# Preface to the Encyclopaedia Subseries on Operator Algebras and Non-Commutative Geometry

The theory of von Neumann algebras was initiated in a series of papers by Murray and von Neumann in the 1930's and 1940's. A von Neumann algebra is a self-adjoint unital subalgebra $M$ of the algebra of bounded operators of a Hilbert space which is closed in the weak operator topology. According to von Neumann's bicommutant theorem, $M$ is closed in the weak operator topology if and only if it is equal to the commutant of its commutant. A *factor* is a von Neumann algebra with trivial centre and the work of Murray and von Neumann contained a reduction of all von Neumann algebras to factors and a classification of factors into types I, II and III.

$C^*$-algebras are self-adjoint operator algebras on Hilbert space which are closed in the norm topology. Their study was begun in the work of Gelfand and Naimark who showed that such algebras can be characterized abstractly as involutive Banach algebras, satisfying an algebraic relation connecting the norm and the involution. They also obtained the fundamental result that a commutative unital $C^*$-algebra is isomorphic to the algebra of complex valued continuous functions on a compact space – its spectrum.

Since then the subject of operator algebras has evolved into a huge mathematical endeavour interacting with almost every branch of mathematics and several areas of theoretical physics.

Up into the sixties much of the work on $C^*$-algebras was centered around representation theory and the study of $C^*$-algebras of type I (these algebras are characterized by the fact that they have a well behaved representation theory). Finite dimensional $C^*$-algebras are easily seen to be just direct sums of matrix algebras. However, by taking algebras which are closures in norm of finite dimensional algebras one obtains already a rich class of $C^*$-algebras – the so-called AF-algebras – which are not of type I. The idea of taking the closure of an inductive limit of finite-dimensional algebras had already appeared in the work of Murray-von Neumann who used it to construct a fundamental example of a factor of type II – the "hyperfinite" (nowadays also called approximately finite dimensional) factor.

One key to an understanding of the class of AF-algebras turned out to be $K$-theory. The techniques of $K$-theory, along with its dual, *Ext*-theory, also found immediate applications in the study of many new examples of $C^*$-algebras that arose in the end of the seventies. These examples include for instance "the noncommutative tori" or other crossed products of abelian $C^*$-algebras by groups of homeomorphisms and abstract $C^*$-algebras generated by isometries with certain relations, now known as the algebras $\mathscr{O}_n$. At the same time, examples of algebras were increasingly studied that codify data from differential geometry or from topological dynamical systems.

On the other hand, a little earlier in the seventies, the theory of von Neumann algebras underwent a vigorous growth after the discovery of a natural infinite family of pairwise nonisomorphic factors of type III and the advent of Tomita-Takesaki theory. This development culminated in Connes' great classification theorems for approximately finite dimensional ("injective") von Neumann algebras.

Perhaps the most significant area in which operator algebras have been used is mathematical physics, especially in quantum statistical mechanics and in the foundations of quantum field theory. Von Neumann explicitly mentioned quantum theory as one of his motivations for developing the theory of rings of operators and his foresight was confirmed in the algebraic quantum field theory proposed by Haag and Kastler. In this theory a von Neumann algebra is associated with each region of space-time, obeying certain axioms. The inductive limit of these von Neumann algebras is a $C^*$-algebra which contains a lot of information on the quantum field theory in question. This point of view was particularly successful in the analysis of superselection sectors.

In 1980 the subject of operator algebras was entirely covered in a single big three weeks meeting in Kingston Ontario. This meeting served as a review of the classification theorems for von Neumann algebras and the success of $K$-theory as a tool in $C^*$-algebras. But the meeting also contained a preview of what was to be an explosive growth in the field. The study of the von Neumann algebra of a foliation was being developed in the far more precise $C^*$-framework which would lead to index theorems for foliations incorporating techniques and ideas from many branches of mathematics hitherto unconnected with operator algebras.

Many of the new developments began in the decade following the Kingston meeting. On the $C^*$-side was Kasparov's $KK$-theory – the bivariant form of $K$-theory for which operator algebraic methods are absolutely essential. Cyclic cohomology was discovered through an analysis of the fine structure of extensions of $C^*$-algebras These ideas and many others were integrated into Connes' vast *Noncommutative Geometry* program. In cyclic theory and in connection with many other aspects of noncommutative geometry, the need for going beyond the class of $C^*$-algebras became apparent. Thanks to recent progress, both on the cyclic homology side as well as on the $K$-theory side, there is now a well developed bivariant $K$-theory and cyclic theory for a natural class of topological algebras as well as a bivariant character taking $K$-theory to cyclic theory. The 1990's also saw huge progress in the classification theory of nuclear $C^*$-algebras in terms of $K$-theoretic invariants, based on new insight into the structure of exact $C^*$-algebras.

On the von Neumann algebra side, the study of subfactors began in 1982 with the definition of the *index* of a subfactor in terms of the Murray-von Neumann theory and a result showing that the index was surprisingly restricted in its possible values. A rich theory was developed refining and clarifying the index. Surprising connections with knot theory, statistical mechanics and quantum field theory have been found. The superselection theory mentioned above turned out to have fascinating links to subfactor theory. The subfactors themselves were constructed in the representation theory of loop groups.

Beginning in the early 1980's Voiculescu initiated the theory of free probability and showed how to understand the free group von Neumann algebras in terms of random matrices, leading to the extraordinary result that the von Neumann algebra $M$ of the free group on infinitelymany generators has full fundamental group, i.e. $pMp$ is isomorphic to $M$ for every non-zero projection $p \in M$. The subsequent introduction of free entropy led to the solution of more old problems in von Neumann algebras such as the lack of a Cartan subalgebra in the free group von Neumann algebras.

