u(t)\| \le \|v_h - v\| + Ch^2\Big(\|v\|_2 + \int_0^t \|u_t\|_2 \,\mathrm{d}s\Big), \quad \textit{for } t \ge 0.$$

Here we require, as usual, that the solution of the continuous problem has the regularity implicitly assumed by the presence of the norms on the right. Note also that for $v_h = I_h v$, (5.31) shows that

$$\|v_h - v\| \le Ch^2\|v\|_2, \tag{10.10}$$

in which case the first term on the right is dominated by the second. The same holds true if $v_h = P_h v$, where P_h denotes the orthogonal projection of L_2 onto S_h, since this choice is the best approximation of v in S_h with respect to the L_2-norm, see (5.39). Another choice of optimal order is $v_h = R_h v$, where R_h is the elliptic (or Ritz) projection onto S_h defined in (5.49) by

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in S_h. \tag{10.11}$$

Thus $R_h v$ is the finite element approximation of the solution of the elliptic problem whose exact solution is v. We recall the error estimates of Theorem 5.5,

$$\|R_h v - v\| + h|R_h v - v|_1 \le Ch^s\|v\|_s, \quad \text{for } s = 1, 2. \tag{10.12}$$

We now turn to the

Proof of Theorem 10.1. In the main step of the proof we shall compare the solution of the semidiscrete problem to the elliptic projection of the exact solution. We write

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta + \rho. \tag{10.13}$$

The second term is easily bounded using (10.12) and obvious estimates by

$$\|\rho(t)\| \le Ch^2\|u(t)\|_2 = Ch^2\Big\|v + \int_0^t u_t \,\mathrm{d}s\Big\|_2 \le Ch^2\Big(\|v\|_2 + \int_0^t \|u_t\|_2 \,\mathrm{d}s\Big).$$

In order to bound θ, we note that

$$
\begin{aligned}
(\theta_t,\chi)+a(\theta,\chi) &= (u_{h,t},\chi)+a(u_h,\chi)-(R_h u_t,\chi)-a(R_h u,\chi)\\
&= (f,\chi)-(R_h u_t,\chi)-a(u,\chi)=(u_t-R_h u_t,\chi),
\end{aligned}
\tag{10.14}
$$

or

$$
(\theta_t,\chi)+a(\theta,\chi)=-(\rho_t,\chi),\quad \forall\chi\in S_h. \tag{10.15}
$$

In this derivation we have used (10.3), (10.2), the definition of R_h in (10.11), and the easily established fact that this operator commutes with time differentiation, i.e., $R_h u_t = (R_h u)_t$. We may now apply the stability estimate (10.5) to (10.15) to obtain

$$
\|\theta(t)\| \le \|\theta(0)\| + \int_0^t \|\rho_t\|\,\mathrm{d}s.
$$

Here

$$
\|\theta(0)\| = \|v_h - R_h v\| \le \|v_h - v\| + \|R_h v - v\| \le \|v_h - v\| + Ch^2\|v\|_2,
$$

and further

$$
\|\rho_t\| = \|R_h u_t - u_t\| \le Ch^2\|u_t\|_2.
$$

Together these estimates prove the theorem. □

We see from the proof of Theorem 10.1 that the error estimate for the semidiscrete parabolic problem is thus a consequence of the stability for this problem combined with the error estimate for the elliptic problem, expressed in terms of $\rho=(R_h-I)u$.

Recalling the maximum principle for parabolic equations, Theorem 8.7, we find at once that, for the solution operator $E(t)$ of the homogeneous case of the initial boundary value problem (10.1), we have $\|E(t)v\|_{\mathcal{C}} \le \|v\|_{\mathcal{C}}$ for $t\ge 0$. The corresponding maximum principle does not hold for the finite element problem, but it may be shown that, if the family $\{\mathcal{T}_h\}$ of triangulations is quasi-uniform, cf. (5.52), then for some $C>1$,

$$
\|E_h(t)v_h\|_{\mathcal{C}} \le C\|v_h\|_{\mathcal{C}},\quad \text{for } t\ge 0.
$$

This may be combined with the error estimate (5.53) for the stationary problem to show a maximum-norm error estimate for the parabolic problem.

