Preface

This book provides an introduction to Lie groups, Lie algebras, and representation theory, aimed at graduate students in mathematics and physics. Although there are already several excellent books that cover many of the same topics, this book has two distinctive features that I hope will make it a useful addition to the literature. First, it treats Lie groups (not just Lie algebras) in a way that minimizes the amount of manifold theory needed. Thus, I neither assume a prior course on differentiable manifolds nor provide a condensed such course in the beginning chapters. Second, this book provides a gentle introduction to the machinery of semisimple groups and Lie algebras by treating the representation theory of $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ in detail before going to the general case. This allows the reader to see roots, weights, and the Weyl group "in action" in simple cases before confronting the general theory.

The standard books on Lie theory begin immediately with the general case: a smooth manifold that is also a group. The Lie algebra is then defined as the space of left-invariant vector fields and the exponential mapping is defined in terms of the flow along such vector fields. This approach is undoubtedly the right one in the long run, but it is rather abstract for a reader encountering such things for the first time. Furthermore, with this approach, one must either assume the reader is familiar with the theory of differentiable manifolds (which rules out a substantial part of one's audience) or one must spend considerable time at the beginning of the book explaining this theory (in which case, it takes a long time to get to Lie theory proper).

My way out of this dilemma is to consider only matrix groups (i.e., closed subgroups of $\mathsf{GL}(n;\mathbb{C})$). (Others before me have taken such an approach, as discussed later.) Every such group is a Lie group, and although not every Lie group is of this form, most of the interesting examples are. The exponential of a matrix is then defined by the usual power series, and the Lie algebra $\mathfrak{g}$ of a closed subgroup G of $\mathsf{GL}(n;\mathbb{C})$ is defined to be the set of matrices X such that $\exp(tX)$ lies in G for all real numbers t. One can show that $\mathfrak{g}$ is, indeed, a Lie algebra (i.e., a vector space and closed under commutators). The usual elementary results can all be proved from this point of view: the image of the exponential mapping contains a neighborhood of the identity; in a connected group, every element is a product of exponentials; every continuous group homomorphism induces a Lie algebra homomorphism. (These results show that every matrix group is a smooth embedded submanifold of $\mathsf{GL}(n;\mathbb{C})$, and hence a Lie group.)

I also address two deeper results: that in the simply-connected case, every Lie algebra homomorphism induces a group homomorphism and that there is a one-to-one correspondence between subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ and connected Lie subgroups H of G. The usual approach to these theorems makes use of the Frobenius theorem. Although this is a fundamental result in analysis, it is not easily stated (let alone proved) and it is not especially Lie-theoretic. My approach is to use, instead, the Baker–Campbell–Hausdorff theorem. This theorem is more elementary than the Frobenius theorem and arguably gives more intuition as to why the above-mentioned results are true. I begin with the technically simpler case of the Heisenberg group (where the Baker–Campbell–Hausdorff series terminates after the first commutator term) and then proceed to the general case.

Appendix C gives two examples of Lie groups that are not matrix Lie groups. Both examples are constructed from matrix Lie groups: One is the universal cover of $\mathsf{SL}(n;\mathbb{R})$ and the other is the quotient of the Heisenberg group by a discrete central subgroup. These examples show the limitations of working with matrix Lie groups, namely that important operations such as the of taking quotients and covers do not preserves the class of matrix Lie groups. In the long run, then, the theory of matrix Lie groups is not an acceptable substitute for general Lie group theory. Nevertheless, I feel that the matrix approach is suitable for a first course in the subject not only because most of the interesting examples of Lie groups are matrix groups but also because all of the theorems I will discuss for the matrix case continue to hold for general Lie groups. In fact, most of the proofs are the same in the general case, *except* that in the general case, one needs to spend a lot more time setting up the basic notions before one can begin.