Many of the topics mentioned in the (obviously incomplete) list above have become large industries in their own right. So it is clear that a conference like the one in Kingston is no longer possible. Nevertheless the subject does retain a certain unity and sense of identity so we felt it appropriate and useful to create a series of encylopaedia volumes documenting the fundamentals of the theory and defining the current state of the subject.

In particular, our series will include volumes treating the essential technical results of $C^*$-algebra theory and von Neumann algebra theory including sections on noncommutative dynamical systems, entropy and derivations. It will include an account of $K$-theory and bivariant $K$-theory with applications and in particular the index theorem for foliations. Another volume will be devoted to cyclic homology and bivariant $K$-theory for topological algebras with applications to index theorems. On the von Neumann algebra side, we plan volumes on the structure of subfactors and on free probability and free entropy. Another volume shall be dedicated to the connections between operator algebras and quantum field theory.

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# Preface

Cyclic homology was introduced in the early eighties independently by Connes and Tsygan. They came from different directions. Connes wanted to associate homological invariants to $K$-homology classes and to describe the index pairing with $K$-theory in that way, while Tsygan was motivated by algebraic $K$-theory and Lie algebra cohomology. At the same time Karoubi had done work on characteristic classes that led him to study related structures, without however arriving at cyclic homology properly speaking.

Many of the principal properties of cyclic homology were already developed in the fundamental article of Connes and in the long paper by Feigin–Tsygan. In the sequel, cyclic homology was recognized quickly by many specialists as a new intriguing structure in homological algebra, with unusual features. In a first phase it was tried to treat this structure as well as possible within the traditional framework of homological algebra. The cyclic homology groups were computed in many examples and new important properties such as product structures, excision for H-unital ideals, or connections with cyclic objects and simplicial topology, were established. An excellent account of the state of the theory after that phase is given in the book of Loday.

This book is an attempt at an account of the present state of cyclic theory that covers in particular a number of more recent results that have changed the face of the theory to a certain extent. An essential feature of cyclic homology which is not so well captured by the ordinary graded cyclic homology groups $\mathrm{HC}_n$ is the $S$-operator, which reflects in fact a filtration – rather than a grading – on cyclic homology, along with the associated periodic cyclic theory introduced by Connes. The most characteristic properties of cyclic homology hold only after stabilization by the $S$-operator. Important steps beyond the classical graded framework were taken by Connes when he introduced the so-called entire theory, and by Goodwillie in his result on nilpotent extensions. The spirit here is to work not with the individual cyclic homology groups in finite dimension, but with the infinite cyclic bicomplex. This global approach also turned out to be the most appropriate for various index theorems. It was put on a new basis by Cuntz–Quillen who used free (or more generally quasifree) resolutions of an algebra and a non-commutative de Rham complex, to describe all the cyclic-type invariants associated with that algebra. This also led to proofs of excision for general extensions in periodic, entire or other cyclic theories. Excision in its bivariant form in turn immediately leads to a natural multiplicative transformation – the bivariant Chern–Connes character – from bivariant $K$-theory, provided that this can be defined, to bivariant cyclic homology. This general transformation contains most previously constructed characters from $K$-theoretic invariants to cyclic theory invariants. The formalism using the non-commutative de Rham complex and free resolutions of an algebra is also the natural basis for cyclic homology theories generalizing the entire theory that can be applied to a wide class of topological algebras, including Banach- and $C^*$-algebras.

Other new techniques in cyclic homology theory were developed by Nest and Tsygan who systematically treat cyclic chains as noncommutative differential forms and Hochschild cochains as noncommutative multivector fields. In this way they recover the standard algebra arising in calculus to a surprisingly large extent. The resulting algebraic structures on the Hochschild and cyclic complexes are applied to prove far-reaching generalizations of the Atiyah–Singer index theorem culminating in the theorem of Bressler–Nest–Tsygan. The algebraic structures were further enriched in the work of Tamarkin and Tamarkin–Tsygan and used to reprove and extend Kontsevich's results on classification of deformation quantizations.

On the other hand, from the beginning, Connes developed cyclic theory into a powerful tool in non-commutative geometry. Striking applications of cyclic homology to geometry appeared already in his work on the transverse fundamental class for actions of discrete groups on a manifold and for transversally oriented foliations, including an interpretation of the Godbillon–Vey class in cyclic cohomology. Another such application is the proof of the Novikov conjecture for hyperbolic groups by Connes–Moscovici using cyclic cohomology and a 'higher' index theorem. More recently, Connes–Moscovici proved two other important index theorems which required new techniques in cyclic homology. The first one is a general abstract local index formula for spectral triples, which works for general algebras sharing some of the characteristic features of the algebra of pseudodifferential operators, and applies to many natural operators in non-commutative geometry. Locality is reflected by the fact that one uses only the Dixmier–Wodzicki trace for the description of the cyclic cocycles determining the index. The application of their local index formula to tranversally hypoelliptic operators on foliations led Connes and Moscovici to the introduction of a new type of cyclic homology for Hopf algebras. The total index in transverse geometry is expressed in terms of a characteristic map from the cyclic homology of a Hopf algebra playing the role of a symmetry group in this situation. The contribution by G. Skandalis in this volume is devoted to an outline of these last two theorems. A much more complete account is planned for another volume in this series.

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