In this regard we mention a variant of the semidiscrete problem (10.2) for which a maximum principle sometimes holds, namely the *lumped mass method*. To define this we replace the matrix B in (10.4) by a diagonal matrix $\bar{B}$, in which the diagonal elements are the row sums of B. One can show that this method can also be defined by

$$
(u_{h,t},\chi)_h + a(u_h,\chi) = (f,\chi),\quad \forall\chi\in S_h,\quad \text{for } t>0, \tag{10.16}
$$

where the inner product in the first term has been obtained by computing the first term in (10.2) by using the nodal quadrature rule (5.64). For this method one may derive a $O(h^2)$ error estimate similar to that of Theorem 10.1. If we now assume that all angles of the triangulations are $\le \pi/2$, then the off-diagonal elements of the stiffness matrix A are nonpositive, and as a result of this one may show that, if $\bar{E}_h(t)$ denotes the solution operator of the modified problem, then

$$\|\bar{E}_h(t)v_h\|_{\mathcal{C}} \le \|v_h\|_{\mathcal{C}}, \quad \text{for } t \ge 0.$$

Returning to the standard Galerkin method (10.1) we now prove the following estimate for the error in the gradient.

Theorem 10.2. *Under the assumptions of Theorem 10.1, we have for* $t \ge 0$,

$$|u_h(t) - u(t)|_1 \le |v_h - v|_1 + Ch\Big\{\|v\|_2 + \|u(t)\|_2 + \Big(\int_0^t \|u_t\|_1^2 \, \mathrm{d}s\Big)^{1/2}\Big\}.$$

Proof. As before we write the error in the form (10.13). Here by (10.12),

$$|\rho(t)|_1 = |R_h u(t) - u(t)|_1 \le Ch\|u(t)\|_2.$$

In order to estimate $\nabla\theta$ we use again (10.15), now with $\chi = \theta_t$. We obtain

$$\|\theta_t\|^2 + \tfrac{1}{2}\frac{d}{dt}|\theta|_1^2 = -(\rho_t, \theta_t) \le \tfrac{1}{2}(\|\rho_t\|^2 + \|\theta_t\|^2),$$

so that

$$\frac{d}{dt}|\theta|_1^2 \le \|\rho_t\|^2,$$

or

$$|\theta(t)|_1^2 \le |\theta(0)|_1^2 + \int_0^t \|\rho_t\|^2 \, \mathrm{d}s \le \big(|v_h - v|_1 + |R_h v - v|_1\big)^2 + \int_0^t \|\rho_t\|^2 \, \mathrm{d}s.$$

Hence, since $a^2 + b^2 \le (|a| + |b|)^2$ and in view of (10.12), we conclude

$$|\theta(t)|_1 \le |v_h - v|_1 + Ch\Big\{\|v\|_2 + \Big(\int_0^t \|u_t\|_1^2 \, \mathrm{d}s\Big)^{1/2}\Big\}, \tag{10.17}$$

which completes the proof. □

Note that if $v_h = I_h v$ or $R_h v$, then

$$|v_h - v|_1 \le Ch\|v\|_2,$$

so that the first term on the right in Theorem 10.2 is dominated by the second.

We make the following observation concerning $\theta = u_h - R_h u$: Assume that we choose $v_h = R_h v$, so that $\theta(0) = 0$. Then in addition to (10.17) we have

$$|\theta(t)|_1 \le \Big(\int_0^t \|\rho_t\|_2 \,\mathrm{d}s\Big)^{1/2} \le Ch^2 \Big(\int_0^t \|u_t\|_2^2 \,\mathrm{d}s\Big)^{1/2}.$$

Hence the gradient of θ is of second order $O(h^2)$, whereas the gradient of the total error is only of order $O(h)$ as $h \to 0$. Thus ∇u_h is a better approximation to $\nabla R_h u$ than is possible to ∇u. This is an example of a phenomenon which is sometimes referred to as *superconvergence*.

The discrete solution operator $E_h(t)$ introduced above also has smoothing properties analogous to the corresponding results in Sect. 8.2 for the continuous problem, such as, for instance

$$|E_h(t)v_h|_1 \le Ct^{-1/2}\|v_h\|, \quad \text{for } t > 0, \quad v_h \in S_h,$$

and

$$\Big\|D_t^k E_h(t)v_h\Big\| = \|\varDelta_h^k E_h(t)v_h\| \le C_k t^{-k}\|v_h\|, \quad \text{for } t > 0,\ v_h \in S_h. \tag{10.18}$$

Such results may be used to show, e.g., the following non-smooth data error estimate for the homogeneous equation.

Theorem 10.3. *Assume that $f = 0$ and let u_h and u be the solutions of* (10.3) *and* (10.1)*, respectively, where now the initial data for* (10.3) *are chosen as $v_h = P_h v$. Then*

$$\|u_h(t) - u(t)\| \le Ch^2 t^{-1}\|v\|, \quad \textit{for } t > 0.$$

The proof is left as an exercise (Problem 10.4). This result shows that the convergence rate is $O(h^2)$ for t bounded away from zero, even when v is only assumed to belong to L_2.

The above theory easily extends to finite elements of higher order, under the appropriate regularity assumptions on the solution. Thus, if the finite element subspace is such that

$$\|R_h v - v\| \le Ch^r\|v\|_r, \quad \forall v \in H^r \cap H_0^1, \tag{10.19}$$

then we may show the following theorem.