In addressing the theory of semisimple groups and Lie algebras, I use representation theory as a motivation for the structure theory. In particular, I work out in detail the representation theory of $\mathsf{SU}(2)$ (or, equivalently, $\mathsf{sl}(2;\mathbb{C})$) and $\mathsf{SU}(3)$ (or, equivalently, $\mathsf{sl}(3;\mathbb{C})$) before turning to the general semisimple case. The $\mathsf{sl}(3;\mathbb{C})$ case (more so than just the $\mathsf{sl}(2;\mathbb{C})$ case) illustrates in a concrete way the significance of the Cartan subalgebra, the roots, the weights, and the Weyl group. In the general semisimple case, I keep the representation theory at the fore, introducing at first only as much structure as needed to state the theorem of the highest weight. I then turn to a more detailed look at root systems, including two- and three-dimensional examples, Dynkin diagrams, and a discussion (without proof) of the classification. This portion of the text includes numerous images of the relevant structures (root systems, lattices of dominant integral elements, and weight diagrams) in ranks two and three.

I take full advantage, in treating the semisimple theory, of the correspondence established earlier between the representations of a simply-connected group and the representations of its Lie algebra. So, although I treat things from the point of view of complex semisimple Lie algebras, I take advantage of the characterization of such algebras as ones isomorphic to the complexification of Lie algebra of a compact simply-connected Lie group K. (Although, for the purposes of this book, we could take this as the definition of a complex semisimple Lie algebra, it is equivalent to the usual algebraic definition.) Having the compact group at our disposal simplifies several issues. First and foremost, it implies the complete reducibility of the representations. Second, it gives a simple construction of Cartan subalgebras, as the complexification of any maximal abelian subalgebra of the Lie algebra of K. Third, it gives a more transparent construction of the Weyl group, as $W = N(T)/T$, where T is a maximal torus in K. This description makes it evident, for example, why the weights of any representation are invariant under the action of W. Thus, my treatment is a mixture of the Lie algebra approach of Humphreys (1972) and the compact group approach of Bröcker and tom Dieck (1985) or Simon (1996).

This book is intended to supplement rather than replace the standard texts on Lie theory. I recommend especially four texts for further reading: the book of Warner (1983) for the manifold side of things and the relationship between Lie groups and Lie algebras, the book of Humphreys (1972) for the Lie algebra approach to representation theory, the book of Bröcker and tom Dieck (1985) for the compact-group approach to representation theory, and the book of Fulton and Harris (1991) for numerous examples of representations of the classical groups. There are, of course, many other books worth consulting; some of these are listed in the Bibliography.

I hope that by keeping the mathematical prerequisites to minimum, I have made this book accessible to students in physics as well as mathematics. Although much of the material in the book is widely used in physics, physics students are often expected to pick up the material by osmosis. I hope that they can benefit from a treatment that is elementary but systematic and mathematically precise. In Appendix A, I provide a quick introduction to the theory of groups (not necessarily Lie groups), which is not as standard a part of the physics curriculum as it is of the mathematics curriculum.

The main prerequisite for this book is a solid grounding in linear algebra, especially eigenvectors and the notion of diagonalizability. A quick review of the relevant material is provided in Appendix B. In addition to linear algebra, only elementary analysis is needed: limits, derivatives, and an occasional use of compactness and the inverse function theorem.

There are, to my knowledge, five other treatments of Lie theory from the matrix group point of view. These are (in order of publication) the book *Linear Lie Groups*, by Hans Freudenthal and H. de Vries, the book *Matrix Groups*, by Morton L. Curtis, the article "Very Basic Lie Theory," by Roger Howe, and the recent books *Matrix Groups: An Introduction to Lie Group Theory*, by Andrew Baker, and *Lie Groups: An Introduction Through Linear Groups*, by Wulf Rossmann. (All of these are listed in the Bibliography.) The book of Freudenthal and de Vries covers a lot of ground, but its unorthodox style and notation make it rather inaccessible. The works of Curtis, Howe, and Baker overlap considerably, in style and content, with the first two chapters of this book, but do not attempt to cover as much ground. For example, none of them treats representation theory or the Baker–Campbell–Hausdorff formula. The book of Rossmann has many similarities with this book, including the use of the Baker–Campbell–Hausdorff formula. However, Rossmann's book is a bit different at the technical level, in that he considers arbitrary subgroups of $\mathsf{GL}(n;\mathbb{C})$, with no restriction on the topology.

Although the organization of this book is, I believe, substantially different from that of other books on the subject, I make no claim to originality in any of the proofs. I myself learned most of the material here from books listed in the Bibliography, especially Humphreys (1972), Bröcker and tom Dieck (1985), and Miller (1972).

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I welcome comments by e-mail at bhall@nd.edu.

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