Preface

These lectures intend to give a self-contained exposure of some techniques for computing the evolution of plane curves. The motions of interest are the so-called motions by curvature. They mean that, at any instant, each point of the curve moves with a normal velocity equal to a function of the curvature at this point. This kind of evolution is of some interest in differential geometry, for instance in the problem of minimal surfaces. The interest is not only theoretical since the motions by curvature appear in the modeling of various phenomena as crystal growth, flame propagation and interfaces between phases. More recently, these equations have also appeared in the young field of image processing where they provide an efficient way to smooth curves representing the contours of the objects. This smoothing is a necessary step for image analysis as soon as the analysis uses some local characteristics of the contours. Indeed, natural images are very noisy and differential features are unreliable if one is not careful before computing them. A solution consists in smoothing the curves to eliminate the small oscillations without changing the global shape of the contours. What kind of smoothing is suitable for such a task? The answer shall be given by an axiomatic approach whose conclusions are that the class of admissible motions is reduced to the motions by curvature. Once this is established, the well-posedness of these equations has to be examined. For certain particular motions, this turns to be true but no complete results are available for the general existence of these motions. This problem shall be turned around by introducing a weak notion of solution using the theory of viscosity solutions of partial differential equations (PDE). A complete theory of existence and uniqueness of those equations will be presented, as self-contained as possible. (Only a technical, though important, lemma will be skipped.) The numerical resolution of the motions by curvature is the next topic of interest. After a rapid review of the most commonly used algorithms, a completely different numerical scheme is presented. Its originality is that it satisfies exactly the same invariance properties as the equations of motion by curvature. It is also inconditionally stable and its convergence can be proved in the sense of viscosity solutions. Moreover, it allows to precisely compute motions by curvature, when the normal velocity is a power of the curvature more than 3, or even 10 in some cases, which seems a priori nearly impossible in a numerical point of view. Many numerical experiments are presented.

Who this volume is addressed to?

We hope that these notes shall interest people from both communities of applied mathematics and image processing. We tried to make them as self-contained as possible. Nevertheless, we skipped the most difficult results since their proof uses techniques that would have led us too far from our main way. Indeed, these lectures are addressed to researchers discovering the common field of mathematics and image processing but also to graduate and PhD students wanting to span a theory from A to Z: from the basic axioms, to mathematical results and numerical applications. The chapters are mostly independent except Chap. 6 that uses results from Chap. 4. The bibliography on every subject we tackle is huge, and we cannot pretend to give exhaustive references on differential geometry, viscosity solutions, mathematical morphology or scale space theory. At the end of most chapters, we give short bibliographical notes detailing in a few words the main steps that produced significant advances in the theory.

Acknowledgements

My first thanks go to Jean-Michel Morel who was my advisor during my thesis and also proposed me to write these notes. It has been a constant pleasure to work in his team in which collaborations are as numerous as fruitful. I would like to thank him for his scientific support but also for all the other qualities he develops in his relations.
I would also like to express my gratitude to Yves Meyer, with whom any discussion is always an opportunity to learn something and to realize we have still so much to learn. I have also been always impressed by his enthusiasm, and thank him sincerely for his encouragements.
I owe also much to Lionel Moisan, who contributed to the redaction of Chap. 6 and whose help, specially for the numerical part, was very precious.
Many thanks also to Jean-Pierre Le Cadre for his careful reading of this text.
I would also like to thank Andrés Almansa, Vicent Caselles, Francine Catté, Agnès Desolneux, Françoise Dibos, Jacques Froment, Yann Gousseau, Frédéric Guichard, Georges Koepfler, François Malgouyres, Simon Masnou, Pascal Monasse, Pablo Musé, Denis Pasquignon, Frédéric Sur for all the discussions we had for the last few years.

Rennes, October 2002 *Frédéric Cao*