Theorem 10.4. *Let u_h and u be the solutions of* (10.3) *and* (10.1)*, respectively, and assume that* (10.19) *holds. Then, for v_h suitably chosen, we have*

$$\|u_h(t) - u(t)\| \le Ch^r\Big(\|v\|_r + \int_0^t \|u_t\|_r \,\mathrm{d}s\Big), \quad \textit{for } t \ge 0.$$

Recall from (5.50) that for $r > 2$ the estimate (10.19) holds for piecewise polynomials of degree $r-1$, but that the regularity assumption $v \in H^r \cap H_0^1$ is then somewhat unrealistic. For a domain $\varOmega$ with a smooth boundary $\varGamma$ special considerations are needed in the boundary layer $\varOmega \setminus \varOmega_h$.

10.2 Some Completely Discrete Schemes

We shall now turn our attention to some simple schemes for discretization also with respect to the time variable, and let S_h be the space of piecewise linear finite element functions as before. We begin with the *backward Euler-Galerkin method*. With k the time step and $U^n \in S_h$ the approximation of $u(t)$ at $t = t_n = nk$, this method is defined by replacing the time derivative in (10.3) by a backward difference quotient, or with $\bar\partial_t U^n = k^{-1}(U^n - U^{n-1})$,

$$(10.20) \qquad \begin{aligned} &(\bar\partial_t U^n, \chi) + a(U^n, \chi) = (f(t_n), \chi), \quad \forall \chi \in S_h,\ n \geq 1,\\ &U^0 = v_h. \end{aligned}$$

Given U^{n-1} this defines U^n implicitly from the discrete elliptic problem

$$(U^n, \chi) + k a(U^n, \chi) = (U^{n-1} + k f(t_n), \chi), \quad \forall \chi \in S_h.$$

Expressing U^n in terms of the basis $\{\Phi\}_{j=1}^{M_h}$ as $U^n(x) = \sum_{j=1}^{M_h} \alpha_j^n \Phi_j(x)$, we may write this equation in the matrix notation introduced in Sect. 10.1 as

$$B\alpha^n + kA\alpha^n = B\alpha^{n-1} + kb^n, \quad \text{for } n \geq 1,$$

where α^n is the vector with components α_j^n, or

$$\alpha^n = (B + kA)^{-1} B\alpha^{n-1} + k(B + kA)^{-1} b^n, \quad \text{for } n \geq 1, \quad \text{with } \alpha^0 = \gamma.$$

We begin our analysis of the backward Euler method by showing that it is unconditionally stable, i.e., that it is stable independently of the relation between h and k. Choosing $\chi = U^n$ in (10.20) we have, since $a(U^n, U^n) \geq 0$,

$$(\bar\partial_t U^n, U^n) \leq \|f^n\|\,\|U^n\|, \quad \text{where } f^n = f(t_n),$$

or

$$\|U^n\|^2 - (U^{n-1}, U^n) \leq k\|f^n\|\,\|U^n\|.$$

Since $(U^{n-1}, U^n) \leq \|U^{n-1}\|\,\|U^n\|$, this shows

$$\|U^n\| \leq \|U^{n-1}\| + k\|f^n\|, \quad \text{for } n \geq 1,$$

and hence, by repeated application,

$$(10.21) \qquad \|U^n\| \leq \|U^0\| + k\sum_{j=1}^{n} \|f^j\|.$$

We shall now prove the following error estimate.

Theorem 10.5. *With U^n and u the solutions of* (10.20) *and* (10.1)*, respectively, and with v_h chosen so that* (10.10) *holds, we have, for $n \geq 0$,*

$$\|U^n - u(t_n)\| \leq Ch^2\Big(\|v\|_2 + \int_0^{t_n} \|u_t\|_2\,\mathrm{d}s\Big) + Ck\int_0^{t_n} \|u_{tt}\|\,\mathrm{d}s.$$

Proof. In analogy with (10.13) we write

$$U^n - u(t_n) = \big(U^n - R_h u(t_n)\big) + \big(R_h u(t_n) - u(t_n)\big) = \theta^n + \rho^n.$$

As before, by (10.12),

$$\|\rho^n\| \le Ch^2 \|u(t_n)\|_2 \le Ch^2 \Big(\|v\|_2 + \int_0^{t_n} \|u_t\|_2 \, \mathrm{d}s \Big).$$

This time, a calculation corresponding to (10.14) yields

$$(\bar\partial_t \theta^n, \chi) + a(\theta^n, \chi) = -(\omega^n, \chi), \tag{10.22}$$

where

$$\omega^n = R_h \bar\partial_t u(t_n) - u_t(t_n) = (R_h - I)\bar\partial_t u(t_n) + (\bar\partial_t u(t_n) - u_t(t_n)) = \omega_1^n + \omega_2^n.$$

