

# PREFACE

*Linear Programming 2* continues where *Linear Programming 1* left off. We assume that the reader has an introductory knowledge of linear programming, for example has read *Linear Programming 1: Introduction* (or its equivalent) and has knowledge of linear algebra (reviewed in the appendices in *Linear Programming 1*). In this volume, we prove all theorems stated and those that were sketched but not proved in *Linear Programming 1*, and we describe various extensions.

*Linear Programming 2* is intended to be an advanced graduate text as well as a reference. Portions of *Linear Programming 1* and *Linear Programming 2* have been used in a graduate-level course that we have taught together. The rest of the discussion here summarizes the contents of this volume.

## OUTLINE OF CHAPTERS

**Chapter 1 (Geometry):** In this chapter we study the geometry and properties of linear inequality systems and how they are related to the Simplex Method, which can be described as a movement along the edges of a convex polyhedral set to obtain a global minimum of the objective function, generate a class of feasible solutions for which the objective  $z \rightarrow -\infty$ , or determine that the convex polyhedral set is infeasible. The important separating hyperplane concepts are also discussed and proved.

**Chapter 2 (Duality and Theorems of the Alternatives):** We provide proofs for the *Weak* and *Strong* Duality Theorems. This is followed by additional theorems on duality; that is, the *Unboundedness Theorem* and the *Primal/Dual Optimality Criteria*. The chapter also discusses complementary slackness and various Theorems of the Alternatives: Gordan's Theorem, Farkas's Lemma, Stiemke's Theorem, Motzkin's Transposition Theorem, Ville's Theorem, and Tucker's Strict Complementary Slackness Theorem.

**Chapter 3 (Early Interior-Point Methods):** In this chapter we trace the early development of interior-point methods. The earliest known method is that attributable to von Neumann [1948], followed by Frisch [1957] (only referenced here), and Dikin [1967]. A theoretical breakthrough was due to Khachian [1979] who developed a polynomial-time ellipsoid algorithm (only referenced

here). This was followed by Karmarkar's [1984] polynomial-time interior-point algorithm.

**Chapter 4 (Interior-Point Methods):** Since the development of Karmarkar's [1984] algorithm several new important practical interior-point algorithms emerged. Among these are the primal logarithmic barrier method, primal-affine algorithm, dual logarithmic barrier method, dual-affine algorithm, and the primal-dual algorithm. All these algorithms are described. The optimal solution obtained by an interior-point method is not necessarily at a vertex; we describe a technique to make it into a vertex.

**Chapter 5 (Degeneracy):** When degeneracy occurs, it is possible for the Simplex Algorithm to have an infinite sequence of iterations with no decrease in the value of  $z$ . The chapter illustrates this with examples due to Hoffman, Beale, and Kuhn. Then various methods for resolving degeneracy are presented: Dantzig's Inductive Methods, Wolfe's Rule, Bland's Rule, and Krishna's Extra Column Rule. This is followed by a technique that attempts to avoid degenerate pivot by making use of an extra objective function and resultant reduced cost calculation.

**Chapter 6 (Variants of the Simplex Method):** Over the years several variants of the Simplex Algorithm have been proposed as a way to reduce the number of iterations. We start by describing an efficient way of determining an incoming column that yields the maximum improvement per iteration. Next we describe the Dual-Simplex Method, Parametric Linear Programming, Self-Dual Parametric Algorithm, Primal-Dual Algorithm, and a Phase I Least-Squares Algorithm.

**Chapter 7 (Transportation Problem and Variations):** The Classical Transportation Problem is stated, and various theorems are proved about it. An example is provided for cycling under degeneracy when the most negative reduced cost is used to select an incoming column. This is followed by a discussion of the Transshipment Problem and transportation problems with bounded partial sums.

**Chapter 8 (Network Flow Theory):** Theorems are proved about the Maximal-Flow problem and the Shortest-Route problem.

**Chapter 9 (Generalized Upper Bounds):** In this chapter we discuss a variation of the Simplex Algorithm to efficiently solve linear programs that have upper bounds on subsets of variables such that each variable appears in at most one subset. Such constraints are called *generalized upper bounds*.

**Chapter 10 (Decomposition):** Decomposition is a term to describe breaking a problem into smaller parts and then using a variant of the Simplex Algorithm to solve the entire problem efficiently. The chapter starts by describing Wolfe's Generalized Linear Program (or a linear program with variable coefficients). The Dantzig-Wolfe Decomposition Principle is described for solving

this class of problems. This is followed by a description of Benders Decomposition which is the Dantzig-Wolfe Decomposition applied to the dual. Benders Decomposition is used to solve Stochastic Programs. Next we describe the application of Dantzig-Wolfe Decomposition to solving of Block-Angular systems. Then staircase structured problems are described; we show how to solve such problems using Dantzig-Wolfe Decomposition and Benders Decomposition. Finally, the possible use of decomposition to solve central planning problems is described.

**Chapter 11 (Stochastic Programming Introduction):** Here we introduce the concept of planning under uncertainty. Simple problems with uncertain demand and uncertain costs respectively are illustrated. This is followed by a discussion of the convexity property of multi-stage problems.

**Chapter 12 (Two-Stage Stochastic Programs):** An important class of optimization problems arise in dynamic systems that describe activities initiated at time  $t$  that have coefficients at time  $t$  and time  $t + 1$ . Such problems, called *dynamic linear programs*, typically have a nonzero submatrix with a staircase structure. The simplest dynamic linear program has only two stages; this is discussed in this chapter.

**Appendix A (Probability Theory Overview):** In this appendix we introduce some basic concepts and notation of probability theory for use in solving stochastic linear programs.

## LINEAR PROGRAMMING 1.

In a graduate course that we have taught together at Stanford, portions of *Linear Programming 1: Introduction* and *Linear Programming 2: Theory & Extensions* have been used.

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