Preface

This book is intended as a course in numerical analysis and approximation theory for advanced undergraduate students or graduate students, and as a reference work for those who lecture or research in this area. Its title pays homage to *Interpolation and Approximation* by Philip J. Davis, published in 1963 by Blaisdell and reprinted by Dover in 1976. My book is less general than Philip Davis's much respected classic, as the qualification "by polynomials" in its title suggests, and it is pitched at a less advanced level.

I believe that no one book can fully cover all the material that *could* appear in a book entitled *Interpolation and Approximation by Polynomials*. Nevertheless, I have tried to cover most of the main topics. I hope that my readers will share my enthusiasm for this exciting and fascinating area of mathematics, and that, by working through this book, some will be encouraged to read more widely and pursue research in the subject. Since my book is concerned with polynomials, it is written in the language of classical analysis and the only prerequisites are introductory courses in analysis and linear algebra.

In deciding whether to include a topic in any book or course of lectures, I always ask myself, Is the proposed item mathematically *interesting*? Paradoxically, utility is a useless guide. For instance, why should we discuss interpolation nowadays? Who uses it? Indeed, how many make direct use of numerical integration, orthogonal polynomials, Bernstein polynomials, or techniques for computing various best approximations? Perhaps the most serious *users* of mathematics are the relatively small number who construct, and the rather larger number who apply, specialist mathematical packages, including those for evaluating standard functions, solving systems of linear equations, carrying out integrations, solving differential equations, drawing surfaces with the aid of CAGD (computer-aided geometrical design) techniques, and so on. However, it is all too easy to make use of such packages without understanding the mathematics on which they are based, or their limitations, and so obtain poor, or even meaningless, results. Many years ago someone asked my advice on a mathematical calculation that he was finding difficult. I gently pointed out that his result was invalid because the series he was summing was divergent. He responded honestly that, faced with an infinite series, his strategy was always to compute the sum of the first hundred terms.

There are many connections between the various chapters and sections in this book. I have sought to emphasize these interconnections to encourage a deeper understanding of the subject. The first topic is interpolation, from its precalculus origins to the insights and advances made in the twentieth century with the study of Lebesgue constants. Unusually, this account of interpolation also pursues the direct construction of the interpolating polynomial by solving the system of linear equations involving the Vandermonde matrix. How could we dream of despising a study of interpolation, when it is so much at the centre of the development of the calculus? Our understanding of the interpolating polynomial leads us naturally to a study of integration rules, and an understanding of Gaussian integration rules requires knowledge of orthogonal polynomials, which are at the very heart of classical approximation theory. The chapter on numerical integration also includes an account of the Euler–Maclaurin formula, in which we can use a series to estimate an integral or vice versa, and the justification of this powerful formula involves some particularly interesting mathematics, with a forward reference to splines. The chapter devoted to orthogonal polynomials is concerned with best approximation, and concentrates on the Legendre polynomials and least squares approximations, and on the Chebyshev polynomials, whose minimax property leads us on to minimax approximations.

One chapter is devoted to Peano kernel theory, which was developed in the late nineteenth century and provides a special calculus for creating and justifying error terms for various approximations, including those generated by integration rules. This account of Peano kernel theory is rather more extensive than that usually given in a textbook, and I include a derivation of the error term for the Euler–Maclaurin formula. The following chapter extends the topic of interpolation to several variables, with most attention devoted to interpolation in two variables. It contains a most elegant generalization of Newton's divided difference formula plus error term to a triangular set of points, and discusses interpolation formulas for various sets of points in a triangle. The latter topic contains material that was first published in the late twentieth century and is justified by geometry dating from the fourth century AD, towards the very end of the golden millennium of Greek mathematics, and by methods belonging to projective geometry, using homogeneous coordinates. Mathematics certainly does not have to be new to be relevant. This chapter contains much material that has not appeared before in a textbook at any level.

There is a chapter on polynomial splines, where we split the interval on which we wish to approximate a function into subintervals. The approximating function consists of a sequence of polynomials, one on each subinterval, that connect together *smoothly*. The simplest and least smooth example of this is a polygonal arc. Although we can detect some ideas in earlier times that remind us of splines, this is a topic that truly belongs to the twentieth century. It is a good example of exciting, relatively new mathematics that worthily stands alongside the best mathematics of any age. Bernstein polynomials, the subject of the penultimate chapter, date from the early twentieth century. Their creation was inspired by the famous theorem stated by Weierstrass towards the end of the nineteenth century that a continuous function on a finite interval of the real line can be approximated by a polynomial with any given precision over the whole interval. Polynomials are simple mathematical objects that are easy to evaluate, differentiate, and integrate, and Weierstrass's theorem justifies their importance in approximation theory.

Several of the processes discussed in this book have special cases where a function is evaluated at equal intervals, and we can scale the variable so that the function is evaluated at the integers. For example, in finite difference methods for interpolation the interpolated function is evaluated at equal intervals, and the same is true of the integrand in the Newton–Cotes integration rules. Equal intervals occur also in the Bernstein polynomials and the uniform B-splines. In our study of these four topics, we also discuss processes in which the function is evaluated at intervals whose lengths are in geometric progression. These can be scaled so that the function is evaluated on the q-integers. Over twenty years ago I was asked to referee a paper by the distinguished mathematician I. J. Schoenberg (1903–1990), who is best known for his pioneering work on splines. Subsequently I had a letter from the editor of the journal saying that Professor Schoenberg wished to know the name of the anonymous referee. Over the following few years I had a correspondence with Professor Schoenberg which I still value very much. His wonderful enthusiasm for mathematics continued into his eighties. Indeed, of his 174 published papers and books, 56 appeared after his retirement in 1973. The above-mentioned paper by Iso Schoenberg was the chief influence on the work done by S. L. Lee and me in applying q-integers to interpolation on triangular regions. The q-integer motif was continued in joint work with Zeynep Koçak on splines and then in my work on the Bernstein polynomials, in which I was joined by Tim Goodman and Halil Oruç. The latter work nicely illustrates variation-diminishing ideas, which take us into the relatively new area of CAGD.