By application of the stability estimate (10.21) to (10.22) we obtain

$$\|\theta^n\| \le \|\theta^0\| + k \sum_{j=1}^n \|\omega_1^j\| + k \sum_{j=1}^n \|\omega_2^j\|.$$

Here, as before, by (10.10) and (10.12),

$$\|\theta^0\| = \|v_h - R_h v\| \le \|v_h - v\| + \|v - R_h v\| \le Ch^2 \|v\|_2.$$

Note now that

$$\omega_1^j = (R_h - I) k^{-1} \int_{t_{j-1}}^{t_j} u_t \, \mathrm{d}s = k^{-1} \int_{t_{j-1}}^{t_j} (R_h - I) u_t \, \mathrm{d}s,$$

whence

$$k \sum_{j=1}^n \|\omega_1^j\| \le \sum_{j=1}^n \int_{t_{j-1}}^{t_j} Ch^2 \|u_t\|_2 \, \mathrm{d}s = Ch^2 \int_0^{t_n} \|u_t\|_2 \, \mathrm{d}s.$$

Further, by Taylor's formula,

$$\omega_2^j = k^{-1}(u(t_j) - u(t_{j-1})) - u_t(t_j) = -k^{-1} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s) \, \mathrm{d}s,$$

so that

$$k \sum_{j=1}^n \|\omega_2^j\| \le \sum_{j=1}^n \Big\| \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s) \, \mathrm{d}s \Big\| \le k \int_0^{t_n} \|u_{tt}\| \, \mathrm{d}s.$$

Together our estimates complete the proof of the theorem. □

Replacing the backward difference quotient with respect to time in (10.20) by a forward difference quotient we arrive at the *forward Euler-Galerkin method*, or with $\partial_t U^n = (U^{n+1} - U^n)/k$,

$$(\partial_t U^n, \chi) + a(U^n, \chi) = (f(t_n), \chi), \quad \forall \chi \in S_h, \; n \geq 1,$$
$$U^0 = v_h.$$

In matrix form this may be expressed as

$$B\alpha^{n+1} = (B - kA)\alpha^n + kb^n, \quad \text{for } n \geq 0,$$

Since B is not a diagonal matrix this method is not explicit. However, if this time discretization method is applied to the lumped mass semidiscrete equation (10.16), and thus B replaced by the diagonal matrix $\bar{B}$, then the corresponding forward Euler method becomes an explicit one.

Using the discrete Laplacian defined in (10.6), the forward Euler method may also be defined by

$$U^{n+1} = (I + k\Delta_h)U^n + kP_h f(t_n), \quad \text{for } n \geq 0, \quad \text{with } U^0 = v_h. \tag{10.23}$$

This method is not unconditionally stable as the backward Euler method, but considering for simplicity only the homogeneous equation, we shall show stability under the condition that the family $\{S_h\}$ is such that

$$\lambda_{M_h,h}\, k \leq 2, \tag{10.24}$$

where $\lambda_{M_h,h}$ is the largest eigenvalue of $-\Delta_h$. Recalling (6.37), we note that this holds, e.g., if the S_h satisfy the inverse inequality (6.36) and if $k \leq 2C^{-1}h^2$, where C is the constant in (6.37), which thus shows conditional stability.

It is clear that (10.23) is stable if and only if $\|(I + k\Delta_h)\chi\| \leq \|\chi\|$ for all $\chi \in S_h$, and since $-\Delta_h$ is symmetric positive definite, this holds if and only if all eigenvalues of $I + k\Delta_h$ belong to $[-1, 1]$. By the positivity of $-\Delta_h$ this is the same as requiring the smallest eigenvalue of $I + k\Delta_h$ to be ≥ -1, or that the largest eigenvalue of $-\Delta_h$ is $\leq 2/k$, which is (10.24). See also Problem 10.3.

Note that because of the non-symmetric choice of the discretization in time, the backward Euler-Galerkin method is only first order accurate in time. We therefore now turn to the *Crank-Nicolson-Galerkin method*, in which the semidiscrete equation is discretized in a symmetric fashion around the point $t_{n-1/2} = (n - \frac{1}{2})k$, which yields method which is second order accurate in time. More precisely, we define $U^n \in S_h$ recursively for $n \geq 1$ by

$$\begin{aligned} &(\bar{\partial}_t U^n, \chi) + a(\tfrac{1}{2}(U^n + U^{n-1}), \chi) = (f(t_{n-1/2}), \chi), \quad \forall \chi \in S_h, \\ &U^0 = v_h. \end{aligned} \tag{10.25}$$