The inclusion of a few rather minor topics in whose development I have been directly involved may cause some eyebrows to be raised. But I trust

I will be forgiven, since my intentions are honourable: I hope that by including a relatively small amount of material on topics in which I have carried out research I can encourage all students of mathematics, and especially those thinking of doing research, to know and understand that the discovery of new ideas in mathematics is not the sole preserve of the most outstanding mathematicians.

I was born in Aberdeen, Scotland. Soon after I began lecturing at the University of St Andrews I learned that the famous Scottish mathematician James Gregory (1638–1675) had also lectured there. Gregory was born in Drumoak, near Aberdeen, and was educated at Aberdeen Grammar School, as I was. Moreover, Gregory's year of birth was exactly three hundred years before mine. There the similarities end, most fortunately for me, in view of the woeful brevity of Gregory's life. James Gregory obtained many important results in the early development of the calculus, including his discovery of the series for the inverse tangent. Indeed, H. W. Turnbull [54], who carried out a most rigorous study of Gregory's publications and unpublished manuscripts and letters, argues that Gregory's mastery of what we call *Maclaurin* series and *Taylor* series, after Colin Maclaurin (1698–1746) and Brook Taylor (1685–1731), entitles him to much more recognition than he has received. Much ahead of his time, Gregory made a bold attempt at showing the transcendental nature of the numbers π and e, results that were completed only at the end of the nineteenth century by C. L .F. Lindemann (1852–1939).

On completing this book it is a pleasure to renew my thanks to my old friend Peter Taylor, from whom I have learned much about numerical analysis. Our meeting as colleagues at the University of Southampton in 1963 was the start of a most valued friendship combined with a fruitful collaboration in mathematics. Our book *Theory and Applications of Numerical Analysis* was first published in 1973 and is still in print as I write this. It has been translated into Chinese and Farsi (Persian), and a second edition was published in 1996. Now, in the fortieth year of our friendship, I am very grateful to Peter Taylor for his most helpful comments on the manuscript of this book. At an earlier stage in the preparation of the manuscript, I particularly remember discussions with Peter concerning the divided difference form for interpolation on a triangular region. This is one of the few significant mathematical conversations I can recall sharing that did not involve writing anything down. We were sitting on a park bench in the Meadows in Edinburgh before attending a concert by the Edinburgh Quartet in the Queen's Hall.

It is also a pleasure to thank my much respected colleague József Szabados, who read the first draft of the manuscript of this book on behalf of my publisher. I am extremely grateful to him for the great care he took in pursuing this task, and for the wisdom of his remarks and suggestions. As a result, I believe I have been able to make some substantial improvements in the text. However, I am solely responsible for the final form of this book.

A few years ago I invited my good friend and former student Halil Oruç to join me in writing a book on approximation theory. We were both very disappointed that he was unable to do this, due to pressure of other work. Different parents produce different children. Indeed, children of the same parents are usually rather different. Therefore, although Halil and I would surely have produced a rather different book together, I hope that he will approve of this one.

This book is my second contribution to the series *CMS Books in Mathematics*, and it is a pleasure to thank the editors, Jonathan and Peter Borwein, for their support and encouragement. I also wish to acknowledge the fine work of those members of the staff of Springer, New York who have been involved with the production of this book. I am especially grateful to the copyeditor, David Kramer. I worked through his suggestions, page by page, with an ever increasing respect for the great care he devoted to his task. I also wish to thank my friend David Griffiths for his help with one of the items in the Bibliography. I must also mention David's book *Learning LaTeX*, written jointly with Desmond J. Higham and published by SIAM. It has been my guide as I prepared this text in LaTeX.

In *Two Millennia of Mathematics*, my first contribution to the CMS series, I expressed my thanks to my early teachers and lecturers and to the many mathematicians, from many countries, who have influenced me and helped me. I will not repeat that lengthy list here, having put it on record so recently. However, having mentioned I. J. Schoenberg, let me write down also, in the order in which I met them, the names of three other approximation theorists, Philip Davis, Ward Cheney, and Ted Rivlin. Their most elegantly written and authoritative books on approximation theory inspired me and taught me a great deal. I would also like to mention the name of my good friend Lev Brutman (1939–2001), whose work I quote in the section on Lebesgue constants. I was his guest in Israel, and he was mine in Scotland, and we corresponded regularly for several years. I admired very much his fine mathematical achievements and his language skills in Russian, Hebrew, and English. He loved to read poetry in these three languages. Lev's favourite poet was Alexander Pushkin (1799–1837), and he also admired Robert Burns (1759–1796), whose work he began reading in Russian translation during his early years in Moscow.

I dedicated my Ph.D. thesis to my dear parents, Betty McArtney Phillips (1910–1961) and George Phillips (1911–1961). With the same measure of seriousness, love, and gratitude, I dedicate this book to my wife, Rona.

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